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An action for the super-5-brane in D=11 supergravity

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Abstract: An alternative path is taken for deriving an action for the supersymmetric 5-brane in 11 dimensions. Selfduality does not follow from the action, but is consistent with the equations of motion for arbitrary supergravity backgrounds. The action involves a 2-form as well as a 5-form world-volume potential; inclusion of the latter makes the action, as well as the non-linear selfduality relation for the 3-form field strength, polynomial. The requirement of invariance under κ-transformations determines the form of the selfduality relation, as well as the action. The formulation is shown to be equivalent to earlier formulations of 5-brane dynamics.

11-dimensional supergravity [1] is believed to be the low-energy effective theory for M-theory. Arising as a strong coupling limit of type IIA superstring theory [2], it is connected to the entire web of dualities relating different vacua of string theory/M-theory (see e.g. refs. [2,3]). The extended objects in 11-dimensional supergravity seem to be closely connected to non-perturbative properties of the theory. The ones carrying non-gravitational charges are a membrane and a 5-brane, and in addition there are the KK (or gravitational) branes, which are a wave and a KK-monopole (6-brane). The field configuration [4] corresponding to the membrane [5], acting as an electric source for the 3-form potential, is singular, and the membrane, though never properly quantised, may prove to constitute the elementary excitations of M-theory [6]. The 5-brane, on the other hand, is a solitonic, non-singular, field configuration, carrying magnetic charge with respect to the 3-form. A world-volume theory for a 5-brane is thought of as a low-energy theory for the moduli of such solitonic solutions.

Due to the fact that the world-volume theory contains chiral bosons, namely a (non-linearly) selfdual field strength, there are difficulties with a lagrangian formulation. There are a number of different routes for circumventing the problem. One is to give up manifest Lorentz invariance [7]. Another is to use one of the existing techniques for lagrangian formulations of theories with chiral bosons [8]. These techniques were applied to the 11-dimensional 5-brane in ref. [9]. The formulation contains auxiliary fields, whose main raison d'être seems to be to parametrise the "non-covariance" of ref. [7]. Although the methods of refs. [7,9] work locally on the world-volume, there may be global obstructions. Another interesting way of deriving the equations of motion is to consider geometric properties of the embedding of the world-volume superspace into target superspace [10,11].

A different approach, which will be followed in this paper, was suggested by Witten [12]. One may write down a lagrangian whose equations of motion do not imply selfduality, but allow it. The allowed form will be unique due to the couplings to background fields. One then needs a prescription for the implementation of this selfduality in a path integral. Ref. [12] discusses this issue in the linearised model, and we will not have anything further to say about that point. We will concentrate on giving the full non-linear version of the action in this approach, and to show, by explicit calculation, that it indeed follows from demanding the correct amount of supersymmetry on the world-volume. The obtained action is quite simple, and might be useful in situations where an effective 5-brane action is needed. Although some of the aspects of 5-brane dynamics are well understood and some may have to await a more profound understanding of M-theory, there are remaining questions, e.g. concerning world-volume solitons [13] and anomaly cancellations [12,14], that should have appropriate answers in the supergravity limit.

Some words on the notation adopted in this paper: the conventions used for forms on superspace are the ordinary ones, where the exterior derivative acts by wedge product from the right. Complex conjugation does not revert the order of fermions, which results in the absence of factors of i, found in most of the literature.

The only antisymmetric tensor field in D=11 supergravity is the 3-form potential C, with field strength $H = dC$. As usual when treating dynamics of "higher-dimensional" branes, it is convenient to include dual field strengths in the background. We therefore also have a 6-form potential $\mathscr{C}$ with field strength $\mathscr{H} = d\mathscr{C} + \frac{1}{2}C\wedge H$, so that its Bianchi identity reads

$$d\mathscr{H} - \tfrac{1}{2}H\wedge H = 0\,. \tag{1}$$

The second term is due to consistency of duality with the presence of the Chern–Simons term $\sim \int C\wedge H\wedge H$. Analogous consistency conditions will be crucial for determining the form of the 5-brane action and its background coupling. The 5-brane is formulated as a 6-dimensional world-volume embedded in 11-dimensional superspace, so the above identities are taken as being forms on superspace. When demonstrating κ-symmetry, we will need to specify the gauge-invariant background quantities carrying at least one spinor index. The constraints containing non-vanishing field components relevant for the κ-variation are (with suitable normalisation)

$$\begin{aligned} T_{\alpha\beta}{}^{a} &= 2(\gamma^{a})_{\alpha\beta}\,,\\ H_{ab\alpha\beta} &= 2(\gamma_{ab})_{\alpha\beta}\,,\\ \mathscr{H}_{abcde\alpha\beta} &= 2(\gamma_{abcde})_{\alpha\beta}\,, \end{aligned} \tag{2}$$

where T is the superspace torsion.

