ESTIMATION AND VARIABLE SELECTION FOR GENERALIZED ADDITIVE PARTIAL LINEAR MODELS

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We study generalized additive partial linear models, proposing the use of polynomial spline smoothing for estimation of nonparametric functions, and deriving quasi-likelihood based estimators for the linear parameters. We establish asymptotic normality for the estimators of the parametric components. The procedure avoids solving large systems of equations as in kernel-based procedures and thus results in gains in computational simplicity. We further develop a class of variable selection procedures for the linear parameters by employing a nonconcave penalized quasi-likelihood, which is shown to have an asymptotic oracle property. Monte Carlo simulations and an empirical example are presented for illustration.

1. Introduction. Generalized linear models (GLM), introduced by Nelder and Wedderburn (1972) and systematically summarized by McCullagh and Nelder (1989), are a powerful tool to analyze the relationship between a discrete response variable and covariates. Given a link function, the GLM expresses the relationship between the dependent and independent variables through a linear functional form. However, the GLM and associated methods may not be flexible enough when analyzing complicated data generated from biological and biomedical research. The generalized additive model (GAM), a generalization of the GLM that replaces linear components by a sum of

smooth unknown functions of predictor variables, has been proposed as an alternative and has been used widely [Hastie and Tibshirani (1990), Wood (2006)]. The generalized additive partially linear model (GAPLM) is a realistic, parsimonious candidate when one believes that the relationship between the dependent variable and some of the covariates has a parametric form, while the relationship between the dependent variable and the remaining covariates may not be linear. GAPLM enjoys the simplicity of the GLM and the flexibility of the GAM because it combines both parametric and nonparametric components.

There are two possible approaches for estimating the parametric component and the nonparametric components in a GAPLM. The first is a combination of kernel-based backfitting and local scoring, proposed by Buja, Hastie and Tibshirani (1989) and detailed by Hastie and Tibshirani (1990). This method may need to solve a large system of equations [Yu, Park and Mammen (2008)], and also makes it difficult to introduce a penalized function for variable selection as given in Section 4. The second is an application of the marginal integration approach [Linton and Nielsen (1995)] to the nonparametric component of the generalized partial linear models. They treated the summand of additive terms as a nonparametric component, which is then estimated as a multivariate nonparametric function. This strategy may still suffer from the "curse of dimensionality" when the number of additive terms is not small [Härdle et al. (2004)].

The kernel-based backfitting and marginal integration approaches are computationally expensive. Marx and Eilers (1998), Ruppert, Wand and Carroll (2003) and Wood (2004) studied penalized regression splines, which share most of the practical benefits of smoothing spline methods, combined with ease of use and reduction of the computational cost of backfitting GAMs. Widely used R/Splus packages gam and mgcv provide a convenient implementation in practice. However, no theoretical justifications are available for these procedures in the additive case. See Li and Ruppert (2008) for recent work in the one-dimensional case.

In this paper, we will use polynomial splines to estimate the nonparametric components. Besides asymptotic theory, we develop a flexible and convenient estimation procedure for GAPLM. The use of polynomial spline smoothing in generalized nonparametric models goes back to Stone (1986), who first obtained the rate of convergence of the polynomial spline estimates for the generalized additive model. Stone (1994) and Huang (1998) investigated polynomial spline estimation for the generalized functional ANOVA model. More recently, Xue and Yang (2006) studied estimation of the additive coefficient model for a continuous response variable using polynomial spline methods. Our models emphasize possibly non-Gaussian responses, and combine both parametric and nonparametric components through a link function. Estimation is achieved through maximizing the quasi-likelihood

with polynomial spline smoothing for the nonparametric functions. The convergence results of the maximum likelihood estimates for the nonparametric parts in this article are similar to those for regression established by Xue and Yang (2006). However, it is very challenging to establish asymptotic normality in our general context, since it cannot be viewed simply as an orthogonal projection, due to its nonlinear structure. To the best of our knowledge, this is the *first attempt* to establish asymptotic normality of the estimators for the parametric components in GAPLM. Moreover, polynomial spline smoothing is a global smoothing method, which approximates the unknown functions via polynomial splines characterized by a linear combination of spline basis. After the spline basis is chosen, the coefficients can be estimated by an efficient one-step procedure of maximizing the quasi-likelihood function. In contrast, kernel-based methods, such as those reviewed above, in which the maximization must be conducted repeatedly at every data point or a grid of values, are more time-consuming. Thus, the application of polynomial spline smoothing in the current context is particularly computationally efficient compared to some of its counterparts.

In practice, a large number of variables may be collected and some of the insignificant ones should be excluded before forming a final model. It is an important issue to select significant variables for both parametric and nonparametric regression models; see Fan and Li (2006) for a comprehensive overview of variable selection. Traditional variable selection procedures such as stepwise deletion and subset selection may be extended to the GAPLM. However, these are also computationally expensive because, for each submodel, we encounter the challenges mentioned above.

To select significant variables in semiparametric models, Li and Liang (2008) adopted Fan and Li's (2001) variable selection procedures for parametric models via nonconcave penalized quasi-likelihood, but their models do not cover the GAPLM. Of course, before developing justifiable variable selection for the GAPLM, it is important to establish asymptotic properties for the parametric components. In this article, we propose a class of variable selection procedures for the parametric component of the GAPLM and study the asymptotic properties of the resulting estimator. We demonstrate how the rate of convergence of the resulting estimate depends on the regularization parameters, and further show that the penalized quasi-likelihood estimators perform asymptotically as an oracle procedure for selecting the model.

The rest of the article is organized as follows. In Section 2, we introduce the GAPLM model. In Section 3, we propose polynomial spline estimators via a quasi-likelihood approach, and study the asymptotic properties of the proposed estimators. In Section 4, we describe the variable selection procedures for the parametric component, and then prove their statistical properties. Simulation studies and an empirical example are presented in Section 5. Regularity conditions and the proofs of the main results are presented in the Appendix.

2. The models. Let Y be the response variable, $\mathbf{X} = (X_1,\ldots,X_{d_1})^{\mathrm{T}} \in R^{d_1}$ and $\mathbf{Z} = (Z_1,\ldots,Z_{d_2})^{\mathrm{T}} \in R^{d_2}$ be the covariates. We assume the conditional density of Y given $(\mathbf{X},\mathbf{Z}) = (\mathbf{x},\mathbf{z})$ belongs to the exponential family

$$(1) \qquad f_{Y|\mathbf{X},\mathbf{Z}}(y|\mathbf{x},\mathbf{z}) = \exp[y\xi(\mathbf{x},\mathbf{z}) - \mathcal{B}\{\xi(\mathbf{x},\mathbf{z})\} + \mathcal{C}(y)]$$

for known functions $\mathcal{B}$ and $\mathcal{C}$, where ξ is the so-called natural parameter in parametric generalized linear models (GLM), is related to the unknown mean response by

$$\mu(\mathbf{x},\mathbf{z}) = E(Y|\mathbf{X} = \mathbf{x},\mathbf{Z} = \mathbf{z}) = \mathcal{B}'\{\xi(\mathbf{x},\mathbf{z})\}.$$

In parametric GLM, the mean function μ is defined via a known link function g by $g\{\mu(\mathbf{x},\mathbf{z})\} = \mathbf{x}^{\mathrm{T}}\boldsymbol{\alpha} + \mathbf{z}^{\mathrm{T}}\boldsymbol{\beta}$, where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are parametric vectors to be estimated. In this article, $g(\mu)$ is modeled as an additive partial linear function

$$(2) \qquad g\{\mu(\mathbf{x},\mathbf{z})\} = \sum_{k=1}^{d_1} \eta_k(x_k) + \mathbf{z}^{\mathrm{T}}\boldsymbol{\beta},$$

where $\boldsymbol{\beta}$ is a d_2-dimensional regression parameter, $\{\eta_k\}_{k=1}^{d_1}$ are unknown and smooth functions and $E\{\eta_k(X_k)\} = 0$ for $1 \le k \le d_1$ for identifiability.

If the conditional variance function $\mathrm{var}(Y|\mathbf{X} = \mathbf{x},\mathbf{Z} = \mathbf{z}) = \sigma^2 V\{\mu(\mathbf{x},\mathbf{z})\}$ for some known positive function V, then estimation of the mean can be achieved by replacing the conditional loglikelihood function $\log\{f_{Y|\mathbf{X},\mathbf{Z}}(y|\mathbf{x},\mathbf{z})\}$ in (1) by a quasi-likelihood function $Q(m,y)$, which satisfies

$$\frac{\partial}{\partial m}Q(m,y) = \frac{y-m}{V(m)}.$$

The first goal of this article is to provide a simple method of estimating $\boldsymbol{\beta}$ and $\{\eta_k\}_{k=1}^{d_1}$ in model (2) based on a quasi-likelihood procedure [Severini and Staniswalis (1994)] with polynomial splines. The second goal is to discuss how to select significant parametric variables in this semiparametric framework.

3. Estimation method.

3.1. *Maximum quasi-likelihood.* Let $(Y_i,\mathbf{X}_i,\mathbf{Z}_i)$, $i = 1,\ldots,n$, be independent copies of $(Y,\mathbf{X},\mathbf{Z})$. To avoid confusion, let $\eta_0 = \sum_{k=1}^{d_1} \eta_{0k}(x_k)$ and $\boldsymbol{\beta}_0$ be the true additive function and the true parameter values, respectively. For simplicity, we assume that the covariate X_k is distributed on a compact interval $[a_k,b_k]$, $k = 1,\ldots,d_1$, and without loss of generality, we take all intervals $[a_k,b_k] = [0,1]$, $k = 1,\ldots,d_1$. Under smoothness assumptions, the η_{0k}'s can be well approximated by spline functions. Let $\mathcal{S}_n$ be the space

of polynomial splines on $[0,1]$ of order $r \geq 1$. We introduce a knot sequence with J interior knots

$$\xi_{-r+1} = \cdots = \xi_{-1} = \xi_0 = 0 < \xi_1 < \cdots < \xi_J < 1 = \xi_{J+1} = \cdots = \xi_{J+r},$$

where $J \equiv J_n$ increases when sample size n increases, where the precise order is given in condition (C5) in Section 3.2. According to Stone (1985), $\mathcal{S}_n$ consists of functions $\hbar$ satisfying:

(i) $\hbar$ is a polynomial of degree $r - 1$ on each of the subintervals $I_j = [\xi_j, \xi_{j+1})$, $j = 0, \ldots, J_n - 1$, $I_{J_n} = [\xi_{J_n}, 1]$;
(ii) for $r \geq 2$, $\hbar$ is $r - 2$ times continuously differentiable on $[0,1]$.

