COMBINATORICS OF OSCILLATING TABLEAUX

A Thesis

by

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ABSTRACT

In this paper, we first introduce the RSK algorithm, which gives a correspondence between integer sequences and standard tableaux. Then we introduce Schensted's theorem and Greene's theorem that describe how the shape of the standard tableau is determined by the sequence. We list four different bijections constructed by using the RSK insertion. The first one is a bijection between vacillating tableaux and pairs (P, T), where P is a set of ordered pairs and T is a standard tableau. The second one is a bijection between set partitions of $[n]$ and vacillating tableaux. The third one is about partial matchings and up-down tableaux and the final one is from sequences to pairs (T, P), where T is still a standard tableau and P is a special oscillating tableau. We analyze some combinatorial statistics via these bijections.

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NOMENCLATURE

f^λ	The number of standard Young tableaux of shape λ
$g_\lambda(n)$	The number of vacillating tableaux of shape λ and length $2n$
$B(n,k)$	The number of partitions of $[n]$ with k blocks distinguished.
cr(P)	Crossing number
ne(P)	Nesting number
m_k^λ	The number of vacillating tableaux of shape λ, length $2k$
$\tilde{f}_k^\lambda$	The number of up-down tableaux of length k, shape μ
$[\|\mu\|]$	The shape of $\|\mu\|$ squares in one row
$[1^{\|\mu\|}]$	The shape of $\|\mu\|$ squares in one column
π_α	The order permutation of α
λ^*	The conjugate of λ
$d_k(\sigma)$	The length of the longest subsequence of σ which has no increasing subsequences of length $k+1$
$a_k(\sigma)$	The length of the longest subsequence of σ which has no decreasing subsequences of length $k+1$
$Des(\pi)$	The set of all descents of π
$d(\pi)$	The descent number of π
$maj(\pi)$	The major index of π
$BS(w)$	The backsteps in a permutation
$Des(P)$	The descent set in a standard tableau

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1. INTRODUCTION

RSK algorithm, named after Robinson, Schensted, and Knuth, is a remarkable combinatorial correspondence between a sequence of integers and a semistandard tableau. Given a sequence $\sigma = (a_1, a_2, ..., a_n)$ of distinct integers, RSK algorithm associates σ to a pair (P, Q) of standard Young tableau (SYT) of the same shape. Such constructions lead to combinatorial proofs of certain identities. For example, if the sequence is a permutation of {1,2,...,n}, then the RSK algorithm proves

$$n! = \sum_{\lambda} (f^{\lambda})^2,$$

where the sum is over all partitions λ of n and f^{λ} is the number of standard Young tableaux of shape λ.

We introduce the RSK algorithm in chapter 2. In the third chapter, we introduce Schensted's theorem, which describes how the tableaux encodes length of the longest increasing and decreasing subsequences of σ. Explicitly, if σ corresponds to a tableau T with shape $\lambda = \{\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_q > 0\}$, then the length of the longest increasing subsequence of σ is λ_1 and the length of the longest decreasing subsequence of σ is q. Greene extended Schented's theorem by giving an interpretation of the rest of the shape of λ.

The main content of this thesis are four different bijections. The first bijection is from vacillating tableaux to pairs (P, T), where P is a set of ordered pairs and T is a standard tableau. This bijection proves the identity

$$g_{\lambda}(n) = B(n, k) f^{\lambda}$$

where $g_{\lambda}(n)$ is the number of vacillating tableaux of shape λ and length $2n$, f^{λ} is the number of standard tableaux of shape λ and content [*k*], and $B(n, k)$ is the number of partitions of [*n*] with k blocks distinguished.

The second bijection is from set partitions P to vacillating tableaux $(\lambda^i)_{i=0}^{N}$ of empty shape. In this case, there is an interesting relation between the crossing number of the set partition cr(P) and the nesting number of the set partition ne(P) and the vacillating tableau. Similar to Schensted's theorem, cr(P) equals to the most number of rows in any λ^i and ne(P) equals to the most number of columns in any λ^i. In addition, if we restrict the partitions to complete matchings, the distribution of descents in matching can be observed in the sequence of tableaux.

The third bijection is from up-down (oscillating) tableaux to a pair (L, Q_μ), where L is a two line array of distinct integers and Q_μ is a standard Young tableau with shape μ. The construction gives a combinatorial proof of the identity:

$$\tilde{f}_k^\mu = \binom{k}{|\mu|}(2r-1)!!f^\mu, \quad \mu \vdash (k-2r)$$

where $\tilde{f}_k^\mu$ is the number of all up-down tableaux of length k and shape μ. Compared with the construction above, we can consider the array L as a partial matching. Once we determine the shape of the oscillating tableau, especially for the two special shapes $[|\mu|]$ or $[1^{|\mu|}]$, we can find the rest part of the matching is determined.

The fourth bijection is from integer sequences in $[n]^r$ to pairs (T, P), where T is a standard tableau and P is a special oscillating tableau named r-vacillating tableau. It proves the identity

$$n^r = \sum_{\lambda \vdash n} f^\lambda m_r^\lambda$$

where m_r^λ is the number of r-vacillating tableaux of shape λ and length $2r$. The bijection also carries the backsteps associated to integer sequences to the descent set on standard tableaux.

2. RSK INSERTION AND RSK ALGORITHM

First, we introduce the RSK algorithm [1].