Following ref. [15], we now introduce one world-volume antisymmetric tensor field for each target space one, generically coupled to the latter with the D-brane type coupling "$F = dA - C$". In the present case, we thus have a 2-form potential A and a 5-form $\mathscr{A}$. When the target space fields obey modified Bianchi identities like eq. (1) for $\mathscr{H}$, so will the world-volume ones (the corresponding case for type IIB was constructed in ref. [15]). Demanding that the world-volume field strengths are invariant under both world-volume and target space gauge transformations determines their form in terms of the potentials (up to trivial rescalings) as

$$\begin{aligned} F &= dA - C\,,\\ \mathscr{F} &= d\mathscr{A} - \mathscr{C} - \tfrac{1}{2}A\wedge H\,, \end{aligned} \tag{3}$$

so that the Bianchi identities read

$$\begin{aligned} dF &= -H\,,\\ d\mathscr{F} &= -\mathscr{H} - \tfrac{1}{2}F\wedge H\,. \end{aligned} \tag{4}$$

It is a straightforward exercise to show that these fields indeed are invariant under

$$
\begin{aligned}
\delta C &= dL\,,\\
\delta\mathscr{C} &= d\mathscr{L}-\tfrac{1}{2}L\wedge H\,,\\
\delta A &= L+dl\,,\\
\delta\mathscr{A} &= \mathscr{L}+d\ell+\tfrac{1}{2}l\wedge H\,.
\end{aligned}
\tag{5}
$$

where L and $\mathscr{L}$ are target superspace 2-form and 5-form gauge parameters, while l and ℓ are world-volume 1- and 4-forms. We use identical symbols for target superspace forms and their pullbacks (which of course are bosonic forms on the world-volume).

When considering κ-transformations, these are induced by local translations of the world-volume in fermionic target space directions as $\delta_\kappa Z^M=\kappa^\alpha E_\alpha{}^M$, which results in the transformation of the pullback of a target space form $\delta_\kappa\Omega=\mathscr{L}_\kappa\Omega\equiv(i_\kappa d+di_\kappa)\Omega$. The parameter κ will, as usual, be constrained to contain only half a spinor worth of independent components. The transformations of the world-volume fields A and $\mathscr{A}$ must be specified so that their field strengths transform gauge-invariantly. These transformations are

$$
\left.\begin{aligned}
\delta_\kappa A &= i_\kappa C\,,\\
\delta_\kappa\mathscr{A} &= i_\kappa\mathscr{C}+\tfrac{1}{2}A\wedge i_\kappa H\,,
\end{aligned}\right\}\quad\Longrightarrow\quad
\left\{\begin{aligned}
\delta_\kappa F &= -i_\kappa H\,,\\
\delta_\kappa\mathscr{F} &= -i_\kappa\mathscr{H}-\tfrac{1}{2}F\wedge i_\kappa H\,.
\end{aligned}\right.
\tag{6}
$$

We want to use the field strengths F and $\mathscr{F}$ to build an action for the 5-brane. The rôle of $\mathscr{F}$ is twofold. It replaces the Wess–Zumino term, which in traditional brane actions gives the electric coupling of a p-brane to a $(p+1)$-form potential. It also renders the action polynomial, as will soon be demonstrated. The entire procedure described in the following could be performed without the $\mathscr{F}$ field and with a Wess–Zumino term. We find the present formalism simpler and more appealing.

Following the procedure of refs. [16,17,18,15], we make an ansatz for the action of the form

$$
S=\int d^6\xi\,\sqrt{-g}\,\lambda\left(1+\Phi(F)-(*\mathscr{F})^2\right)\,,
\tag{7}
$$

where Φ is some yet undetermined function of F (with the lower indices of the F's contracted by inverse metrics), whose form will be determined from symmetry arguments. We might have included a numerical constant in front of the last term; it would then be determined to 1 for our actual normalisation of the fields in eq. (3). The field λ is a scalar Lagrange multiplier[†], whose equation of motion enforces

$$
*\mathscr{F}=\pm\sqrt{1+\Phi(F)}\,.
\tag{8}
$$

† The factor $\sqrt{-g}$ is inessential for the dynamics of the 5-brane, and can be omitted if one uses a scalar density Lagrange multiplier. In ref. [18], a specific choice for the g-dependence was made, whose sole consequence was that a tensionless string was included in the spectrum.

