

## Consistent $S^2$ Pauli Reduction of Six-dimensional Chiral Gauged Einstein-Maxwell Supergravity

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### ABSTRACT

Six-dimensional  $N = (1, 0)$  Einstein-Maxwell gauged supergravity is known to admit a  $(\text{Minkowski})_4 \times S^2$  vacuum solution with four-dimensional  $N = 1$  supersymmetry. The massless sector comprises a supergravity multiplet, an  $SU(2)$  Yang-Mills vector multiplet, and a scalar multiplet. In this paper it is shown that, remarkably, the six-dimensional theory admits a fully consistent dimensional reduction on the 2-sphere, implying that all solutions of the four-dimensional  $N = 1$  supergravity can be lifted back to solutions in six dimensions. This provides a striking realisation of the idea, first proposed by Pauli, of obtaining a theory that includes Yang-Mills fields by dimensional reduction on a coset space. We address the cosmological constant problem within this model, and find that if the Kaluza-Klein mass scale is taken to be  $10^{-3}$  eV (as has recently been suggested) then four-dimensional gauge-coupling constants for bulk fields must be of the order of  $10^{-31}$ . We also suggest a link between a modification of the model with 3-branes, and a five-dimensional model based on an  $S^1/Z_2$  orbifold.

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# 1 Introduction

Dimensional reductions can be divided into two types. The first includes the original  $S^1$  reduction of Kaluza [1], and the group-manifold reductions pioneered by DeWitt [2]. All these are characterised by the fact that the reduction ansätze for the metric and the other higher-dimensional fields are invariant under a transitively-acting group of isometries in the internal space;  $U(1)$  in the case of the Kaluza  $S^1$  reduction, and  $G_L$  (the left action of  $G$ ) in the case of a DeWitt reduction on the group manifold  $G$ . The group invariance of the ansatz ensures that the reduction will necessarily be *consistent*, in the sense that all solutions of the reduced lower-dimensional equations of motion will correspond to solutions of the higher-dimensional equations of motion. The transitivity of the group action implies that there will be just a finite number of fields in the lower-dimensional theory. These include a  $U(1)$  gauge boson in the Kaluza reduction, and Yang-Mills fields with gauge group  $G_R$  in the DeWitt reduction.

A second type of dimensional reduction was first proposed by Pauli [3, 4], ten years before DeWitt's non-abelian generalisation of Kaluza's circle reduction. Pauli's specific example was a reduction on  $S^2$ , and more generally one can consider a reduction on any coset space  $G/H$ . In a small-fluctuation analysis there will always be Yang-Mills fields whose gauge group is the isometry of the coset space (which we can generally take to be  $G_R$ ). If these could be present also in a full non-linear reduction, it would provide a more economical way of obtaining Yang-Mills fields, since the number of extra dimensions needed to obtain the gauge group  $G$  from a coset  $G/H$  can be much smaller than the number needed for a DeWitt reduction on  $G$ . However, in a Pauli reduction on  $G/H$  it is clear that an ansatz that retains the Yang-Mills  $G_R$  gauge fields cannot be invariant under any transitively-acting group of isometries, and so there is no straightforward group-theoretic reason why such a reduction should be consistent. In fact in general such coset reductions are guaranteed to be *inconsistent*. (Pauli never exhibited a concrete example with a consistent reduction.) What is quite remarkable, however, is that there do exist exceptional cases where Pauli reductions can be consistent. This gives a considerable impetus to the search for consistent Pauli reductions.

One of the first examples of a consistent Pauli reduction was the reduction of eleven-dimensional supergravity on  $S^7$ , to give  $SO(8)$ -gauged  $N = 8$  supergravity in four dimensions. In fact this example is of immense complexity, and although a proof of its consistency is presented in [5], no complete and fully explicit reduction ansatz has been given. A simpler, but still highly non-trivial, example is the  $S^4$  reduction of eleven-dimensional supergravity

to give gauged  $SO(5)$  supergravity in seven dimensions [6]. A variety of other explicit examples have also been encountered, including a consistent reduction of the  $SL(2, \mathbb{R})$ -singlet sector of type IIB supergravity on  $S^5$ , giving an  $SO(6)$  gauge theory in five dimensions [7], and a consistent Pauli reduction of the  $D$ -dimensional bosonic string on  $S^3$  or  $S^{D-3}$  [8].