Definition 1. *An* integer partition *of a nonnegative integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ of positive integers satisfying $\sum_{1 \le i \le l} \lambda_i = n$. Empty partition is included and denoted by $\emptyset$. We let Par $:= \bigcup_{n \ge 0}$ Par(n).*

Definition 2. *Let $\lambda = \lambda_1 ... \lambda_r \in$Par with $r \le n$. A* semistandard tableau (SST) T *over $\{1, 2, ..., n\}$ of shape λ is a scheme*

$$T = \begin{matrix} T_{11} & T_{12} & \dots & T_{1\lambda_1} \\ T_{21} & T_{22} & \dots & T_{2\lambda_2} \\ \dots & & & \\ T_{r1} & \dots & T_{r\lambda_r} & \end{matrix} \qquad T_{ij} \in \{1, 2, .., n\}$$

with $T_{i1} \le T_{i2} \le ... \le T_{i\lambda_i}$ for the rows and $T_{1j} < T_{2j} < ...$ for the columns. Set $\mu_k = \#\{T_{ij} : T_{ij} = k\}$; then $\mu = (\mu_1, \mu_2, .., \mu_n)$ is called the type *of T.*

Algorithm 1 The RSK Insertion

The elementary operation is the insertion of an element x into a given semistandard tableau T, T$\leftarrow x$:

1. Let R be the first row of T.
2. While x is less than some element in R, do

   A Let y be the smallest element of R greater than x;

   B Replace $y \in R$ with x;

   C Let $x := y$ and let R be the next row.
3. Place x at the end of R.

Given a matrix $A = (a_{i,j})_{i,j\geq 1}$ over $\mathbb{N}_0$ with finitely many nonzero elements and $\sum_{i,j} a_{i,j} = n$, we can define a $2 \times n$-scheme associated to A as:

$$\begin{pmatrix} 1 & ... & 1 & 2 & ... & 2 & 3 & ... & 3 & ... \\ 1 & ... & & 1 & ... & & 1 & ... & & \end{pmatrix}$$

where column $\binom{i}{j}$ appears a_{ij} times. Obviously, it is a bijection, so we can identify the matrix with its associated $2 \times n$-scheme.

$$A = \begin{pmatrix} i_1 & i_2 & ... & i_n \\ j_1 & j_2 & ... & j_n \end{pmatrix}$$

Algorithm 2 The RSK Algorithm

The RSK Algorithm associates to a given matrix A a pair (P, Q) of SSTs by inserting elements step by step.

1. $P(0) = Q(0) = \emptyset$.
2. Suppose $(P(t), Q(t))$ has been constructed, then:
   A $P(t+1)$=$P(t)\leftarrow j_{t+1}$
   B $Q(t+1)$ arises from $Q(t)$ by putting i_{t+1} in that position such that $Q(t+1)$ has the same shape as $P(t+1)$. The other elements of $Q(t)$ remain unchanged.

Theorem 1. *The RSK algorithm gives a bijection between the matrices A over $\mathbb{N}_0$ and the ordered pairs (P, Q) of semistandard tableaux of the same shape, where $type(P)$=col(A), $type(Q)$=row(A).*

The key point of the proof is the inverse construction $(P, Q) \to A$. Let Q_{rs} be the rightmost entry of the largest element in Q which is the position where the last insertion path in $P(t)$. P_{rs} was bumped by the rightmost entry in row $r-1$. From row $r-1$ we go back to $r-2$ until we find the element a that was inserted in $P(t-1)$ and set $Q(t-1) = Q(t)\backslash Q_{rs}$, and continue.

Theorem 2. *The RSK algorithm has a symmetry property:*

$$A \xrightarrow{RSK} (P,Q) \textit{ implies } A^T \xrightarrow{RSK} (Q,P)$$

Definition 3. *A semistandard tableau T over {1,2,...,n} is called a standard tableau (or Young tableau) SYT if every $k \in \{1,...,n\}$ appears exactly once in T.*

We can find that the entries in a standard tableau are distinct integers and each row and column forms an increasing sequence. Given a sequence $\alpha = a_1a_2...a_n$ of positive integers, we can get the *order permutation* of α, $\pi_\alpha = r(a_1)r(a_2)...r(a_n)$.

Example 1.

$$S = \begin{pmatrix} 112223566 \\ 231156144 \end{pmatrix} \rightarrow \tilde{S} = \begin{pmatrix} 123456789 \\ 451289367 \end{pmatrix}$$

The matrix $\tilde{A}$ belonging to $\tilde{S}$ is thus an $n \times n$ permutation matrix and the semistandard tableau turns to be a standard tableau.

The construction is used to prove the symmetry of RSK insertion. In addition, all the following contents in this paper are based on standard tableaux.

3. SCHENSTED'S THEOREM

Theorem 3 (Schensted [2])**.** *If σ^* is obtained from σ by writing the sequence $a_1, a_2, ..., a_n$ in reverse order, then $P(\sigma^*) = P(\sigma)^T$(the transpose of $P(\sigma)$). If $P(\sigma)$ has shape λ, then $P(\sigma^*)$ has shape λ^*(the conjugate of λ).*

Given a sequence of distinct integers $\sigma = a_1a_2...a_n$, Schensted described a method for computing the length of the longest increasing and decreasing subsequences of σ. We can get a pair of Young tableaux(P, Q) associating to σ. Assume the shape of tableau P is defined by a partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_q\}$ and $\lambda^* = \{\lambda_1^* \geq \lambda_2^* \geq ... \geq \lambda_r^*\}$ is the partition conjugate to λ.

Theorem 4 (Schensted's Theorem)**.** *The length of the longest increasing subsequence of σ is λ_1 (the number of columns of P), and the length of the longest decreasing subsequence is λ_1^* (the number of rows of P).*

3.1 Greene's Theorem

Greene extended Schensted's Theorem by interpreting the rest of the shape of λ [2]. For each $k \leq n$, $d_k(\sigma)$ denotes the length of the longest subsequence of σ which has no increasing subsequences of length $k+1$ and $a_k(\sigma)$ denotes the length of the longest subsequence of σ which has no decreasing subsequences of length $k+1$.

Lemma 1. *Let $(X, \leq)$ be a finite partially ordered set, and let r be the largest size of a chain. Then X can be partitioned into r but no fewer antichains.*

Lemma 2 (Dilworth's Theorem)**.** *Let $(X, \leq)$ be a finite partially ordered set, and let m be the largest size of an antichain. Then X can be partitioned into m but no fewer chains.*

According to these two lemmas, the definitions of $d_k(\sigma)$ and $a_k(\sigma)$ are equivalent to the length of the longest subsequence consisting of k descending subsequences and the length of the longest subsequence consisting of k ascending subsequences.

