

# Optimal estimation of joint parameters in phase space

M. G. Genoni,<sup>1</sup> M. G. A. Paris,<sup>2</sup> G. Adesso,<sup>3</sup> H. Nha,<sup>4</sup> P. L. Knight,<sup>1</sup> and M. S. Kim<sup>1</sup>

<sup>1</sup>*QOLS, Blackett Laboratory, Imperial College London, SW7 2AZ, United Kingdom*

<sup>2</sup>*Dipartimento di Fisica, Università degli Studi di Milano, I-20133, Milano, Italy*

<sup>3</sup>*School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, United Kingdom*

<sup>4</sup>*Department of Physics, Texas A & M University at Qatar, Doha, Qatar*

(Dated: May 22, 2018)

We address the joint estimation of the two defining parameters of a displacement operation in phase space. In a measurement scheme based on a Gaussian probe field and two homodyne detectors, it is shown that both conjugated parameters can be measured below the standard quantum limit when the probe field is entangled. We derive the most informative Cramér-Rao bound, providing the theoretical benchmark on the estimation and observe that our scheme is nearly optimal for a wide parameter range characterizing the probe field. We discuss the role of the entanglement as well as the relation between our measurement strategy and the generalized uncertainty relations.

PACS numbers: 42.50.Xa, 03.65.Ta

## I. INTRODUCTION

One of the most promising avenues for quantum technology is the use of quantum resources to improve the sensitivity in the estimation of (not directly observable) relevant physical parameters, *e.g.* for applications in metrology and sensing [1, 2]. While the estimation of a single parameter has been extensively studied from both theoretical and experimental viewpoints [1, 3], the *joint* estimation of multiple parameters has not received enough attention (notable contributions are [4–15]). Here we take a further step forward in this direction by providing optimal quantum-enhanced strategies to estimate the two conjugated parameters characterizing a paradigmatic and ubiquitous quantum operation, phase-space displacement. Quantum states of single-mode bosonic CV systems can be described by quasiprobability distributions on a two-dimensional real phase space [16]. The operation of displacing a state by a phase space vector  $(q_0, p_0)$  is represented by the Weyl displacement operator

$$\hat{D}(q_0, p_0) = \exp(ip_0\hat{q} - iq_0\hat{p}), \quad (1)$$

where  $\hat{q}$  and  $\hat{p}$  are the two quadrature operators satisfying the canonical commutation relation  $[\hat{q}, \hat{p}] = i\mathbb{1}$ . A relevant question is, starting with a reference state  $\varrho_0$  that undergoes an unknown displacement, how accurately can we jointly estimate the two conjugate parameters  $q_0$  and  $p_0$  of the displacement operator with a measurement on the displaced state  $\varrho = \hat{D}(q_0, p_0)\varrho_0\hat{D}^\dagger(q_0, p_0)$ ? One possibility is certainly to use coherent states as initial probe states, followed by heterodyne detection as a measurement strategy at the output of the displacement transformation [4]. This has in fact been the standard technique to estimate and measure displacement, as it naturally complies with the generalized Heisenberg uncertainty relation [17]. On the other hand, one may ask whether entanglement, in the form of Einstein-Podolsky-Rosen (EPR) correlations [18], could lead to a better estimation precision for this task, as it was suggested in [19],

as it happens in the case of phase estimation in quantum metrology [1] and for the joint estimation of the temperature and coupling constant in a bosonic dissipative channel [14].

In this paper we propose and analyze a measurement scheme where CV entanglement is used to improve the estimation precision for this particular and relevant problem. For this purpose, we limit the analysis to Gaussian states and operations [20]. An introduction on LQE for multiple parameters will be given in Sec. II, while in Sec. III we will derive the bounds for the displacement estimation by considering single- and two-mode Gaussian states. In Sec. IV we show a simple measurement scheme involving two-mode squeezed thermal states which achieves the ultimate bound for different values of squeezing and thermal photons and beats the standard quantum limit for this kind of estimation. We also discuss how the performances and the bounds change when one has a-priori information about the parameters to be estimated, and when the displacement operation presents an inner uncertainty. In Sec. V we discuss the role of entanglement, showing how it is always necessary to beat the classical optimal strategy, and also necessary for symmetric probe states. In Sec. VI we discuss the relationship between this multi-parameter estimation and the generalized uncertainty relations, and finally we end the paper with some concluding remarks.

## II. MULTI-PARAMETER LOCAL ESTIMATION THEORY

How can we know that a given measurement scheme, able to estimate certain parameters, is optimal? Is it possible to estimate these parameters with a better precision? In order to answer this question, we make use of tools derived from local quantum estimation (LQE) theory [2, 21–23]. The purpose of LQE theory is indeed to determine the ultimate precision achievable and the corresponding optimal measurement for the estimation

of parameters characterizing a physical quantum system. In particular it has been applied for the estimation of different quantities, including the quantum phase of a harmonic oscillator [24, 25], CV Gaussian unitary parameters [26], the amount of quantum correlations of bipartite quantum states [27, 28], and the coupling constants of different kinds of interactions [13, 14, 29–37]. While most studies so far have been limited to single parameter estimation, the case of multiple parameters is more complex as different bounds can be derived for the same setting. Moreover, as we will point out later, these bounds are not always achievable, in particular when one deals with conjugate variables for which a Heisenberg-type uncertainty relation applies.

