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Abstract

Magical supergravities are a very special class of supergravity theories whose symmetries and matter content in various dimensions correspond to symmetries and underlying algebraic structures of the remarkable geometries of the Magic Square of Freudenthal, Rozenfeld and Tits. These symmetry groups include the exceptional groups and some of their special subgroups. In this paper, we study the general gaugings of these theories in six dimensions which lead to new couplings between vector and tensor fields. We show that in the absence of hypermultiplet couplings the gauge group is uniquely determined by a maximal set of commuting translations within the isometry group $SO(n_T, 1)$ of the tensor multiplet sector. Moreover, we find that in general the gauge algebra allows for central charges that may have nontrivial action on the hypermultiplet scalars. We determine the new minimal couplings, Yukawa couplings and the scalar potential.

Contents

1 Introduction

There exists a remarkable class of supergravity theories in $D = 3, 4, 5, 6$, known as magical supergravities, whose geometries and symmetries correspond to those the Magic Square of Freudenthal, Rozenfeld and Tits [1, 2]. In five dimensions these theories describe the coupling of $N = 2$ supergravity to 5, 8 , 14 and 26 vector multiplets, respectively, and are the unique *unified* Maxwell-Einstein supergravity theories with symmetric target spaces. In $D = 6$ they describe the coupling of a fixed number of vector multiplets as well as tensor multiplets to supergravity [3]. The scalar fields of these theories parametrize certain symmetric spaces in $D = 3, 4, 5$ [1] that were later referred to as very special quaternionic Kähler, very special Kähler and very special real, respectively. See [4] for a review of these geometries and their relation to $6D$ theories. The magical theories in $D = 6$ are parent theories from which all the magical supergravities in $D = 3, 4, 5$ can be obtained by dimensional reduction. The scalar coset spaces in all magical supergravities are collected in table 1. Stringy origins and constructions of some of the magical supergravity theories in various dimensions, with or without additional hypermultiplet couplings, are known [5, 6, 7, 8, 9, 10].

$D = 6$		$D = 5$		$D = 4$		$D = 3$
$\frac{SO(9,1)}{SO(9)}$	$\longrightarrow$	$\frac{E_{6(-26)}}{F_4}$	$\longrightarrow$	$\frac{E_{7(-25)}}{E_6 \times SO(2)}$	$\longrightarrow$	$\frac{E_{8(-24)}}{E_7 \times SU(2)}$
$\frac{SO(5,1)}{SO(5)}$	$\longrightarrow$	$\frac{SU^*(6)}{USp(6)}$	$\longrightarrow$	$\frac{SO^*(12)}{U(6)}$	$\longrightarrow$	$\frac{E_{7(-5)}}{SO(12) \times SU(2)}$
$\frac{SO(3,1)}{SO(3)}$	$\longrightarrow$	$\frac{SL(3,\mathbb{C})}{SO(3)}$	$\longrightarrow$	$\frac{SU(3,3)}{SU(3) \times SU(3) \times U(1)}$	$\longrightarrow$	$\frac{E_{6(+2)}}{SU(6) \times SU(2)}$
$\frac{SO(2,1)}{SO(2)}$	$\longrightarrow$	$\frac{SL(3,\mathbb{R})}{SO(3)}$	$\longrightarrow$	$\frac{Sp(6,\mathbb{R})}{U(3)}$	$\longrightarrow$	$\frac{F_{4(+4)}}{USp(6) \times USp(2)}$

Table 1: Scalar target spaces of magical supergravities in $6, 5, 4$ and 3 dimensions.

Gaugings of magical supergravities have been investigated in $D = 5$ [11, 12, 13] as well as in 4 and 3 dimensions [14, 15, 16, 17, 18]. However, the gaugings associated with the isometries of the scalar cosets in $D = 6$ listed above have not been studied so far. In this paper, we aim to close this gap. The gauging phenomenon is especially interesting in this case since it involves tensor as well as vector multiplets such that the corresponding tensor and vector fields transform in the vector and spinor representations of the isometry group $SO(n_T, 1)$, respectively. Furthermore, the vector

multiplets do not contain any scalar fields. Including the coupling of hypermultiplet couplings introduces additional subtleties with regard to the nature of full gauge group that are allowed by supersymmetry.

We determine the general gauging of magical supergravities in six dimensions and show that in the absence of hypermultiplet couplings the gauge group is uniquely determined by the maximal set of (n_T-1) commuting translations within the isometry group $SO(n_T, 1)$. In addition, a linear combination of these generators may act on the fermion fields as a $U(1)_R$ generator of the R-symmetry group $Sp(1)_R$. In the general case, the gauge algebra allows for central charges that may have nontrivial action on the hypermultiplet scalars. We show that the emergence of central charges can be explained by the fact that the gauge group is a diagonal subgroup of $(n_T - 1)$ translational isometries and $(n_T - 1)$ Abelian gauge symmetries of the vector fields.

The plan of the paper is as follows. In the next section, we give a review of the magical supergravity theories in six dimensions. In section 3 we determine the possible gauge groups and non-abelian tensor gauge transformations using the embedding tensor formalism and show that they are characterized by the choice of a constant spinor of $SO(n_T, 1)$. We also give a study of the relevant spinor orbits. In section 4, we elaborate on the structure of the gauge group by embedding the symmetries of the $6D$ magical theories in the corresponding $5D$ magical supergravities. We then choose a particular basis and evaluate the gauge group generators in the vector/tensor- and hypersector. We then work out the Yukawa couplings and the scalar potential induced by the gauging. We conclude with comments on salient features of our results and open problems as well as a discussion of the stringy origins of magical supergravity theories in section 5.

2   Ungauged $6D$ Magical Supergravity Theories

2.1   Field Content of $6D$ Magical Supergravity Theories

We consider the minimal chiral $N = (1, 0)$ supergravity in 6D coupled to n_T tensor multiplets, n_V vector multiplets and n_H hypermultiplets [19, 20, 21, 22]. We shall group together the single 2-form potential of pure supergravity that has self-dual field strength, with n_T 2-form potentials of the tensor multiplets that have anti-selfdual

field strengths, and label them collectively as $B_{\mu\nu}^I$. Thus, the field content is

$$\text{supergravity and tensor multiplets}: \quad \{e_\mu^m, \psi_\mu^i, B_{\mu\nu}^I, \chi^{ai}, L^I\},$$

$$\text{vector multiplets}: \quad \{A_\mu^A, \lambda^{Ai}\},$$

$$\text{hypermultiplets}: \quad \{\phi^X, \psi^r\}, \tag{2.1}$$

with

$$\begin{aligned}
I &= 0, 1, \ldots, n_T, \\
a &= 1, \ldots, n_T, \\
A &= 1, \ldots, n_V, \\
X &= 1, \ldots, 4n_H, \\
r &= 1, \ldots, 2n_H.
\end{aligned} \tag{2.2}$$

The gravitino, tensorino and gaugino in addition carry the doublet index of the R-symmetry group $Sp(1)_R$ labeled by $i = 1, 2$. All fermions are symplectic Majorana-Weyl, where $(\psi_\mu^i, \lambda^{Ai})$ have positive chirality and (χ^{ai}, ψ^r) have negative chirality. L^I denotes a representative of the coset space $\mathcal{M}_T = SO(n_T, 1)/SO(n_T)$ parametrized by n_T real scalars. It has the tangent space group $SO(n_T)$ with respect to which the tensorinos transform as a vector. The scalars ϕ^X parametrize a general quaternionic manifold $\mathcal{M}_Q$. We will discuss the structure of the scalar target spaces in the next two subsections.

Magical supergravities exist for the particular values $n_T = 2, 3, 5, 9$ with the vectors transforming in the spinor representation of $SO(n_T, 1)$, see table 2 for details and their explicit reality properties. A defining property of these theories is the existence of an $SO(n_T, 1)$ invariant tensor Γ_{AB}^I (the Dirac Γ-matrices for $n_T = 2, 3$, and Van der Waerden symbols for $n_T = 5, 9$, respectively), giving rise to non-trivial couplings between vector and tensor fields, and satisfying the well-known identity

$$\Gamma_{I(AB}\Gamma_{C)D}^I = 0. \tag{2.3}$$

These are the Fierz identities of supersymmetric Yang-Mills theories in the critical dimensions.

2.2 The Tensor Multiplet Scalars

The n_T scalars in the model parametrize the coset $\mathcal{M} = SO(n_T, 1)/SO(n_T)$. It is convenient to introduce the coset representatives in the $n_T + 1$ dimensional representation of the isometry group. We denote them by (L_I, L_I^a) and they obey the

G_T	$\mathcal{R}_{\rm v}$	A_μ^A	Γ_{AB}^I	$\mathcal{R}_{\rm ten}$
$SO(9,1)$	$\mathbf{16}_c$	MW	Γ_{AB}^I	$\mathbf{10}$
$SO(5,1) \times USp(2)$	$(\mathbf{4}_c, \mathbf{2})$	SMW, $A = (\alpha r)$	$\Gamma_{\alpha r, \beta s}^I = \Gamma_{\alpha\beta}^I \epsilon_{rs}$	$(\mathbf{6}, \mathbf{1})$
$SO(3,1) \times U(1)$	$(\mathbf{2}, \mathbf{1})_+ + (\mathbf{1}, \mathbf{2})_-$	W, $A = \{\alpha, \dot\beta\}$	$\begin{pmatrix} 0 & \Gamma_{\alpha\dot\beta}^I \\ \bar\Gamma_{\dot\alpha\beta}^I & 0 \end{pmatrix}$	$(\mathbf{2}, \mathbf{2})_{\mathbf{0}}$
$SO(2,1)$	$\mathbf{2}$	M	Γ_{AB}^I	$\mathbf{3}$

Table 2: The first column shows the full global symmetry groups of the magical super-gravities, the second column gives the representation content of the vector fields under these groups, whose reality properties are listed in the third column: Majorana (M), Weyl (W), Majorana-Weyl (MW), symplectic Majorana-Weyl (SMW). The last column gives the two-form representation content.

relations [3]

$$L^I L_I = -1 \,, \qquad L_a^I L_{Ib} = \delta_{ab} \,, \qquad L^I L_{Ia} = 0 \,,$$
$$I = 0, 1, \ldots, n_T \,, \quad a = 1, \ldots, n_T \,. \tag{2.4}$$

Raising and lowering of the $SO(n_T, 1)$ indices is done with the Lorentzian metric η_{IJ}, and for the $SO(n_T)$ vector indices with δ_{ab}. Equation (2.4) can be equivalently stated as

$$-L_I L_J + L_I^a L_J^a \;=\; \eta_{IJ} \,, \tag{2.5}$$

and the coset representative can be used to define the metric

$$g_{IJ} \;=\; L_I L_J + L_I^a L_J^a \,, \tag{2.6}$$

and the tensors

$$m_{AB} \;\equiv\; L_I \Gamma_{AB}^I \,, \qquad m_{AB}^a \;\equiv\; L_I^a \Gamma_{AB}^I \,, \tag{2.7}$$

with Γ_{AB}^I from table 2, which will be used to parametrize the various couplings in the action. Note that $m^{AB} \equiv -L_I \Gamma^{IAB}$ is the inverse matrix of m_{AB}. Next, we define the scalar current and $SO(n_T)$ composite connection as

$$L^{Ia} \partial_\mu L_I = P_\mu^a \,, \qquad L^{I[a} \partial_\mu L_I^{b]} = Q_\mu^{ab} \,, \tag{2.8}$$

where the covariant derivative in $D_\mu P_\nu^a$ involves the connection Q_μ^{ab}. Integrability relations state that

$$D_{[\mu} P_{\nu]}^a = 0 \ , \qquad Q_{\mu\nu}^{ab} \equiv 2\partial_{[\mu} Q_{\nu]}^{ab} + 2Q_{[\mu}^{ac} Q_{\nu]}^{cb} = -2P_{[\mu}^a P_{\nu]}^b \ . \qquad (2.9)$$

It also follows from (2.8) that

$$\partial_\mu L_I = P_\mu^a L_I^a \ , \qquad D_\mu L_I^a = P_\mu^a L_I \ . \qquad (2.10)$$

A parametrization of the coset representative which is convenient for the following can be given according to the decomposition (3-grading)

$$\mathfrak{so}(n_T, 1) \ \longrightarrow \ N_{(n_T-1)}^- \oplus \left(\mathfrak{so}(n_T - 1) \oplus \mathfrak{so}(1, 1) \right) \oplus N_{(n_T-1)}^+ \ , \qquad (2.11)$$

where the $\pm$ superscript refers to the $\mathfrak{so}(1, 1)$ charges. We choose

$$L \ = \ e^{\varphi^\alpha N_\alpha} e^{\sigma\Delta} \ , \qquad (2.12)$$

with the $(n_T - 1)$ nilpotent generators $N_\alpha \in N_{(n_T-1)}^+$, and the $\mathfrak{so}(1, 1)$ generator Δ, normalized such that $[\Delta, N_\alpha] = N_\alpha$. With this parametrization, we obtain

$$P_\mu^\alpha \ = \ e^{-\sigma} \partial_\mu \varphi^\alpha \ , \qquad P_\mu^1 \ = \ \partial_\mu \sigma \ , \qquad Q_\mu^{1\alpha} \ = \ e^{-\sigma} \partial_\mu \varphi^\alpha \ , \qquad (2.13)$$

where the index a has been split into $a \to \{1, \alpha\}$, with $\alpha = 2, \ldots, n_T$.