Theorem 5 (Greene's Theorem [2]). *For each $k \leq n$,*

$$a_k(\sigma) = \lambda_1 + \lambda_2 + ... + \lambda_k$$
$$d_k(\sigma) = \lambda_1^* + \lambda_2^* + ... + \lambda_k^*$$

I introduce the proof briefly. First, Knuth gave a complete characterization for two sequences σ_1 and σ_2 with the same tableau, that is, $P(\sigma_1) = P(\sigma_2)$.

Theorem 6 (Knuth). *Suppose $x < y < z$. Let σ_1 be a sequence which contains three adjacent terms of one of the following four types: $(y, x, z), (y, z, x), (x, z, y)$ or (z, x, y). If σ_2 is obtained from σ_1 by interchanging x and z, then $P(\sigma_1) = P(\sigma_2)$.*

Every sequence can be transformed into a sequence $\tilde{\sigma}$ obtained by listing the rows of $P(\sigma)$ in order starting from the bottom. $\tilde{\sigma}$ can give the tableau back again, so it can be thought of as a *canonical form* for sequence σ. Then we can assume the sequence σ with the tableau $P(\sigma)$

$$\sigma = (s(q,1), s(q,2), .., s(q,\lambda_q), .., s(1,1), ..., s(1,\lambda_1))$$

$$P(\sigma) = \begin{matrix} s(1,1) & s(1,2) & .. & .. & s(1,\lambda_1) \\ .. & .. & .. & s(2,\lambda_2) & \\ s(k,1) & .. & .. & & \\ .. & .. & .. & & \\ s(q,1) & ... & s(q,\lambda_q) & & \end{matrix}$$

Give a sequence of distinct integers $\sigma = (a_1, a_2, ..., a_n)$ and a subsequence of it, say, $\gamma = (a_{i_1}, a_{i_2}, ..., a_{i_q})$. If γ contains no increasing subsequences of length $k+1$, we call γ a *k-decreasing subsequence of σ*. We can find that every *k-decreasing subsequence* is the union of k decreasing subsequences. Similarly, we can define a *k-increasing subsequence of σ*.

Consider the case of canonical form. First, we prove $a_k(\sigma) = \lambda_1 + \lambda_2 + ... + \lambda_k$. Assume $k \leq q$, then the subsequence starting with $s(k, 1)$ and containing all subsequent elements is *k-increasing* and has length $\lambda_1 + \lambda_2 + ... + \lambda_k$. We find that σ can be partitioned into $r = \lambda_1$ decreasing subsequences, if we read each column from bottom to top.

Since a *k-increasing* sequence can intersect a decreasing sequence at most k times, it follows that

$$\lambda_1 + \lambda_2 + ... + \lambda_k \leq a_k(\sigma) \leq \sum_{i=1}^{r} \min\{k, \lambda_i^*\}$$

Here λ_i^* is the number of entries in the *i-th* column. That means we add up all the bold entries, i.e. the first k rows:

$$P(\sigma) = \begin{matrix} \mathbf{s(1,1)} & \mathbf{s(1,2)} & .. & .. & \mathbf{s(1,}\lambda_1\mathbf{)} \\ .. & .. & .. & .. & \\ \mathbf{s(k,1)} & \mathbf{s(k,2)} & .. & \mathbf{s(k,}\lambda_k\mathbf{)} & \\ .. & .. & .. & & \\ s(q,1) & ... & s(q,\lambda_q) & & \end{matrix}$$

Hence, we can get $a_k(\sigma) = \lambda_1 + \lambda_2 + ... + \lambda_k$. To get the conclusion of $d_k(\sigma)$, we consider the union of the first *k-decreasing* sequences above whose total length is $\lambda_1^* + \lambda_2^* + ... + \lambda_k^*$. Dually, the rows of $P(\sigma)$ partition σ into increasing subsequences and we can find similarly:

$$\lambda_1^* + \lambda_2^* + ... + \lambda_k^* \leq d_k(\sigma) \leq \sum_{i=1}^{r} \min\{k, \lambda_i\}$$

Hence, $d_k(\sigma) = \lambda_1^* + \lambda_2^* + ... + \lambda_k^*$. Finally, we prove the theorem.

Greene's theorem shows us that we can obtain the shape of the associated tableau without going through the whole sequence by RSK insertion.

4. INSERTIONS ON SEQUENCES OF TABLEAUX

4.1 Basic notations

Definition 4. *A* set partition *of* $[n]$ *is a set of nonempty subsets of* $[n]$ *such that every element* i *in* $[n]$ *is in exactly one of these subsets.*

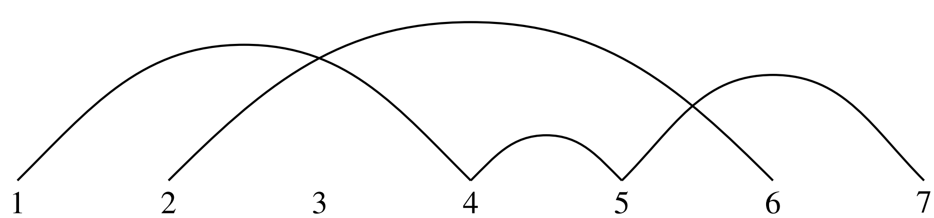
Definition 5. *A (complete) matching on* $[2n]$=$\{1, 2, ..., 2n\}$ *is a set partition of* $[2n]$ *of type* $(2, 2, ...2)$. *It can be represented by listing its* n *blocks, as* $\{(i_1, j_1), .., (i_n, j_n)\}$, *where* $i_r < j_r$. *Two blocks* (i_r, j_r) *and* (i_s, j_s) *form a* crossing *if* $i_r < i_s < j_r < j_s$ *and form a* nesting *if* $i_r < i_s < j_s < j_r$.