Let us start by considering the general case, that is a family of quantum states  $\varrho_{\mathbf{z}}$  which depend on a set of  $d$  different parameters  $\mathbf{z} = \{z_\mu\}$ ,  $\mu = 1, \dots, d$ . One can define the so called Symmetric Logarithmic Derivative (SLD) and Right Logarithmic Derivative (RLD) operators for each one of the parameters involved, respectively as

$$\frac{\partial \varrho}{\partial z_\mu} = \frac{L_\mu^{(S)} \varrho + \varrho L_\mu^{(S)}}{2} \quad \text{SLD.} \quad (2)$$

$$\frac{\partial \varrho}{\partial z_\mu} = \varrho L_\mu^{(R)} \quad \text{RLD.} \quad (3)$$

Then one can define the two matrices

$$\mathbf{H}_{\mu\nu} = \text{tr} \left[ \varrho_{\mathbf{z}} \frac{L_\mu^{(S)} L_\nu^{(S)} + L_\nu^{(S)} L_\mu^{(S)}}{2} \right], \quad (4)$$

$$\mathbf{J}_{\mu\nu} = \text{tr}[\varrho_{\mathbf{z}} L_\nu^{(R)} L_\mu^{(R)\dagger}]. \quad (5)$$

By defining the covariance matrix elements  $V(\mathbf{z})_{\mu\nu} = E[z_\mu z_\nu] - E[z_\mu]E[z_\nu]$  and a weight (positive definite) matrix  $\mathbf{G}$ , two different Cramér-Rao bounds holds

$$\text{tr}[\mathbf{G}\mathbf{V}] \geq \frac{1}{M} \text{tr}[\mathbf{G}(\mathbf{H})^{-1}], \quad (6)$$

$$\text{tr}[\mathbf{G}\mathbf{V}] \geq \frac{\text{tr}[\mathbf{G}\text{Re}(\mathbf{J}^{-1})] + \text{tr}[|\mathbf{G}\text{Im}(\mathbf{J}^{-1})|]}{M}, \quad (7)$$

where  $\text{tr}[A]$  is the trace operation on a finite dimensional matrix  $A$  and  $M$  is the number of measurements performed. We observe that if we choose  $\mathbf{G} = \mathbb{1}$ , we obtain the two bounds on the sum of the variances of the parameters involved

$$\sum_{\mu} \text{Var}(z_\mu) \geq \frac{\mathbf{B}_S}{M} := \frac{1}{M} \text{tr}[\mathbf{H}^{-1}], \quad (8)$$

$$\sum_{\mu} \text{Var}(z_\mu) \geq \frac{\mathbf{B}_R}{M} := \frac{\text{tr}[\text{Re}(\mathbf{J}^{-1})] + \text{tr}[|\text{Im}(\mathbf{J}^{-1})|]}{M}. \quad (9)$$

The matrices  $\mathbf{H}$  and  $\mathbf{J}$  are called respectively the Symmetric Logarithmic Derivative (SLD) [22] and Right Logarithmic Derivative (RLD) [4, 6–9] quantum Fisher information matrices. Neither the SLD bound  $\mathbf{B}_S$ , nor the

RLD bound  $\mathbf{B}_R$  on the sum of the variances are in general achievable [5]. The first one could not be achievable because it corresponds to the bound obtained by measuring optimally and simultaneously each single parameter and this is not possible when the optimal measurements do not commute. At the same time the RLD bound could not be achievable because the optimal estimator does not always correspond to a proper quantum measurement (that is a proper positive operator valued measure). Moreover which one of these bounds is more informative, that is which one is higher and then tighter, depends strongly on the estimation problem considered. One can then define the *most informative* Cramér-Rao bound,

$$\mathbf{B}_{\text{MI}} = \max\{\mathbf{B}_S, \mathbf{B}_R\},$$

obtaining the single inequality

$$\sum_{\mu} \text{Var}(z_\mu) \geq \frac{\mathbf{B}_{\text{MI}}}{M}.$$

### A. Cramér-Rao bounds with a-priori information

Similar bounds can be obtained in the case where one has a certain a-priori information regarding the distribution on the parameters one wants to estimate. Let us assume that the a-priori information is described by a probability distribution  $\mathcal{P}_{\text{prior}}(\mathbf{z})$ . One can define a Fisher-information matrix of the a-priori distribution as

$$\mathbf{A}_{\mu\nu} = \int d\mathbf{z} \mathcal{P}_{\text{prior}}(\mathbf{z}) \left( \frac{\partial \log \mathcal{P}_{\text{prior}}(\mathbf{z})}{\partial z_\mu} \right) \left( \frac{\partial \log \mathcal{P}_{\text{prior}}(\mathbf{z})}{\partial z_\nu} \right). \quad (10)$$

The new Cramér-Rao bounds that take into account this a-priori information, will read

$$\text{tr}[\mathbf{G}\mathbf{V}] \geq \frac{1}{M} \text{tr}[\mathbf{G}(\mathbf{H} + \mathbf{A})^{-1}], \quad (11)$$

$$\text{tr}[\mathbf{G}\mathbf{V}] \geq \frac{1}{M} (\text{tr}[\mathbf{G}\text{Re}((\mathbf{J} + \mathbf{A})^{-1})] + \text{tr}[|\mathbf{G}\text{Im}((\mathbf{J} + \mathbf{A})^{-1})|]) \quad (12)$$

and, for  $\mathbf{G} = \mathbb{1}$ ,

$$\sum_{\mu} \text{Var}(z_\mu) \geq \frac{\mathbf{B}_S(\mathbf{\Delta})}{M} := \frac{1}{M} \text{tr}[(\mathbf{H} + \mathbf{A})^{-1}], \quad (13)$$

$$\sum_{\mu} \text{Var}(z_\mu) \geq \frac{\mathbf{B}_R(\mathbf{\Delta})}{M} := \frac{1}{M} (\text{tr}[\text{Re}((\mathbf{J} + \mathbf{A})^{-1})] + \text{tr}[|\text{Im}((\mathbf{J} + \mathbf{A})^{-1})|]) \quad (14)$$

Here  $\mathbf{\Delta}$  denotes a vector of parameters characterizing the prior information at our disposal.