2.3 Hypermultiplet scalars

Supersymmetry requires the hyperscalar manifold $\mathcal{M}_Q$ to be quaternionic Kähler [23]. Let us review the basic properties of quaternionic Kähler manifolds, following [24]. They have the tangent space group $Sp(n_H) \times Sp(1)_R$, and one can introduce the vielbeins V_X^{ri} and their inverse V_{ri}^X satisfying

$$g_{XY} V_{ri}^X V_{sj}^Y = \Omega_{rs}\epsilon_{ij} \ , \qquad V_{ri}^X V^{Yrj} + \ (X \leftrightarrow Y) = g^{XY} \delta_i^j \ , \qquad (2.14)$$

where g_{XY} is the target space metric. An $Sp(n_H) \times Sp(1)_R$ valued connection is defined through the vanishing torsion condition [1]

$$\partial_X V_{Yri} + A_{Xr}{}^s V_{Ysi} + A_{Xi}{}^j V_{Yrj} \ - (X \leftrightarrow Y) = 0 \ . \qquad (2.15)$$

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[1] $Sp(n)$ refers to the compact symplectic group of rank n which is denoted as $USp(2n)$ in some of the physics literature.

From the fact that the vielbein V_{ri}^X is covariantly constant, one derives that[2]

$$R_{XYZT}V_{ri}^T V_{sj}^Z \ = \ \epsilon_{ij}\,F_{XYrs} + \Omega_{rs}\,F_{XYij} \ , \tag{2.16}$$

where F_{ij} and F_{rs} are the curvature two-forms of the $Sp(1)_R$ and $Sp(n_H)$ connection, respectively.

The manifold has a quaternionic Kähler structure characterized by three locally defined $(1,1)$ tensors $J^x{}_X{}^Y$ $(x,y,z=1,2,3)$ satisfying the quaternion algebra

$$J^x{}_X{}^Y J^y{}_Y{}^Z = -\delta^{xy}\delta_X^Z + \epsilon^{xyz}J^z{}_X{}^Z \ . \tag{2.17}$$

In terms of the vielbein, these tensors can be expressed as

$$J^x{}_X{}^Y \ = \ -i(\sigma^x)_i{}^j\,V_X^{ri}V_{rj}^Y \ , \tag{2.18}$$

with Pauli matrices σ^x. We can define a triplet of two-forms $J_{XY}^x = J^x{}_X{}^Z g_{ZY}$, and these are covariantly constant as follows

$$\nabla_X J_{YZ}^x + \epsilon^{xyz}A_X^y J_{YZ}^z = 0 \ , \tag{2.19}$$

with $A_X^x \equiv \frac{i}{2}(\sigma^x)_i{}^j A_{Xj}{}^i$. For $n_H > 1$, quaternionic Kähler manifolds are Einstein spaces, i.e. $R_{XY} = \lambda g_{XY}$. It follows, using (2.19), that [24]

$$F_{XY}^x = \frac{\lambda}{n_H + 1}J_{XY}^x \ . \tag{2.20}$$

Local supersymmetry relates λ to the gravitational coupling constant (which we will set to one), and in particular requires that $\lambda < 0$ [23], explicitly $\lambda = -(n_H + 1)$. For $n_H = 1$ all Riemannian 4-manifolds are quaternionic Kähler. Sometimes (2.20) is used to extend the definition of quaternionic Kähler to 4D, which restricts the manifold to be Einstein and self-dual [24].

Homogeneous quaternionic Kähler manifolds were classified by Wolf [25] and Alekseevski [26]. For $\lambda > 0$, they are the well known compact symmetric spaces, and for $\lambda < 0$ they are noncompact analogs of these symmetric spaces, and non-symmetric spaces found by Alekseevskii [26]. There exists an infinite family of homogeneous quaternionic Kähler spaces that are not in Alekseevskii's classification. As was shown in [27] this infinite family of quaternionic Kähler spaces arises as the scalar manifolds of $3D$ supergravity theories obtained by dimensionally reducing the generic non-Jordan family of $5D$, $N = 2$ Maxwell-Einstein supergravities discovered in [28].

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[2]In our conventions, $[\nabla_X, \nabla_Y]\,X_Z = R_{XYZ}{}^T\,X_T$.

8

Choosing $\lambda = -(n_H + 1)$, and using $M_i{}^j = -i(\sigma^x)_i{}^j M^x$ for any triplet M^x, we have the relation

$$F_{XYi}{}^j = -2V_{[X}^{jr}V_{Y]ir} \ . \tag{2.21}$$

Substitution of this relation into (2.16) and use of curvature cyclic identity gives [23]

$$F_{XYrs} = V_{[X}^{pi}V_{Y]i}^q \left(-2\Omega_{pr}\Omega_{qs} + \Omega_{pqrs}\right) \ , \tag{2.22}$$

where Ω_{pqrs} is a totally symmetric tensor defined by this equation.

For any isometry on the quaternionic Kähler manifold defined by a Killing vector field K^X, one can define the triplet of moment maps [29]

$$C^x \ \equiv \ \frac{1}{4n_H} J^x{}_Y{}^X \nabla_X K^Y \ , \tag{2.23}$$

satisfying

$$D_X C^x \ \equiv \ \partial_X C^x + \epsilon^{xyz} A_X^y C^z \ = \ J^x{}_{XY} K^Y \ , \tag{2.24}$$

where in particular we have used (2.16). Using (2.18), we can write $C_i{}^j = -i(\sigma^x)_i{}^j C^x$ as

$$C_i{}^j = -\frac{1}{2n_H} V_{ri}^X V_Y^{rj} \nabla_X K^Y \ . \tag{2.25}$$

As usual, these functions will later parametrize the Yukawa couplings and the scalar potential of the gauged theory. For later use, let us also define the function

$$C_r{}^s \ \equiv \ -\tfrac{1}{2} V_{ri}^X V_Y^{si} \nabla_X K^Y \ , \tag{2.26}$$

for a given Killing vector field K^Y, which satisfies $\nabla_X C_r{}^s = -F_{XYr}{}^s K^Y$, which may be shown in analogy with (2.24).

We conclude this section by defining the notation

$$P_\mu^{ri} \ = \ \partial_\mu \phi^X V_X^{ri} \ , \qquad Q_\mu^{ij} \ = \ \partial_\mu \phi^X A_X^{ij} \ , \qquad Q_\mu^{rs} \ = \ \partial_\mu \phi^X A_X^{rs} \ , \tag{2.27}$$

which will be used in the following sections together with the relations

$$D_{[\mu} P_{\nu]}^{ri} = 0 \ , \quad Q_{\mu\nu i}{}^j = 2P_{[\mu}^{rj} P_{\nu]ri} \ , \quad Q_{\mu\nu rs} = P_{[\mu}^{pi} P_{\nu]i}^q \left(-2\Omega_{pr}\Omega_{qs} + \Omega_{pqrs}\right) \ . \tag{2.28}$$

2.4 The Field Equation and Supersymmetry Transformations

The bosonic field equation of the full theory including the hypermultiplets are given up to fermionic contributions by [20, 22]

$$0 = G^+_{\mu\nu\rho} \ , \tag{2.29}$$

$$0 = G^{a-}_{\mu\nu\rho} \ , \tag{2.30}$$

$$0 = R_{\mu\nu} - \frac{1}{4}g_{IJ}G_{\mu\rho\sigma}{}^I G_\nu{}^{\rho\sigma J} - P^a_\mu P_{\nu a} - 2P^{ri}_\mu P_{\nu ri}$$

$$- 2m_{AB}\left(F^A_{\mu\rho}F_\nu{}^{\rho B} - \frac{1}{8}g_{\mu\nu}F^A_{\rho\sigma}F^{\rho\sigma B}\right) \ , \tag{2.31}$$

$$0 = D_\mu P^{\mu a} - \frac{1}{2}m^a_{AB}F^A_{\mu\nu}F^{\mu\nu B} - \frac{1}{6}G^a_{\mu\nu\rho}G^{\mu\nu\rho} \ , \tag{2.32}$$

$$0 = D_\mu P^{\mu ri} \ , \tag{2.33}$$

$$0 = D_\nu\left(m_{AB}F^{\mu\nu B}\right) + \left(m_{AB}G^{\mu\nu\rho}_- + m^a_{AB}G^{\mu\nu\rho\,a}_+\right)F^B_{\nu\rho} \ , \tag{2.34}$$

where we have defined the 3-form and 2-form field strengths

$$G^I_{\mu\nu\rho} = 3\partial_{[\mu}B^I_{\nu\rho]} + 3\Gamma^I_{AB}F^A_{[\mu\nu}A^B_{\rho]} \ ,$$

$$F^A_{\mu\nu} = 2\partial_{[\mu}A^A_{\nu]} \ . \tag{2.35}$$

and the projected field strengths

$$G_{\mu\nu\rho} = G^I_{\mu\nu\rho}L_I \ , \qquad G^a_{\mu\nu\rho} = G^I_{\mu\nu\rho}L^a_I \ , \tag{2.36}$$

and the superscripts $\pm$ in (2.29), (2.30) refer to the (anti-)selfdual part of the projected field strengths. The covariant derivatives acting on objects carrying the tangent space indices of the tensor and hyperscalar manifolds are defined as

$$D_\mu X^a = \partial_\mu X^a + Q^{ab}_\mu X_b \ ,$$

$$D_\mu X^{ri} = \partial_\mu X^{ri} + Q^{rs}_\mu X^i_s + Q^{ij}_\mu X^r_j \ , \tag{2.37}$$

10

with the connections from (2.8), (2.27). The fermionic field equations, to linear order in fermionic fields, take the form

$$
\begin{aligned}
0 &= \gamma^{\mu\nu\rho}D_\nu\psi^i_\rho - \frac{1}{2}G^{\mu\nu\rho}\gamma_\nu\psi^i_\rho - \frac{1}{2}\gamma^\nu\gamma^\mu\chi^{ai}P^a_\mu + \gamma^\nu\gamma^\mu\psi_r P^{ri}_\mu \\
&\quad + \frac{1}{2}m_{AB}\left(\gamma^{\rho\sigma}\gamma^\mu\lambda^{Ai}F^B_{\rho\sigma}\right) + \frac{1}{4}H^a_{\mu\nu\rho}\gamma^{\nu\rho}\chi^{ai} \; ,
\end{aligned}
\tag{2.38}
$$

$$
\begin{aligned}
0 &= \gamma^\mu D_\mu\chi^a - \frac{1}{24}\gamma^{\mu\nu\rho}\chi^a G_{\mu\nu\rho} - \frac{1}{2}m^a_{AB}\gamma^{\mu\nu}\lambda^A F^B_{\mu\nu} \\
&\quad + \frac{1}{4}G^a_{\mu\nu\rho}\gamma_{\mu\nu}\psi_\rho - \frac{1}{2}\gamma^\mu\gamma^\nu\psi_\mu P^a_\nu \; ,
\end{aligned}
\tag{2.39}
$$

$$
0 = \gamma^\mu D_\mu\psi^r + \frac{1}{24}\gamma^{\mu\nu\rho}\psi^r G_{\mu\nu\rho} - \gamma^\mu\gamma^\nu\psi_{\mu i}P^{ri}_\mu \; ,
\tag{2.40}
$$

$$
\begin{aligned}
0 &= m_{AB}\gamma^\mu D_\mu\lambda^B + \frac{1}{4}m_{aAB}\gamma^{\mu\nu}\chi^a F^B_{\mu\nu} + \frac{1}{24}m_{aAB}\gamma^{\mu\nu\rho}\lambda^B G^a_{\mu\nu\rho} \\
&\quad + \frac{1}{2}m_{aAB}\gamma^\mu\lambda^B P^a_\mu + \frac{1}{4}m_{AB}\gamma^\mu\gamma^{\nu\rho}\psi_\mu F^B_{\nu\rho} \; ,
\end{aligned}
\tag{2.41}
$$

where we have suppressed the $Sp(1)_R$ indices. The supersymmetry transformation rules, up to cubic fermion terms, are

$$
\delta e^m_\mu = \bar{\epsilon}\gamma^m\psi_\mu \; ,
$$

$$
\delta\psi_\mu = D_\mu\epsilon + \frac{1}{48}\gamma^{\rho\sigma\tau}\gamma_\mu\epsilon G_{\rho\sigma\tau} \; ,
$$

$$
\delta_{\text{cov.}}B_{\mu\nu}{}^I = -2\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]}L^I + \bar{\epsilon}\gamma_{\mu\nu}\chi^a L^I_a \; ,
$$

$$
\delta\chi^a = \frac{1}{2}\gamma^\mu\epsilon P^a_\mu - \frac{1}{24}\gamma^{\mu\nu\rho}\epsilon G^a_{\mu\nu\rho} \; ,
$$

$$
\delta L_I = \bar{\epsilon}\chi^a L^a_I \; ,
\tag{2.42}
$$

$$
\delta A^A_\mu = \bar{\epsilon}\gamma_\mu\lambda^A \; ,
$$

$$
\delta\lambda^A = -\frac{1}{4}\gamma^{\mu\nu}\epsilon F^A_{\mu\nu} \; ,
$$

$$
\delta\phi^X = V^X_{ri}\bar{\epsilon}^i\psi^r \; ,
$$

$$
\delta\psi^r = P^{ri}_\mu\gamma^\mu\epsilon_i \; .
$$

11

The covariant derivative of the supersymmetry parameter carries the Lorentz algebra valued spin connection and the $Sp(1)_R$ connection, and the covariant variation of the 2-form potential is defined as

$$\delta_{\text{cov.}} B^I_{\mu\nu} = \delta B_{\mu\nu}{}^I - 2\Gamma^I_{AB} A^A_{[\mu} \delta A^B_{\nu]} \,, \tag{2.43}$$

such that we have the general variation formula

$$\delta G^I_{\mu\nu\rho} = 3\partial_{[\mu} \delta_{\text{cov.}} B^I_{\nu\rho]} + 6\Gamma^I_{AB} F^A_{[\mu\nu} \delta A^B_{\rho]} \,. \tag{2.44}$$