Definition 6. *A k-crossing of a set partition is a subset* $\{(i_1, j_1), (i_2, j_2), ..., (i_k, j_k)\}$ *of its standard representation where* $i_1 < i_2 < ... < i_k < j_1 < j_2 < ... < j_k$. *A k-nesting of a set partition is a subset* $\{(i_1, j_1), (i_2, j_2), ..., (i_k, j_k)\}$ *of its standard representation where* $i_1 < i_2 < ... < i_k < j_k < j_{k-1} < ... < j_1$.

Definition 7. *Let cr(*P*) be the maximal* i *such that* P *has an* i*-crossing, ne(*P*) the maximal* j *such that* P *has a* j*-nesting.*

A block $\{i_1, i_2, ..., i_b\}$, $i_1 < i_2 < ... < i_b$, of a set partition is represented by the set of pairs $\{(i_1, i_2), (i_2, i_3), ..., (i_{b-1}, i_b)\}$. If a set partition is represented by the union of all sets of pairs and the union is taken over all its blocks, it is called the *standard representation*. Given a partition P of $[n]$, it can be represented by a graph on the vertex set $[n]$ whose edge set consists of arcs connecting the elements of each block in numerical order.

Example 2. *The standard representation of* 1457-26-3 [3] is $\{(1, 4), (4, 5), (5, 7), (2, 6)\}$:



Definition 8. *A sequence* $\emptyset = \lambda^0, \lambda^1, ..., \lambda^n = \lambda$ *of partitions is called an* oscillating tableau (up-down tableau) of shape λ *if either* $\lambda^{i-1} \subseteq \lambda^i$ *or* $\lambda^{i-1} \supseteq \lambda^i$ *and* λ^{i-1} *and* λ^i *differ by exactly one square,* $i = 1, 2, ..., n$*. The number* n *is called the length of the oscillating tableau.*

Definition 9. *A* vacillating tableau V_λ^{2n} *of shape* λ *and length* $2n$ *is a sequence* λ_0,λ_1,...,λ_{2n} *of integer partitions such that (i)*$\lambda_0 = \emptyset$ *and* $\lambda_{2n} = \lambda$*, (ii)*λ_{2i+1} *is obtained from* λ_{2i} *by doing nothing or deleting a square, (iii)*λ_{2i} *is obtained from* λ_{2i-1} *by doing nothing or adding a square.*

Example 3. *An example of an oscillating tableau of shape* $11(\begin{smallmatrix}\square\\\square\end{smallmatrix})$ *and length 6 is given by:*

$$\left(\emptyset \quad \square \quad \square\square \quad \square \quad \begin{smallmatrix}\square\\\square\end{smallmatrix} \quad \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \quad \begin{smallmatrix}\square\\\square\end{smallmatrix} \right)$$

denoted by

$$\emptyset, 1, 2, 1, 11, 21, 11$$

For a vacillating tableau of shape $11(\begin{smallmatrix}\square\\\square\end{smallmatrix})$ *and length 10:*

$$\left(\emptyset \quad \emptyset \quad \square \quad \square \quad \square \quad \square \quad \square\square \quad \square\square \quad \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \quad \begin{smallmatrix}\square\\\square\end{smallmatrix} \quad \begin{smallmatrix}\square\\\square\end{smallmatrix} \right)$$

denoted by

$$\emptyset, \emptyset, 1, 1, 1, 1, 2, 2, 21, 11, 11$$

4.2 Bijection from vacillating tableaux to pairs (P, T)

Theorem 7. *Let* $g_\lambda(n)$ *be the number of vacillating tableaux of shape* $\lambda \vdash k$ $(\sum \lambda = k)$ *and length* $2n$*.We have:*

$$g_\lambda(n) = B(n, k) f^\lambda,$$

where f^λ *is the number of standard tableaux of shape* λ *and content* [k]*, and* $B(n, k)$ *is the number of set partitions of* [n] *with* k *blocks distinguished.*

The theorem comes from [3]. Given a vacillating tableau V of shape λ and length $2n$, we can define a sequence (P_i, T_i) where P_i is a set of ordered pairs of integers in [n], and T_i is a standard tableau of shape λ_i. P_0 is the empty set and T_0 is the empty tableau.

A1 If $\lambda^i = \lambda^{i-1}$, then $(P_i, T_i) = (P_{i-1}, T_{i-1})$.

A2 If $\lambda^i \supset \lambda^{i-1}$, then $i = 2k$ for some integer $k \in [n]$. Let $P_i = P_{i-1}$ and T_i is obtained from T_{i-1} by adding the entry k in the square $\lambda^i \backslash \lambda^{i-1}$.

A3 If $\lambda^i \subset \lambda^{i-1}$, then $i = 2k-1$ for some integer $k \in [n]$. Let T_i be the unique tableau of shape λ_i such that T_{i-1} is obtained from T_i by row (RSK)-inserting some number j. Let P_i be obtained from P_{i-1} by adding the ordered pair (j, k).

The construction above can be reversed. Given a pair (P, T_{2n}) and the standard representation of P, denoted by $E(P)$, we reconstruct the preceding tableaux and get the sequence of shapes. If we have the T_{2k} for some $k \le n$, we can get the tableaux T_{2k-1}, T_{2k-2} by the following rules:

a1 $T_{2k-1} = T_{2k}$ if the integer k does not appear in T_{2k}. Otherwise T_{2k-1} is obtained from T_{2k} by deleting the square containing k.

a2 $T_{2k-2} = T_{2k-1}$ if $E(P)$ does not have an edge of the form (i, k). Otherwise there is a unique $i < k$ such that $(i, k) \in E(P)$. In that case let T_{2k-2} be obtained from T_{2k-1} by row-inserting i.