## B. Evaluation of the RLD Fisher information

In the following we give some details about the derivation of the RLD Fisher information when the RLD operator cannot be evaluated directly. Let us suppose that the derivative with respect to every parameter has the following form

$$\frac{\partial \varrho}{\partial z_\mu} = \varrho L_\mu^{(a)} + B_\mu \varrho. \quad (15)$$

To obtain the RLD operator  $L_\mu = L_\mu^{(a)} + L_\mu^{(b)}$  as defined in Eq. (3) we have to find the operator  $L_\mu^{(b)}$  such that (assuming that  $\varrho^{-1}$  exists)

$$B_\mu \varrho = \varrho L_\mu^{(b)}. \quad (16)$$

$$L_\mu^{(b)} = \varrho^{-1} B_\mu \varrho. \quad (17)$$

Then, after some algebra, we can express the elements of the RLD Fisher information matrix as

$$\begin{aligned} J_{\mu\nu} &= \text{tr}[\varrho L_\nu L_\mu^\dagger], \\ &= \text{tr}[\varrho L_\nu^{(a)} (L_\mu^{(a)})^\dagger] + \text{tr}[\varrho B_\mu^\dagger L_\nu^{(a)}] + \\ &\quad + \text{tr}[\varrho (L_\mu^{(a)})^\dagger B_\nu] + \text{tr}[B_\nu \varrho^2 B_\mu^\dagger \varrho^{-1}]. \end{aligned} \quad (18)$$

## III. CRAMÉR-RAO BOUNDS FOR DISPLACEMENT ESTIMATION

Let us now consider a generic probe state (pure or mixed)  $\varrho_0$ , which is displaced by the operator  $\hat{D}(q_0, p_0)$  to the state  $\varrho = \hat{D}(q_0, p_0) \varrho_0 \hat{D}^\dagger(q_0, p_0)$ . In the following we derive explicit formulas for the SLD and RLD Fisher information matrices. Let us start by considering the SLD Fisher information for a given probe state whose diagonal form reads  $\varrho_0 = \sum_n p_n |\phi_n\rangle\langle\phi_n|$ . One proves that in our case the SLD operator in Eq. (2) satisfies the property  $L_\mu^{(S)} = \hat{D}(q_0, p_0) \mathcal{L}_\mu \hat{D}^\dagger(q_0, p_0)$ , where

$$\mathcal{L}_\mu = 2i \sum_{n \neq m} \langle G_\mu \rangle_{nm} \frac{p_n - p_m}{p_n + p_m} |\phi_n\rangle\langle\phi_m| \quad (19)$$

with  $\mu, \nu = \{q_0, p_0\}$ ,  $\langle G_\mu \rangle_{nm} = \langle \phi_n | G_\mu | \phi_m \rangle$  and where  $G_{q_0} = \hat{p}$ ,  $G_{p_0} = -\hat{q}$  are the generators of the two orthogonal displacements. Then the SLD Fisher information matrix elements read

$$H_{\mu\nu} = \frac{1}{2} \text{tr}[\varrho_0 (\mathcal{L}_\mu \mathcal{L}_\nu + \mathcal{L}_\nu \mathcal{L}_\mu)], \quad (20)$$

$$= 2 \sum_{s \neq t} p_s \left( \frac{p_s - p_t}{p_s + p_t} \right)^2 (\langle G_\mu \rangle_{st} \langle G_\nu \rangle_{ts} + \langle G_\nu \rangle_{st} \langle G_\mu \rangle_{ts}). \quad (21)$$

Let us consider now more in detail the case of the RLD Fisher information. By differentiating  $\varrho$  with respect to

the parameters  $\mu = \{q_0, p_0\}$ , we obtain formulas resembling Eq. (15), where  $L_\mu^{(a)} = B_\mu^\dagger = -iG_\mu$ . Then, starting from Eq. (18), and by observing that

$$\hat{D}^\dagger(\mu) \hat{p} \hat{D}(\mu) = \hat{p} - p_0 \quad (22)$$

$$\hat{D}^\dagger(\mu) \hat{q} \hat{D}(\mu) = \hat{q} + q_0, \quad (23)$$

we can express, after some algebra, the elements of the Fisher information matrix in terms of the generators of the displacement as

$$J_{\mu\nu} = \text{tr}[G_\nu \varrho_0^2 G_\mu \varrho_0^{-1}] + \text{tr}[\varrho_0 G_\nu G_\mu] - 2\text{tr}[\varrho_0 G_\mu G_\nu].$$

We notice that the Fisher matrices do not depend on the values of the parameters to be estimated and that the only elements that are involved are the probe state and the generators of the two transformations.

### A. Most informative bounds for single- and two-mode probe states

The most general single-mode Gaussian state with zero initial displacement can be written as  $\varrho_0 = S(r) \nu_N S(r)^\dagger$  where  $\nu_N = \frac{1}{N+1} \sum_n \frac{N}{N+1} |n\rangle\langle n|$  is a thermal state and  $S(r) = \exp\{-\frac{r}{2}(a^\dagger{}^2 - a^2)\}$  is the single-mode squeezing operator. Notice that every single-mode squeezed state evolving in a noisy dissipative channel can always be written in this form, which makes this treatment important for actual implementations [38]. In this case the two bounds  $B_S$  and  $B_R$  are evaluated and the most-informative for the single-mode case  $B_{\text{MI}}^{(1)}$  is found to be equal to the RLD bound, yielding

$$B_{\text{MI}}^{(1)}(r, N) = (2N + 1) \cosh 2r + 1. \quad (24)$$

For zero squeezing the results obtained by Yuen and Lax is recovered [4]. Moreover one can verify that this bound is achieved for any value of squeezing and thermal photons by performing an heterodyne measurement. In general we observe that the bound grows with  $N$  and  $r$ . It is thus clear that single-mode squeezing is not useful for displacement estimation and the optimal measurement setup involving single-mode Gaussian probe states and heterodyne detection corresponds to using the vacuum (or any coherent state) as a probe field. The corresponding bound is denoted by

$$B_{\text{sql}} = B_{\text{MI}}^{(1)}(0, 0) = 2, \quad (25)$$

as to the standard quantum limit (SQL). We note that the SQL does not depend at all on the mean energy of the probe coherent state: by increasing the mean photon number of the coherent states one does not obtain any enhancement in the estimation precision.