The field strengths are invariant under the gauge transformations

$$\begin{aligned} \delta A^A_\mu &= \partial_\mu \Lambda^A \,, \\ \delta_{\text{cov.}} B^I_{\mu\nu} &= 2\partial_{[\mu} \Lambda^I_{\nu]} - 2\Gamma^I_{AB} \Lambda^A F^B_{\mu\nu} \,. \end{aligned} \tag{2.45}$$

2.5 The Action

The field equations described above are derivable from the following action [21, 30] [3]

$$\begin{aligned}
e^{-1}\mathcal{L} = {}& R - \frac{1}{12} g_{IJ} G^I_{\mu\nu\rho} G^{\mu\nu\rho J} - \frac{1}{4} P^a_\mu P^{\mu a} - \frac{1}{2} P^{ri}_\mu P_{\mu ri} \\
& - \frac{1}{4} m_{AB} F^A_{\mu\nu} F^{\mu\nu B} - \frac{1}{8} \varepsilon^{\mu\nu\rho\sigma\lambda\tau} \Gamma_{IAB} B^I_{\mu\nu} F^A_{\rho\sigma} F^B_{\lambda\tau} \\
& + \frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{2} \bar{\chi}^a \gamma^\mu D_\mu \chi^a - \frac{1}{2} \bar{\psi}^r \gamma^\mu D_\mu \psi_r \\
& - m_{AB} \bar{\lambda}^A \gamma^\mu D_\mu \lambda^B + \frac{1}{2} \bar{\psi}_\mu \gamma_\nu \gamma^\mu \chi^a P^\nu_a - (\bar{\psi}^i_\mu \gamma_\nu \gamma^\mu \psi^r) P^\nu_{ri} \\
& + \frac{1}{48} G_{\mu\nu\rho} \left(\bar{\psi}^\lambda \gamma_{[\lambda} \gamma^{\mu\nu\rho} \gamma_{\tau]} \psi^\tau + \bar{\chi}^a \gamma^{\mu\nu\rho} \chi^a - \bar{\psi}^r \gamma^{\mu\nu\rho} \psi_r \right) \\
& + \frac{1}{24} G^a_{\mu\nu\rho} \left(\bar{\psi}_\lambda \gamma^{\mu\nu\rho} \gamma^\lambda \chi^a - m_{aAB} \bar{\lambda}^A \gamma^{\mu\nu\rho} \lambda^B \right) \\
& - \frac{1}{2} F^A_{\mu\nu} \left(m_{AB} \bar{\psi}_\lambda \gamma^{\mu\nu} \gamma^\lambda \lambda^B - m_{aAB} \bar{\chi}^a \gamma^{\mu\nu} \lambda^B \right) \,,
\end{aligned} \tag{2.46}$$

<hr>

[3]In our conventions, the Minkowski metric is given by $\eta_{mn} = \text{diag.}(-,+,+,+,+,+)$, and the Clifford algebra is generated by $\{\gamma_m, \gamma_n\} = 2\eta_{mn}$. The Ricci tensor is defined as $R_\mu{}^m = R_{\mu\nu}{}^{mn} e^\nu_n$. The $Sp(1)_R$ indices are raised and lowered as $\lambda^i = \epsilon^{ij}\lambda_j$, $\lambda_j = \lambda^i \epsilon_{ij}$, with $\epsilon_{ij}\epsilon^{ik} = \delta^k_j$, and the $SO(n_T)$ indices are raised and lowered with δ_{ab}. Often we will suppress the $Sp(1)_R$ indices, and use the notation $\bar{\psi}\chi = \bar{\psi}^i \chi_i$. The fermionic bilinears have the symmetry $\bar{\psi}\gamma_{\mu_1\dots\mu_n}\chi = (-1)^n \bar{\chi}\gamma_{\mu_n\dots\mu_1}\psi$, with the $Sp(1)_R$ index contraction suppressed.

provided that the (anti-)selfduality conditions (2.29), (2.30) are imposed *after* the variation of the action with respect to the 2-form potential. In particular, the 2-form field equation, upon projections with L_I and L_I^a yields

$$\nabla_\mu G^{\mu\rho\sigma} + P_\mu^a G_a^{\mu\rho\sigma} + \frac{1}{4}\varepsilon^{\rho\sigma\lambda\tau\mu\nu} m_{AB} F_{\lambda\tau}^A F_{\mu\nu}^B = 0 \ ,$$

$$\nabla_\mu G^{\mu\rho\sigma\, a} + P_\mu^a G^{\mu\rho\sigma} - \frac{1}{4}\varepsilon^{\rho\sigma\lambda\tau\mu\nu} m_{AB}^a F_{\lambda\tau}^A F_{\mu\nu}^B = 0 \ , \tag{2.47}$$

up to fermionic contributions. These equations, in turn, agree with the results that follow from taking the divergence of the (anti-)selfduality equations (2.29) and (2.30).

The presence of the $B \wedge F \wedge F$ term in the action is noteworthy. Since the 3-form field strength is Chern-Simons modified, normally it is not expected to arise in the action because in this case the 2-form potential transforms under Yang-Mills gauge transformations which typically do not leave invariant a term of the form $B \wedge F \wedge F$ in the action. However, this term is allowed in magical supergravities due to the identity (2.3).

3   Gauging a Subgroup of the Global Symmetry Group

We begin with the building blocks needed for the gauging of a subgroup G_0 of the global symmetry group of the Lagrangian that utilizes a suitable subset of the n_V vector fields that is dictated by the so called embedding tensor [31, 32, 33], which is subject to certain constraints. The global symmetry group of the Lagrangian (2.46) and hence the equations of motion that follow from it obviously contains the isometry group $SO(n_T, 1)$ of the tensor scalars. For the magical theories with $n_T = 5$ and $n_T = 3$ it comprises an additional factor $USp(2)$ and $U(1)$, respectively, exclusively acting on the vector multiplets. In addition, all these theories have an $Sp(1)_R$ R-symmetry group and $U(1)^{n_V}$ Abelian symmetry groups.

Most of the formulas presented are very similar to the structures encountered in the gauging of the maximal supergravity in six dimensions [34], we shall see however that in contrast to the maximal case, the construction for the magical theories allows only for a very limited choice of possible gauge groups.

13

3.1 Embedding Tensor and the Tensor Hierarchy

The key ingredient in the construction is the general covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - A_\mu{}^A X_A \ , \tag{3.1}$$

where

$$X_A = \Theta_A{}^{IJ} t_{IJ} + \Theta_A{}^{\mathcal{X}} t_{\mathcal{X}} + \Theta_A{}^{\mathcal{A}} t_{\mathcal{A}} \ , \tag{3.2}$$

showing that the gauge group is parametrized by the choice of the embedding tensors $\Theta_A{}^{IJ}$, $\Theta_A{}^{\mathcal{X}}$, and $\Theta_A{}^{\mathcal{A}}$. Here, $t_{IJ} = t_{[IJ]}$ are the $SO(n_T, 1)$ generators satisfying the algebra

$$[t_{IJ}, t_{KL}] \;=\; 4 \left(\eta_{I[K} \, t_{L]J} - \eta_{J[K} \, t_{L]I} \right) \ , \tag{3.3}$$

while the generators $t_{\mathcal{X}}$ span the additional symmetries $USp(2)$ and $U(1)$ for $n_T = 5$ and $n_T = 3$, respectively. The generators $t_{\mathcal{A}}$ denote the isometries of the quaternionic Kähler manifold parametrized by the hyperscalars, including the $Sp(1)_R$ R-symmetry. We will denote the group with generators (t_{IJ}, t_χ) by G_T (see Table 1) and the group with generators $t_{\mathcal{A}}$ by G_H.

For transparency of the presentation we will first discuss the case of gauge groups that do not involve the hyperscalars, i.e. set $\Theta_A{}^{\mathcal{A}} = 0$, and extend the construction to the general case with $\Theta_A{}^{\mathcal{A}} \neq 0$ in section 4.2.

Closure of the gauge algebra imposes the conditions [33, 35]

$$[X_A, X_B] \;=\; -X_{[AB]}{}^C X_C \ , \qquad X_{(AB)}{}^C X_C \;=\; 0 \ , \tag{3.4}$$

where the "structure constants" $X_{AB}{}^C \equiv (X_A)_B{}^C$ are obtained from the generator (3.2) evaluated in the representation $\mathcal{R}_v$ of the vector fields and are in general not antisymmetric in A and B. The proper non-abelian field strength transforming co-variantly under gauge transformations is given by the combination [32, 35]

$$\mathcal{G}_{\mu\nu}^A = \mathcal{F}_{\mu\nu}^A + X_{(BC)}{}^A B_{\mu\nu}^{BC} \ , \tag{3.5}$$

where

$$\mathcal{F}_{\mu\nu}^A = 2\partial_{[\mu} A_{\nu]}^A + X_{[BC]}{}^A A_\mu^B A_\nu^C \ . \tag{3.6}$$

The two-forms $B_{\mu\nu}^{AB} = B_{\mu\nu}^{(AB)}$ transform in the symmetric tensor product of two vector representations $(\mathcal{R}_v \otimes \mathcal{R}_v)_{\mathrm{sym}}$, and the non-abelian gauge transformations are

$$\begin{aligned}
\delta A_\mu^A &= \mathcal{D}_\mu \Lambda^A - X_{(BC)}{}^A \Lambda_\mu^{(BC)} \ , \\
\delta B_{\mu\nu}^{AB} &= 2\mathcal{D}_{[\mu} \Lambda_{\nu]}^{AB} - 2\Lambda^{(A} \mathcal{G}_{\mu\nu}^{B)} + 2A_{[\mu}^{(A} \delta A_{\nu]}^{B)} \ .
\end{aligned} \tag{3.7}$$

14

Consistency of the construction imposes that the additional two-forms $B_{\mu\nu}^{(AB)}$ required in (3.5) for closure of the non-abelian gauge algebra on the vector fields form a subset of the n_T two-forms present in the theory. In other words, it is necessary that the intertwining tensor $X_{(BC)}{}^A$ factors according to

$$X_{(BC)}{}^A \;\; = \;\; \Gamma_{BC}^I \, \theta_I^A \;, \tag{3.8}$$

with a constant tensor θ_I^A such that with the identification $B_{\mu\nu}^I = \Gamma_{AB}^I B_{\mu\nu}^{AB}$ the system of gauge transformations (3.7) takes the form

$$
\begin{aligned}
\delta_\Lambda A_\mu^A &\;\; = \;\; \mathcal{D}_\mu \Lambda^A - \theta_I^A \Lambda_\mu^I \;, \\
\delta_{\mathrm{cov},\Lambda} B_{\mu\nu}^I &\;\; = \;\; 2\mathcal{D}_{[\mu} \Lambda_{\nu]}^I - 2\Gamma_{AB}^I \Lambda^A \mathcal{G}_{\mu\nu}^B \;,
\end{aligned}
\tag{3.9}
$$

with δ_{cov} defined in (2.43), and provides a proper covariantization of the *Abelian* system (2.45). This shows how the gauging of the theory in general not only corresponds to covariantizing the derivatives according to (3.1) but also induces a nontrivial deformation of the 2-form tensor gauge transformations. In particular, 2-forms start to transform by (Stückelberg)-shift under the gauge transformations of the 1-forms. Pushing the same reasoning to the three-form potential and the associated gauge transformations, leads to the following set of covariant field strengths

$$
\begin{aligned}
\mathcal{G}_{\mu\nu}^A &\;\; \equiv \;\; 2\partial_{[\mu} A_{\nu]}^A + X_{[BC]}{}^A A_\mu^B A_\nu^C + B_{\mu\nu}^I \theta_I^A \;, \\[2mm]
\mathcal{H}_{\mu\nu\rho}^I &\;\; \equiv \;\; 3\,\mathcal{D}_{[\mu} B_{\nu\rho]}^I + 6\,\Gamma_{AB}^I \, A_{[\mu}{}^A \left(\partial_\nu A_{\rho]}{}^B + \tfrac{1}{3} X_{[CD]}{}^B A_\nu{}^C A_\rho{}^D \right) + \theta^{IA} \, C_{\mu\nu\rho\,A} \;, \\[2mm]
\mathcal{G}_{\mu\nu\rho\sigma\,A} &\;\; \equiv \;\; 4\mathcal{D}_{[\mu} C_{\nu\rho\sigma]\,A} - (\Gamma_I)_{AB} \Big(6 B_{\mu\nu}^I \mathcal{G}_{\rho\sigma}^B + 6\theta^{BJ} B_{[\mu\nu}^I B_{\rho\sigma]J} \\[1mm]
&\qquad\qquad\qquad + 8\Gamma_{CD}^I A_{[\mu}^B A_\nu^C \partial_\rho A_{\sigma]}^D + 2\Gamma_{CD}^I X_{EF}{}^D A_{[\mu}{}^B A_\nu{}^C A_\rho{}^E A_{\sigma]}{}^F \Big) \;,
\end{aligned}
\tag{3.10}
$$

with three-form fields $C_{\mu\nu\rho\,A}$. While the construction so far is entirely off-shell, the equations of motion will impose (anti-)self-duality of the dressed field strengths $L_I \mathcal{G}_{\mu\nu\rho}^I$ and $L_I^a \mathcal{G}_{\mu\nu\rho}^I$, respectively, whereas the three-form fields $C_{\mu\nu\rho\,A}$ are on-shell dual to the vector fields by means of a first order equation

$$e\,\theta_I^A \mathcal{G}_A^{\mu\nu\rho\lambda} \;\; = \;\; -\frac{1}{2}\,\epsilon^{\mu\nu\rho\lambda\sigma\tau} \, m_{AB}\, \mathcal{G}_{\sigma\tau}^A \theta_I^B \;, \tag{3.11}$$

with the metric m_{AB} from (2.7). In particular, the three-form fields transform in the contragredient representation under the global symmetry group. Similar to the above

15

construction, their presence in the first equation of (3.10) is required for closure of the algebra on the two-forms.[4] The hierarchy of p-forms may be continued to four-forms and five-forms which are on-shell dual to the scalar fields and the embedding tensor, respectively, see [35, 36], but none of these fields will enter the covariantized action and the tensor hierarchy can consistently be truncated to (3.10).