According to the bijection $\Phi(V) = (P, T_{2n})$, we can proof the **Theorem 7** easily. If we consider the vacillating tableaux of empty shape, then we get the conclusion that

$$g_\emptyset(n) = B(n)$$

Moreover, any walk from $\emptyset$ to $\emptyset$ in $m+n$ steps can be viewed as a walk from $\emptyset$ to some shape λ in n steps, then followed by the reverse of a walk from $\emptyset$ to λ in m steps, which gives us

$$\sum_\lambda g_\lambda(n) g_\lambda(m) = g_\emptyset(m+n) = B(m+n)$$

4.3 Bijection ϕ from partitions to vacillating tableaux

There is a bijection between the set of vacillating tableaux of empty shape and the partition set of $[n]$ [3]. Given a partition $P \in \Pi_n$ with the standard representation, we construct the vacillating tableau $\phi(P)$ as follows:

Start from the empty standard Young Tableau by letting $T_{2n} = \emptyset$, read the number $j \in [n]$ one by one from n to 1, and define T_{2j-1}, T_{2j-2} for each j.

B1 If j is the righthand endpoint of an arc (i, j), but not a lefthand endpoint, first do nothing, then insert i (by RSK algorithm) into the tableau.

B2 If j is the lefthand endpoint of an arc (j, k), but not a righthand endpoint, first remove j, then do nothing.

B3 If j is an isolated point, do nothing twice.

B4 If j is the righthand endpoint of an arc (i, j), and the lefthand endpoint of another arc (j, k), then delete j first, and then insert i.

The vacillating tableau $\phi(P)$ is the sequences of shaped of these SYT's.

Example 4. *Given the partition 1457-26-3, we can get the sequence of SYT's:*

$\emptyset$	$\emptyset$	1	1	1	1	1	2	24	2	2	5	5	$\emptyset$	$\emptyset$
				2	2	2				5				

and the vacillating tableau is

$\emptyset$	$\emptyset$	□	□	□	□	□	□	□□	□	□	□	□	$\emptyset$	$\emptyset$
				□	□	□				□				

which is denoted by

$$\emptyset \quad \emptyset \quad 1 \quad 1 \quad 11 \quad 11 \quad 11 \quad 1 \quad 2 \quad 1 \quad 11 \quad 1 \quad 1 \quad \emptyset \quad \emptyset$$

There are some interesting statistics. The relation between cr(P), ne(P) and the vacillating tableaux is given in the following theorem.

Theorem 8. *Let $P \in \Pi_n$ and $\phi(P) = (\emptyset = \lambda^0, \lambda^1, ..., \lambda^{2n} = \emptyset)$. Then* cr(P) *is the most number of rows in any λ^i, and* ne(P) *is the most number of columns in any λ^i.*

The theorem is general enough to hold in other cases.

4.3.1 Descent numbers and major indices on complete matching

Restricting to the case that every block has two numbers and each number appears only once, the set partitions of $[n]$ turn to be a complete matching and the map above only has two cases, **B**1 and **B**2. we can omit the "do nothing" operation and start from the empty SYT $T_0 = \emptyset$. We find that there is an one-to-one correspondence between oscillating tableaux with shape empty, length n and the set of matchings on $[n]$.

Kim finds some properties of distribution of descents in matchings and those properties are proved by such construction [4].

Definition 10. *A permutation $\pi \in S_n$ has a* descent *at position i if $\pi(i) > \pi(i+1)$, where $i = 1, ..., n-1$. The set of all descents of π is denoted by $Des(\pi)$ and the* descent number *of π is defined as $d(\pi) := |Des(\pi)| + 1$.*

Definition 11. *Given a a permutation $\pi \in S_n$, the* major index *of π is the sum of the descent position of π, i.e.*

$$maj(\pi) = \sum_{i \in Des(\pi)} i$$

Define a sequence of Young tableaux of length $2n$. Starting with $T_0 = \emptyset$, given T_{i-1}, we define T_i as follows:

B1 If i is the smaller element in its block $\{i < j\}$ in π, let T_i be obtained from T_{i-1} by row-inserting j.

B2 If i is the larger element in its block $\{i < j\}$ in π, let T_{i-1} contains i, T_i is obtained by removing i and sliding the hole out(jeu de taquin)

In a general permutation π, it has a basic property of symmetry.

$$\sum_{\pi \in \sigma_n} p^{d(\pi)} = \sum_i a_i p^i$$

where $a_i = a_{n+1-i}$. For major indices, it has similar conclusion. Considering the permutation π' with reverse order of π, the positions of descents in π and π' are complementary, which gives us the symmetry. However, for the matchings, it doesn't hold since the reverse order of a matching might not be a matching again.

Kim finds the following theorem about symmetry which holds for matchings.

Theorem 9. *Let $\pi \in S_{2n}$ be a matching and define π' by $\pi' = T^{-1}(T(\pi)')$. Then,*

$$d(\pi) + d(\pi') = 2(n+1)$$

and

$$\text{maj}(\pi) + \text{maj}(\pi') = 2n^2$$

Example 5. *Consider the matching $\sigma = (14)(23)(56) \in S_6$. The sequence of tableaux associated with σ is*

$$\begin{pmatrix} \emptyset & \boxed{4} & \begin{array}{c}\boxed{3}\\ \boxed{4}\end{array} & \boxed{4} & \emptyset & \boxed{6} & \emptyset \end{pmatrix}$$

and the associated oscillating tableau is

$$T(\sigma) = \begin{pmatrix} \emptyset & \square & \begin{array}{c}\square\\ \square\end{array} & \square & \emptyset & \square & \emptyset \end{pmatrix}$$

The conjugate of $T(\sigma)$ is

$$T(\sigma)' = \left(\emptyset \quad \square \quad \square\square \quad \square \quad \emptyset \quad \square \quad \emptyset \right)$$