Equally-spaced knots are used in this article for simplicity of proof. However other regular knot sequences can also be used, with similar asymptotic results.

We will consider additive spline estimates $\widehat{\eta}$ of η_0. Let $\mathcal{G}_n$ be the collection of functions η with the additive form $\eta(\mathbf{x}) = \sum_{k=1}^{d_1} \eta_k(x_k)$, where each component function $\eta_k \in \mathcal{S}_n$ and $\sum_{i=1}^{n} \eta_k(X_{ik}) = 0$. We seek a function $\eta \in \mathcal{G}_n$ and a value of $\boldsymbol{\beta}$ that maximize the quasi-likelihood function

$$(3) \qquad L(\eta, \boldsymbol{\beta}) = n^{-1} \sum_{i=1}^{n} Q[g^{-1}\{\eta(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\beta}\}, Y_i].$$

For the kth covariate x_k, let $b_{j,k}(x_k)$ be the B-spline basis functions of order r. For any $\eta \in \mathcal{G}_n$, write $\eta(\mathbf{x}) = \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{b}(\mathbf{x})$, where $\mathbf{b}(\mathbf{x}) = \{b_{j,k}(x_k), j = 1, \ldots, J_n + r, k = 1, \ldots, d_1\}^{\mathrm{T}}$ is the collection of the spline basis functions, and $\boldsymbol{\gamma} = \{\gamma_{j,k}, j = 1, \ldots, J_n + r, k = 1, \ldots, d_1\}^{\mathrm{T}}$ is the spline coefficient vector. Thus, the maximization problem in (3) is equivalent to finding $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ to maximize

$$(4) \qquad \ell(\boldsymbol{\gamma}, \boldsymbol{\beta}) = n^{-1} \sum_{i=1}^{n} Q[g^{-1}\{\boldsymbol{\gamma}^{\mathrm{T}}\mathbf{b}(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\beta}\}, Y_i].$$

We denote the maximizer as $\widehat{\boldsymbol{\beta}}$ and $\widehat{\boldsymbol{\gamma}} = \{\widehat{\gamma}_{j,k}, j = 1, \ldots, J_n + r, k = 1, \ldots, d_1\}^{\mathrm{T}}$. Then the spline estimator of η_0 is $\widehat{\eta}(\mathbf{x}) = \widehat{\boldsymbol{\gamma}}^{\mathrm{T}}\mathbf{b}(\mathbf{x})$, and the centered spline component function estimators are

$$\widehat{\eta}_k(x_k) = \sum_{j=1}^{J_n+r} \widehat{\gamma}_{j,k}b_{j,k}(x_k) - n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{J_n+r} \widehat{\gamma}_{j,k}b_{j,k}(X_{ik}), \qquad k = 1, \ldots, d_1.$$

The above estimation approach can be easily implemented because this approximation results in a generalized linear model. However, theoretical justification for this estimation approach is very challenging [Huang (1998)].

Let $N_n = J_n + r - 1$. We adopt the normalized B-spline space $\mathcal{S}_n^0$ introduced in Xue and Yang (2006) with the following normalized basis

$$B_{j,k}(x_k) = \sqrt{N_n}\left\{ b_{j+1,k}(x_k) - \frac{E(b_{j+1,k})}{E(b_{1,k})} b_{1,k}(x_k) \right\},$$

(5)
$$1 \leq j \leq N_n, 1 \leq k \leq d_1,$$

which is convenient for asymptotic analysis. Let $\mathbf{B}(\mathbf{x}) = \{B_{j,k}(x_k), j = 1, \ldots, N_n, \ k = 1, \ldots, d_1\}^{\mathrm{T}}$ and $\mathbf{B}_i = \mathbf{B}(\mathbf{X}_i)$. Finding $(\boldsymbol{\gamma}, \boldsymbol{\beta})$ that maximizes (4) is mathematically equivalent to finding $(\boldsymbol{\gamma}, \boldsymbol{\beta})$ which maximizes

$$n^{-1} \sum_{i=1}^{n} Q[g^{-1}\{\mathbf{B}_i^{\mathrm{T}}\boldsymbol{\gamma} + \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\beta}\}, Y_i].$$

Then the spline estimator of η_0 is $\widehat{\eta}(\mathbf{x}) = \widehat{\boldsymbol{\gamma}}^{\mathrm{T}}\mathbf{B}(\mathbf{x})$, and the centered spline estimators of the component functions are

$$\widehat{\eta}_k(x_k) = \sum_{j=2}^{N_n} \widehat{\gamma}_{j,k} B_{j,k}(x_k) - n^{-1} \sum_{i=1}^{n} \sum_{j=2}^{N_n} \widehat{\gamma}_{j,k} B_{j,k}(X_{ik}), \qquad k = 1, \ldots, d_1.$$

We show next that estimators of both the parametric and nonparametric components have nice asymptotic properties.

3.2. *Assumptions and asymptotic results.* Let v be a positive integer and $\alpha \in (0,1]$ such that $p = v + \alpha > 2$. Let $\mathcal{H}(p)$ be the collection of functions g on $[0,1]$ whose vth derivative, $g^{(v)}$, exists and satisfies a Lipschitz condition of order α, $|g^{(v)}(m^*) - g^{(v)}(m)| \leq C|m^* - m|^\alpha$, for $0 \leq m^*, m \leq 1$, where C is a positive constant. Following the notation of Carroll et al. (1997), let $\rho_\ell(m) = \{dg^{-1}(m)/dm\}^\ell / V\{g^{-1}(m)\}$ and $q_\ell(m, y) = \partial^\ell/\partial m^\ell Q\{g^{-1}(m), y\}$, so that

$$q_1(m, y) = \partial/\partial m Q\{g^{-1}(m), y\} = \{y - g^{-1}(m)\}\rho_1(m),$$

$$q_2(m, y) = \partial^2/\partial m^2 Q\{g^{-1}(m), y\} = \{y - g^{-1}(m)\}\rho_1'(m) - \rho_2(m).$$

For simplicity of notation, write $\mathbf{T} = (\mathbf{X}, \mathbf{Z})$ and $\mathbf{A}^{\otimes 2} = \mathbf{A}\mathbf{A}^{\mathrm{T}}$ for any matrix or vector $\mathbf{A}$. We make the following assumptions:

(C1) The function $\eta_0''(\cdot)$ is continuous and each component function $\eta_{0k}(\cdot) \in \mathcal{H}(p)$, $k = 1, \ldots, d_1$.

(C2) The function $q_2(m, y) < 0$ and $c_q < |q_2^\nu(m, y)| < C_q$ $(\nu = 0, 1)$ for $m \in R$ and y in the range of the response variable.

(C3) The distribution of $\mathbf{X}$ is absolutely continuous and its density f is bounded away from zero and infinity on $[0,1]^{d_1}$.

(C4) The random vector $\mathbf{Z}$ satisfies that for any unit vector $\boldsymbol{\omega} \in R^{d_2}$

$$c \leq \boldsymbol{\omega}^{\mathrm{T}} E(\mathbf{Z}^{\otimes 2} | \mathbf{X} = \mathbf{x})\boldsymbol{\omega} \leq C.$$

(C5) The number of knots $n^{1/(2p)} \ll N_n \ll n^{1/4}$.

REMARK 1. The smoothness condition in (C1) describes a requirement on the best rate of convergence that the functions $\eta_{0k}(\cdot)$'s can be approximated by functions in the spline spaces. Condition (C2) is imposed to ensure the uniqueness of the solution; see, for example, Condition 1a of Carroll et al. (1997) and Condition (i) of Li and Liang (2008). Condition (C3) requires a boundedness condition on the covariates, which is often assumed in asymptotic analysis of nonparametric regression problems; see Condition 1 of Stone (1985), Assumption (B3)(ii) of Huang (1999) and Assumption (C1) of Xue and Yang (2006). The boundedness assumption on the support can be replaced by a finite third moment assumption, but this will add much extra complexity to the proofs. Condition (C4) implies that the eigenvalues of $E(\mathbf{Z}^{\otimes 2}|\mathbf{X} = \mathbf{x})$ are bounded away from 0 and ∞. Condition (C5) gives the rate of growth of the dimension of the spline spaces relative to the sample size.

For measurable functions φ_1, φ_2 on $[0,1]^{d_1}$, define the empirical inner product and the corresponding norm as

$$\langle \varphi_1, \varphi_2 \rangle_n = n^{-1} \sum_{i=1}^{n} \{\varphi_1(\mathbf{X}_i)\varphi_2(\mathbf{X}_i)\}, \qquad \|\varphi\|_n^2 = n^{-1} \sum_{i=1}^{n} \varphi^2(\mathbf{X}_i).$$

If φ_1 and φ_2 are L^2-integrable, define the theoretical inner product and corresponding norm as

$$\langle \varphi_1, \varphi_2 \rangle = E\{\varphi_1(\mathbf{X})\varphi_2(\mathbf{X})\}, \qquad \|\varphi\|_2^2 = E\varphi^2(\mathbf{X}).$$

Let $\|\varphi\|_{nk}^2$ and $\|\varphi\|_{2k}^2$ be the empirical and theoretical norm of φ on $[0,1]$, defined by

$$\|\varphi\|_{nk}^2 = n^{-1} \sum_{i=1}^{n} \varphi^2(X_{ik}), \qquad \|\varphi\|_{2k}^2 = E\varphi^2(X_k) = \int_0^1 \varphi^2(x_k) f_k(x_k)\, dx_k,$$

where $f_k(\cdot)$ is the density function of X_k.

Theorem 1 describes the rates of convergence of the nonparametric parts.