The field strengths (3.5) and (3.10) transform covariantly under the full set of non-abelian gauge transformations

$$
\begin{aligned}
\delta_\Lambda A_\mu^A &= \mathcal{D}_\mu \Lambda^A - \theta_I^A \Lambda_\mu^I , \\[4pt]
\delta_{\mathrm{cov},\Lambda} B_{\mu\nu}^I &= 2\,\mathcal{D}_{[\mu}\Lambda_{\nu]}^I - 2\Gamma_{AB}^I \Lambda^{(A}\mathcal{G}_{\mu\nu}^{B)} - \theta^{AI}\,\Lambda_{\mu\nu\,A} , \\[4pt]
\delta_{\mathrm{cov},\Lambda} C_{\mu\nu\rho\,A} &= 3\,\mathcal{D}_{[\mu}\Lambda_{\nu\rho]\,A} + 6\Gamma_{AB}^I\,\mathcal{G}_{[\mu\nu}^B\,\Lambda_{\rho]I} + 2\Gamma_{AB}^I\,\Lambda^B\,\mathcal{H}_{\mu\nu\rho\,I} ,
\end{aligned}
\tag{3.12}
$$

with gauge parameters Λ^A, Λ_μ^I, $\Lambda_{\mu\nu\,A}$, the covariant variation $\delta_{cov.}B_{\mu\nu}^I$ as defined in (2.43), and

$$
\delta_{\mathrm{cov}.}C_{\mu\nu\rho\,A} \equiv \delta C_{\mu\nu\rho\,A} - 6\,\Gamma_{AB}^I\,B_{[\mu\nu\,I}\,\delta A_{\rho]}{}^B - 2\Gamma_{AB}^I(\Gamma_I)_{CD}\,A_{[\mu}^B A_\nu^C\,\delta A_{\rho]}^D . \tag{3.13}
$$

The modified Bianchi identities are given by

$$
\mathcal{D}_{[\mu}\mathcal{G}_{\nu\rho]}^A = \tfrac{1}{3}\theta_I^A \mathcal{H}_{\mu\nu\rho}^I , \tag{3.14}
$$

$$
\mathcal{D}_{[\mu}\mathcal{H}_{\nu\rho\sigma]}^I = \tfrac{3}{2}\Gamma_{AB}^I\,\mathcal{G}_{[\mu\nu}^A\mathcal{F}_{\rho\sigma]}^B + \tfrac{1}{4}\theta^{IA}\mathcal{G}_{\mu\nu\rho\sigma\,A} . \tag{3.15}
$$

Just as consistency of the gauge algebra on the vector fields above gave rise to the constraint (3.8), an analogous constraint follows from closure of the algebra on the two-forms:

$$
(X_A)_I{}^J = 2\left(\theta_I^B\Gamma_{AB}^J - \theta^{JB}(\Gamma_I)_{AB}\right) . \tag{3.16}
$$

Otherwise a consistent gauge algebra would require the presence of more than the (available) n_V three-forms $C_{\mu\nu\rho\,A}$. Recalling that the generator on the l.h.s. is defined by (3.2), this constraint translates into the relation

$$
\Theta_A{}^{IJ} = -\Gamma_{AB}^{[I}\,\theta^{J]B} , \tag{3.17}
$$

---

[4]Strictly speaking, also the four-form field strength $\mathcal{F}_{\mu\nu\rho\sigma\,A}$ needs to be corrected by a Stückelberg type term carrying explicit four forms that are on-shell duals to the scalar fields. For our present purpose we will ignore these terms as they are projected out from all the relevant equations of motion.

between the various components of the embedding tensor. Putting this together with (3.8) and the fact that Γ^I_{AB} is an invariant tensor, one explicitly obtains the magical Γ-matrix identity (2.3). This shows in particular, that for values of n_T different from $2, 3, 5, 9$, the non-abelian gauge algebra does not close (in accordance with the appearance of the classical gauge anomaly in the action).[5] Furthermore, this calculation gives rise to the linear relation

$$\Gamma^I_{DC} X_{[AB]}{}^D = 2\Gamma^I_{D[A}\Gamma^J_{B]C}\theta^D_J - \tfrac{4}{3}(\Gamma_J)_{D[A}\Gamma^J_{B]C}\theta^{ID} + \Gamma^I_{D[C}X_{AB]}{}^D \; . \tag{3.18}$$

Using all the linear constraints (3.8), (3.16), (3.18), the quadratic constraint (3.4) finally translates into the following set of relations

$$\theta^{IA}\eta_{IJ}\theta^{JB} = 0 \;, \quad \theta^{IA}\Gamma^{[J}_{AB}\theta^{K]B} = 0 \;, \quad X_{[AB]}{}^C\theta^{IB} \;=\; \Gamma^J_{AD}\,\theta^C_J\theta^{ID} \;,$$

$$X_{[AB]}{}^D X_{[CD]}{}^E + X_{[CA]}{}^D X_{[BD]}{}^E + X_{[BC]}{}^D X_{[AD]}{}^E \;=\; \Gamma^I_{D[A}\,X_{BC]}{}^D\,\theta^E_I \; . \tag{3.19}$$

The first two relations turn out to be very restrictive. In particular, the first equation implies that θ^{IA} is a matrix of mutually orthogonal null vectors in (n_T+1)-dimensional Minkowski space, which requires that they are all proportional, i.e. θ^{IA} factorizes as

$$\theta^{IA} = \zeta^A\xi^I \;, \tag{3.20}$$

with an unconstrained (commuting) spinor ζ^A and $\xi^I\xi_I = 0$. The second equation of (3.19) then has the unique solution (up to irrelevant normalization) $\xi^I = \Gamma^I_{AB}\zeta^A\zeta^B$ which defines a null vector by virtue of the identity (2.3). The same identity implies that the tensor θ^{IA} is Γ-traceless:

$$\Gamma^I_{AB}\theta^B_I = 0 \; . \tag{3.21}$$

From these results one can already deduce some important facts on the structure of the gauge group. Let us decompose the gauge group generators as

$$X_A = \hat{X}_A + \mathring{X}_A \;, \tag{3.22}$$

according to (3.2) into the part acting within the isometry group $SO(n_T, 1)$ and the contribution of generators t_χ, respectively. Using (3.2), (3.17) and (3.20) together with the Γ-matrix algebra and the magical identity (2.3) we find

$$\begin{aligned} \hat{X}_{(BC)}{}^A &=& -\zeta^D\zeta^E\zeta^F\,(\Gamma_{IJ})_{(B}{}^A\Gamma^I_{C)F}\Gamma^J_{DE} \\ &=& \tfrac{1}{2}\zeta^D\zeta^E\zeta^F\,(\Gamma_J)_{DE}(\Gamma^I\Gamma^J)_F{}^A(\Gamma_I)_{BC} \;=\; \Gamma^I_{BC}\theta^{IA} \; . \end{aligned} \tag{3.23}$$

---

[5]Note that this argument singling out once more the magical cases does not even rely on the existence of a supersymmetric action.

Comparing this to the linear constraint (3.8) thus implies $\mathring{X}_{(BC)}{}^A = 0$. Some closer inspection then shows that this furthermore implies

$$\mathring{X}_{BC}{}^A = 0 \ . \tag{3.24}$$

Thus, within the vector/tensor sector, the gauge group entirely lives within the scalar isometry group $SO(n_T, 1)$, even in the cases $n_T = 3$ and $n_T = 5$, where the full global symmetry group of the action possesses additional factors. All other equations of (3.19) can then be shown to be identically satisfied. Summarizing, the possible gaugings of the magical theories are entirely determined by the choice of a constant spinor ζ^A of the isometry group $SO(n_T, 1)$. In the following we will discuss the possibility of inequivalent choices of ζ^A and subsequently study the structure of the resulting gauge group.

3.2 Spinor orbits

With the gauging determined by the choice of a spinor, one may wonder whether there are different orbits of the action of $SO(n_T, 1)$ on the spinorial representation which would represent inequivalent gaugings. The orbits of spinors up to 12 dimensions were studied long ago by Igusa [37]. More recently the spinors in critical dimensions were studied by Bryant [38, 39], whose study uses heavily the connection between spinors in critical dimensions and the four division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.

Understanding of the structure of gauge groups we obtain as well as the connection to the work of Bryant on orbits are best achieved by studying the embedding of the symmetries of the $6D$ magical theories in the corresponding $5D$ supergravity theories obtained by dimensional reduction. Hence we shall first review briefly the $5D$ magical supergravity theories.

Ungauged magical supergravity theories in five dimensions are Maxwell-Einstein supergravities that describe the coupling of pure $N = 2$ supergravity to 5, 8, 14 and 26 vector multiplets , respectively. They are uniquely defined by simple Euclidean Jordan algebras , $J_3^{\mathbb{A}}$, of degree three generated by 3×3 Hermitian matrices over the four division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$(quaternions), $\mathbb{O}$(octonions). The vector fields in these theories , including the graviphoton, are in one-to-one correspondence with the elements of the underlying simple Jordan algebras. Their scalar manifolds are symmetric spaces of the form:

$$\mathcal{M}_5 = \frac{Str_0(\mathbb{A})}{Aut(\mathbb{A})} \tag{3.25}$$

where $Str_0(\mathbb{A})$ and $Aut(\mathbb{A})$ are the reduced structure group and the automorphism group of $J_3(\mathbb{A})$, respectively,which we list below [1, 2]

$$
\begin{aligned}
\mathcal{M}_5 &= \frac{SL(3,\mathbb{R})}{SO(3)} \\
\mathcal{M}_5 &= \frac{SL(3,\mathbb{C})}{SU(3)} \\
\mathcal{M}_5 &= \frac{SU^*(6)}{USp(6)} \\
\mathcal{M}_5 &= \frac{E_{6(-26)}}{F_4}
\end{aligned}
\tag{3.26}
$$

They can be truncated to theories belonging to the so-called generic Jordan family generated by reducible Jordan algebras ($\mathbb{R} \oplus J_2^{\mathbb{A}}$) where $J_2^{\mathbb{A}}$ are the Jordan algebras generated by 2×2 Hermitian matrices over $\mathbb{A}$. The isometry groups of the scalar manifolds of the $5D$ theories resulting from the truncation are as follows:

$$
\begin{aligned}
Str_0[\mathbb{R} \oplus J_2^{\mathbb{R}}] &= SO(1,1) \times Spin\,(2,1) \subset SL\,(3,\mathbb{R}) \\
Str_0[\mathbb{R} \oplus J_2^{\mathbb{C}}] &= SO(1,1) \times Spin\,(3,1) \subset SL\,(3,\mathbb{C}) \\
Str_0[\mathbb{R} \oplus J_2^{\mathbb{H}}] &= SO(1,1) \times Spin\,(5,1) \subset SU^*\,(6) \\
Str_0[\mathbb{R} \oplus J_2^{\mathbb{O}}] &= SO(1,1) \times Spin\,(9,1) \subset E_{6(-26)}
\end{aligned}
\tag{3.27}
$$

These truncated theories descend from $6D$ supergravity theories with $n_T = 2, 3, 5$ and $n_T = 9$ tensor multiplets and no vector multiplets. A general element of the Jordan algebras $J_3^{\mathbb{A}}$ of degree three can be decomposed with respect to its Jordan subalgebra $J_2^{\mathbb{A}}$ as

$$
X = \begin{pmatrix} J_2^{\mathbb{A}} & \psi(\mathbb{A}) \\ \psi^\dagger(\mathbb{A}) & \mathbb{R} \end{pmatrix}
\tag{3.28}
$$

where $\psi(\mathbb{A})$ is a two component spinor over $\mathbb{A}$

$$
\psi(\mathbb{A}) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}
\tag{3.29}
$$

and † represents transposition times conjugation in the underlying division algebra $\mathbb{A}$. Using this decomposition it was shown in [40] that the Fierz identities for supersymmetric Yang-Mills theories in critical dimensions follow from the adjoint identities satisfied by the elements of $J_3^{\mathbb{A}}$ that define the magical supergravity theories in five

dimensions [1, 2]. In the $6D$ magical supergravity theories the tensor fields correspond to the elements of $J_2^{\mathbb{A}}$ and the vector fields are represented by the elements[6]

$$\begin{pmatrix} 0 & \psi(\mathbb{A}) \\ \psi^\dagger(\mathbb{A}) & 0 \end{pmatrix} \tag{3.30}$$