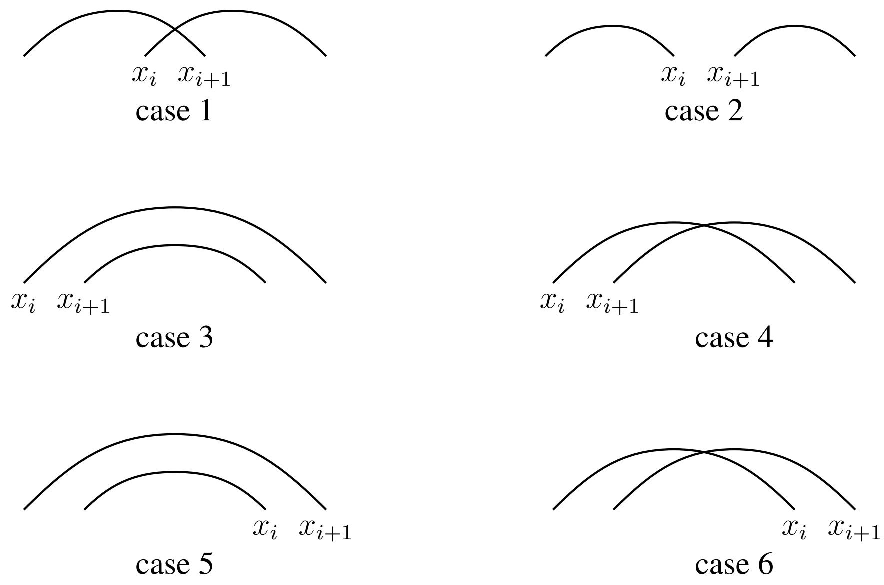
The associated matching is $\pi' = T^{-1}(T(\pi)') = (13)(24)(56)$. π has descents at position 1,2,3,5 and π' has descents at position 2,5. $d(\pi) = 5$, $d(\pi') = 3$, $d(\pi) + d(\pi') = 2(3+1)$ and $maj(\pi) = 11$, $maj(\pi') = 7$, $maj(\pi) + maj(\pi') = 2 * 3^2$.

Given a matching $\pi \in S_{2n}$, let

$$T(\pi) = (\lambda_0, ..., \lambda_{2n})$$

For any $1 \leq i \leq 2n-1$, consider the size of λ_{i-1}, λ_i, and λ_{i+1}. We can see that there are six cases:

1. If $|\lambda_{i-1}| < |\lambda_i|$ and $|\lambda_i| > |\lambda_{i+1}|$, then π has a descent at position i.
2. If $|\lambda_{i-1}| > |\lambda_i|$ and $|\lambda_i| < |\lambda_{i+1}|$, then π doesn't have a descent at position i.
3. If $|\lambda_{i-1}| < |\lambda_i| < |\lambda_{i+1}|$ and the box added the second time is in a strictly lower row than the box added the first time, then π has a descent at position i.
4. If $|\lambda_{i-1}| < |\lambda_i| < |\lambda_{i+1}|$ and the box added the second time is in a weakly higher row than the box added the first time, then π doesn't have a descent at position i.
5. If $|\lambda_{i-1}| > |\lambda_i| > |\lambda_{i+1}|$ and the box removed the first time is in a strictly lower row than the box removed the second time, then π has a descent at position i.
6. If $|\lambda_{i-1}| > |\lambda_i| > |\lambda_{i+1}|$ and the box removed the first time is in a weakly higher row than the box removed the second time, then π doesn't have a descent at position i.



These six cases cover all the possibilities of sequences of three standard tableaux and they are used to prove **Theorem 9**. According to **Theorem 9**, there is a perfect symmetry for both descent numbers and major indices.

Proposition 1. *The descent numbers and major indices form a kind of special generating functions named* Palindromic Function*, which is*

$$\sum_{\pi \in S_{2n}} p^{d(\pi)} q^{maj(\pi)} = \sum a_{i,j} p^i q^j$$

where $a_{i,j} = a_{(2n+1)-i,2n^2-j}$.

4.4 Map between oscillating tableau and partial matching

Let $\tilde{f}_k^\mu$ denote the cardinality of the set F_k^μ of all up-down tableaux of length k, shape μ, and f^μ denote the number of standard Young tableaux of shape μ. To prove the identity:

$$\tilde{f}_k^\mu = \binom{k}{|\mu|}(2r-1)!!f^\mu, \quad \mu \vdash (k-2r)$$

Sundaram [5] sets up a bijection between up-down tableaux S_μ^k and pairs (L, Q_μ), where Q_μ is a SYT of shape μ and L is a two-line array

$$L = \begin{pmatrix} j_1 & \dots & j_r \\ i_1 & \dots & i_r \end{pmatrix}$$

with j's in the top row written in increasing order and the i's in the bottom row are such that $j_k > i_k$ for each $k = 1, .., r$ and the j's and i's are all distinct and $\{entries\ in\ Q_\mu\} \cup \{entries\ in\ L\} = [k]$.

Given the sequence S_μ^k, we build up an associated sequence of tableaux T_i for each each shape μ^i of the sequence:

C1 If the sequence is increasing at step j, given the SYT T_{j-1} with shape μ^{j-1} and μ^j is one box larger than μ^{j-1}, then T_j is the SYT obtained by adding a j to T_{j-1} in the position of the added box.

C2 If μ^j is one box less than μ^{j-1}, to get T_j we do the following:

(a) Bump out the extra entry of T_{j-1} *by columns*(inverse Schensted column-insertion) toget a tableau T_j of shape μ^j, and a letter x.

(b) Record the fact that a removal occurred at step j by putting the pair (j, x) into a two-line array L, with j on top.

Since the x was bumped out at step j, it must have been inserted in an earlier step, so $x < j$. Then we know that the process reverses.

Given the pair (L, Q_μ), Q_μ is the kth step of the sequence and we reconstruct the preceding tableaux and get the sequence of shapes. Given SYT T_j, we try to get T_{j-1}:

D1 If j does not appear in the top row of the two-line array L, then we delete the box labelled j from T_j.

D2 If j does appear in the top row of L, which means the pair (j, i) is in L, then we insert i, $T_{j-1} = (i \stackrel{RSK}{\longrightarrow} T_j)$.