THEOREM 1. *Under conditions* (C1)–(C5), *for* $k = 1, \ldots, d_1$, $\|\widehat{\eta} - \eta_0\|_2 = O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\}$; $\|\widehat{\eta} - \eta_0\|_n = O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\}$; $\|\widehat{\eta}_k - \eta_{0k}\|_{2k} = O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\}$ *and* $\|\widehat{\eta}_k - \eta_{0k}\|_{nk} = O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\}$.

Let $m_0(\mathbf{T}) = \eta_0(\mathbf{X}) + \mathbf{Z}^{\mathrm{T}}\boldsymbol{\beta}_0$ and define

(6) $\qquad \Gamma(\mathbf{x}) = \dfrac{E[\mathbf{Z}\rho_2\{m_0(\mathbf{T}))\}|\mathbf{X} = \mathbf{x}]}{E[\rho_2\{m_0(\mathbf{T})\}|\mathbf{X} = \mathbf{x}]}, \qquad \widetilde{\mathbf{Z}} = \mathbf{Z} - \Gamma^{\mathrm{add}}(\mathbf{X}),$

where

$$(7) \qquad \Gamma^{\mathrm{add}}(\mathbf{x}) = \sum_{k=1}^{d_1} \Gamma_k(x_k)$$

is the projection of Γ onto the Hilbert space of theoretically centered additive functions with a norm $\|f\|^2_{\rho_2,m_0} = E[f(\mathbf{X})^2 \rho_2\{m_0(\mathbf{T})\}]$. To obtain asymptotic normality of the estimators in the linear part, we further impose the conditions:

(C6) The additive components in (7) satisfy that $\Gamma_k(\cdot) \in \mathcal{H}(p)$, $k = 1, \ldots, d_1$.

(C7) For ρ_ℓ, we have

$$|\rho_\ell(m_0)| \le C_\rho \quad \text{and} \quad |\rho_\ell(m) - \rho_\ell(m_0)| \le C^*_\rho |m - m_0|$$

$$\text{for all } |m - m_0| \le C_m, \ell = 1, 2.$$

(C8) There exists a positive constant C_0, such that $E[\{Y - g^{-1}(m_0(\mathbf{T}))\}^2 | \mathbf{T}] \le C_0$, almost surely.

The next theorem shows that the maximum quasi-likelihood estimator of $\boldsymbol{\beta}_0$ is root-n consistent and asymptotically normal, although the convergence rate of the nonparametric component $\boldsymbol{\eta}_0$ is of course slower than root-n.

THEOREM 2. *Under conditions* (C1)–(C8), $\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \to \text{Normal}(0, \boldsymbol{\Omega}^{-1})$, *where* $\boldsymbol{\Omega} = E[\rho_2\{m_0(\mathbf{T})\}\widetilde{\mathbf{Z}}^{\otimes 2}]$.

The proofs of these theorems are given in the Appendix.

It is worthwhile pointing out that taking the additive structure of the nuisance parameter into account leads to a smaller asymptotic variance than that of the estimators which ignore the additivity [Yu and Lee (2010)]. Carroll et al. (2009) had the same observation for a special case with repeated measurement data when g is the identity function.

4. Selection of significant parametric variables. In this section, we develop variable selection procedures for the parametric component of the GAPLM. We study the asymptotic properties of the resulting estimator, illustrate how the rate of convergence of the resulting estimate depends on the regularization parameters, and further establish the oracle properties of the resulting estimate.

4.1. *Penalized likelihood.* Building upon the quasi-likelihood given in (3), we define the penalized quasi-likelihood as

$$(8) \qquad \mathcal{L}(\eta, \boldsymbol{\beta}) = \sum_{i=1}^{n} Q[g^{-1}\{\eta(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\beta}\}, Y_i] - n \sum_{j=1}^{d_2} p_{\lambda_j}(|\beta_j|),$$

where $p_{\lambda_j}(\cdot)$ is a prespecified penalty function with a regularization parameter λ_j. The penalty functions and regularization parameters in (8) are not necessarily the same for all j. For example, we may wish to keep scientifically important variables in the final model, and therefore do not want to penalize their coefficients. In practice, λ_j can be chosen by a data-driven criterion, such as cross-validation (CV) or generalized cross-validation [GCV, Craven and Wahba (1979)].

Various penalty functions have been used in variable selection for linear regression models, for instance, the L_0 penalty, in which $p_{\lambda_j}(|\beta|) = 0.5\lambda_j^2 I(|\beta| \neq 0)$. The traditional best-subset variable selection can be viewed as a penalized least squares with the L_0 penalty because $\sum_{j=1}^{d_2} I(|\beta_j| \neq 0)$ is essentially the number of nonzero regression coefficients in the model. Of course, this procedure has two well known and severe problems. First, when the number of covariates is large, it is computationally infeasible to do subset selection. Second, best subset variable selection suffers from high variability and instability [Breiman (1996), Fan and Li (2001)].

The Lasso is a regularization technique for simultaneous estimation and variable selection [Tibshirani (1996), Zou (2006)] that avoids the drawbacks of the best subset selection. It can be viewed as a penalized least squares estimator with the L_1 penalty, defined by $p_{\lambda_j}(|\beta|) = \lambda_j|\beta|$. Frank and Friedman (1993) considered bridge regression with an L_q penalty, in which $p_{\lambda_j}(|\beta|) = \lambda_j|\beta|^q$ $(0 < q < 1)$. The issue of selection of the penalty function has been studied in depth by a variety of authors. For example, Fan and Li (2001) suggested using the SCAD penalty, defined by

$$p'_{\lambda_j}(\beta) = \lambda_j \left\{ I(\beta \leq \lambda_j) + \frac{(a\lambda_j - \beta)_+}{(a-1)\lambda_j} I(\beta > \lambda_j) \right\}$$

for some $a > 2$ and $\beta > 0$,

where $p_{\lambda_j}(0) = 0$, and λ_j and a are two tuning parameters. Fan and Li (2001) suggested using $a = 3.7$, which will be used in Section 5.

Substituting η by its estimate in (8), we obtain a penalized likelihood

$$(9) \qquad \mathcal{L}_P(\boldsymbol{\beta}) = \sum_{i=1}^{n} Q[g^{-1}\{\mathbf{B}_i^{\mathrm{T}}\widehat{\boldsymbol{\gamma}} + \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\beta}\}, Y_i] - n\sum_{j=1}^{d_2} p_{\lambda_j}(|\beta_j|).$$

Maximizing $\mathcal{L}_P(\boldsymbol{\beta})$ in (9) yields a maximum penalized likelihood estimator $\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}$. The theorems established below demonstrate that $\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}$ performs asymptotically as well as an oracle estimator.

4.2. *Sampling properties.* We next show that with a proper choice of λ_j, the maximum penalized likelihood estimator $\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}$ has an asymptotic oracle

property. Let $\boldsymbol{\beta}_0 = (\beta_{10}, \ldots, \beta_{d_2 0})^{\mathrm{T}} = (\boldsymbol{\beta}_{10}^{\mathrm{T}}, \boldsymbol{\beta}_{20}^{\mathrm{T}})^{\mathrm{T}}$, where $\boldsymbol{\beta}_{10}$ is assumed to consist of all nonzero components of $\boldsymbol{\beta}_0$ and $\boldsymbol{\beta}_{20} = \mathbf{0}$ without loss of generality. Similarly we write $\mathbf{Z} = (\mathbf{Z}_1^{\mathrm{T}}, \mathbf{Z}_2^{\mathrm{T}})^{\mathrm{T}}$. Denote $w_n = \max_{1 \le j \le d_2}\{|p_{\lambda_j}''(|\beta_{j0}|)|,$ $\beta_{j0} \neq 0\}$ and

$$(10) \qquad a_n = \max_{1 \le j \le d_2}\{|p_{\lambda_j}'(|\beta_{j0}|)|, \beta_{j0} \neq 0\}.$$

THEOREM 3. *Under the regularity conditions given in Section 3.2, and if $a_n \to 0$ and $w_n \to 0$ as $n \to \infty$, then there exists a local maximizer $\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}$ of $\mathcal{L}_P(\boldsymbol{\beta})$ defined in (9) such that its rate of convergence is $O_P(n^{-1/2} + a_n)$, where a_n is given in (10).*

Next, define $\boldsymbol{\xi}_n = \{p_{\lambda_1}'(|\beta_{10}|)\operatorname{sgn}(\beta_{10}), \ldots, p_{\lambda_s}'(|\beta_{s0}|)\operatorname{sgn}(\beta_{s0})\}^{\mathrm{T}}$ and a diagonal matrix $\boldsymbol{\Sigma}_\lambda = \operatorname{diag}\{p_{\lambda_1}''(|\beta_{10}|), \ldots, p_{\lambda_s}''(|\beta_{s0}|)\}$, where s is the number of nonzero components of $\boldsymbol{\beta}_0$. Define $\mathbf{T}_1 = (\mathbf{X}, \mathbf{Z}_1)$ and $m_0(\mathbf{T}_1) = \eta_0(\mathbf{X}) + \mathbf{Z}_1^{\mathrm{T}}\boldsymbol{\beta}_{10}$, and further let

$$\Gamma_1(\mathbf{x}) = \frac{E[\mathbf{Z}_1 \rho_2\{m_0(\mathbf{T}_1)\}|\mathbf{X} = \mathbf{x}]}{E[\rho_2\{m_0(\mathbf{T}_1)\}|\mathbf{X} = \mathbf{x}]}, \qquad \widetilde{\mathbf{Z}}_1 = \mathbf{Z}_1 - \Gamma_1^{\mathrm{add}}(\mathbf{X}),$$

where Γ_1^{add} is the projection of Γ_1 onto the Hilbert space of theoretically centered additive functions with the norm $\|f\|_{\rho_2, m_0}^2$.