The isometry group of the scalar manifold of a $6D$ magical supergravity is given by the reduced structure group $Str_0(J_2^{\mathbb{A}})$ of $J_2^{\mathbb{A}}$. They are well known to be isomorphic to the linear fractional groups $SL(2, \mathbb{A})$ for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$:

$$\begin{aligned} Str_0(J_2^{\mathbb{R}}) &= Spin(2, 1) = SL(2, \mathbb{R}) \\ Str_0(J_2^{\mathbb{C}}) &= Spin(3, 1) = SL(2, \mathbb{C}) \\ Str_0(J_2^{\mathbb{H}}) &= Spin(5, 1) = SL(2, \mathbb{H}) \end{aligned} \tag{3.31}$$

The isometry group $Spin(9, 1)$ of the octonionic theory can be similarly interpreted using the Jordan algebraic formulation

$$Str_0(J_2^{\mathbb{O}}) = Spin(9, 1) = SL(2, \mathbb{O}) \tag{3.32}$$

Orbits of the spinors appearing in Table 1 corresponding to the vector fields of $6D$ magical supergravity theories under the action of the isometry groups $Str_0(J_2^{\mathbb{A}})$ of their scalar manifolds were studied by Bryant using their realizations as two component spinors $\psi(\mathbb{A})$ over the underlying division algebras $\mathbb{A}$. According to Bryant the entire spinor space $\psi(\mathbb{O})$ corresponding to a Majorana-Weyl spinor forms a single orbit under the action of $Spin(9, 1)$ with the isotropy group $Spin(7)\circledS T_8$ [38, 39]:

$$Orbit(\psi(\mathbb{O})) = \frac{Spin(9, 1)}{Spin(7)\circledS T_8} \tag{3.33}$$

where T_8 denotes the eight dimensional translations. Similarly, the quaternionic spinor $\psi(\mathbb{H})$ corresponding to a symplectic Majorana-Weyl spinor forms a single orbit under the action of $Spin(5, 1)$:

$$Orbit(\psi(\mathbb{H})) = \frac{Spin(5, 1)}{SU(2)\circledS T_4} \tag{3.34}$$

---

[6] The singlet vector field of the $5D$ theory represented by $\mathbb{R}$ corresponds to the vector field that comes from the $6D$ graviton. The bare graviphoton of the $5D$ Maxwell-Einstein supergravity is a linear combination of this vector field and the vector field that descends from the gravitensor of the $6D$ theory corresponding to the identity element of the Jordan algebra of degree three.

Complex two component spinor $\psi(\mathbb{C})$ is a Weyl spinor and forms a single orbit of $SL(2, \mathbb{C}) = Spin(3, 1)$

$$Orbit(\psi(\mathbb{C})) = \frac{SL(2, \mathbb{C})}{T_2} \tag{3.35}$$

Restricting to real spinors one finds

$$Orbit(\psi(\mathbb{R})) = \frac{SL(2, \mathbb{R})}{T_1} \tag{3.36}$$

To summarize, in all cases there is a single spinor orbit, such that different choices of the spinor ζ^A lead to equivalent gaugings.

4 Structure of the gauge group, new couplings and anomalies

4.1 The gauge group in the vector/tensor sector

With the above results, the gauge group generators (3.2) in the vector/tensor sector take the explicit form

$$
\begin{aligned}
(\hat{X}_A)_B{}^C &= (\bar{\zeta}\Gamma^I\zeta)(\Gamma^J\zeta)_A(\Gamma_{IJ})_B{}^C \,, \\
(\hat{X}_A)^{IJ} &= 4\,(\bar{\zeta}\Gamma^{[I}\zeta)(\Gamma^{J]}\zeta)_A \,,
\end{aligned} \tag{4.1}
$$

in terms of the spinor ζ^A. A simple calculation shows that

$$
\begin{aligned}
(\hat{X}_A\hat{X}_B)_C{}^D &= 0 \,, \\
(\hat{X}_A\hat{X}_B\hat{X}_C)_I{}^J &= 0 \,,
\end{aligned} \tag{4.2}
$$

i.e. these generators span an $(n_T - 1)$-dimensional nilpotent abelian algebra[7] and gauge $(n_T - 1)$ translations. Remarkably, this seems to be the only possible gauge group. *Furthermore these $(n_T - 1)$ translations can not lie strictly within the isometry group $\mathfrak{so}(n_T, 1)$, cf. (2.11) due to the appearance of central extensions of the gauge algebra as we explain below.*

Having seen above, that there is a single spinor orbit, all different choices of the spinor ζ^A lead to equivalent gaugings and it will be useful to give a presentation of

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[7]The cubic nilpotency of the generators in the vector representation can also be seen by identifying this representation in the tensor product of two spinor representations.

the generators (4.1) in an explicit basis. We will proceed with the analysis of the maximal case $n_T = 9$ from which all the lower magical theories can be obtained by truncation. For $SO(9,1)$, according to (3.33) the compact part of the little group of a spinor is an $SO(7)$ under which the fundamental representations decompose as

$$\zeta^A : \mathbf{16}_c \;\rightarrow\; \mathbf{8}_+ + \mathbf{7}_- + \mathbf{1}_- \,, \qquad \xi^I : \mathbf{10} \;\rightarrow\; \mathbf{8}_0 + \mathbf{1}_{+2} + \mathbf{1}_{-2} \,,$$
$$A \;\rightarrow\; (\alpha, t, 0) \,, \qquad\qquad\qquad I \;\rightarrow\; (\alpha, +, -) \,, \qquad\qquad (4.3)$$

with the subscripts referring to the $SO(1,1)$ charges in the decomposition (2.11). This basis corresponds to the spinor ζ^A pointing in a given direction $\zeta^A = \frac{1}{2}(\vec{0}, \vec{0}, g^{1/3})$, such that the gauge group generators (4.1) take the explicit form

$$\text{in the } \mathbf{16} \;:\; (\hat{X}_\alpha)_\beta{}^t = g\gamma^t_{\alpha\beta} \,, \quad (\hat{X}_\alpha)_\beta{}^0 = g\delta_{\alpha\beta} \,,$$

$$\text{in the } \mathbf{10} \;:\; (\hat{X}_\alpha)_\beta{}^- = -(\hat{X}_\alpha)_+{}^\beta = g\delta_{\alpha\beta} \,, \qquad\qquad (4.4)$$

with (antisymmetric) $SO(7)$ gamma matrices $\gamma^t_{\alpha\beta}$ ($t = 1, \dots, 7$), and all other components vanishing. We have introduced an explicit coupling constant g that carries charge -3 under $SO(1,1)$.

In this basis, the full non-semisimple (nilpotent) structure of the gauge algebra (3.4) becomes explicitly:

$$[X_\alpha, X_\beta] = -g\gamma^t_{\alpha\beta} X_t \,, \qquad [X_\alpha, X_t] = 0 = [X_t, X_u] \,, \qquad X_0 = 0 \,. \qquad (4.5)$$

The generators X_t thus act as central extensions of the algebra which vanish when evaluated on vector or tensor fields: $(X_t)_I{}^J = 0 = (X_t)_A{}^B$, but which may have a non-trivial action in the hypermultiplet sector, as we shall discuss in section 4.2[8].

The structure of the centrally extended Abelian nilpotent gauge group is best understood by studying the embedding of the gauge group into the U-duality group of the corresponding ungauged $5D$ Maxwell-Einstein supergravity. For the exceptional supergravity $5D$ U-duality group is $E_{6(-26)}$ whose Lie algebra has a 3-graded decomposition with respect to the isometry group $SO(9,1)$ of $6D$ theory:

$$E_{6(-26)} = K_{\mathbf{16_c}} \oplus SO(9,1) \times SO(1,1)_D \oplus T_{\mathbf{16_c}} \qquad\qquad (4.6)$$

where $T_{\mathbf{16_c}}$ denotes the 16 dimensional translational symmetries corresponding to Abelian gauge symmetries of the vector fields of $6D$ theory. The generator Δ that determines the 3-grading of $SO(9,1)$ (see (2.11)) leads to a 5-grading of $E_{6(-26)}$

$$T_{\mathbf{8^c}} \qquad\qquad\qquad\qquad\qquad\qquad T_{\mathbf{8^s}}$$

$$\tilde{N}_{\mathbf{8^v}} \quad -\,-\,-\,- \quad (SO(8) \times SO(1,1)_\Delta \times SO(1,1)_D) \quad -\,-\,-\,- \quad N_{\mathbf{8^v}}$$

$$K_{\mathbf{8^s}} \qquad\qquad\qquad\qquad\qquad\qquad K_{\mathbf{8^c}}$$
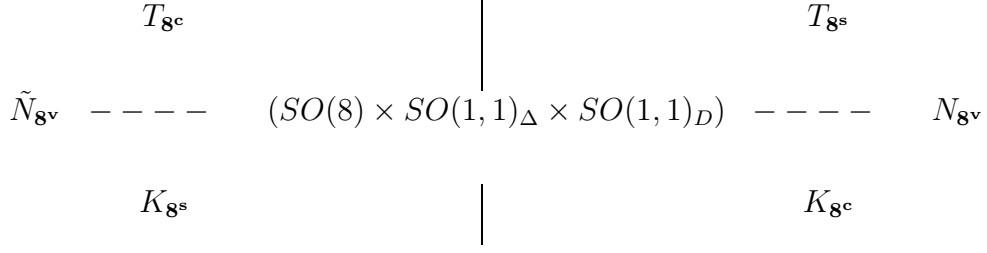
Table 3: Above we give the 5 by 3 grading of $E_{6(-26)}$ with respect to the generators Δ and D respectively. Eight dimensional representations that are in triality are denoted as $\mathbf{8^v}, \mathbf{8^c}$ and $\mathbf{8^s}$.

so that D and Δ determine a 5 by 3 grading of $E_{6(-26)}$ with respect to its $SO(8) \times SO(1,1)_\Delta \times SO(1,1)_D$ subgroup as shown in Table 3.

Restricting to the $Spin(7)$ subgroup such that $SO(8)$ irreps decompose as

$$\mathbf{8^v} = \mathbf{8} \tag{4.7}$$
$$\mathbf{8^s} = \mathbf{7} + \mathbf{1}$$
$$\mathbf{8^c} = \mathbf{8}$$

one finds that the 8 generators $(T_\alpha + N_\alpha)$, transforming in the spinor representation of $Spin(7)$, form a centrally extended nilpotent Abelian subalgebra with 7 generators T_t acting as its central elements. They generate a nilpotent subgroup of $F_{4(-20)}$ which is a subgroup of $5D$ U-duality group $E_{6(-26)}$. $F_{4(-20)}$ admits a 5-grading of the form:

$$F_{4(-20)} = \mathbf{7_{-2}} \oplus \mathbf{8_{-1}} \oplus \ Spin(7) \times SO(1,1) \oplus \mathbf{8_{+1}} \oplus \mathbf{7_{+2}} \ . \tag{4.8}$$

Thus the generators that gauge the centrally extended Abelian subalgebra in the embedding tensor formalism can be uniquely identified with the generators of this nilpotent subalgebra of $F_{4(-20)}$

$$X_\alpha \equiv T_\alpha + N_\alpha \ , \qquad X_t \equiv T_t \ , \tag{4.9}$$

where $\alpha = 1, 2, \cdots, 8$ and $t = 1, 2, \cdots, 7$.

The quaternionic $6D$ magical supergravity has $SU^*(6)$ as its $5D$ U-duality group which has a 3-grading with respect to its $SU^*(4) \times SU(2)$ subgroup. The vector fields

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[8]We should point out that a similar phenomenon of central extension of gauge groups arises also in $4D$ supergravity theories obtained by dimensional reduction of $5D$, $N = 2$ Yang-Mills -Einstein supergravity theories coupled to tensor fields[15].

of the $6D$ theory transform as a symplectic Majorana-Weyl spinor of $SU^*(4) \times SU(2)$. The analog of $F_{4(-20)}$ is the $USp(4,2)$ subgroup of $SU^*(6)$. The centrally extended nilpotent Abelian translation gauge group sits inside $USp(4,2)$ which has the 5-graded decomposition

$$USp(4,2) = \mathbf{3_{-2}} \oplus \mathbf{4_{-1}} \oplus SU(2) \times SO(1,1) \oplus \mathbf{4_{+1}} \oplus \mathbf{3_{+2}} \ . \tag{4.10}$$

As for the complex magical theory the $5D$ U-duality group $SL(3,\mathbb{C})$ has a 3-grading with respect to its $SL(2,\mathbb{C}) \times U(1)$ under which the vector fields of $6D$ theory transform as a pair of complex Weyl spinors. The nilpotent gauge group sits inside the $SU(2,1)$ subgroup of $SL(3,\mathbb{C})$ and has the 5-grading

$$SU(2,1) = \mathbf{1_{-2}} \oplus \mathbf{2_{-1}} \oplus U(1) \times SO(1,1) \oplus \mathbf{2_{+1}} \oplus \mathbf{1_{+2}} \ . \tag{4.11}$$

The simple groups in which the centrally extended Abelian gauge groups of octonionic, quaternionic and complex magical theories can be minimally embedded satisfy the following chain of inclusions

$$F_{4(-20)} \supset USp(4,2) \times USp(2) \supset SU(2,1) \times U(1) \tag{4.12}$$

The fact that the method of embedding tensor formalism leads to unique centrally extended Abelian gauge groups for each of the magical supergravity theories is quite remarkable. Even though the "central charges" act trivially on the vector and tensor fields as must be evident from the above analysis they may have nontrivial action on the hyperscalars as will be discussed in the next section.