Example 6. *Here is an example [5]. For an oscillating tableau:*

$$1, 11, 21, 211, 111, 11, 21, 22, 221, 211,$$

we can get a pair (L, Q_μ)*:*

$$L = \left(\left\{ \begin{array}{ccc} 5 & 6 & 10 \\ 2 & 4 & 3 \end{array} \right\}, \begin{array}{cc} 1 & 7 \\ 8 & \\ 9 & \end{array} \right)$$

4.4.1 Combinatorial statistics on partial matching

As we know, there is a bijection between an up-down tableau of shape $\emptyset$, length $2n$ and a complete matching S_{2n}. Some combinatoric statistics of the matchings can be observed in the tableaux, such as crossing number and nesting number. According to Sundaram's identity, we know that there is a bijection between each up-down tableau of length k, shape μ and a pair of (L, Q_μ). It is a general case since it ends at shape μ rather than an empty shape. We can consider L as a partial matching and all the entries in Q_μ represent the lefthand endpoints of the arcs. **Theorem 8** and the property of descents still hold.

Assume that we can complement the partial matching, which means we have $|\mu|$ more steps $k+1$, $k+2$,.., $k+|\mu|$ and all of them are righthand endpoints. Generally, there are more than one cases for these extra steps to match those lefthand endpoints, i.e. those numbers in the standard Young tableau Q_μ. Then we consider two special cases that the standard Young tableaux Q_μ have shape $[|\mu|]$ or $[1^{|\mu|}]$

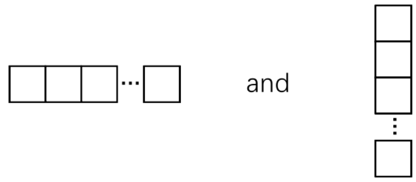


Figure 4.1: Two special shapes of SYTs

According to the rule for the standard Young tableau, once the shape and the number of the tableau are determined, there is only one case to complement the matching. Given the numbers $a_1 < a_2 < ... < a_{|\mu|}$ which are those lefthand endpoints without righthand endpoints, then

1. In the case of shape $[|\mu|]$, the remaining matching are $(a_1, k+1)$,$(a_2, k+2)$,...,$(a_{|\mu|}, k+|\mu|)$ and they form a $|\mu|$-$crossing$.

2. In the case of shape $[1^{|\mu|}]$, the remaining matching are $(a_1, k+|\mu|)$,$(a_2, k+|\mu|-1)$,...,$(a_{|\mu|}, k+1)$ and they form a $|\mu|$-$crossing$.

Ignoring the complement above, we add the infinite point for the partial matching which is considered as the righthand endpoint for those remaining lefthand endpoints. According to the definition of k-$crossing$ and k-$nesting$, these new arcs don't form any crossings or nestings and only one of them helps form crossing or nesting with original arcs.

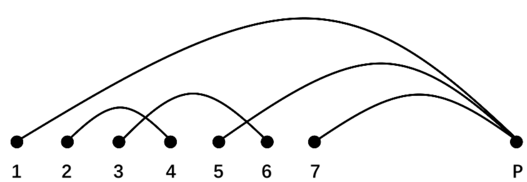


Figure 4.2: Infinite point

In this case, up-down tableaux P of shape $[|\mu|]$ have the property that the number of rows in any μ^i is equal to *ne(P)* and up-down tableaux P of shape $[1^{|\mu|}]$ have the property that the number of columns in any μ^i is equal to *cr(P)*.

5. INTEGER SEQUENCES AND R-VACILLATING TABLEAUX

To prove the identity:

$$n^r = \sum_{\lambda \vdash n} f^\lambda m_r^\lambda$$

Halverson [6] constructs an invertible map that turns a sequence $(i_1, ..., i_r)$ of numbers in the range $1 \leq i_j \leq n$ into a pair (T_λ, P_λ) consisting of a standard tableaux T_λ of shape λ and a special type of oscillating tableaux P_λ of shape λ and length $2r$ for some $\lambda \vdash n$, which is called r-vacillating tableaux. The definition of r-vacillating tableaux is:

Definition 12. *A* r-vacillating tableau *of shape λ and length $2r$ is a sequence of partitions,*

$$((n) = \lambda^{(0)}, \lambda^{(\frac{1}{2})}, \lambda^{(1)}, ..., \lambda^{(r-\frac{1}{2})}, \lambda^{(r)} = \lambda),$$

satisfying:

1. $\lambda^{(i)} \vdash n$ *and* $\lambda^{(i+\frac{1}{2})} \vdash (n-1)$*,*
2. $\lambda^{(i)} \supseteq \lambda^{(i+\frac{1}{2})}$ *and* $|\lambda^{(i)}/\lambda^{(i+\frac{1}{2})}|$*=1,*
3. $\lambda^{(i+\frac{1}{2})} \subseteq \lambda^{(i+1)}$ *and* $|\lambda^{(i+1)}/\lambda^{(i+\frac{1}{2})}|$*=1,*

The bijection uses jeu de taquin and RSK insertion [7]. First, let me introduce the former process.

Given a standard tableau of shape $\lambda \vdash n$, jeu de taquin provides an algorithm for removing the box containing x from T and producing a standard tableau S of shape $\mu \vdash (n-1)$ with $\mu \subseteq \lambda$ and $\{1, ..., n\} \backslash \{x\}$. Let $S_{i,j}$ denote the entry of S in row i and column j, and a *corner* of S is a box whose removal leaves the Young diagram of a partition. Then the process $S = (x \xleftarrow{jdt} T)$.

E1 Let $c = S_{i,j}$ be the box containing x

E2 While c is not a corner, do

(a) Let c' be the box containing min$\{S_{i+1,j}, S_{i,j+1}\}$

(b) Exchange the positions of c and c'

E3 Delete c

Given $i_1, ..., i_r$, with $1 \leq i_j \leq n$, we will produce a pair (T_λ, P_λ) consisting of a standard tableaux T_λ and r-vacillating tableaux P_λ. Initialize the first tableau to be the standard tableaux of shape (n):

$$T^{(0)} = \boxed{1}\boxed{2}...\boxed{n}$$

Then recursively define standard tableaux $T^{(j+\frac{1}{2})}$ and $T^{(j+1)}$ by:

$$T^{(j+\frac{1}{2})} = (i_{j+1} \xleftarrow{jdt} T^{(j)}), \quad 0 \leq j \leq k-1.$$
$$T^{(j+1)} = (i_{j+1} \xrightarrow{RSK} T^{(j+\frac{1}{2})}), \quad 0 \leq j \leq k-1.$$