THEOREM 4. *Suppose that the regularity conditions given in Section 3.2 hold, and that $\liminf_{n \to \infty} \liminf_{\beta_j \to 0^+} \lambda_{jn}^{-1} p_{\lambda_{jn}}'(|\beta_j|) > 0$. If $\sqrt{n}\lambda_{jn} \to \infty$ as $n \to \infty$, then the root-n consistent estimator $\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}$ in Theorem 3 satisfies $\widehat{\boldsymbol{\beta}}_2^{\mathrm{MPL}} = \mathbf{0}$, and $\sqrt{n}(\boldsymbol{\Omega}_s + \boldsymbol{\Sigma}_\lambda)\{\widehat{\boldsymbol{\beta}}_1^{\mathrm{MPL}} - \boldsymbol{\beta}_{10} + (\boldsymbol{\Omega}_s + \boldsymbol{\Sigma}_\lambda)^{-1}\boldsymbol{\xi}_n\} \to \mathrm{Normal}(\mathbf{0}, \boldsymbol{\Omega}_s)$, where $\boldsymbol{\Omega}_s = [\rho_2\{m_0(\mathbf{T}_1)\}\widetilde{\mathbf{Z}}_1^{\otimes 2}]$.*

4.3. *Implementation.* As pointed out by Li and Liang (2008), many penalty functions, including the L_1 penalty and the SCAD penalty, are irregular at the origin and may not have a second derivative at some points. Thus, it is often difficult to implement the Newton–Raphson algorithm directly. As in Fan and Li (2001), Hunter and Li (2005), we approximate the penalty function locally by a quadratic function at every step in the iteration such that the Newton–Raphson algorithm can be modified for finding the solution of the penalized likelihood. Specifically, given an initial value $\boldsymbol{\beta}^{(0)}$ that is close to the maximizer of the penalized likelihood function, the penalty $p_{\lambda_j}(|\beta_j|)$ can be locally approximated by the quadratic function as $\{p_{\lambda_j}(|\beta_j|)\}' = p_{\lambda_j}'(|\beta_j|)\operatorname{sgn}(\beta_j) \approx \{p_{\lambda_j}'(|\beta_j^{(0)}|)/|\beta_j^{(0)}|\}\beta_j$, when $\beta_j^{(0)}$ is

not very close to 0; otherwise, set $\widehat{\beta}_j = 0$. In other words, for $\beta_j \approx \beta_j^{(0)}$, $p_{\lambda_j}(|\beta_j|) \approx p_{\lambda_j}(|\beta_j^{(0)}|) + (1/2)\{p'_{\lambda_j}(|\beta_j^{(0)}|)/|\beta_j^{(0)}|\}(\beta_j^2 - \beta_j^{(0)2})$. For instance, this local quadratic approximation for the L_1 penalty yields

$$|\beta_j| \approx (1/2)|\beta_j^{(0)}| + (1/2)\beta_j^2/|\beta_j^{(0)}| \qquad \text{for } \beta_j \approx \beta_j^{(0)}.$$

Standard error formula for $\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}$. We follow the approach in Li and Liang (2008) to derive a sandwich formula for the estimator $\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}$. Let

$$\ell'(\boldsymbol{\beta}) = \frac{\partial \ell(\widehat{\boldsymbol{\gamma}}, \boldsymbol{\beta})}{\partial \boldsymbol{\beta}}, \qquad \ell''(\boldsymbol{\beta}) = \frac{\partial^2 \ell(\widehat{\boldsymbol{\gamma}}, \boldsymbol{\beta})}{\partial \boldsymbol{\beta} \, \partial \boldsymbol{\beta}^{\mathrm{T}}};$$

$$\boldsymbol{\Sigma}_\lambda(\boldsymbol{\beta}) = \mathrm{diag}\left\{\frac{p'_{\lambda_1}(|\beta_1|)}{|\beta_1|}, \ldots, \frac{p'_{\lambda_{d_2}}(|\beta_{d_2}|)}{|\beta_{d_2}|}\right\}.$$

A sandwich formula is given by

$$\widehat{\mathrm{cov}}(\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}) = \{n\ell''(\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}) - n\boldsymbol{\Sigma}_\lambda(\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}})\}^{-1}\widehat{\mathrm{cov}}\{\ell'(\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}})\}$$
$$\times \{n\ell''(\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}) - n\boldsymbol{\Sigma}_\lambda(\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}})\}^{-1}.$$

Following conventional techniques that arise in the likelihood setting, the above sandwich formula can be shown to be a consistent estimator and will be shown in our simulation study to have good accuracy for moderate sample sizes.

Choice of λ_j*'s.* The unknown parameters (λ_j) can be selected using data-driven approaches, for example, generalized cross validation as proposed in Fan and Li (2001). Replacing $\boldsymbol{\beta}$ in (4) with its estimate $\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}$, we maximize $\ell(\boldsymbol{\gamma}, \widehat{\boldsymbol{\beta}}^{\mathrm{MPL}})$ with respect to $\boldsymbol{\gamma}$. The solution is denoted by $\widehat{\boldsymbol{\gamma}}^{\mathrm{MPL}}$, and the corresponding estimator of η_0 is defined as

(11) $$\widehat{\boldsymbol{\eta}}^{\mathrm{MPL}}(\mathbf{x}) = (\widehat{\boldsymbol{\gamma}}^{\mathrm{MPL}})^{\mathrm{T}}\mathbf{B}(\mathbf{x}).$$

Here the GCV statistic is defined by

$$\mathrm{GCV}(\lambda_1, \ldots, \lambda_{d_2}) = \frac{\sum_{i=1}^n D[Y_i, g^{-1}\{\widehat{\boldsymbol{\eta}}^{\mathrm{MPL}}(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}\}]}{n\{1 - e(\lambda_1, \ldots, \lambda_{d_2})/n\}^2},$$

where $e(\lambda_1, \ldots, \lambda_{d_2}) = \mathrm{tr}[\{\ell''(\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}}) - n\boldsymbol{\Sigma}_\lambda(\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}})\}^{-1}\ell''(\widehat{\boldsymbol{\beta}}^{\mathrm{MPL}})]$ is the effective number of parameters and $D(Y, \mu)$ is the deviance of Y corresponding to fitting with $\boldsymbol{\lambda}$. The minimization problem over a d_2-dimensional space is difficult. However, Li and Liang (2008) conjectured that the magnitude of λ_j should be proportional to the standard error of the unpenalized maximum pseudo-partial likelihood estimator of β_j. Thus, we suggest taking

$\lambda_j = \lambda \mathrm{SE}(\widehat{\beta}_j)$ in practice, where $\mathrm{SE}(\widehat{\beta}_j)$ is the estimated standard error of $\widehat{\beta}_j$, the unpenalized likelihood estimate defined in Section 3. Then the minimization problem can be reduced to a one-dimensional problem, and the tuning parameter can be estimated by a grid search.

5. Numerical studies.

5.1. *A simulation study.* We simulated 100 data sets consisting of $n = 100$, 200 and 400 observations, respectively, from the GAPLM:

$$(12) \qquad \mathrm{logit}\{\mathrm{pr}(Y = 1)\} = \eta_1(X_1) + \eta_2(X_2) + \mathbf{Z}^{\mathrm{T}}\boldsymbol{\beta},$$

where

$$\eta_1(x) = \sin(4\pi x),$$
$$\eta_2(x) = 10\{\exp(-3.25x) + 4\exp(-6.5x) + 3\exp(-9.75x)\}$$

and the true parameters $\boldsymbol{\beta} = (3, 1.5, 0, 0, 0, 0, 2, 0)^{\mathrm{T}}$. X_1 and X_2 are independently uniformly distributed on $[0, 1]$. Z_1 and Z_2 are normally distributed with mean 0.5 and variance 0.09. The random vector $(Z_1, \ldots, Z_6, X_1, X_2)$ has an autoregressive structure with correlation coefficient $\rho = 0.5$.

In order to determine the number of knots in the approximation, we performed a simulation with 1,000 runs for each sample size. In each run, we fit, without any variable selection procedure, all possible spline approximations with 0–7 internal knots for each nonparametric component. The internal knots were equally spaced quantiles from the simulated data. We recorded the combination of the numbers of knots used by the best approximation, which had the smallest prediction error (PE), defined as

$$(13) \qquad \mathrm{PE} = \frac{1}{n}\sum_{i=1}^{n}\{\mathrm{logit}^{-1}(\mathbf{B}_i^{\mathrm{T}}\widehat{\boldsymbol{\gamma}} + \mathbf{Z}_i^{\mathrm{T}}\widehat{\boldsymbol{\beta}}) - \mathrm{logit}^{-1}(\eta(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\beta})\}^2.$$

$(2, 2)$ and $(5, 3)$ are most frequently chosen for sample sizes 100 and 400, respectively. These combinations were used in the simulations for the variable selection procedures.

The proposed selection procedures were applied to this model and B-splines were used to approximate the two nonparametric functions. In the simulation and also the empirical example in Section 5.2, the estimates from ordinary logistic regression were used as the starting values in the fitting procedure.

To study model fit, we also defined model error (ME) for the parametric part by

$$(14) \qquad \mathrm{ME}(\widehat{\boldsymbol{\beta}}) = (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})^{\mathrm{T}} E(ZZ^{\mathrm{T}})(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}).$$

TABLE 1

*Results from the simulation study in Section 5.1. C, I and MRME stand for the average
number of the five zero coefficients correctly set to* 0*, the average number of the three
nonzero coefficients incorrectly set to* 0*, and the median of the relative model errors. The
model errors are defined in (14)*

n	Method	C	I	MRME
100	ORACLE	5	0	0.27
	SCAD	4.29	0.93	0.60
	Lasso	3.83	0.67	0.51
	BIC	4.53	0.95	0.54
400	ORACLE	5	0	0.33
	SCAD	4.81	0.27	0.49
	Lasso	3.89	0.10	0.67
	BIC	4.90	0.35	0.46

The relative model error is defined as the ratio of the model error between
the fitted model using variable selection methods and using ordinary logistic
regression.

The simulation results are reported in Table 1, in which the columns
labeled with "C" give the average number of the five zero coefficients cor-
rectly set to 0, the columns labeled with "I" give the average number of the
three nonzero coefficients incorrectly set to 0, and the columns labeled with
"MRME" give the median of the relative model errors.