4.2   The gauge group in the hypersector

We will now also allow for isometries of the quaternionic Kähler manifold to be gauged, i.e. consider the full generator (3.2) with non-vanishing $\Theta_A{}^{\mathcal{A}}$. The generators $t_{\mathcal{A}}$ denote the isometries on the quaternionic Kähler manifold acting by a Killing vector field $K_{\mathcal{A}}^X$

$$t_{\mathcal{A}} \cdot \phi^X \ = \ K_{\mathcal{A}}^X(\phi) \ , \tag{4.13}$$

on the hyperscalars. It is straightforward to derive that under this transformation the $Sp(1)_R \times Sp(n_H)$ connections transform as

$$t_{\mathcal{A}} \cdot Q_{\mu i}{}^j \ = \ \partial_\mu \phi^X D_X(\mathcal{S}_{\mathcal{A}})_i{}^j \ , \qquad t_{\mathcal{A}} \cdot Q_{\mu r}{}^s \ = \ \partial_\mu \phi^X D_X(\mathcal{S}_{\mathcal{A}})_r{}^s \ , \tag{4.14}$$

with

$$(S_\mathcal{A})_i{}^j \equiv K_\mathcal{A}^X A_{Xi}{}^j + C_{\mathcal{A}i}{}^j \,, \qquad (S_\mathcal{A})_r{}^s \equiv K_\mathcal{A}^X A_{Xr}{}^s + C_{\mathcal{A}r}{}^s \,, \qquad (4.15)$$

in terms of the functions $C_{\mathcal{A}i}{}^j$ and $C_{\mathcal{A}r}{}^s$ defined in (2.25), (2.26), respectively, for the Killing vector field $K_\mathcal{A}$. From this, we conclude that the fermion fields transform as

$$t_\mathcal{A} \cdot \chi_i^a = -(S_\mathcal{A})_i{}^j \chi_j^a \,, \qquad t_\mathcal{A} \cdot \psi_r = -(S_\mathcal{A})_r{}^s \psi_s \,, \qquad \text{etc.} \qquad (4.16)$$

Upon gauging, the gauge covariant derivative of the hyperscalars is given by

$$\mathcal{D}_\mu \phi^X = \partial_\mu \phi^X - g A_\mu^A \mathcal{K}_A^X \,, \qquad \mathcal{K}_A^X \equiv \Theta_A{}^\mathcal{A} K_\mathcal{A}^X \,, \qquad (4.17)$$

and the gauge covariant derivatives of the fermion fields by

$$\begin{aligned}
\mathcal{D}_\mu \psi_\nu^i &= \nabla_\mu \psi_\nu^i + \mathcal{Q}_\mu{}^{ij} \psi_{\nu j} \,, \\
\mathcal{D}_\mu \chi^{ai} &= \nabla_\mu \chi^{ai} + \mathcal{Q}_\mu^{ab} \chi_b^i + \mathcal{Q}_\mu{}^{ij} \chi_j^a \,, \\
\mathcal{D}_\mu \psi^r &= \nabla_\mu \psi^r + \mathcal{Q}_\mu{}^{rs} \psi_s \,, \\
\mathcal{D}_\mu \lambda^{Ai} &= \nabla_\mu \lambda^{Ai} + \mathcal{Q}_\mu{}^{ij} \lambda_j^A - A_\mu^C X_{CB}{}^A \lambda^{Bi} \,,
\end{aligned} \qquad (4.18)$$

with[9]

$$\mathcal{Q}_\mu^{ab} = L^{I[a} \mathcal{D}_\mu L_I^{b]} \,, \qquad (4.19)$$

$$\begin{aligned}
\mathcal{Q}_\mu{}^{ij} &= \mathcal{D}_\mu \phi^X A_X{}^{ij} + A_\mu^A \mathcal{S}_A{}^{ij} \\
&= \partial_\mu \phi^X A_X{}^{ij} + A_\mu^A \mathcal{C}_A{}^{ij} \,,
\end{aligned} \qquad (4.20)$$

$$\begin{aligned}
\mathcal{Q}_\mu{}^{rs} &= \mathcal{D}_\mu \phi^X A_X{}^{rs} + A_\mu^A \mathcal{S}_A{}^{rs} \\
&= \partial_\mu \phi^X A_X{}^{rs} + A_\mu^A \mathcal{C}_A{}^{rs} \,,
\end{aligned} \qquad (4.21)$$

with the following definitions

$$\begin{aligned}
\mathcal{S}_A{}^{ij} &= \mathcal{K}_A^X A_X{}^{ij} + \mathcal{C}_A{}^{ij} \,, \\
\mathcal{C}_A{}^{ij} &= -\frac{1}{2n_H} V_{ri}^X V_Y^{rj} \nabla_X \mathcal{K}_A^Y \,,
\end{aligned} \qquad (4.22)$$

---

[9]See [41] for a description of the $\mathcal{S}$-functions in the context of G/H coset sigma models in which an arbitrary subgroup of G is gauged.

and similarly

$$\mathcal{S}_A{}^{rs} = \mathcal{K}_A^X A_X{}^{rs} + \mathcal{C}_A{}^{rs} \,,$$
$$\mathcal{C}_A{}^{rs} = -\tfrac{1}{2} V_{ri}^X V_Y^{si} \nabla_X \mathcal{K}_A^Y \,.$$

(4.23)

The constraint analysis in the vector/tensor sector remains unchanged in presence of the $t_{\mathcal{A}}$, such that the gauge group in this sector still reduces to the set of $(n_T - 1)$ Abelian translations as we have derived above. Gauge invariance of the new components $\Theta_A{}^{\mathcal{A}}$ of the embedding tensor on the other hand implies that

$$[\Theta_A{}^{\mathcal{A}} t_{\mathcal{A}}, \Theta_B{}^{\mathcal{B}} t_{\mathcal{B}}] \;\; = \;\; -X_{AB}{}^C \Theta_C{}^{\mathcal{A}} t_{\mathcal{A}} \,,$$

(4.24)

with the same structure constants $X_{AB}{}^C$ encountered in (3.4). It follows from (3.4) that

$$\theta^{IA} \Theta_A{}^{\mathcal{A}} = 0 \,.$$

(4.25)

Furthermore, using the explicit form (4.5) of the structure constants $X_{AB}{}^C$, we find that $\Theta_0{}^{\mathcal{A}} = 0$ and the gauging in the hypersector of the octonionic magical supergravity corresponds to selecting 8+7 Killing vector fields $\mathcal{K}_\alpha \equiv \Theta_\alpha{}^{\mathcal{A}} K_{\mathcal{A}}$, $\mathcal{K}_t \equiv \Theta_t{}^{\mathcal{A}} K_{\mathcal{A}}$ (not necessarily linearly independent), which satisfy the algebra

$$[\mathcal{K}_\alpha, \mathcal{K}_\beta] = -g\gamma_{\alpha\beta}^t \mathcal{K}_t \,, \qquad [\mathcal{K}_\alpha, \mathcal{K}_t] = 0 = [\mathcal{K}_t, \mathcal{K}_u] \,.$$

(4.26)

Thus, the generators associated with the full gauge group in the magical supergravities, including both the vector-tensor and hyper sectors, are

$$X_A = \{\widehat{X}_\alpha + \mathcal{K}_\alpha \,, \mathcal{K}_t\} \,,$$

(4.27)

The existence of a combination of Killing vectors that satisfy this algebra in its maximal form, i.e. with none of the generators set to zero, is a nontrivial constraint, since such an algebra does not necessarily lie in the isometry of the hyperscalar manifold. A trivial solution to the constraints (4.26) is given by setting $\mathcal{K}_\alpha = 0 = \mathcal{K}_t$ in which case the gauge group simply does not act in the hypersector, and has the generators X_α. A less trivial option is the choice $\mathcal{K}_t = 0$ in which case the gauge algebra generators consist of $\widehat{X}_\alpha + \mathcal{K}_\alpha$, and $\mathcal{K}_\alpha$ can be chosen to be any set of up to 8 commuting (compact, noncompact or nilpotent) isometries, which may in particular include $U(1)_R$ subgroup of the $Sp(1)_R$ R-symmetry group.

In general, and in contrast to the vector/tensor sector, in the hypersector the generators $\mathcal{K}_t$ may act as nontrivial central charges of the gauge algebra. We can solve the constraints (4.26) by selecting an ideal $\mathcal{I}$ inside the algebra $\mathcal{A}$ defined by (4.26),

26

representing all generators in $\mathcal{I}$ by zero, and embedding the quotient $\mathcal{A}/\mathcal{I}$ into the isometry algebra of the quaternionic manifold. (The solution considered above where $\mathcal{K}_t$ is set to zero is a particular example of this procedure). In this case the generators will be embedded among the positive root generators. For a coset manifold and its representative in the corresponding triangular gauge, their action does not induce a compensating transformation acting on the fermions, implying that the matrices $(\mathcal{S}_A)_i{}^j$, $(\mathcal{S}_A)_r{}^s$ of (4.15), (4.16) vanish.

As an illustration we give some examples of embeddings of the nilpotent gauge groups with nontrivial central charges into simple quaternionic Lie groups

- As we discussed above, the nilpotent gauge algebra (4.26) of the octonionic magical theory with all seven $\mathcal{K}_t$ non-vanishing can be embedded into the Lie algebra of the group $F_{4(-20)}$ which admits a five grading according to (4.8)

$$\mathbf{7_{-2}} \oplus \mathbf{8_{-1}} \oplus Spin(7)_0 \oplus SO(1,1)_0 \oplus \mathbf{8_{+1}} \oplus \mathbf{7_{+2}} \ , \tag{4.28}$$

with the obvious embedding of (4.26) as the generators of positive grading. The group $F_{4(-20)}$ may be embedded into the isometry group of the coset space $E_{6(-26)}/F_4$ and via the chain along the first line of table 1 further into the isometry group of the quaternionic Kähler manifold $E_{8(-24)}/(E_7 \times SU(2))$. With hyperscalars in this particular quaternionic Kähler manifold, the algebra (4.26) can thus be realized. Interestingly, this manifold is precisely the moduli space of this magical theory without hypers upon dimensional reduction to $D = 3$ dimensions, cf. table 1. The corresponding $6D$ theory is anomaly free as we will discuss later and the scalar manifold of the resulting ungauged $3D$ theory is doubly exceptional

$$\mathcal{M}_3 = [\frac{E_{8(-24)}}{(E_7 \times SU(2))}] \times [\frac{E_{8(-24)}}{(E_7 \times SU(2))}] \tag{4.29}$$

The quaternionic symmetric space of minimal dimension whose isometry group has $F_{4(-20)}$ as a subgroup is

$$\frac{E_{7(-5)}}{SO(12) \times SU(2)} \tag{4.30}$$

However, $E_{7(-5)}$ does not have $E_{6(-26)}$ as a subgroup and the $6D$ octonionic magical theory coupled to hypermultiplets with this target manifold is not anomaly free.

- An example of a non-maximal realization of (4.26) for the octonionic magical theory, is given by selecting one of the central charges, i.e. splitting $\{\mathcal{K}_t\} = \{\mathcal{K}, \mathcal{K}_{\tilde{t}}\}$ and setting the ideal $\mathcal{I}$ spanned by the six $\mathcal{K}_{\tilde{t}}$ to zero. According to the structure of the $SO(7)$ gamma matrix in the structure constants (upon breaking $SO(7)$ down to the $SO(6)$ defined by $\mathcal{K}$) the quotient $\mathcal{A}/\mathcal{I}$ is given by the algebra

$$[\mathcal{K}_a, \mathcal{K}_b] \;=\; 0 \;=\; [\mathcal{K}^a, \mathcal{K}^b] \,, \qquad [\mathcal{K}_a, \mathcal{K}^b] \;=\; g\delta_a^b\, \mathcal{K} \,. \tag{4.31}$$

This algebra can e.g. easily be embedded into the quaternionic Kähler manifold $SU(4,2)/S(U(4)\times U(2))$, whose isometry group admits a five grading according to

$$\mathbf{1_{-2}} \oplus (\mathbf{4} + \bar{\mathbf{4}})_{-\mathbf{1}} \oplus U(3,1)_0 \oplus O(1,1)_0 \oplus (\mathbf{4} + \bar{\mathbf{4}})_{+\mathbf{1}} \oplus \mathbf{1_{+2}} \,, \tag{4.32}$$

with the obvious embedding of (4.31) as the generators of positive grading.