A further class of consistent Pauli reductions found in [8] consisted of  $S^2$  reductions of an Einstein-Maxwell-dilaton system in any dimension  $D$ , with a specific value of the dilaton-coupling constant  $a$  in the Maxwell kinetic term  $-\frac{1}{4}e^{a\phi} F^2$ . This value is in fact precisely the one that allows the Einstein-Maxwell-dilaton theory to be itself derived as a Kaluza reduction of pure Einstein gravity in  $(D + 1)$  dimensions. This fact was exploited in [9] where it was shown that the consistent Pauli reduction on  $S^2$  could be derived by starting from the necessarily consistent DeWitt reduction of  $(D + 1)$ -dimensional Einstein gravity on  $S^3$ , and reinterpreting it as first a Kaluza reduction to give the Einstein-Maxwell-dilaton theory in  $D$  dimensions, followed by the Pauli reduction on  $S^2$ . An immediate generalisation shows that *any* theory obtainable from a Kaluza  $S^1$  reduction of some yet higher-dimensional theory can then be consistently Pauli reduced on  $G/U(1)$  for any  $G$  [9]. A further generalisation expressing a DeWitt reduction on  $G$  as a DeWitt reduction on  $H$  followed by a Pauli reduction on  $G/H$  can also be given [9]. Thus in these types of Pauli reductions, one can after all find a group-theoretic understanding for their consistency [9].

Not every consistent Pauli reduction admits such an interpretation in terms of a DeWitt reduction from a yet higher dimension, however. For example, there is no explanation for the consistent  $S^7$  reduction of eleven-dimensional supergravity in terms of an  $SO(7)$  DeWitt reduction of some 32-dimensional theory. In fact, none of the other consistent reductions on  $S^n$  with  $n \geq 3$  described above seem to admit any explanation in terms of DeWitt reductions.

In this paper, we find a new and very remarkable consistent Pauli reduction with an  $S^2$  internal space. Specifically, we consider the six-dimensional chiral gauged supergravity studied in [10], which was shown to admit a supersymmetric  $(\text{Minkowski})_4 \times S^2$  vacuum with  $N = 1$  four-dimensional supersymmetry. The six-dimensional theory is a particular case of a more general class of supergravities constructed in [11]. We find that because of intricate “conspiracies” between the structure of the six-dimensional theory and the properties of the 2-sphere, there exists a consistent reduction that yields the  $N = 1$  supergravity coupled to an  $SU(2)$  Yang-Mills multiplet and a scalar multiplet in four dimensions. This reduction is all the more remarkable because the six-dimensional gauged supergravity itself seems to have no seven-dimensional origin via a Kaluza  $S^1$  reduction. Thus it provides the first example

of a consistent  $S^2$  reduction that has no apparent underlying group-theoretic explanation. As such, it can perhaps be considered to provide the most striking realisation of Pauli's original idea.

We begin in section 2 by presenting our  $S^2$  reduction of the bosonic sector of the chiral six-dimensional gauged Einstein-Maxwell supergravity, indicating how the verification of the consistency of the reduction proceeds. In section 3 we consider the fermionic sector of the theory, showing how one can use the six-dimensional supersymmetry transformation rules in order to derive the fermionic reduction ansatz. In section 4, we consider the lift to six dimensions of black holes in the reduced four-dimensional theory. In section 5 we examine some of the implications of our results for four-dimensional physics, and we make a comparison with previous results in the literature [12, 13]. The model requires dimensionless Yang-Mills coupling constants for bulk gauge fields of the order of  $10^{-31}$  if, as suggested in [13], the Kaluza-Klein mass scale is taken to be of order  $10^{-3}$  eV. We also find that 3-brane modifications of the model may have bulk gauge fields with Yang-Mills coupling constants of order unity, but this requires fine-tuning, which can be achieved with needle-shaped internal spaces (rather than rugby balls), which are effectively one-dimensional  $S^1/Z_2$  orbifolds. Section 6 contains discussion and conclusions. In an appendix, we give some details of the curvature calculations for the class of metric reduction ansatz that we use in this paper.