Summarizing Table 1, we conclude that BIC performs the best in terms
of correctly identifying zero coefficients, followed by SCAD and LASSO. On
the other hand, BIC is also more likely to set nonzero coefficients to zero,
followed by SCAD and LASSO. This indicates that BIC most aggressively
reduce the model complexity, while LASSO tends to include more variables
in the models. SCAD is a useful compromise between these two procedures.
With an increase of sample sizes, both SCAD and BIC nearly perform as
if they had Oracle property. The MRME values of the three procedures are
comparable. Results of the cases not depicted here have characteristics sim-
ilar to those shown in Table 1. Readers may refer to the online supplemental
materials.

We also performed a simulation with correlated covariates. We generated
the response Y from model (12) again but with $\boldsymbol{\beta} = (3.00, 1.50, 2.00)$. The
covariates Z_1, Z_2, X_1 and X_2 were marginally normal with mean zero and
variance 0.09. In order, (Z_1, Z_2, X_1, X_2) had autoregressive correlation coef-
ficient ρ, while Z_3 is Bernoulli with success probability 0.5. We considered
two scenarios: (i) moderately correlated covariates ($\rho = 0.5$) and (ii) highly
correlated ($\rho = 0.7$) covariates. We did 1,000 simulation runs for each case
with sample sizes $n = 100, 200$ and 400. From our simulation, we observe that
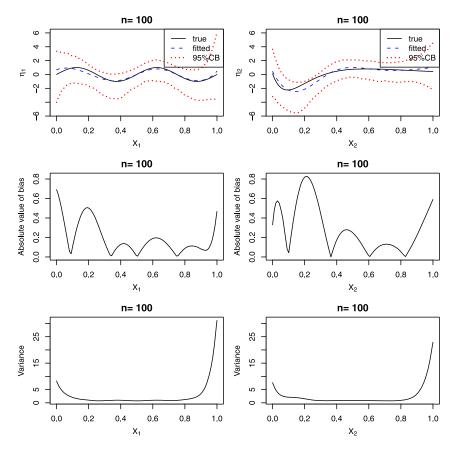the estimator becomes more unstable when the correlation among covariates

FIG. 1.   *The mean, absolute value of the bias and variance of the fitted nonparametric functions when $n = 100$ and $\rho = 0.5$ [the left panel for $\eta_1(x_1)$ and the right for $\eta_2(x_2)$]. 95% CB stands for the 95% confidence band.*

is higher. In scenario (i), all simulation runs converged. However, there were 6, 3 and 7 cases of nonconvergence over the 1,000 simulation runs for sample sizes $100, 200$ and 400, respectively, in scenario (ii). In addition, the variance and bias of the fitted functions in scenario (ii) were much larger than those in scenario (i), especially on the boundaries of the covariates' support. This can be observed in Figures 1 and 2, which present the mean, absolute value of bias and variance of the fitted nonparametric functions for $\rho = 0.5$ and $\rho = 0.7$ with sample size $n = 100$. Similar results are obtained for sample sizes $n = 200$ and 400, but are not given here.

5.2. *An empirical example.* We now apply the GAPLM and our variable selection procedure to a data set from the Pima Indian diabetes study [Smith et al. (1988)]. This data set is obtained from the UCI Repository of Machine Learning Databases, and is selected from a larger data set held by
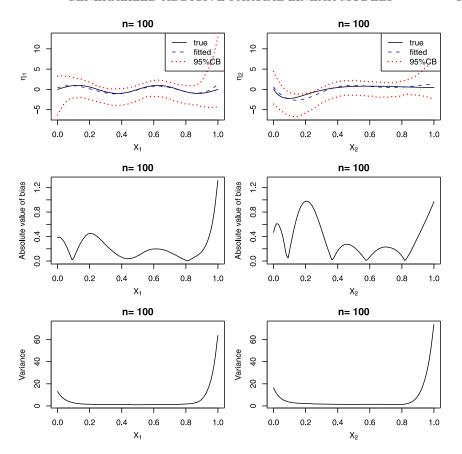
FIG. 2.   *The mean, absolute value of the bias and variance of the fitted nonparametric functions when $n = 100$ and $\rho = 0.7$. The left panel is for $\eta_1(x_1)$ and the right panel is for $\eta_2(x_2)$. Here 95% CB stands for the 95% confidence band.*

the National Institutes of Diabetes and Digestive and Kidney Diseases. All patients in this database are Pima Indian women at least 21 years old and living near Phoenix, Arizona. The response Y is the indicator of a positive test for diabetes. Independent variables from this data set include: *NumPreg*, the number of pregnancies; *DBP*, diastolic blood pressure (mmHg); *DPF*, diabetes pedigree function; *PGC*, the plasma glucose concentration after two hours in an oral glucose tolerance test; *BMI*, body mass index [weight in kg/(height in m)^2]; and *AGE* (years). There are in total 724 complete observations in this data set.

In this example, we explore the impact of these covariates on the probability of a positive test. We first fit the data set using a linear logistic regression model: the estimated results are listed in the left panel of Table 2. These results indicate that *NumPreg*, *DPF*, *PGC* and *BMI* are statistically significant, while *DBP* and *AGE* are not statistically significant.

TABLE 2

Results for the Pima study. Left panel: estimated values, associated standard errors and P-values by using GLM. Right panel: Estimates, associated standard errors using the GAPLM with the proposed variable selection procedures

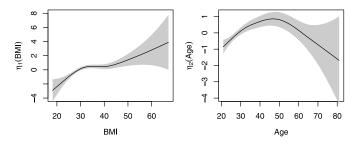|	GLM				GAPLM		
	Est.	**s.e.**	z **value**	$\Pr(>|z|)$	**SCAD (s.e.)**	**LASSO (s.e.)**	**BIC (s.e.)**
NumPreg	0.118	0.033	3.527	0	0 (0)	0.021 (0.019)	0 (0)
DBP	−0.009	0.009	−1.035	0.301	0 (0)	−0.006 (0.005)	0 (0)
DPF	0.961	0.306	3.135	0.002	0.958 (0.312)	0.813 (0.262)	0.958 (0.312)
PGC	0.035	0.004	9.763	0	0.036 (0.004)	0.034 (0.003)	0.036 (0.004)
BMI	0.091	0.016	5.777	0			
AGE	0.017	0.01	1.723	0.085			

FIG. 3.   *The patterns of the nonparametric functions of BMI and Age (solid lines) with* ± *s.e. (shaded areas) using the R function, gam, for the Pima study.*

However, a closer investigation shows that the effect of *AGE* and *BMI* on the logit transformation of the probability of a positive test may be nonlinear, see Figure 3. Thus, we employ the following GAPLM for this data analysis,

$$\text{logit}\{P(Y=1)\} = \eta_0 + \beta_1 NumPreg + \beta_2 DBP + \beta_3 DPF$$

(15)

$$+ \beta_4 PGC + \eta_1(BMI) + \eta_2(AGE).$$

Using B-splines to approximate $\eta_1(BMI)$ and $\eta_2(AGE)$, we adopt 5-fold cross-validation to select knots and find that the approximation with no internal knots performs well for the both nonparametric components.

We applied the proposed variable selection procedures to the model (15), and the estimated coefficients and their standard errors are listed in the right panel of Table 2. Both SCAD and BIC suggest that *DPF* and *PGC* enter the model, whereas *NumPreg* and *DBP* are suggested not to enter. However, the LASSO suggests an inclusion of *NumPreg* and *DBP*. This may be because LASSO admits many variables in general, as we observed in the simulation studies. The nonparametric estimators of $\eta_1(BMI)$ and $\eta_2(AGE)$, which are obtained by using the SCAD-based procedure, are similar to the solid lines

in Figure 3. It is worth pointing that the effect of AGE on the probability of a positive test shows a concave pattern, and women whose age is around 50 have the highest probability of developing diabetes. Importantly, the linear logistic regression model does not reveal this significant effect.

It is interesting that the variable $NumPreg$ is statistically insignificant when we fit the data using GAPLM with the proposed variable selection procedure, but shows a statistically significant impact when we use GLM. One might reasonably conjecture that this phenomenon might be due to model misspecification. To test this, we conducted a simulation as follows. We generated the response variables using the estimates and functions obtained by GAPLM with the SCAD. Then we fit a GLM for the generated data set. We repeated the generation and fitting procedures 5,000 times and found that $NumPreg$ is identified positively significant 67.42% percent of the time at level 0.05 in the GLMs. For DBP, DPF, PGC, BMI and AGE, the percentages that they are identified as statistically significant at the level 0.05 are 4.52%, 90.36%, 100% and 99.98% and 56.58%, respectively. This means that $NumPreg$ can incorrectly enter the model, with more than 65% probability, when a wrong model is used, while DBP, DPF, PGC, BMI and AGE seem correctly to be classified as insignificant and significant covariates even with this wrong GLM model.

6. Concluding remarks. We have proposed an effective polynomial spline technique for the GAPLM, then developed variable selection procedures to identify which linear predictors should be included in the final model fitting. The contributions we made to the existing literature can be summarized in three ways: (i) the procedures are computationally efficient, theoretically reliable, and intuitively appealing; (ii) the estimators of the linear components, which are often of primary interest, are asymptotically normal; and (iii) the variable selection procedure for the linear components has an asymptotic oracle property. We believe that our approach can be extended to the case of longitudinal data [Lin and Carroll (2006)], although the technical details are by no means straightforward.

An important question in using GAPLM in practice is which covariates should be included in the linear component. We suggest proceeding as follows. The continuous covariates are put in the nonparametric part and the discrete covariates in the parametric part. If the estimation results show that some of the continuous covariate effects can be described by certain parametric forms such as a linear form, either by formal testing or by visualization, then a new model can be fit with those continuous covariate effects moved to the parametric part. The procedure can be iterated several times if needed. In this way, one can take full advantage of the flexible exploratory analysis provided by the proposed method. However, developing a more efficient and automatic criterion warrants future study. It is worth pointing out the proposed procedure may be instable for high-dimensional

data, and may encounter collinear problems. Addressing these challenging
questions is part of ongoing work.

APPENDIX

Throughout the article, let $\| \cdot \|$ be the Euclidean norm and $\|\varphi\|_\infty =$ $\sup_m |\varphi(m)|$ be the supremum norm of a function φ on $[0,1]$. For any matrix $\mathbf{A}$, denote its L_2 norm as $\|\mathbf{A}\|_2 = \sup_{\|\mathbf{x}\|\neq 0} \|\mathbf{A}\mathbf{x}\|/\|\mathbf{x}\|$, the largest eigenvalue.