Let us focus now on the more interesting two-mode case, where the displacement operator is applied only on one

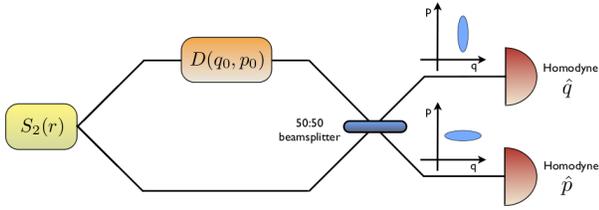


FIG. 1. (Color on-line) Measurement scheme for the estimation of the displacement given a two-mode squeezed probe state. After the displacement operation, the modes are mixed in a balanced beam splitter and then orthogonal homodyne measurements are performed on the output modes.

of the two-modes. The probe state corresponds to a two-mode squeezed thermal state, which is an archetype of the (possibly noisy) Gaussian entangled states:

$$\varrho_0 = \hat{S}_2(r)(\nu_N \otimes \nu_N)\hat{S}_2^\dagger(r), \quad (26)$$

where  $\hat{S}_2(r) = \exp\{r(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b})\}$  is the two-mode squeezing operator. The two bounds can also be straightforwardly evaluated, obtaining

$$\mathbf{B}_S^{(2)}(r, N) = \frac{2N+1}{\cosh 2r} \quad (27)$$

$$\mathbf{B}_R^{(2)}(r, N) = \frac{4N(1+N)}{(2N+1)\cosh 2r - 1}. \quad (28)$$

Both are increasing functions of the average number of thermal photons  $N$  and decreasing functions of the squeezing parameter  $r$  (and thus of the entanglement of the probe state). In this case which bound is the most informative depends on the actual values of  $r$  and  $N$ . Comparing Eqs (36) and (37), when  $\cosh(2r) < 2N+1$ ,  $\mathbf{B}_S < \mathbf{B}_R$ . Thus we define a threshold value for the squeezing as  $r_{\text{ths}} = \frac{1}{2} \cosh^{-1}(2N+1)$ , and the most informative bound reads

$$\mathbf{B}_{\text{MI}}^{(2)}(r, N) = \begin{cases} \mathbf{B}_R^{(2)}(r, N) & \text{for } r < r_{\text{ths}} \\ \mathbf{B}_S^{(2)}(r, N) & \text{for } r \geq r_{\text{ths}}. \end{cases} \quad (29)$$

We notice that for  $N=0$  the most informative bound coincides with the SLD bound, while if we increase the value of  $N$  and for small values of the squeezing parameter  $r$ , the most informative bound turns out to be the RLD bound. By inspecting the most informative bound  $\mathbf{B}_{\text{MI}}^{(2)}$ , we notice that for different values of the parameters the bound is smaller than the SQL bound  $\mathbf{B}_{\text{SQL}}$ . One may then wonder if by using entangled probe states one can achieve a better result; in the next section we indeed present a simple measurement scheme, outperforming the classical single-mode strategy.

#### IV. NEAR-OPTIMAL MEASUREMENT SCHEME

As pointed out in the previous section, if we consider coherent states as probe states and then, after the displacement operation, we perform a heterodyne measurement, we achieve the SQL  $\mathbf{B}_{\text{SQL}}$ . On the other hand, the bounds obtained for entangled probe states suggest that the SQL can be in principle overcome. Indeed we now illustrate the two-mode measurement scheme able to beat bound  $\mathbf{B}_{\text{SQL}}$  and to achieve the optimality for different values of the parameters characterizing the probe state. The scheme is pictured in Fig. 1. It clearly resembles the CV version of the dense coding protocol [39] and was already suggested for the estimation of displacement [19]. The probe state corresponds to a two-mode squeezed thermal state (26). The displacement operator is applied on one mode after which the two modes are mixed at a balanced beam splitter. Then the output fields of the beam splitter are described by the density operator

$$\varrho' = \hat{U}_{\text{bs}}\varrho\hat{U}_{\text{bs}}^\dagger = \varrho'_1 \otimes \varrho'_2 \quad (30)$$

where  $\hat{U}_{\text{bs}} = \exp\{\frac{\pi}{4}(\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b})\}$  is the beam splitter operator [16],  $\varrho = \hat{D}(q_0, p_0)\varrho_0\hat{D}^\dagger(q_0, p_0)$  is the state after the displacement, and

$$\varrho'_1 = \hat{D}(q'_0, p'_0)\hat{S}(r)\nu_N\hat{S}^\dagger(r)\hat{D}^\dagger(q'_0, p'_0) \quad (31)$$

$$\varrho'_2 = \hat{D}(q'_0, p'_0)\hat{S}(-r)\nu_N\hat{S}^\dagger(-r)\hat{D}^\dagger(q'_0, p'_0) \quad (32)$$

with  $q'_0 = \frac{q_0}{\sqrt{2}}$ ,  $p'_0 = \frac{p_0}{\sqrt{2}}$  [40]. The output state is a tensor product of two states squeezed in orthogonal directions and both displaced by the rescaled values  $q'_0$  and  $p'_0$ . One performs a homodyne measurement of the quadrature  $\hat{p}$  on the state  $\varrho'_1$  and of the quadrature  $\hat{q}$  on  $\varrho'_2$ , obtaining respectively the parameter values  $q_0$  and  $p_0$ . As the states are squeezed in orthogonal directions, the two variances approach exponentially to zero by increasing the squeezing parameter  $r$  as  $\text{Var}(q_0) = \text{Var}(p_0) = (2N+1)e^{-2r}$ : the higher is the squeezing the more precise is the estimation. The sum of the two variances is

$$\text{Var}(q_0) + \text{Var}(p_0) = 2(2N+1)e^{-2r} \geq \mathbf{B}_{\text{MI}}^{(2)}(r, N) \quad (33)$$