## 2 Pauli $S^2$ Reduction of the Bosonic Sector

The bosonic sector of the six-dimensional  $N = (1, 0)$  gauged Einstein-Maxwell supergravity is described by the Lagrangian [10]

$$\mathcal{L} = \hat{R} \hat{*}\mathbf{1} - \frac{1}{4} \hat{*}d\hat{\phi} \wedge d\hat{\phi} - \frac{1}{2} e^{\hat{\phi}} \hat{*}\hat{H}_{(3)} \wedge \hat{H}_{(3)} - \frac{1}{2} e^{\frac{1}{2}\hat{\phi}} \hat{*}\hat{F}_{(2)} \wedge \hat{F}_{(2)} - 8g^2 e^{-\frac{1}{2}\hat{\phi}} \hat{*}\mathbf{1}, \quad (2.1)$$

where  $\hat{F}_{(2)} = d\hat{A}_{(1)}$ ,  $\hat{H}_{(3)} = d\hat{B}_{(2)} + \frac{1}{2}\hat{F}_{(2)} \wedge \hat{A}_{(1)}$ , and we place hats on all six-dimensional fields. (We use conventions where  $\hat{*}\hat{\omega} \wedge \hat{\omega} = (1/p!) \hat{\omega}^{M_1 \dots M_p} \hat{\omega}_{M_1 \dots M_p} \hat{*}\mathbf{1}$  for any  $p$ -form  $\hat{\omega}$ .) Here  $g$  is the gauge-coupling constant, and the fermions all carry charge  $g$  in their minimal coupling to the  $U(1)$  gauge field  $\hat{A}$ . The bosonic equations of motion following from (2.1) are

$$\begin{aligned} \hat{R}_{MN} &= \frac{1}{4} \partial_M \hat{\phi} \partial_N \hat{\phi} + \frac{1}{2} e^{\frac{1}{2}\hat{\phi}} (\hat{F}_{MN}^2 - \frac{1}{8} \hat{F}^2 \hat{g}_{MN}) + \frac{1}{4} e^{\hat{\phi}} (\hat{H}_{MN}^2 - \frac{1}{6} \hat{H}^2 \hat{g}_{MN}) \\ &\quad + 2g^2 e^{-\frac{1}{2}\hat{\phi}} \hat{g}_{MN}, \\ \hat{\square} \hat{\phi} &= \frac{1}{4} e^{\frac{1}{2}\hat{\phi}} \hat{F}^2 + \frac{1}{6} e^{\hat{\phi}} \hat{H}^2 - 8g^2 e^{-\frac{1}{2}\hat{\phi}}, \end{aligned}$$

$$\begin{aligned}
d(e^{\frac{1}{2}\hat{\phi}} \hat{*}\hat{F}_{(2)}) &= e^{\hat{\phi}} \hat{*}\hat{H}_{(3)} \wedge \hat{F}_{(2)}, \\
d(e^{\hat{\phi}} \hat{*}\hat{H}_{(3)}) &= 0.
\end{aligned}
\tag{2.2}$$

Note that the dimensionful coupling constant  $g$  can be rescaled at will by adding a constant to  $\hat{\phi}$ , together with compensating rescalings of the other fields. Thus it is really the quantity  $g e^{-\frac{1}{4}\hat{\phi}_0}$ , where  $\hat{\phi}_0$  is the expectation value of  $\hat{\phi}$ , that has physical significance in the theory. We could, conversely, without loss of generality set  $\hat{\phi}_0 = 0$ .

It has long been known that this theory admits a solution of the form  $(\text{Minkowski})_4 \times S^2$ , and furthermore, that this solution has  $N = 1$  supersymmetry in the four-dimensional spacetime [10]. The spectrum of four-dimensional massless fields has been discussed at the linearised level in [10, 12], and according to [12] it comprises the  $N = 1$  supergravity multiplet, a Yang-Mills  $SU(2)$  triplet of vector multiplets, and a singlet scalar multiplet. The  $SU(2)$  Yang-Mills fields have a natural origin in the isometry group of the compactifying 2-sphere.

In the reduction of a generic theory on a coset space such as  $S^2 = SO(3)/SO(2)$ , the massless fields that one finds in a linearised analysis of the harmonic expansion will not decouple from the massive fields associated with the higher harmonics. In other words, one finds that when the full nonlinear structure of the theory is taken into account, the field equations for the massive fields will have source terms built purely from the massless fields. These sources prevent one from consistently setting the massive fields to zero. In the present case, however, it turns out that some very remarkable conspiracies imply that there is an exact decoupling of the massive fields, allowing us to find a consistent reduction on  $S^2$  that yields well-defined four-dimensional equations of motion for the massless sector of the full dimensionally-reduced theory. In particular, we obtain the  $SU(2)$  Yang-Mills fields associated with the isometry group of the round 2-sphere, despite the fact that the reduction ansatz will (necessarily) not be invariant under the  $SU(2)$  action of the round sphere.