A.1. Technical lemmas. In the following, let $\mathcal{F}$ be a class of measurable functions. For probability measure Q, the $L_2(Q)$-norm of a function $f \in \mathcal{F}$ is defined by $(\int |f|^2 \, dQ)^{1/2}$. According to van der Vaart and Wellner (1996), the δ-covering number $\mathcal{N}(\delta, \mathcal{F}, L_2(Q))$ is the smallest value of $\mathcal{N}$ for which there exist functions $f_1, \ldots, f_{\mathcal{N}}$, such that for each $f \in \mathcal{F}$, $\|f - f_j\| \leq \delta$ for some $j \in \{1, \ldots, \mathcal{N}\}$. The δ-covering number with bracketing $\mathcal{N}_{[\cdot]}(\delta, \mathcal{F}, L_2(Q))$ is the smallest value of $\mathcal{N}$ for which there exist pairs of functions $\{[f_j^L, f_j^U]\}_{j=1}^{\mathcal{N}}$ with $\|f_j^U - f_j^L\| \leq \delta$, such that for each $f \in \mathcal{F}$, there is a $j \in \{1, \ldots, \mathcal{N}\}$ such that $f_j^L \leq f \leq f_j^U$. The δ-entropy with bracketing is defined as $\log \mathcal{N}_{[\cdot]}(\delta, \mathcal{F}, L_2(Q))$. Denote $\mathcal{J}_{[\cdot]}(\delta, \mathcal{F}, L_2(Q)) = \int_0^\delta \sqrt{1 + \log \mathcal{N}_{[\cdot]}(\varepsilon, \mathcal{F}, L_2(Q))} \, d\varepsilon$. Let Q_n be the empirical measure of Q. Denote $G_n = \sqrt{n}(Q_n - Q)$ and $\|G_n\|_{\mathcal{F}} = \sup_{f \in \mathcal{F}} |G_n f|$ for any measurable class of functions $\mathcal{F}$.

We state several preliminary lemmas first, whose proofs are included in the supplemental materials. Lemmas 1–3 will be used to prove the remaining lemmas and the main results. Lemmas 4 and 5 are used to prove Theorems 1–3.

LEMMA 1 [Lemma 3.4.2 of van der Vaart and Wellner (1996)]. *Let M_0 be a finite positive constant. Let $\mathcal{F}$ be a uniformly bounded class of measurable functions such that $Qf^2 < \delta^2$ and $\|f\|_\infty < M_0$. Then*

$$E_Q^* \|G_n\|_{\mathcal{F}} \leq C_0 \mathcal{J}_{[\cdot]}(\delta, \mathcal{F}, L_2(Q)) \left\{ 1 + \frac{\mathcal{J}_{[\cdot]}(\delta, \mathcal{F}, L_2(Q))}{\delta^2 \sqrt{n}} M_0 \right\},$$

where C_0 is a finite constant independent of n.

LEMMA 2 [Lemma A.2 of Huang (1999)]. *For any $\delta > 0$, let*

$$\Theta_n = \{\eta(\mathbf{x}) + \mathbf{z}^{\mathrm{T}}\boldsymbol{\beta}; \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\| \leq \delta, \eta \in \mathcal{G}_n, \|\eta - \eta_0\|_2 \leq \delta\}.$$

Then, for any $\varepsilon \leq \delta$, $\log \mathcal{N}_{[\cdot]}(\delta, \Theta_n, L_2(P)) \leq cN_n \log(\delta/\varepsilon)$.

For simplicity, let

$$(16) \qquad \mathbf{D}_i = (\mathbf{B}_i^{\mathrm{T}}, \mathbf{Z}_i^{\mathrm{T}}), \qquad \mathbf{W}_n = n^{-1} \sum_{i=1}^{n} \mathbf{D}_i^{\mathrm{T}} \mathbf{D}_i.$$

LEMMA 3. *Under conditions* (C1)–(C5), *for the above random matrix* $\mathbf{W}_n$, *there exists a positive constant* C *such that* $\|\mathbf{W}_n^{-1}\|_2 \leq C$, *a.s.*

According to a result of de Boor [(2001), page 149], for any function $g \in \mathcal{H}(p)$ with $p < r - 1$, there exists a function $\widetilde{g} \in \mathcal{S}_n^0$, such that $\|\widetilde{g} - g\|_\infty \leq CN_n^{-p}$, where C is some fixed positive constant. For η_0 satisfying (C1), we can find $\widetilde{\boldsymbol{\gamma}} = \{\widetilde{\gamma}_{j,k}, j = 1, \ldots, N_n, k = 1, \ldots, d_1\}^{\mathrm{T}}$ and an additive spline function $\widetilde{\eta} = \widetilde{\boldsymbol{\gamma}}^{\mathrm{T}} \mathbf{B}(\mathbf{x}) \in \mathcal{G}_n$, such that

(17) $$\|\widetilde{\eta} - \eta_0\|_\infty = O(N_n^{-p}).$$

Let

(18) $$\widetilde{\boldsymbol{\beta}} = \arg\max_{\boldsymbol{\beta}} n^{-1} \sum_{i=1}^{n} Q[g^{-1}\{\widetilde{\eta}(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\beta}\}, Y_i].$$

In the following, let $m_{0i} \equiv m_0(\mathbf{T}_i) = \eta_0(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\beta}_0$ and $\varepsilon_i = Y_i - g^{-1}(m_{0i})$. Further let

$$\widetilde{m}_0(\mathbf{t}) = \widetilde{\eta}(\mathbf{x}) + \mathbf{z}^{\mathrm{T}}\boldsymbol{\beta}_0, \qquad \widetilde{m}_{0i} \equiv \widetilde{m}_0(\mathbf{T}_i) = \widetilde{\eta}(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\beta}_0.$$

LEMMA 4. *Under conditions* (C1)–(C5), $\sqrt{n}(\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \to$ Normal$(\mathbf{0}, \mathbf{A}^{-1} \times \boldsymbol{\Sigma}_1 \mathbf{A}^{-1})$, *where* $\widetilde{\boldsymbol{\beta}}$ *is in (18)*, $\mathbf{A} = E[\rho_2\{m_0(\mathbf{T})\}\mathbf{Z}^{\otimes 2}]$ *and* $\boldsymbol{\Sigma}_1 = E[q_1^2\{m_0(\mathbf{T})\}\mathbf{Z}^{\otimes 2}]$.

In the following, denote $\widetilde{\boldsymbol{\theta}} = (\widetilde{\boldsymbol{\gamma}}^{\mathrm{T}}, \widetilde{\boldsymbol{\beta}}^{\mathrm{T}})^{\mathrm{T}}$, $\widehat{\boldsymbol{\theta}} = (\widehat{\boldsymbol{\gamma}}^{\mathrm{T}}, \widehat{\boldsymbol{\beta}}^{\mathrm{T}})^{\mathrm{T}}$ and

(19) $$\widetilde{m}_i \equiv \widetilde{m}(\mathbf{T}_i) = \widetilde{\eta}(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}\widetilde{\boldsymbol{\beta}} = \mathbf{B}_i^{\mathrm{T}}\widetilde{\boldsymbol{\gamma}} + \mathbf{Z}_i^{\mathrm{T}}\widetilde{\boldsymbol{\beta}}.$$

LEMMA 5. *Under conditions* (C1)–(C5),

$$\|\widehat{\boldsymbol{\theta}} - \widetilde{\boldsymbol{\theta}}\| = O_P\{N_n^{1/2-p} + (N_n/n)^{-1/2}\}.$$

A.2. Proof of Theorem 1. According to Lemma 5,

$$\|\widehat{\eta} - \widetilde{\eta}\|_2^2 = \|(\widehat{\boldsymbol{\gamma}} - \widetilde{\boldsymbol{\gamma}})^{\mathrm{T}}\mathbf{B}\|_2^2 = (\widehat{\boldsymbol{\gamma}} - \widetilde{\boldsymbol{\gamma}})^{\mathrm{T}} E\left[n^{-1} \sum_{i=1}^{n} \mathbf{B}_i^{\otimes 2}\right](\widehat{\boldsymbol{\gamma}} - \widetilde{\boldsymbol{\gamma}})$$

$$\leq C\|\widehat{\boldsymbol{\gamma}} - \widetilde{\boldsymbol{\gamma}}\|_2^2,$$

thus $\|\widehat{\eta} - \widetilde{\eta}\|_2 = O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\}$ and

$$\|\widehat{\eta} - \eta_0\|_2 \leq \|\widehat{\eta} - \widetilde{\eta}\|_2 + \|\widetilde{\eta} - \eta_0\|_2 = O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\} + O_P(N_n^{-p})$$

$$= O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\}.$$

By Lemma 1 of Stone (1985), $\|\widehat{\eta}_k - \eta_{0k}\|_{2k} = O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\}$, for each $1 \le k \le d_1$. Equation (17) implies that $\|\widehat{\eta} - \widetilde{\eta}\|_n = O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\}$. Then

$$\|\widehat{\eta} - \eta_0\|_n \le \|\widehat{\eta} - \widetilde{\eta}\|_n + \|\widetilde{\eta} - \eta_0\|_n$$
$$= O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\} + O_P(N_n^{-p})$$
$$= O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\}.$$

Similarly,

$$\sup_{\eta_1,\eta_2 \in \mathcal{S}_n^0} \left| \frac{\langle \eta_1, \eta_2 \rangle_n - \langle \eta_1, \eta_2 \rangle}{\|\eta_1\|_2 \|\eta_2\|_2} \right| = O_P\{(\log(n)N_n/n)^{1/2}\}$$

and $\|\widehat{\eta}_k - \eta_{0k}\|_{nk} = O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\}$, for any $k = 1, \ldots, d_1$.