One can observe that we obtain for the two-parameter estimation the same optimal scaling in terms of the degree of squeezing, as the one obtained for the single-parameter displacement estimation in [41]; in particular for a pure two-mode squeezed state ( $N=0$ ), one achieves for large squeezing the Heisenberg limit scaling  $1/\bar{N}$ , where  $\bar{N} = \sinh^2 r$  denotes the mean number of photons. Comparing Eq. (33) with Eq. (25), it is clear that this scheme can outperform the single-mode strategy. For  $N=0$ , as long as squeezing is non-zero, we can estimate the parameters better than the SQL suggests. For  $N \neq 0$ , if the field exhibits two-mode squeezing, that is, if it is squeezed stronger than the following threshold

$$r > r_{\text{SQL}}(N) = \frac{1}{4} \ln(1 + 4N + 4N^2) \quad (34)$$

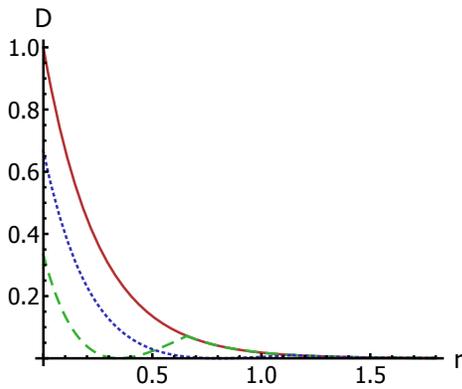


FIG. 2. (color on-line) Renormalized difference  $D(r, N)$  between the sum of the variances for the estimation of displacement with the double-homodyne scheme and the most informative bound  $B_{\text{MI}}^{(2)}(r, N)$ , as a function of the squeezing parameter  $r$  and for different values of thermal photons: continuous-red line  $N = 0$ ; dashed-green line  $N = 0.5$ ; dotted-blue line  $N = 2$ .

we can beat the SQL.

If we now compare the obtained results to the most-informative bounds derived in the previous section, we observe that the sum of the variances  $E(r, N) = \text{Var}(q_0) + \text{Var}(p_0)$  in Eq. (33) is, as expected, bounded from below by  $B_{\text{MI}}^{(2)}(r, N)$ . However one may wonder if, in some range of parameters, the scheme becomes (nearly) optimal, that is if the bound is almost saturated. In Fig. 2 we plot the quantity

$$D(r, N) = \frac{E(r, N) - B_{\text{MI}}^{(2)}(r, N)}{B_{\text{MI}}^{(2)}(r, N)}$$

for different values of  $N$  and as a function of the squeezing  $r$ . One observes that by increasing the squeezing parameter  $r$ , our scheme is optimal with  $D(r, N) \simeq 0$ . For lower values of  $r$ , we notice that  $D(r, N)$  is not always monotonically decreasing; this is because the most informative bound changes between the RLD bound  $B_{\text{R}}^{(2)}(r, N)$  and the SLD bound  $B_{\text{S}}^{(2)}(r, N)$ , as explained before. As remarked before, the measurement scheme we present resembles the CV version of the dense coding protocol [39]; however, while the dense coding protocol requires more than 4 dB of squeezing to outperform single-mode strategies, our estimation strategy outperforms the SQL for *any* value of two-mode squeezing at the input.

### A. Estimation with a-priori information

Let us consider now the case where we have some prior distribution on the parameters we want to estimate. In particular, for the sake of simplicity we consider the parameters taken randomly with the following a-priori probability distribution:

$$\mathcal{P}_{\text{prior}}(q_0, p_0) = \mathcal{G}_{0, \Delta}(q_0) \mathcal{G}_{0, \Delta}(p_0),$$

where  $\mathcal{G}_{\mu, \sigma^2}(x)$  denotes a Gaussian distribution, centered at  $\mu$  and with variance  $\sigma^2$ . The objective is to minimize the average precision one gets on the estimation of these random parameters; we can then evaluate the bounds in Eqs. (13) and (14), for different input states and strategies. By considering coherent states as the input, the most-informative RLD bound will be equal to

$$B_{\text{SQL}}(\Delta) := \frac{2\Delta^2}{1 + \Delta^2}. \quad (35)$$

If we rather consider a two-mode squeezed thermal state  $\varrho_0 = S_2(r)\nu_N \otimes \nu_N S_2^\dagger(r)$  as the probe, we obtain the following bounds:

$$B_{\text{S}}^{(2)}(r, N, \Delta) = \frac{2(2N + 1)\Delta^2}{2N + 1 + 2\Delta^2 \cosh 2r} \quad (36)$$

$$B_{\text{R}}^{(2)}(r, N, \Delta) = \frac{4N(1 + N)\Delta^2}{2N(1 + N) + \Delta^2[(2N + 1) \cosh 2r - 1]}. \quad (37)$$

All these bounds decrease with the decreasing  $\Delta$ . We note that all the bounds discussed before can be re-obtained by taking the limit of flat a-priori distribution ( $\Delta \rightarrow \infty$ ). If one fixes the value of  $\Delta$ , one can define the most-informative bound  $B_{\text{MI}}^{(2)} = \max\{B_{\text{S}}^{(2)}, B_{\text{R}}^{(2)}\}$ , and compare it with the corresponding SQL bound. One could then ask which is the optimal measurement strategy, and if the precision obtained saturates the most informative bounds for different possible probe states. Let us start by considering input coherent states: as proved in [4], the optimal cheating strategy corresponds to multiply the heterodyne outcomes by a factor

$$K_c = \frac{\Delta^2}{1 + \Delta^2}. \quad (38)$$

It is indeed easy to check that with this choice the obtained averaged variances are equal to the SQL limit  $B_{\text{SQL}}(\Delta)$  derived in Eq. (35) (notice that this is also the optimal choice used in [42, 43] to derive the classical benchmark for teleportation of coherent states). Let us consider the general case where, for given values of  $q_0$  and  $p_0$ , the variances obtained with a certain measurement strategy are equal and do not depend on the parameters themselves, that is