We arrived at the correct reduction ansatz by a process of trial and error, and adjusting of certain parameters. We shall now simply present the result, and then indicate how one establishes that it does indeed work. The six-dimensional metric will be written as

$$ds^2 = e^{\frac{1}{2}\phi} ds_4^2 + e^{-\frac{1}{2}\phi} g_{mn} (dy^m + 2g A^i K_i^m)(dy^n + 2g A^j K_j^n),
\tag{2.3}$$

where  $g_{mn}$  is the metric on the round  $S^2$ , normalised to  $R_{mn} = 8g^2 g_{mn}$ , where  $g$  is the gauge coupling constant in (2.1). The quantities  $K_i = K_i^m \partial/\partial y^m$  are the three Killing vectors on the round 2-sphere. The four-dimensional metric  $ds_4^2 = g_{\mu\nu} dx^\mu dx^\nu$ , the Yang-Mills

gauge potentials  $A^i = A^i_\mu dx^\mu$ , and the scalar field  $\phi$  all depend on the four-dimensional coordinates  $x^\mu$  only, and are independent of the coordinates  $y^m$  of the 2-sphere. It will prove convenient to introduce an orthonormal basis  $\hat{e}^A$  for the six-dimensional metric, which we do by defining

$$\hat{e}^\alpha = e^{\frac{1}{4}\phi} e^\alpha, \quad \hat{e}^a = e^{-\frac{1}{4}\phi} (e^a + 2g A^i K_i^a), \quad (2.4)$$

where  $e^\alpha$  is an orthonormal basis for  $ds_4^2$ , and  $e^a$  is an orthonormal basis for the metric  $g_{mn} dy^m dy^n$  on the round 2-sphere. The quantities  $K_i^a$  are the orthonormal frame components of the Killing vectors on  $S^2$ :  $K_i^a = K_i^m e_m^a$ .

We find that the appropriate ansatz for the other six-dimensional fields is as follows:

$$\begin{aligned} \hat{F}_{(2)} &= 2g e^{\frac{1}{2}\phi} \epsilon_{ab} \hat{e}^a \wedge \hat{e}^b - \mu_i F^i, \\ \hat{H}_{(3)} &= H_{(3)} - 2g F^i \wedge K_i^a (e^a + 2g A^j K_j^a), \\ \hat{\phi} &= -\phi, \end{aligned} \quad (2.5)$$

where the Yang-Mills field strengths are defined by  $F^i = dA^i + g \epsilon_{ijk} A^j \wedge A^k$ . Here, the three scalars  $\mu_i$  are the  $SU(2)$  triplet of lowest non-trivial harmonics on  $S^2$ . It will prove convenient for some purposes to have an explicit representation for the metric of the 2-sphere. Bearing in mind that the round  $S^2$  in (2.3) is normalised so that  $R_{mn} = 8g^2 g_{mn}$ , it follows that we may write it as

$$g_{mn} dy^m dy^n = \frac{1}{8g^2} (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2.6)$$

In terms of these coordinates, we can write the scalar harmonics  $\mu_i$  as

$$\mu_1 = \sin \theta \cos \varphi, \quad \mu_2 = \sin \theta \sin \varphi, \quad \mu_3 = \cos \theta. \quad (2.7)$$

One can easily construct the Killing vectors  $K_i^m$  from these, as

$$K_i^m = \frac{1}{8g^2} \epsilon^{mn} \partial_n \mu_i, \quad (2.8)$$

giving

$$K_1 = -\sin \varphi \frac{\partial}{\partial \theta} - \cot \theta \cos \varphi \frac{\partial}{\partial \varphi}, \quad K_2 = \cos \varphi \frac{\partial}{\partial \theta} - \cot \theta \sin \varphi \frac{\partial}{\partial \varphi}, \quad K_3 = \frac{\partial}{\partial \varphi}. \quad (2.9)$$