A.3. Proof of Theorem 2. We first verify that

$$(20) \qquad n^{-1} \sum_{i=1}^n \rho_2(m_{0i}) \widetilde{\mathbf{Z}}_i \Gamma(\mathbf{X}_i)^{\mathrm{T}} (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) = o_P(n^{-1/2}),$$

$$(21) \qquad n^{-1} \sum_{i=1}^n \{(\widehat{\eta} - \eta_0)(\mathbf{X}_i)\} \rho_2(m_{0i}) \widetilde{\mathbf{Z}}_i = o_P(n^{-1/2}),$$

where $\widetilde{\mathbf{Z}}$ is defined in (6).

Define

$$(22) \qquad \mathcal{M}_n = \{m(\mathbf{x}, \mathbf{z}) = \eta(\mathbf{x}) + \mathbf{z}^{\mathrm{T}} \boldsymbol{\beta} : \eta \in \mathcal{G}_n\}.$$

Noting that ρ_2 is a fixed bounded function under (C7), we have $E[(\widehat{\eta} - \eta_0)(\mathbf{X})\rho_2(m_0)\widetilde{Z}_l]^2 \le O(\|\widehat{m} - m_0\|_2^2)$, for $l = 1, \ldots, d_2$. By Lemma 2, the logarithm of the ε-bracketing number of the class of functions

$$\mathcal{A}_1(\delta) = \{\rho_2\{m(\mathbf{x}, \mathbf{z})\}\{\mathbf{z} - \Gamma(\mathbf{x})\} : m \in \mathcal{M}_n, \|m - m_0\| \le \delta\}$$

is $c\{N_n \log(\delta/\varepsilon) + \log(\delta^{-1})\}$, so the corresponding entropy integral

$$\mathcal{J}_{[\cdot]}(\delta, \mathcal{A}_1(\delta), \|\cdot\|) \le c\delta\{N_n^{1/2} + \log^{1/2}(\delta^{-1})\}.$$

According to Lemmas 4 and 5 and Theorem 1, $\|\widehat{m} - m_0\|_2 = O_P\{N_n^{1/2-p} + (N_n/n)^{1/2}\}$. By Lemma 7 of Stone (1986), we have $\|\widehat{\eta} - \eta_0\|_\infty \le cN_n^{1/2}\|\widehat{\eta} - \eta_0\|_2 = O_P(N_n^{1-p} + N_n n^{-1/2})$, thus

$$(23) \qquad \|\widehat{m} - m_0\|_\infty = O_P(N_n^{1-p} + N_n n^{-1/2}).$$

Thus by Lemma 1 and Theorem 1, for $r_n = \{N_n^{1/2-p} + (N_n/n)^{1/2}\}^{-1}$,

$$E\left|n^{-1}\sum_{i=1}^{n}\{(\widehat{\eta}-\eta_0)(\mathbf{X}_i)\}\rho_2(m_{0i})\widetilde{\mathbf{Z}}_i - E[(\widehat{\eta}-\eta_0)(\mathbf{X})\rho_2\{m_0(\mathbf{T})\}\widetilde{\mathbf{Z}}]\right|$$

$$\leq n^{-1/2}Cr_n^{-1}\{N_n^{1/2} + \log^{1/2}(r_n)\}\left[1 + \frac{cr_n^{-1}\{N_n^{1/2} + \log^{1/2}(r_n)\}}{r_n^{-2}\sqrt{n}}M_0\right]$$

$$\leq O(1)n^{-1/2}r_n^{-1}\{N_n^{1/2} + \log^{1/2}(r_n)\},$$

where $r_n^{-1}\{N_n^{1/2} + \log^{1/2}(r_n)\} = o(1)$ according to condition (C5). By the definition of $\widetilde{\mathbf{Z}}$, for any measurable function ϕ, $E[\phi(\mathbf{X})\rho_2\{m_0(\mathbf{T})\}\widetilde{\mathbf{Z}}] = \mathbf{0}$. Hence (21) holds. Similarly, (20) follows from Lemmas 1 and 5.

According to condition (C6), the projection function $\Gamma^{\mathrm{add}}(\mathbf{x}) = \sum_{k=1}^{d_1}\Gamma_k(x_k)$, where the theoretically centered function $\Gamma_k \in \mathcal{H}(p)$. By the result of de Boor [(2001), page 149], there exists an empirically centered function $\widehat{\Gamma}_k \in \mathcal{S}_n^0$, such that $\|\widehat{\Gamma}_k - \Gamma_k\|_\infty = O_P(N_n^{-p})$, $k = 1,\ldots,d_1$. Denote $\widehat{\Gamma}^{\mathrm{add}}(\mathbf{x}) = \sum_{k=1}^{d_1}\widehat{\Gamma}_k(x_k)$ and clearly $\widehat{\Gamma}^{\mathrm{add}} \in \mathcal{G}_n$. For any $\boldsymbol{\nu} \in R^{d_2}$, define $\widehat{m}_{\boldsymbol{\nu}} = \widehat{m}(\mathbf{x},\mathbf{z}) + \boldsymbol{\nu}^{\mathrm{T}}\{\mathbf{z} - \widehat{\Gamma}^{\mathrm{add}}(\mathbf{x})\} = \{\widehat{\eta}(\mathbf{x}) - \boldsymbol{\nu}^{\mathrm{T}}\widehat{\Gamma}^{\mathrm{add}}(\mathbf{x})\} + (\widehat{\boldsymbol{\beta}} + \boldsymbol{\nu})^{\mathrm{T}}\mathbf{z} \in \mathcal{M}_n$, where $\mathcal{M}_n$ is given in (22). Note that $\widehat{m}_{\boldsymbol{\nu}}$ maximizes the function $\widehat{l}_n(m) = n^{-1}\sum_{i=1}^{n}Q[g^{-1}\{m(\mathbf{T}_i)\},Y_i]$ for all $m \in \mathcal{M}_n$ when $\boldsymbol{\nu} = \mathbf{0}$, thus $\frac{\partial}{\partial\boldsymbol{\nu}}\widehat{l}_n(\widehat{m}_{\boldsymbol{\nu}})|_{\boldsymbol{\nu}=\mathbf{0}} = \mathbf{0}$. For simplicity, denote $\widehat{m}_i \equiv \widehat{m}(\mathbf{T}_i)$, and we have

$$(24)\qquad \mathbf{0} \equiv \left.\frac{\partial}{\partial\boldsymbol{\nu}}\widehat{l}_n(\widehat{m}_{\boldsymbol{\nu}})\right|_{\boldsymbol{\nu}=\mathbf{0}} = n^{-1}\sum_{i=1}^{n}q_1(\widehat{m}_i,Y_i)\widetilde{\mathbf{Z}}_i + O_P(N_n^{-p}).$$

For the first term in (24), we get

$$n^{-1}\sum_{i=1}^{n}q_1(\widehat{m}_i,Y_i)\widetilde{\mathbf{Z}}_i = n^{-1}\sum_{i=1}^{n}q_1(m_{0i},Y_i)\widetilde{\mathbf{Z}}_i$$

$$+ n^{-1}\sum_{i=1}^{n}q_2(m_{0i},Y_i)(\widehat{m}_i - m_{0i})\widetilde{\mathbf{Z}}_i$$

$$(25)$$

$$+ n^{-1}\sum_{i=1}^{n}q_2'(\bar{m}_i,Y_i)(\widehat{m}_i - m_{0i})^2\widetilde{\mathbf{Z}}_i$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

We decompose II into two terms II_1 and II_2 as follows:

$$\mathrm{II} = n^{-1}\sum_{i=1}^{n}q_2(m_{0i},Y_i)\widetilde{\mathbf{Z}}_i\{(\widehat{\eta}-\eta_0)(\mathbf{X}_i)\} + n^{-1}\sum_{i=1}^{n}q_2(m_{0i},Y_i)\widetilde{\mathbf{Z}}_i\mathbf{Z}_i^{\mathrm{T}}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)$$

$$= \mathrm{II}_1 + \mathrm{II}_2.$$

We next show that

(26) $$\mathrm{II}_1 = \mathrm{II}_1^* + o_P(n^{-1/2}),$$

where $\mathrm{II}_1^* = -n^{-1} \sum_{i=1}^n \rho_2(m_{0i}) \widetilde{\mathbf{Z}}_i \{ (\widehat{\eta} - \eta_0)(\mathbf{X}_i) \}$. Using an argument similar to the proof of Lemma 5, we have

$$(\widehat{\eta} - \eta_0)(\mathbf{X}_i) = \mathbf{B}_i^{\mathrm{T}} \mathbf{K} \mathbf{V}_n^{-1} \left\{ n^{-1} \sum_{i=1}^n q_1(m_{0i}, Y_i) \mathbf{D}_i^{\mathrm{T}} + o_P(N_n^{-p}) \right\},$$

where $\mathbf{K} = (\mathbf{I}_{N_n d_1}, \mathbf{0}_{(N_n d_1) \times d2})$ and $\mathbf{I}_{N_n d_1}$ is a diagonal matrix. Note that the expectation of the square of the sth column of $n^{-1/2}(\mathrm{II}_1 - \mathrm{II}_1^*)$ is

$$E\left[n^{-1/2} \sum_{i=1}^n \{ q_2(m_{0i}, Y_i) + \rho_2(m_{0i}) \} \widetilde{Z}_{is} (\widehat{\eta} - \eta_0)(\mathbf{X}_i) \right]^2$$

$$= n^{-1} \sum_{i=1}^n \sum_{j=1}^n E\{ \varepsilon_i \varepsilon_j \rho_1'(m_{0i}) \rho_1'(m_{0j}) \widetilde{Z}_{is} \widetilde{Z}_{js} (\widehat{\eta} - \eta_0)(\mathbf{X}_i)(\widehat{\eta} - \eta_0)(\mathbf{X}_j) \}$$

$$= n^{-3} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n E\{ \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l \rho_1'(m_{0i}) \rho_1'(m_{0j}) \rho_1(m_{0k}) \rho_1(m_{0l})$$

$$\times \widetilde{Z}_{is} \widetilde{Z}_{js} \mathbf{B}_i^{\mathrm{T}} \mathbf{K} \mathbf{V}_n^{-1} \mathbf{D}_i^{\mathrm{T}} \mathbf{B}_j^{\mathrm{T}} \mathbf{K} \mathbf{V}_n^{-1} \mathbf{D}_j^{\mathrm{T}} \}$$

$$+ o(n N_n^{-2p}) = o(1), \qquad s = 1, \ldots, d_2.$$

Thus, (26) holds by Markov's inequality. Based on (21), we have $\mathrm{II}_1^* = o_P(n^{-1/2})$. Using similar arguments and (20) and (21), we can show that

$$\mathrm{II}_2 = -n^{-1} \sum_{i=1}^n \rho_2(m_{0i}) \widetilde{\mathbf{Z}}_i \mathbf{Z}_i^{\mathrm{T}} (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) + o_P(n^{-1/2})$$

$$= -n^{-1} \sum_{i=1}^n \rho_2(m_{0i}) \widetilde{\mathbf{Z}}_i^{\otimes 2} (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) + o_P(n^{-1/2}).$$