The two solutions correspond to selfduality or anti-selfduality for F and charge plus or minus 1, i.e., a 5-brane or an anti-5-brane.

We should remark that the inclusion of the 6-form field strength is by no means necessary — one could as well work with an action with a square root type kinetic term plus a Wess–Zumino term. In ref. [17], a parallel was drawn between the existence of world-volume forms of maximal rank and the the possibility of branes to possess boundaries. It was noted that the absence of boundaries for the 5-brane would seem to provide an argument against the type of formulation given in this paper. It would be interesting to understand whether or not the presence of the 6-form has more profound implications; at the moment we do not know, but simply use it because it is calculationally convenient.

The equations of motion for the world-volume potentials are

$$d(\lambda{*}K)-\lambda(*\mathscr{F})H=0\,, \tag{9a}$$

$$d(\lambda{*}\mathscr{F})=0\,, \tag{9b}$$

where we have introduced the field $K=\frac{\partial\Phi}{\partial F}$, and where the dependence of $\mathscr{F}$ on A is responsible for the second term in the equation of motion (9a) for A. Taking the constraint following from variation of λ into consideration, the equation of motion (9b) for $\mathscr{A}$ simply determines λ up to a constant factor. A crucial point is now to compare the equation of motion for A to its Bianchi identity, thereby determining what kind of selfduality is consistent with the ansatz*. It follows immediately that consistency of the identification of Bianchi identities and equations of motion demands the selfduality condition to read

$$*K=-(*\mathscr{F})F\,. \tag{10}$$

K will typically be given as $F+$(higher terms), and one way to continue would be to look for explicit expressions for K that turn eq. (10) into a self-consistent relation, i.e., one that expresses one of the linear chirality components of F in terms of the other one. We will however take another direction — by considering κ-symmetry, the function Φ in the action, and thereby the selfduality relation, will be determined. The so determined selfduality will be consistent.

It is amusing to note that although the action is not specified with respect to the dependence on the field F, any variation (including variations w.r.t. the metric) will contain K, which by the selfduality relation (10) may be reexpressed in terms of F. The plan is therefore to perform a κ-variation, express the variation of the action entirely in F and $\mathscr{F}$, and look for conditions on these fields that makes the variation vanish. We may actually restrict the variation to the factor $\Psi\equiv 1+\Phi(F)-(*\mathscr{F})^2$, since variation of the factors in front of it will yield terms proportional to Ψ that vanish due to the constraint $\Psi\approx 0$ (so called 1.5 order formalism), or compensated by the appropriate transformation of λ.

* The corresponding problem, without the introduction of a world-volume 5-form potential, was formulated and solved to lowest order in F in ref. [19].

Performing the κ-variation and using the known relation between the number of F's and g's in Φ gives

$$\begin{aligned}\delta\Psi = \tfrac{1}{6}K^{ijk}\delta F_{ijk} - \tfrac{1}{4}F^{i}{}_{kl}K^{jkl}\delta g_{ij}\\ + \tfrac{2}{6!}\mathscr{F}^{ijklmn}\delta\mathscr{F}_{ijklmn} - \tfrac{1}{5!}\mathscr{F}^{iklmnp}\mathscr{F}^{j}{}_{klmnp}\delta g_{ij}\,.\end{aligned} \qquad (11)$$

Inserting the background fields of eq. (2) gives the explicit expression in terms of the spinor κ:

$$\delta_\kappa\Psi = 4(*\mathscr{F})\bar{E}_i\left[\tfrac{1}{5!\sqrt{-g}}\varepsilon^{ijklmn}\gamma_{jklmn} + \tfrac{1}{2}{*F^{ijk}}\gamma_{jk} + \tfrac{1}{4}F^{(i}{}_{kl}{*F^{j)kl}}\gamma_j + (*\mathscr{F})\gamma^i\right]\kappa\,. \qquad (12)$$

The task is now to find a matrix $M^\alpha{}_\beta$ of half rank, so that (12) vanishes for $\kappa = M\zeta$. An explicit calculation, where eq. (12) is expanded in antisymmetric products of γ-matrices of different ranks, using the most general ansatz for M, shows that one necessarily must have

$$\kappa = \left[*\mathscr{F} - \tfrac{1}{\sqrt{-g}}\varepsilon^{ijklmn}\left(\tfrac{1}{6!}\gamma_{ijklmn} - \tfrac{1}{2(3!)^2}F_{ijk}\gamma_{lmn}\right)\right]\zeta \qquad (13)$$

for some spinor ζ. Two other necessary conditions come out of the calculation, namely the form of the selfduality relation,

$$-(*\mathscr{F}){*F_{ijk}} = F_{ijk} - \tfrac{1}{2}q_{[i}{}^{l}F_{jk]l} + \tfrac{1}{12}\mathrm{tr}k F_{ijk}\,, \qquad (14)$$

where[†] $k_{ij} = \frac{1}{2}F_i{}^{kl}F_{jkl}$, $q_{ij} = k_{ij} - \frac{1}{6}g_{ij}\mathrm{tr}k$, and

$$(*\mathscr{F})^2 = 1 + \tfrac{1}{24}\mathrm{tr}q^2 - \tfrac{1}{144}(\mathrm{tr}k)^2\,. \qquad (15)$$