- According to (4.10) the nilpotent gauge algebra of the quaternionic magical theory can be embedded into the Lie algebra of $USp(4,2)$ which is quaternionic real. Therefore the quaternionic symmetric space of minimal dimension whose isometry group includes the nilpotent gauge group with all three central charges is

$$\frac{USp(4,2)}{USp(4) \times USp(2)} \tag{4.33}$$

If we require the isometry group of the hypermanifold to have the corresponding $5D$ isometry group $SU^*(6)$ as a subgroup we can follow the chain along the second line of table 1 further to the target manifold

$$\frac{E_{7(-5)}}{SO(12) \times SU(2)} \tag{4.34}$$

The scalar manifold of the ungauged quaternionic magical theory coupled to this hypermatter has the double exceptional isometry group

$$\frac{E_{7(-5)}}{SO(12) \times SU(2)} \times \frac{E_{7(-5)}}{SO(12) \times SU(2)} \tag{4.35}$$

in three dimensions.

- For the complex magical theory according to (4.11) the nilpotent gauge group embeds into $SU(2,1)$ which is quaternionic real. Thus the minimal hypermanifold in this case is

$$\frac{SU(2,1)}{U(2)} \tag{4.36}$$

  Going along the third row of table 1 we can also couple the theory to hypermultiplets with the target manifold

$$\frac{E_{6(2)}}{SU(6) \times SU(2)} \tag{4.37}$$

  Again the corresponding $3D$ target space with this hypersector is doubly exceptional.

4.3 Gauging of R-symmetry

In presence of hypermultiplets, the R-symmetry $Sp(1)_R$ is embedded into the isometries on the quaternionic Kähler manifold and its gauging is a particular case of the construction discussed in the previous section. In absence of hypermultiplets, the R-symmetry acts exclusively on the fermions and may be included in the gauging by extending the gauge group generators (3.2) to

$$X_A \;\; = \;\; \hat{X}_A + \Theta_A{}^{ij} t_{ij} \;\; \equiv \;\; \hat{X}_A + \xi_A \;, \tag{4.38}$$

with t_{ij} representing the $Sp(1)_R$ generators. In complete analogy to the calculation leading to (4.24) one derives the conditions

$$[\xi_A, \xi_B] \;\; = \;\; -X_{AB}{}^C \xi_C \;, \tag{4.39}$$

from which we find that the most general gauging in absence of hypermultiplets is given by

$$X_\alpha \;\; = \;\; \hat{X}_\alpha + \xi_\alpha c^{ij} t_{ij} \;, \qquad X_t \;\; = \;\; 0 \;\; = \;\; X_0 \;, \tag{4.40}$$

with $\hat{X}_\alpha$ from (4.4) and constant ξ_α and c^{ij} selecting a $U(1)$ generator within $Sp(1)_R$. All formulas of the previous section apply, in particular the connections on the fermion fields are still given by (4.18), upon setting all hyperscalars to zero and with constant $C_{\alpha i}{}^j \equiv \xi_\alpha c_i{}^j$. The gauge algebra is still of the form (4.5) but with X_t set to zero, i.e. $[X_\alpha, X_\beta] = 0$, with X_α from (4.40).

4.4 Gauged Magical Supergravities

Putting together the ingredients described in previous sections, and following the standard Noether procedure, we find that the action for gauged magical supergravity, up to quartic fermion terms, is given by

$$
\begin{aligned}
e^{-1}\mathcal{L} = {} & R - \frac{1}{12}g_{IJ}\mathcal{H}^I_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho J} - \frac{1}{4}\mathcal{P}^a_\mu\mathcal{P}^{\mu a} - \frac{1}{2}\mathcal{P}^{ri}_\mu\mathcal{P}_{\mu ri} - \frac{1}{4}m_{AB}\mathcal{G}^A_{\mu\nu}\mathcal{G}^{\mu\nu B} \\
& + \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho - \frac{1}{2}\bar{\chi}^a\gamma^\mu\mathcal{D}_\mu\chi^a - \frac{1}{2}\bar{\psi}^r\gamma^\mu\mathcal{D}_\mu\psi_r \\
& - m_{AB}\bar{\lambda}^A\gamma^\mu\mathcal{D}_\mu\lambda^B + \frac{1}{2}\bar{\psi}_\mu\gamma_\nu\gamma^\mu\chi^a\mathcal{P}^\nu_a - (\bar{\psi}^i_\mu\gamma_\nu\gamma^\mu\psi^r)\mathcal{P}^\nu_{ri} \\
& + \frac{1}{48}\mathcal{H}_{\mu\nu\rho}\left(\bar{\psi}^\lambda\gamma_{[\lambda}\gamma^{\mu\nu\rho}\gamma_{\tau]}\psi^\tau + \bar{\chi}^a\gamma^{\mu\nu\rho}\chi^a - \bar{\psi}^r\gamma^{\mu\nu\rho}\psi_r\right) \\
& + \frac{1}{24}\mathcal{H}^a_{\mu\nu\rho}\left(\bar{\psi}_\lambda\gamma^{\mu\nu\rho}\gamma^\lambda\chi^a - m_{aAB}\bar{\lambda}^A\gamma^{\mu\nu\rho}\lambda^B\right) \\
& - \frac{1}{2}\mathcal{G}^A_{\mu\nu}\left(m_{AB}\,\bar{\psi}_\lambda\gamma^{\mu\nu}\gamma^\lambda\lambda^B - m_{aAB}\,\bar{\chi}^a\gamma^{\mu\nu}\lambda^B\right) + \mathcal{L}_{\text{top}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{pot}}\,,
\end{aligned}
\tag{4.41}
$$

where $\mathcal{L}_{\text{top}}$ is the gauge invariant completion of the $B \wedge F \wedge F$ term and its variation is

$$
\begin{aligned}
\delta\mathcal{L}_{\text{top}} = {} & \tfrac{1}{6}\epsilon^{\mu\nu\rho\sigma\lambda\tau}\,\Gamma^I_{AB}\left(\mathcal{H}_{\mu\nu\rho\,I}\mathcal{G}^A_{\sigma\lambda}\,\delta_{\text{cov}}A^B_\tau - \tfrac{3}{4}\mathcal{G}^A_{\mu\nu}\mathcal{G}^B_{\rho\sigma}\,\delta_{\text{cov}}B_{\lambda\tau\,I}\right) \\
& + \tfrac{1}{48}\epsilon^{\mu\nu\rho\sigma\lambda\tau}\,\theta^A_I\left(\mathcal{G}_{\mu\nu\rho\sigma\,A}\,\delta_{\text{cov}}B_{\lambda\tau\,I} - \tfrac{4}{3}\mathcal{H}^I_{\mu\nu\rho}\,\delta_{\text{cov}}C_{\sigma\lambda\tau\,A}\right)\,.
\end{aligned}
\tag{4.42}
$$

and the Yukawa couplings and the scalar potential are given by[10]

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\text{Yukawa}} = {} & \bar{\psi}^i_\mu\gamma^\mu\lambda^A_i\,\theta^{IB}m_{AB}L_I - \bar{\psi}_{\mu i}\gamma^\mu\lambda^A_j\,\mathcal{C}^{ij}_A + \bar{\chi}^{ai}\lambda^A_i\,\theta^{IB}L_I m_{a\,AB} \\
& - \bar{\chi}^a_i\lambda^A_j\,m_{a\,AC}m^{CB}\mathcal{C}^{ij}_B - 2\bar{\psi}_r\lambda^A_i V^{ri}_X\mathcal{K}^X_A\,,
\end{aligned}
\tag{4.43}
$$

$$
e^{-1}\mathcal{L}_{\text{potential}} = -\frac{1}{4}\left(\theta^{IA}\theta^{JB}m_{AB}g_{IJ} + \mathcal{C}_{Aij}\mathcal{C}^{ij}_B m^{AB}\right)\,.
\tag{4.44}
$$

The functions g_{IJ}, m_{AB} and $\mathcal{C}^{ij}_A$ are defined in (2.6), (2.7) and (4.22), respectively, and the gauge covariant derivatives of the fermions in (4.18).

---

[10] For the $n_T = 5$ theory and in absence of hypermultiplets these couplings have been obtained in [42] by truncation from gaugings of the maximal theory [34].

The local supersymmetry transformations of the gauged theory, up to cubic fermions terms, are

$$\delta e_\mu^m = \bar\epsilon \gamma^m \psi_\mu \ ,$$

$$\delta \psi_\mu = \mathcal{D}_\mu \epsilon + \frac{1}{48} \gamma^{\rho\sigma\tau} \gamma_\mu \epsilon \mathcal{H}_{\rho\sigma\tau} \ ,$$

$$\delta_{\rm cov.} B_{\mu\nu}{}^I = -2\bar\epsilon \gamma_{[\mu}\psi_{\nu]} L^I + \bar\epsilon \gamma_{\mu\nu}\chi^a L_a^I \ ,$$

$$\delta_{\rm cov.} C_{\mu\nu\rho A} = -\bar\epsilon \gamma_{\mu\nu\rho}\lambda_A \ ,$$

$$\delta \chi^a = \frac{1}{2}\gamma^\mu \epsilon \mathcal{P}_\mu^a - \frac{1}{24}\gamma^{\mu\nu\rho}\epsilon \mathcal{G}^a_{\mu\nu\rho} \ ,$$

$$\delta L_I = \bar\epsilon \chi^a L_I^a \ ,$$

$$\delta A_\mu^A = \bar\epsilon \gamma_\mu \lambda^A \ ,$$

$$\delta \lambda^A = -\frac{1}{4}\gamma^{\mu\nu}\epsilon \mathcal{G}^A_{\mu\nu} - \tfrac{1}{2}\theta^{IA} L_I \, \epsilon_i - \tfrac{1}{2}m^{AB}\mathcal{C}_{Bij}\, \epsilon^j \ ,$$

$$\delta \phi^X = V_{ri}^X \, \bar\epsilon^i \psi^r \ ,$$

$$\delta \psi^r = \mathcal{P}_\mu^{ri}\gamma^\mu \epsilon_i \ .$$

$$(4.45)$$

In establishing the supersymmetry of the action, it is important to recall that the (anti)self-duality equations $\mathcal{H}^+_{\mu\nu\rho} = 0$ and $\mathcal{H}^{a-}_{\mu\nu\rho} = 0$ are to be used after the varying the action. In carrying out the Noether procedure, it is also useful to note that the gauge covariant scalar currents

$$\mathcal{P}_\mu^a = L^{Ia}\mathcal{D}_\mu L_I \ , \qquad \mathcal{P}_\mu^{ri} = \mathcal{D}_\mu \phi^X V_X^{ri} \ , \tag{4.46}$$

satisfy the relations

$$\mathcal{D}_{[\mu}\mathcal{P}_{\nu]}^a = -\frac{1}{2}\mathcal{F}_{\mu\nu}^A \mathcal{C}_A{}^a \ , \qquad \mathcal{C}_A{}^a = \widehat{X}_A^{IJ} L_I^a L_J \ , \tag{4.47}$$

$$\mathcal{D}_{[\mu}\mathcal{P}_{\nu]}^{ri} = -\mathcal{F}_{\mu\nu}^A \mathcal{C}_A^{ri} \ , \qquad \mathcal{C}_A^{ri} = \Theta_A{}^{\mathcal{A}} K_{\mathcal{A}}^X V_X^{ri} \ . \tag{4.48}$$

Also encountered in the Noether procedure are the curvatures associated with the connections defined in (4.19) and (4.20) which take the form

$$\mathcal{Q}_{\mu\nu}^{ab} = -2\mathcal{P}_{[\mu}^a \mathcal{P}_{\nu]}^b - \mathcal{F}_{\mu\nu}^A \mathcal{C}_A{}^{ab} \ , \qquad \mathcal{C}_A{}^{ab} = \widehat{X}_A^{IJ} L_I^a L_J^b \ , \tag{4.49}$$

$$\mathcal{Q}_{\mu\nu i}{}^j = 2\mathcal{P}_{[\mu}^{rj}\mathcal{P}_{\nu]ri} + \mathcal{F}_{\mu\nu}^A \mathcal{S}_{Ai}{}^j \ , \tag{4.50}$$

31

with $\mathcal{S}_{Ai}{}^j$ from (4.22). In the course of Noether procedure, it is also useful to note that the curvature $\mathcal{F}^A_{\mu\nu}$ occurring in equations (4.47)–(4.50) can be replaced by $\mathcal{G}^A_{\mu\nu}$, by exploiting the constraint (4.25).

As emphasized above, the (anti)self-duality equations on the projections of $\mathcal{H}^I_{\mu\nu\rho}$ are to be imposed *after* the Euler-Lagrange variation of the action with respect to all fields. Indeed, the field equation for $C_{\mu\nu\rho A}$, which has no kinetic term and it appears in the action only through $\mathcal{L}_{\text{top}}$ and the $\mathcal{G}^A_{\mu\nu}$ dependent terms, is identically satisfied provided that the stated (anti)self-duality equations are used. Varying the action with respect to $B^I_{\mu\nu}$, on the other hand, again modulo the (anti)self-duality conditions and their consequences, give precisely the duality equation (3.11). Another salient feature of the action is that the scalar potential is a positive definite expression. Using the explicit parametrization of $SO(n_T, 1)$ given in (2.12) and the basis (4.4), the scalar potential takes the form

$$
e^{-1}\mathcal{L}_{\text{potential}} \;=\; -\tfrac{1}{16}g^2 e^{-3\sigma} - \tfrac{1}{8}e^{-\sigma}\left(\mathcal{C}^{ij}_\alpha + \gamma^t_{\alpha\beta}\varphi^\beta \mathcal{C}^{ij}_t\right)^2 - \tfrac{1}{8}e^\sigma\,\mathcal{C}_{t\,ij}\mathcal{C}^{ij}_t \;, \quad (4.51)
$$

with the functions $\mathcal{C}^{ij}_\alpha$, $\mathcal{C}^{ij}_t$ from (2.25) defined for the Killing vector fields satisfying the algebra (4.26). It follows immediately from the e^σ powers, that for $\mathcal{C}^{ij}_t = 0$ this potential does not admit extremal points.