$$\text{Var}(q_0) = \text{Var}(p_0) := \text{Var}_0.$$

One can prove that the scaling factor that minimizes the average sum of the variances is equal to

$$K_{\text{min}} = \frac{\Delta^2}{\text{Var}_0 + \Delta^2}, \quad (39)$$

and the obtained result is

$$\langle \text{Var}_K(q_0) + \text{Var}_K(p_0) \rangle = \frac{2\text{Var}_0 \Delta^2}{\text{Var}_0 + \Delta^2},$$

where  $\langle \cdot \rangle$  denotes the average on the a-priori distribution. This is also the case for the two-mode squeezed thermal states considered before. Of course the scaling factor in this case depends on the probe state parameters, since

$$\text{Var}_0 = (2N + 1)e^{-2r}. \quad (40)$$

If this information is not available, one can always adopt the coherent states optimal strategy and use the scaling factor  $K_c$  in Eq. (38), which does not depend on the input state, obtaining

$$\langle \text{Var}_K(q_0) + \text{Var}_K(p_0) \rangle = \frac{\Delta^2(1 + \Delta^2 \text{Var}_0)}{(1 + \Delta^2)^2}. \quad (41)$$

The different results for two-mode squeezed thermal

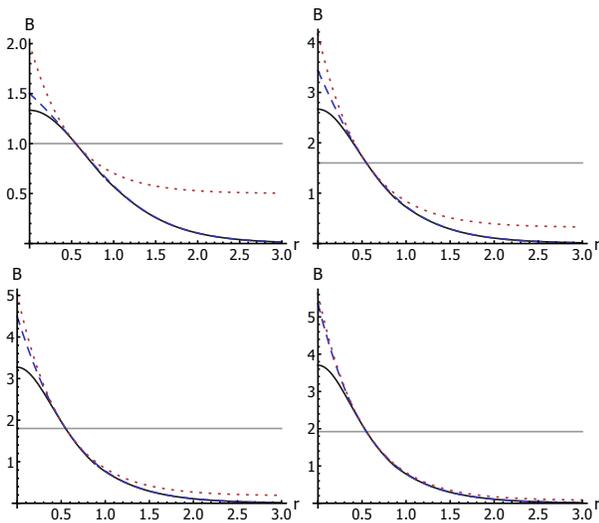


FIG. 3. (color online) Dashed-blue line: average sum of the variances for two-mode squeezed thermal probe states with  $N = 1$  mean thermal photons and by adopting the optimal scaling factor  $K_{\min}$ . Dotted-red line: average sum of the variances for two-mode squeezed thermal probe states with  $N = 1$  mean thermal photons and by adopting the coherent state scaling factor  $K_c$ . Solid-black line: most-informative bound for two mode squeezed thermal state (with  $N = 1$ ). Solid-grey line: standard quantum limit  $\mathcal{B}_{\text{SQL}}$ . All the plots are functions of the squeezing parameter  $r$  and for different values of the a-priori uncertainty  $\Delta$ . Top-left:  $\Delta = 1$ ; top-right:  $\Delta = 2$ ; bottom-left:  $\Delta = 3$ ; bottom-right:  $\Delta = 5$ .

states are shown in Fig. 3: one observes that by using the scaling factor  $K_{\min}$ , the estimation strategy is nearly optimal, that is the most-informative bound is saturated for a wide range of parameters of the probe state, and for different values of the a-priori uncertainty  $\Delta$ . One also observes that by using the simpler scaling factor  $K_c$ , one still beats the SQL limit by increasing the squeezing parameter; in particular, for zero thermal photons ( $N = 0$ ), the entangled assisted strategy always beat the SQL for any value of the squeezing parameter  $r$ . On the other hand this strategy is far to be optimal for low values of  $\Delta$  and for large values of the squeezing parameter  $r$ .

## B. Estimation of imperfect displacement operations

Let us consider the case where the displacement operation is imperfect. We thus have an additional uncertainty on the parameters we want to estimate. We assume that the two corresponding values are distributed according to a certain probability distribution  $\mathcal{P}_{\text{err}}(q', p')$  which has mean values  $q_0$  and  $p_0$ . The output state, after the displacement operation, can thus be written as

$$\varrho = \int dq' dp' \mathcal{P}_{\text{err}}(q', p') \hat{D}(q', p') \varrho \hat{D}^\dagger(q', p'). \quad (42)$$

For the sake of simplicity, let us consider that the error probability is a product of two Gaussian independent probability distributions, *i.e.*

$$\mathcal{P}_{\text{err}}(q', p') = \mathcal{G}_{q_0, \Delta_q^2}(q') \mathcal{G}_{p_0, \Delta_p^2}(p').$$

Using our entanglement-assisted estimation strategy, we obtain the following result for the variances of the estimated parameters:

$$\text{Var}(q_0) + \text{Var}(p_0) = 2(2N + 1)e^{-2r} + \Delta_q^2 + \Delta_p^2. \quad (43)$$

It is clear that the additional uncertainties are simply added to the previous results, giving, as expected, a worse performance in terms of estimation precision.

## V. THE ROLE OF ENTANGLEMENT

In our measurement scheme, we make use of entangled Gaussian states showing EPR correlations, as probe states. One may then ask whether the entanglement of these states is necessary (or even sufficient) to beat the SQL bound obtained by means of the single-mode strategy. For this purpose, we consider a generic two-mode Gaussian state, without local squeezing, *i.e.* with  $\langle (\Delta \hat{q}_i)^2 \rangle = \langle (\Delta \hat{p}_i)^2 \rangle$ ,  $i = 1, 2$ . This is a reasonable choice because we know that local squeezing does not help in our scheme. Given a generic two-mode quantum state, the corresponding quadrature operators  $\hat{q}_i$  and  $\hat{p}_i$  and an arbitrary (nonzero) real number  $a$ , if we define the operators  $\hat{u}$  and  $\hat{v}$  as  $\hat{u} = |a| \hat{q}_1 + \frac{1}{a} \hat{q}_2$ ,  $\hat{v} = |a| \hat{p}_1 - \frac{1}{a} \hat{p}_2$ , Duan et al. [44] proved that, the condition

$$\langle (\Delta u)^2 \rangle + \langle (\Delta v)^2 \rangle < a^2 + \frac{1}{a^2} \quad (44)$$

is a sufficient condition for inseparability.