According to (23) and condition (C5), we have

$$\mathrm{III} = n^{-1} \sum_{i=1}^n q_2'(\bar{m}_i, Y_i)(\widehat{m}_i - m_{0i})^2 \widetilde{\mathbf{Z}}_i$$

$$\leq C \| \widehat{m} - m_0 \|_\infty^2 = O_p\{ N_n^{2(1-p)} + N_n^2 n^{-1} \}$$

$$= o_P(n^{-1/2}).$$

Combining (24) and (25), we have

$$\mathbf{0} = n^{-1} \sum_{i=1}^n q_1(m_{0i}, Y_i) \widetilde{\mathbf{Z}}_i + \{ E[\rho_2\{m_0(\mathbf{T})\} \widetilde{\mathbf{Z}}^{\otimes 2}] + o_P(1) \} (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) + o_P(n^{-1/2}).$$

Note that

$$E[\rho_1^2\{m_0(\mathbf{T})\}\varepsilon^2\widetilde{\mathbf{Z}}^{\otimes 2}] = E[E(\varepsilon^2|\mathbf{T})\rho_1^2\{m_0(\mathbf{T})\}\widetilde{\mathbf{Z}}^{\otimes 2}] = E[\rho_2\{m_0(\mathbf{T})\}\widetilde{\mathbf{Z}}^{\otimes 2}].$$

Thus the desired distribution of $\widehat{\boldsymbol{\beta}}$ follows.

A.4. Proof of Theorem 3. Let $\tau_n = n^{-1/2} + a_n$. It suffices to show that for any given $\zeta > 0$, there exists a large constant C such that

$$(27) \qquad \mathrm{pr}\Big\{ \sup_{\|\mathbf{u}\|=C} \mathcal{L}_P(\boldsymbol{\beta}_0 + \tau_n\mathbf{u}) < \mathcal{L}_P(\boldsymbol{\beta}_0) \Big\} \geq 1 - \zeta.$$

Denote

$$U_{n,1} = \sum_{i=1}^{n}[Q\{g^{-1}(\widehat{\eta}^{\mathrm{MPL}}(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}(\boldsymbol{\beta}_0 + \tau_n\mathbf{u})), Y_i\}$$

$$- Q\{g^{-1}(\widehat{\eta}^{\mathrm{MPL}}(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\beta}_0), Y_i\}]$$

and $U_{n,2} = -n\sum_{j=1}^{s}\{p_{\lambda_n}(|\beta_{j0} + \tau_n v_j|) - p_{\lambda_n}(|\beta_{j0}|)\}$, where s is the number of components of $\boldsymbol{\beta}_{10}$. Note that $p_{\lambda_n}(0) = 0$ and $p_{\lambda_n}(|\beta|) \geq 0$ for all β. Thus, $\mathcal{L}_P(\boldsymbol{\beta}_0 + \tau_n\mathbf{u}) - \mathcal{L}_P(\boldsymbol{\beta}_0) \leq U_{n,1} + U_{n,2}$. Let $\widehat{m}_{0i}^{\mathrm{MPL}} = \widehat{\eta}^{\mathrm{MPL}}(\mathbf{X}_i) + \mathbf{Z}_i^{\mathrm{T}}\boldsymbol{\beta}_0$. For $U_{n,1}$, note that

$$U_{n,1} = \sum_{i=1}^{n}[Q\{g^{-1}(\widehat{m}_{0i}^{\mathrm{MPL}} + \tau_n\mathbf{u}^{\mathrm{T}}\mathbf{Z}_i), Y_i\} - Q\{g^{-1}(\widehat{m}_{0i}^{\mathrm{MPL}}), Y_i\}].$$

Mimicking the proof for Theorem 2 indicates that

$$(28) \qquad U_{n,1} = \tau_n\mathbf{u}^{\mathrm{T}}\sum_{i=1}^{n}q_1(m_{0i}, Y_i)\widetilde{\mathbf{Z}}_i + \frac{n}{2}\tau_n^2\mathbf{u}^{\mathrm{T}}\boldsymbol{\Omega}\mathbf{u} + o_P(1),$$

where the orders of the first term and the second term are $O_P(n^{1/2}\tau_n)$ and $O_P(n\tau_n^2)$, respectively. For $U_{n,2}$, by a Taylor expansion and the Cauchy–Schwarz inequality, $n^{-1}U_{n,2}$ is bounded by $\sqrt{s}\tau_n a_n\|\mathbf{u}\| + \tau_n^2 w_n\|\mathbf{u}\|^2 = C\tau_n^2(\sqrt{s} + w_n C)$. As $w_n \to 0$, both the first and second terms on the right-hand side of (28) dominate $U_{n,2}$, by taking C sufficiently large. Hence, (27) holds for sufficiently large C.

A.5. Proof of Theorem 4. The proof of $\widehat{\boldsymbol{\beta}}_2^{\mathrm{MPL}} = 0$ is similar to that of Lemma 3 in Li and Liang (2008). We therefore omit the details and refer to the proof of that lemma.

Let $\widehat{m}^{\mathrm{MPL}}(\mathbf{x}, \mathbf{z}_1) = \widehat{\eta}^{\mathrm{MPL}}(\mathbf{x}) + \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\beta}_{10}$, for $\widehat{\eta}^{\mathrm{MPL}}$ in (11), and $m_0(\mathbf{T}_{1i}) = \boldsymbol{\eta}_0^{\mathrm{T}}(\mathbf{X}_i) + \mathbf{Z}_{i1}^{\mathrm{T}}\boldsymbol{\beta}_{10}$. Define $\mathcal{M}_{1n} = \{m(\mathbf{x}, \mathbf{z}_1) = \eta(\mathbf{x}) + \mathbf{z}_1^{\mathrm{T}}\boldsymbol{\beta}_1 : \eta \in \mathcal{G}_n\}$. For any $\boldsymbol{\nu}_1 \in R^s$, where s is the dimension of $\boldsymbol{\beta}_{10}$, define

$$\widehat{m}_{\boldsymbol{\nu}_1}^{\mathrm{MPL}}(\mathbf{t}_1) = \widehat{m}(\mathbf{x}, \mathbf{z}_1) + \boldsymbol{\nu}_1^{\mathrm{T}}\widetilde{\mathbf{z}}_1 = \{\widehat{\eta}^{\mathrm{MPL}}(\mathbf{x}) - \boldsymbol{\nu}_1^{\mathrm{T}}\Gamma_1(\mathbf{x})\} + (\widehat{\boldsymbol{\beta}}_1^{\mathrm{MPL}} + \boldsymbol{\nu}_1)^{\mathrm{T}}\mathbf{z}_1.$$

Note that $\widehat{m}_{\boldsymbol{\nu}_1}^{\mathrm{MPL}}$ maximizes $\sum_{i=1}^{n} Q[g^{-1}\{m_0(\mathbf{T}_{1i})\}, Y_i] - n\sum_{j=1}^{s} p_{\lambda_{jn}}(|\widehat{\beta}_{j1}^{\mathrm{MPL}} + v_{j1}|)$ for all $m \in \mathcal{M}_{1n}$ when $\boldsymbol{\nu}_1 = \mathbf{0}$. Mimicking the proof for Theorem 2 indicates that

$$
\begin{aligned}
\mathbf{0} = {} & n^{-1}\sum_{i=1}^{n} q_1\{m_0(\mathbf{T}_{1i}), Y_i\}\widetilde{\mathbf{Z}}_{1i} + \{p'_{\lambda_{jn}}(|\beta_{j0}|)\operatorname{sign}(\beta_{j0})\}_{j=1}^{s} + o_P(n^{-1/2}) \\
& + \{E[\rho_2\{m_0(\mathbf{T}_1)\}\widetilde{\mathbf{Z}}_1^{\otimes 2}] + o_P(1)\}(\widehat{\boldsymbol{\beta}}_1^{\mathrm{MPL}} - \boldsymbol{\beta}_{10}) \\
& + \left\{\sum_{j=1}^{s} p''_{\lambda_{jn}}(|\beta_{j0}|) + o_P(1)\right\}(\widehat{\boldsymbol{\beta}}_{j1}^{\mathrm{MPL}} - \beta_{j0}).
\end{aligned}
$$

Thus, asymptotic normality follows because

$$
\begin{aligned}
\mathbf{0} = {} & n^{-1}\sum_{i=1}^{n} q_1\{m_0(\mathbf{T}_{1i}), Y_i\}\widetilde{\mathbf{Z}}_{1i} + \xi_n + o_P(n^{-1/2}) \\
& + \{\boldsymbol{\Omega}_s + \boldsymbol{\Sigma}_\lambda + o_P(1)\}(\widehat{\boldsymbol{\beta}}_1^{\mathrm{MPL}} - \boldsymbol{\beta}_{10}),
\end{aligned}
$$

$$
E[\rho_1^2\{m_0(\mathbf{T}_1)\}\{Y - m_0(\mathbf{T}_1)\}^2\widetilde{\mathbf{Z}}_1^{\otimes 2}] = E[\rho_2\{m_0(\mathbf{T}_1)\}\widetilde{\mathbf{Z}}_1^{\otimes 2}].
$$

SUPPLEMENTARY MATERIAL

Detailed proofs and additional simulation results of: Estimation and variable selection for generalized additive partial linear models
(DOI: 10.1214/11-AOS885SUPP; .pdf). The supplemental materials contain detailed proofs and additional simulation results.

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