We can gain more insight to the nature of the new couplings, by observing that in the normalization of generators given in (4.5) the covariant field strengths (3.5) take the explicit form

$$
\begin{aligned}
\mathcal{G}^\alpha_{\mu\nu} &= 2\partial_{[\mu}A^\alpha_{\nu]}\;, \\
\mathcal{G}^t_{\mu\nu} &= 2\partial_{[\mu}A^t_{\nu]} + g\gamma^t_{\alpha\beta}\,A_\mu{}^\alpha A_\nu{}^\beta\;, \\
\mathcal{G}^0_{\mu\nu} &= 2\partial_{[\mu}A^0_{\nu]} + gB^+_{\mu\nu}\;,
\end{aligned}
\qquad (4.52)
$$

and the minimal couplings of vectors to tensors are explicitly given by

$$
\begin{aligned}
\mathcal{D}_\mu B^\alpha_{\nu\rho} &= \partial_\mu B^\alpha_{\nu\rho} + gA^\alpha_\mu B^+_{\nu\rho}\;, \\
\mathcal{D}_\mu B^-_{\nu\rho} &= \partial_\mu B^-_{\nu\rho} - gA^\alpha_\mu B^\alpha_{\nu\rho}\;, \\
\mathcal{D}_\mu B^+_{\nu\rho} &= \partial_\mu B^+_{\nu\rho}\;.
\end{aligned}
\qquad (4.53)
$$

In the explicit parametrization (2.12) and in the basis defined by (4.4), the various

components of the kinetic matrices g_{IJ} and m_{AB} take the explicit form

$$
\begin{aligned}
g_{\alpha\beta} &= \delta_{\alpha\beta} + 2e^{-2\sigma}\varphi_\alpha\varphi_\beta \,, & g_{\alpha-} &= e^{-2\sigma}\varphi_\alpha \,, & g_{\alpha+} &= \left(1 + e^{-2\sigma}\varphi^\beta\varphi_\beta\right)\varphi_\alpha \,, \\
g_{+-} &= e^{-2\sigma}\varphi^\alpha\varphi_\alpha \,, & g_{++} &= \left(e^\sigma + e^{-\sigma}\varphi^\alpha\varphi_\alpha\right)^2 \,, & g_{--} &= e^{-2\sigma} \,, \\
m_{\alpha\beta} &= \tfrac{1}{2}\delta_{\alpha\beta}\left(e^\sigma + e^{-\sigma}\varphi^\alpha\varphi_\alpha\right) \,, & m_{tu} &= \tfrac{1}{2}e^{-\sigma}\delta_{tu} \,, & m_{00} &= \tfrac{1}{2}e^{-\sigma} \,, \\
m_{t\alpha} &= \tfrac{1}{2}\gamma^t_{\alpha\beta}\,e^{-\sigma}\varphi^\beta \,, & m_{0\alpha} &= \tfrac{1}{2}e^{-\sigma}\varphi_\alpha \,,
\end{aligned}
\tag{4.54}
$$

respectively. The modified scalar currents (4.46) also take a simple form

$$
\mathcal{P}^\alpha_\mu = e^{-\sigma}\left(\partial_\mu\varphi^\alpha - g A^\alpha_\mu\right) = \mathcal{Q}^{1\alpha}_\mu \,, \qquad \mathcal{P}^1_\mu = \partial_\mu\sigma \,,
\tag{4.55}
$$

and the modified integrability condition (4.47) and the curvature (4.49) read

$$
\mathcal{D}_{[\mu}\mathcal{P}^\alpha_{\nu]} = -\tfrac{1}{2}g e^{-\sigma}\mathcal{G}^\alpha_{\mu\nu} \,,
$$

$$
\mathcal{Q}^{1\alpha}_{\mu\nu} = 2\mathcal{P}^\alpha_{[\mu}\,\partial_{\nu]}\sigma - g e^{-\sigma}\mathcal{G}^\alpha_{\mu\nu} \,,
\tag{4.56}
$$

The above formulae exhibit intricate couplings of vector and tensor fields, and the nature of the shift symmetries that have been gauged.

5 Conclusions and discussion

In this paper we have determined the possible gaugings in magical supergravities in $6D$, which are supergravities with 8 real supersymmetries coupled to a fixed number of vector and tensor multiplets, and arbitrary number of hypermultiplets. We have employed the embedding tensor formalism which determines in a systematic fashion the appropriate combination of vector fields that participate in the gauging process. It turns out that the allowed gauge group is uniquely determined in each case and the underlying Lie algebra, displayed in (4.5), is nilpotent generated by $(n_T - 1)$ Abelian translations with $(n_T - 2)$ central charges. Due to these central charges, the translation generators can not lie strictly in the isometry group $\mathfrak{so}(n_T, 1)$ acting on the scalar fields of the tensor multiplets. The central charges do not act on the vector/tensor sector , but may act nontrivially in the hypersector. We analysed the possible embeddings of the nilpotent gauge group into the isometry groups of the quaternionic Kähler manifolds of the hyperscalars. Since R-symmetry $Sp(1)_R$ is part of the isometry group of the hyperscalars, the embedding of the gauge group into the isometries on the quaternionic Kähler manifold determines whether $U(1)_R$ subgroup

33

of $Sp(1)_R$ can be gauged such that it acts nontrivially on the fermions. In absence of hypermultiplets, the R-symmetry acts exclusively on the fermions and one can use a linear combination of the Abelian gauge fields to gauge $U(1)_R$ such that it acts nontrivially on the fermions.

Despite the simultaneous appearance of both Chern-Simons modified 3-form field strengths as well as generalized Chern-Simons terms, the gauged magical supergravity theories we have presented are truly gauge invariant. While arbitrary number of hypermultiplets are allowed, the special number of vector and tensor multiplets is crucial for this invariance. Indeed, coupling of any additional vector (and/or tensor) multiplets would impose stringent constraints on the Chern-Simons coupling of the vectors to tensors. Existence and construction of theories satisfying these constraints that can be interpreted as extensions of magical supergravities remains to be investigated. The failure to satisfy these constraints would give rise to classical anomalies which then should satisfy the Wess-Zumino consistency conditions.

Turning to the magical gauged $6D$ supergravities we have constructed here, while truly gauge invariant, they may still have gravitational, gauge and mixed anomalies at the quantum level, owing to the presence of chiral fermions and self-dual 2-form potentials. As is well known, the gravitational anomalies are encoded in an 8-form anomaly polynomial which, in general, contains terms of the form $(\mathrm{tr}R^4)$ and $(\mathrm{tr}R^2)^2$. The first kind of terms must necessarily be absent for anomaly freedom. In presence of n_V vector multiplets and n_H hypermultiplets, it is well known that this imposes the condition $n_H = 273 + n_V - 29n_T$. From magical supergravities, this condition is satisfied with multiplicities (n_T, n_V, n_H) given by $(9, 16, 28)$, $(5, 8, 136)$, $(3, 4, 190)$ and $(2, 2, 217)$, respectively. Once the condition for the absence of the $(\mathrm{tr}R^4)$ terms is satisfied, the total gravitational anomaly polynomial (in conventions described in [43]) becomes $\Omega_8 = \frac{1}{128}(n_T - 9)(\mathrm{tr}\,R^2)^2$, with $\mathrm{tr}\,R^2 \equiv \mathrm{tr}\,R \wedge R$. The full gravitational anomaly vanishes identically for the octonionic magical supergravity, with (n_T, n_V, n_H) multiplicities given by $(9, 16, 28)$. However, in presence of gaugings, there will still be gauge and mixed anomalies. In the gauged magical supergravities with $n_T = 2, 3, 5$, the purely gravitational anomaly will be present as well. The determination of the full set of anomalies and the possible elimination by suitable Green-Schwarz type mechanisms in gauged magical supergravities is beyond the scope if this paper, and will be treated elsewhere. A detailed analysis is expected to contain elements similar to those encountered in [44] in their treatment of anomalies in gauged $N = 1$ supergravities in $4D$ in which the embedding tensor plays a key role as well.

Since their discovery higher dimensional and/or stringy origins of magical supergravity theories , which are invariant under 8 real supersymmetries, have been of great interest. Largest magical supergravity defined by the exceptional Jordan algebra $J_3^{\mathbb{O}}$

has groups of the E series as its U-duality group in $5, 4$ and 3 dimensions just like the maximal supergravity with 32 supersymmetries , but are of different real forms. The maximal supergravity and the largest magical supergravity defined by the octonionic Jordan algebra $J_3^{\mathbb{O}}$ have a common sector which is the magical supergravity theory defined by the quaternionic Jordan algebra $J_3^{\mathbb{H}}$. As was pointed out in [6, 10]. the low energy effective theory of one of the dual pairs of compactifications of IIB superstring to $4D$ studied by Sen and Vafa [5] is precisely the magical $N = 2$, $4D$ Maxwell-Einstein supergravity theory defined by the quaternionic Jordan algebra $J_3^{\mathbb{H}}$ without any hypermultiplets[11]. Since the construction of dual pairs in Sen and Vafa's work uses orbifolding on $T^4 \times S^1 \times S^1$, one can use their methods to construct the $6D$ quaternionic magical theory from IIB superstring directly. Whether one can obtain the gauged quaternionic magical theory constructed above by turning on fluxes is an interesting open problem. The complex magical supergravity defined by $J_3^{\mathbb{C}}$ can be obtained by truncation of the quaternionic theory to a subsector singlet under a certain $U(1)$ subgroup. In [7] some hypermultiplet-free $N = 2$, $4D$ string models based on asymmetric orbifolds with world-sheet superconformal symmetry using $2D$ fermionic construction were given. Two of these models correspond to the magical supergravity theories in $4D$ defined by the complex Jordan algebra $J_3^{\mathbb{C}}$ with the moduli space

$$\mathcal{M}_4 = \frac{SU(3,3)}{SU(3) \times SU(3) \times U(1)} \tag{5.1}$$

and the quaternionic $J_3^{\mathbb{H}}$ theory with the $4D$ moduli space

$$\mathcal{M}_4 = \frac{SO^*(12)}{U(6)} \tag{5.2}$$

Direct orbifold construction of the exceptional supergravity theory from superstring theory without hypermultiplets has so far proven elusive. In [6, 10] it was argued that an exceptional self-mirror Calabi-Yau 3-fold must exist such that type II superstring theory compactified on it leads to the exceptional supergravity coupled to hypermultiplets parametrizing the quaternionic symmetric space $E_{8(-24)}/E_7 \times SU(2)$. This was based on the observation [45] that there exists a six dimensional $(1,0)$ supergravity theory, which is free from gravitational anomalies, with 16 vector multiplets, 9 tensor multiplets and 28 hypermultiplets, parametrizing the exceptional quaternionic symmetric space $E_{8(-24)}/E_7 \times SU(2)$, which reduces to the $4D$ supergravity with scalar

---

[11]This follows from the fact that the bosonic sector of the $N = 2$ supersymmetric compactification , in question, coincides with that of $N = 6$ supergravity. Unique $4D$ Maxwell-Einstein supergravity theory with that property is the quaternionic magical theory [1].

manifold

$$\mathcal{M}_V \times \mathcal{M}_H = \frac{E_{7(-25)}}{E_6 \times U(1)} \times \frac{E_{8(-24)}}{E_7 \times SU(2)} \tag{5.3}$$

and the fact that the moduli space of the FHSV model [46] is a subspace of this doubly exceptional moduli space. The authors of [8] reconsidered the string derivation of FHSV model over the Enriques Calabi-Yau manifold, which corresponds to a $6D$, $(1,0)$ supergravity theory with $n_T = 9$, $n_H = 12$ and $n_V = 0$, and argued that the octonionic magical theory defined by $J_3^{\mathbb{O}}$ admits a string interpretation closely related to the Enriques model and 16 Abelian vectors of the octonionic magical supergravity theory in $6D$ is related to the rank of Type I and heterotic strings. In mathematics literature, Todorov [9] gave a construction of a Calabi-Yau 3-fold such that Type IIB superstring theory compactified over it leads to the magical $4D$ Maxwell-Einstein supergravity theory defined by the complex Jordan algebra $J_3^{\mathbb{C}}$ coupled to $(h^{(1,1)}+1) = 30$ hypermultiplets. Whether his construction can be extended to obtain a Calabi-Yau 3-fold that would lead to the $4D$ exceptional Maxwell-Einstein supergravity theory defined by $J_3^{\mathbb{O}}$ coupled to hypermultiplets is an open problem. If such an exceptional Calabi-Yau 3-fold exists and is elliptically fibered, then F-theory compactified over it is expected to be described by the $6D$ octonionic magical supergravity theory coupled to hypermultiplets.

Acknowledgments

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