One can easily notice that, the inseparability condition is the same as  $\text{Var}(q_0) + \text{Var}(p_0) < \mathcal{B}_{\text{sql}}$ , assuming  $a = 1$ , which gives the lowest bound in Eq. (44). This clearly shows that the entanglement of the probe state is a necessary condition if we are to beat the SQL obtained using coherent states and heterodyne measurements. Moreover, for symmetric states, such as the two-mode squeezed thermal state  $\varrho = \hat{S}_2(r)(\nu_N \otimes \nu_N) \hat{S}_2^\dagger(r)$ ,

it is proved that the condition (44) with  $a = 1$  is a necessary and sufficient condition for inseparability [44]. As a consequence, for this class of states, entanglement is not only necessary but also sufficient to beat the SQL. It is also straightforward to find a counterexample in order to prove that in the asymmetric case, entanglement is only necessary but not sufficient. Let us consider an asymmetric two-mode squeezed thermal state  $\varrho = \hat{S}_2(r)\nu_{N_1} \otimes \nu_{N_2} \hat{S}_2^\dagger(r)$ , with  $N_1 \neq N_2$ ; if we set  $N_1 = 0$  the state is always entangled for  $r \neq 0$ , but to beat the SQL on the estimation of displacement, one can show that  $N_2$  has to be moderately low (one can derive the threshold value as a function of squeezing parameter  $r$ ). It is worth stressing the fact that the state must be entangled before the application of the displacement operator. If we consider the case where a two-mode squeezer is applied after the action of the displacement operator on a thermal state  $\nu_N$ , no enhancement in the precision estimation can be achieved. Here, this squeezing operation can be thought as a part of the measurement process. The ultimate precision in this case coincides with the results described for single-mode states and has to comply with the SQL. This result is in fact related to the security of the CV quantum key distribution protocol with coherent states [45].

## VI. PARAMETER ESTIMATION AND UNCERTAINTY RELATIONS

We have observed that it is possible to measure the two conjugate parameters below the SQL. This seems to contradict with the (generalized) Heisenberg uncertainty relations. Nevertheless, if one looks carefully at the setup, one notices that the fundamental uncertainty relations are never violated: the variances corresponding to the true quantum quadrature operators  $\hat{q}_i$  and  $\hat{p}_i$ , on each mode involved and at every step of the measurement setup always satisfy the uncertainty relation, as it ought to be. The generalized uncertainty relations derived in [17] show that an inherent and unavoidable extra noise has to be taken into account if one wants to estimate two conjugate parameters by means of a joint measurement. However that analysis did not take into account the possibility of having a two-mode entangled state as the initial probe as described in the previous scheme [46]. In fact, in our setup the pre-existent entanglement is exploited in order to perform precise measurements on different modes, and thus on commuting observables. Specifically, if we consider the product of the corresponding variances on the estimation of the parameters  $q_0$  and  $p_0$ , we are

led to conclude that the generalized uncertainty relation seems to be violated when

$$\text{Var}(q_0)\text{Var}(p_0) < 1. \quad (45)$$

If  $\text{Var}(q_0) = \text{Var}(p_0)$ , as it is always the case by considering  $\varrho_0 = \hat{S}_2(r)\nu_N \otimes \nu_N \hat{S}_2^\dagger(r)$  as a probe state and our measurement setting, one can clearly observe that the condition (45) is equivalent to beating the SQL bound. Then, as described in the previous section, entanglement is always necessary and, in the symmetric case, also sufficient to violate the generalized uncertainty relation on the conjugate parameters by means of the proposed setup.

## VII. REMARKS

The estimation of the two conjugate parameters of a displacement operation is important both for applications and fundamental reasons. Displacement operations are indeed ubiquitous in most of the quantum protocols for CV systems. On the other hand, as we remarked earlier, this estimation follows from the uncertainty relations, and thus from the foundational properties of quantum mechanics. In this Letter we have presented a measurement scheme which estimates accurately the two real parameters characterizing the unitary operation of displacement in phase space, by using Gaussian entangled probe states and homodyne detections. We have further derived the ultimate quantum bounds on the multiparameter estimation for single and two-mode input Gaussian states, showing that our setup is optimal for a large range of parameter values characterizing the probe states. We have discussed the role of entanglement, showing that in our setup its presence is always necessary, and in symmetric cases also sufficient, to beat the standard quantum limit achievable by using coherent input states and heterodyne detection. Finally we have analyzed in detail the relationship between our results and the generalized Heisenberg uncertainty relation for conjugate parameters.

## ACKNOWLEDGMENTS

The authors acknowledge useful discussions with M. Barbieri, S. Braunstein, R. Demkowicz-Dobrzanski, M. Guta, F. Illuminati, A. Monras, S. Olivares and A. Serafini. This work was supported by the UK EPSRC, the IT-MIUR (FIRB RBF10YQ3H), and the NPRP 4-554-1-084 from Qatar National Research Fund.

[1] V. Giovannetti, S. Lloyd and L. Maccone, *Science* **306**, 1330 (2004).

[2] M. G. A. Paris, *Int. J. Quant. Inf.* **7**, 125 (2009).

[3] M. Kacprowicz, R. Demkowicz-Dobrzanski, W. Wasilewski, K. Banaszek and I. A. Walmsley, *Nature Photonics* **4**, 357 (2010).