Let $\lambda^{(j)}$ be the shape of $T^{(j)}$, and let $\lambda^{(j+\frac{1}{2})}$ be the shape of $T^{(j+\frac{1}{2})}$. Then let

$$P_\lambda = (\lambda^{(0)}, \lambda^{(\frac{1}{2})}, \lambda^{(1)}, \lambda^{(1\frac{1}{2})}, ..., \lambda^{(k)})$$
$$T_\lambda = T^{(k)}$$

The iterative delete-insert process that associates the pair to the sequence can be denoted by:

$$(i_1, ..., i_k) \xrightarrow{DI} (T_\lambda, P_\lambda)$$

Example 7. *Given sequence* $(2, 4, 3)$*, which satisfies* $1 \leq i_j \leq 6$ *[6]. At the beginning, the tableau is:*

$$T^{(0)} = \left(\boxed{1} \quad \boxed{2} \quad \boxed{3} \quad \boxed{4} \quad \boxed{5} \quad \boxed{6} \right)$$

$2 \xleftarrow{jdt} T^{(0)}$

$$T^{(\frac{1}{2})} = \begin{pmatrix} \boxed{1} & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} \end{pmatrix}$$

$2 \xrightarrow{RSK} T^{(\frac{1}{2})}$

$$T^{(1)} = \begin{pmatrix} \boxed{1} & \boxed{2} & \boxed{4} & \boxed{5} & \boxed{6} \\ \boxed{3} & & & & \end{pmatrix}$$

$4 \xleftarrow{jdt} T^{(1)}$

$$T^{(1\frac{1}{2})} = \begin{pmatrix} \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} \\ \boxed{3} & & & \end{pmatrix}$$

$4 \xrightarrow{RSK} T^{(1\frac{1}{2})}$

$$T^{(2)} = \begin{pmatrix} \boxed{1} & \boxed{2} & \boxed{4} & \boxed{6} \\ \boxed{3} & \boxed{5} & & \end{pmatrix}$$

$3 \xleftarrow{jdt} T^{(2)}$

$$T^{(2\frac{1}{2})} = \begin{pmatrix} \boxed{1} & \boxed{2} & \boxed{4} & \boxed{6} \\ \boxed{5} & & & \end{pmatrix}$$

$4 \xrightarrow{RSK} T^{(2\frac{1}{2})}$

$$T^{(3)} = \begin{pmatrix} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{6} \\ \boxed{4} & & & \\ \boxed{5} & & & \end{pmatrix}$$

Finally, we get the pair:

$$(2,4,3) \xrightarrow{DI} \left(\begin{array}{cccc} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{6} \\ \boxed{4} & & & \\ \boxed{5} & & & \end{array} , (6,5,51,41,42,41,411) \right)$$

Lemma 3. *Consider two successive row-insertions, first row-inserting x in a tableau T and then row-inserting x' in the resulting tableau $T \leftarrow x$, giving rise to two routes R and R' and two new boxes B and B'.*

1. If $x \leq x'$, then R is strictly left of R', and B is strictly left of and weakly below B'.

2. If $x > x'$, then R' s weakly left of R and B' is weakly left of and strictly below B.

Halverson's insertion gives a bijection between a sequence and a pair (T_λ, P_λ), which carries the backsteps associated to integer sequences to the descent set on standard tableaux [7]. Map sequences in $\{1, ..., n\}^r$ to permutations in S_n using following surjection

$$\{1, ..., n\}^r \rightarrow S_n$$

$$a = (a_1, ..., a_r) \mapsto w_a = \text{RT}(1, 2, .., n, a_1, ..., a_r)$$

where RT$(1, 2, .., n, a_1, ..., a_r)$ is the permutation consisting of the rightmost occurrence of each integer in {1,...,n}.

Definition 13. *The* backsteps *in a permutation $w = (w_1, ..., w_n) \in S_n$ are*

$$BS(w) = \{i | i+1 \textit{ is to the left of } i \textit{ in } w = (w_1, ..., w_n)\}$$

We have defined the descent set. If P is a standard tableau, then the descent set of P is

$$Des(P) = \{i | i+1 \textit{ is in a lower row than } i \textit{ in } P\}$$

Proposition 2. *If $a \in \{1,..,n\}^r$ and $a \xrightarrow{DI} (P_a, Q_a)$, where P_a is a standard tableau of shape $\lambda \vdash n$ and Q_a is an r-vacillating tableau starting from specific shape, then*

$$BS(w_a) = Des(P_a)$$

In the example above, the sequence is (2,4,3) and the associated permutation is

$$RT(1,2,3,4,5,6,2,4,3) = (1,5,6,2,4,3),$$

We find the backsteps in this permutation is $(3,4)$. In the tableau

$$T^{(3)} = \begin{pmatrix} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{6} \\ \boxed{4} & & & \\ \boxed{5} & & & \end{pmatrix},$$

the descent set is $(3,4)$. That is, $BS(w_a) = Des(P_a)$.

6. CONCLUSIONS

RSK insertion gives a bijection between sequences of distinct integers and standard tableaux. Schensted proved the length of the longest increasing and decreasing subsequences equals to the number of columns and rows separately. Then, Greene found a global description of the shape of the tableau.

In this paper, besides the introduction of RSK insertion and Schensted's theorem, I summarize four different bijections constructed by using RSK insertion. All these bijections can be used to prove corresponding combinatorial identities.

In the case of complete matchings, Kim found the property of symmetry which forms a special generating function named Palindromic Function. We also list all the six possibilities of sequences of three standard tableaux. Three of them form a descent and others don't.

In the case of partial matching, we consider two special shapes of standard tableaux. After adding an infinite point for the partial matching, we find that the theorem about crossing number and nesting number holds partially.

The last bijection is from a sequence of integers to a special type of oscillating tableau called r-vacillating tableau. It has an interesting property that the backstep in the permutation constructed from the sequence equals to the descent set of the standard tableau.

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