Due to selfduality, the matrix q fulfills $q^2 = \frac{1}{6}\mathrm{tr}q^2$, and the the antisymmetrisation made explicit in eq. (14) is automatic. By comparing the form of the selfduality relation to the condition (10), the function Φ in the action is determined:

$$\Phi = \tfrac{1}{6}\mathrm{tr}k - \tfrac{1}{24}\mathrm{tr}q^2 + \tfrac{1}{144}(\mathrm{tr}k)^2\,, \qquad (16)$$

so that, by the equation of motion for the Lagrange multiplier,

$$(*\mathscr{F})^2 = \left(1 + \tfrac{1}{12}\mathrm{tr}k\right)^2 - \tfrac{1}{24}\mathrm{tr}q^2\,. \qquad (17)$$

† The matrix k looks formally similar to the one denoted by the same symbol in ref. [20] (denoted r below), but since our F is non-linearly selfdual, it contains a trace in addition to the traceless part.

Consistency of these two expressions for $*\mathscr{F}$ (eqs. (15) and (17)) clearly demands the identity

$$\mathrm{tr}q^2 = 2\mathrm{tr}k + \tfrac{1}{6}(\mathrm{tr}k)^2 \tag{18}$$

to hold. This relation follows from the selfduality relation (14) by contracting it with F^{ijk}. Thus the on-shell value of $*\mathscr{F}$ simplifies to

$$(*\mathscr{F})^2 = 1 + \tfrac{1}{12}\mathrm{tr}k\,. \tag{19}$$

There are two more (related) consistency checks to be performed, that concern chirality: the internal consistency of the selfduality relation (14) and the chirality projection implied by eqn. (13). By applying the selfduality relation twice, one obtains a constraint on $(*\mathscr{F})^2$ which is identical to eqn. (17). The half-rank property of the matrix in eq. (13) is also guaranteed by selfduality and the actual value of $(*\mathscr{F})^2$. This concludes the demonstration of κ-symmetry and the internal consistency of the action, in the sense that it allows the consistent restriction to a chiral self-interacting supersymmetric sector.

It should be stressed that neither eq. (18) nor any other identities following from the selfduality relation may be reinserted in the lagrangian. The field equations would then change, so would the selfduality relation, and the entire intricate web of consistency relations between equations of motion, selfduality and κ-symmetry would break down.

Finally, we would like to show how our fields are related to the ones in other formulations, where non-linear selfduality relations also are encountered. We start by searching for a 3-form field h, formed from F, which is linearly chiral. For simplicity, we chose positive chirality, $*h = h$, and set $*\mathscr{F} = -N$ with N positive. An ansatz

$$h_{ijk} = \varrho\big(F_{ijk} + \alpha q_i{}^l F_{jkl}\big) \equiv \varrho m_i{}^l F_{jkl}\,, \tag{20a}$$

$$F_{ijk} = \varrho^{-1}(m^{-1})_i{}^l h_{jkl}\,, \tag{20b}$$

gives $\alpha = -\frac{1}{2N(N+1)}$. Note that eqn. (20$b$) is identical, up to normalisation, to the relation between the linearly selfdual field strength and the closed one established in ref. [20], provided that ϱ is a constant and that $m_{ij} = g_{ij} - \frac{1}{2}h_i{}^{kl}h_{jkl}$. Calculating the matrix $r_{ij} \equiv \frac{1}{2}h_i{}^{kl}h_{jkl}$, one finds using eqs. (18) and (19) that

$$r = \varrho^2 q\big(1 + \tfrac{1}{6}\alpha^2\mathrm{tr}q^2 + \tfrac{1}{3}\alpha\mathrm{tr}k\big) = \frac{2\varrho^2}{N(N+1)}q = -4\varrho^2\alpha q\,. \tag{21}$$

This provides a (highly non-trivial) check that the matrix m has the desired form for a constant value of ϱ, $\varrho = \pm\frac{1}{2}$, reflecting only a difference in the normalisation of F with respect to the background tensor field.

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