- [4] H. P. Yuen and M. Lax, IEEE Trans Inf Theory **IT-19**, 740 (1973).
- [5] C. W. Helstrom and R. S. Kennedy, IEEE Trans. Inf. Theory **20**, 16 (1974)
- [6] V. P. Belavkin, Theoret. Math. Physics **26**, 316 (1976).
- [7] A. S. Holevo, *Probabilistic and statistical aspects of quantum theory* (North-Holland, Amsterdam, 1982).
- [8] H. Nagaoka, IEICE Technical Report **IT89-42**, 9-14 (1989).
- [9] A. Fujiwara, METR 94-9 (1994).
- [10] G. Chiribella, G. M. D'Ariano and M. F. Sacchi, J. Phys. A: Math. Gen. **39**, 2127 (2006).
- [11] K. C. Young, M. Sarovar, R. Kosut and K. B. Whaley, Phys. Rev. A **79**, 062301 (2009).
- [12] Y. Watanabe, T. Sagawa and M. Ueda, Phys. Rev. Lett. **104**, 020401 (2010).
- [13] A. Monras and F. Illuminati, Phys. Rev. A **81**, 062326 (2010).
- [14] A. Monras and F. Illuminati, Phys. Rev. A **83**, 012315 (2011).
- [15] P. J. D. Crowley, A. Datta, M. Barbieri and I. A. Walmesley, arXiv:1206.0043 [quant-ph].
- [16] S. M. Barnett and P. M. Radmore, *Methods in theoretical quantum optics* (Oxford, 1997).
- [17] E. Arthurs and J. L. Kelly Jr., Bell. Syst. Tech. J. **44**, 725 (1965); E. Arthurs and M. S. Goodman, Phys. Rev. Lett. **60**, 2447 (1988).
- [18] M. D. Reid, P. D. Drummond, W. P. Bowen, E. G. Cavalcanti, P. K. Lam, H. A. Bachor, U. L. Andersen and G. Leuchs, Rev. Mod. Phys. **81**, 1727 (2009).
- [19] M. D'Ariano, P. Lo Presti and M. G. A. Paris, Phys. Rev. Lett. **87**, 270404 (2001).
- [20] A. Ferraro, S. Olivares and M. G. A. Paris, *Gaussian States in Quantum Information*, (Bibliopolis, Napoli, 2005);
- [21] H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, NJ, 1946); C. R. Rao, *Information and the accuracy attainable in the estimation of statistical parameters*, Bull. Calcutta Math. Soc. **37**, 8189 (1945).
- [22] C. W. Helstrom, *Quantum detection and estimation theory* (Academic Press, New York, 1976).
- [23] S. Braunstein and C. Caves, Phys. Rev. Lett. **72**, 3439 (1994); S. Braunstein, C. Caves, and G. Milburn. Ann. Phys. **247**, 135 (1996).
- [24] A. Monras, Phys. Rev. A **73**, 033821 (2006).
- [25] M. G. Genoni, S. Olivares and M. G. A. Paris, Phys. Rev. Lett. **106**, 153603 (2011)
- [26] R. Gaiba and M. G. A. Paris, Phys. Lett. A **373**, 934 (2009); M. G. Genoni, C. Invernizzi and M. G. A. Paris, Phys. Rev. A **80**, 033842 (2009).
- [27] M. G. Genoni, P. Giorda and M. G. A. Paris, Phys. Rev. A **78**, 032303 (2008).
- [28] G. Brida, I. Degiovanni, A. Florio, M. Genovese, P. Giorda, A. Meda, M. G. A. Paris and A. Shurupov, Phys. Rev. Lett. **104**, 100501 (2010).
- [29] M. Sarovar and G. Milburn, J. Phys. A **39**, 8487 (2006).
- [30] M. Hotta, T. Karasawa and M. Ozawa, Phys. Rev. A **72**, 052334 (2006).
- [31] A. Monras and M. G. A. Paris, Phys. Rev. Lett. **98**, 160401 (2007).
- [32] G. Adesso, F. Dell'Anno, S. De Siena, F. Illuminati, and L. A. M. Souza, Phys. Rev. A **79**, 040305(R) (2009).
- [33] M. Aspachs, G. Adesso, and I. Fuentes, Phys. Rev. Lett. **105**, 151301 (2010).
- [34] A. Fujiwara, Phys. Rev. A **63**, 042304 (2001).
- [35] Z. Ji, G. Wang, R. Duan, Y. Feng and M. Ying, IEEE Transf. Inf. Theory **54**, 5172 (2008).
- [36] S. Boixo and A. Monras, Phys. Rev. Lett. **100**, 100503 (2008).
- [37] B. M. Escher, R. L. de Matos Filho and L. Davidovich, Nature Phys. **7**, 406 (2011).
- [38] A. Serafini, M. G. A. Paris, F. Illuminati, and S. De Siena, J. Opt. B: Quantum Semiclass. Opt. **7**, R19 (2005).
- [39] S. L. Braunstein and H. J. Kimble, Phys. Rev. A **61**, 042302 (2000).
- [40] M. S. Kim and B. C. Sanders, Phys. Rev. A **53**, 3694 (1996).
- [41] W. J. Munro, K. Nemoto, G. J. Milburn and S. L. Braunstein, Phys. Rev. A **66**, 023819 (2002).
- [42] S. L. Braunstein, C. A. Fuchs and H. J. Kimble, J. Mod. Opt. **47**, 267 (2000).
- [43] K. Hammerer, M. M. Wolf, E. S. Polzik and J. I. Cirac, Phys. Rev. Lett. **94**, 150503 (2005).
- [44] L.-M. Duan, G. Giedke, J. I. Cirac and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000)
- [45] F. Grosshans and P. Grangier, Phys. Rev. Lett. **88**, 057902 (2002).
- [46] In the derivation of the *generalized uncertainty relations* [17], joint measurements of conjugated observables are realized in an enlarged Hilbert space comprising the degrees of freedom of an auxiliary ancillary system. A basic assumption is made: signal and ancilla are prepared in a factorized (uncorrelated) state. If this assumption is dropped, as it is the case if signal and ancilla are prepared in an entangled state, the generalized uncertainty relations are no longer valid and lower bounds can be obtained.