They satisfy the algebra  $[K_i, K_j] = -\epsilon_{ijk} K_k$ . Some useful *lemmata* are

$$\begin{aligned} K_i &= \frac{1}{8g^2} \epsilon_{ijk} \mu_j d\mu_k, & dK_i &= 8g^2 \epsilon_{ijk} K_j \wedge K_k = \frac{1}{4g^2} \mu_i \Omega_{(2)}, \\ g^{mn} \partial_m \mu_i \partial_n \mu_j &= 8g^2 (\delta_{ij} - \mu_i \mu_j), & \epsilon^{mn} \partial_m \mu_i \partial_n \mu_j &= 8g^2 \epsilon_{ijk} \mu_k, \end{aligned} \quad (2.10)$$

where  $\Omega_{(2)} \equiv \sin \theta d\theta \wedge d\varphi$  is the volume form of the unit  $S^2$ .

In performing the substitution of the ansatz into the six-dimensional field equations, it is useful to note that the six-dimensional Hodge duals of  $\hat{F}_{(2)}$  and  $\hat{H}_{(3)}$  are given by

$$\begin{aligned}\hat{*}\hat{F}_{(2)} &= 4g e^{\frac{3}{2}\phi} *1 - \frac{1}{2} \mu_i \epsilon_{ab} *F^i \wedge \hat{e}^a \wedge \hat{e}^b, \\ \hat{*}\hat{H}_{(3)} &= \frac{1}{2} e^{-\frac{1}{2}\phi} \epsilon_{ab} *H_{(3)} \wedge \hat{e}^a \wedge \hat{e}^b + \frac{1}{4g} *F^i \wedge D\mu_i,\end{aligned}\quad (2.11)$$

where an unhatted  $*$  denotes the Hodge dual in the four-dimensional metric  $ds_4^2$ , and

$$D\mu_i \equiv d\mu_i + 2g \epsilon_{ijk} A^j \mu_k. \quad (2.12)$$

The results from substituting the ansatz into the six-dimensional Bianchi identities and field equations are as follows. First, one can see from (2.5) that  $\hat{F}_{(2)}$  can be rewritten as  $\hat{F}_{(2)} = (2g)^{-1} \Omega_{(2)} - d(\mu_i A^i)$ , which shows that  $d\hat{F}_{(2)} = 0$  is satisfied identically. After some algebra one finds that the Bianchi identity  $d\hat{H}_{(3)} = \frac{1}{2} \hat{F}_{(2)} \wedge \hat{F}_{(2)}$  implies the four-dimensional equation

$$dH_{(3)} = \frac{1}{2} F^i \wedge F^i. \quad (2.13)$$

The six-dimensional equation  $d(e^{\hat{\phi}} \hat{*}\hat{H}_{(3)}) = 0$  implies the two four-dimensional equations

$$\begin{aligned}D(e^{-\phi} *F^i) &= e^{-2\phi} *H_{(3)} \wedge F^i, \\ d(e^{-2\phi} *H_{(3)}) &= 0,\end{aligned}\quad (2.14)$$

while the six-dimensional equation  $d(e^{\frac{1}{2}\hat{\phi}} \hat{*}\hat{F}^i) = e^{\hat{\phi}} \hat{*}\hat{H}_{(3)} \wedge \hat{F}_{(2)}$  gives again the first of the equations in (2.14). The six-dimensional dilaton equation of motion yields the four-dimensional equation

$$d*d\phi = \frac{1}{2} e^{-\phi} *F^i \wedge F^i + e^{-2\phi} *H_{(3)} \wedge H_{(3)}. \quad (2.15)$$

Finally, using the expressions for the Ricci tensor given in (A.6), we find that the six-dimensional Einstein equations involving  $\hat{R}_{\alpha\beta}$ ,  $\hat{R}_{\alpha b}$  and  $\hat{R}_{ab}$  respectively yield the four-dimensional equations

$$\begin{aligned}R_{\alpha\beta} &= \frac{1}{2} \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{4} \square \phi \eta_{\alpha\beta} + \frac{1}{2} e^{-\phi} (F_{\alpha\gamma}^i F_{\beta}^{i\gamma} - \frac{1}{8} F_{\gamma\delta}^i F^{i\gamma\delta} \eta_{\alpha\beta}) \\ &\quad + \frac{1}{4} e^{-2\phi} (H_{\alpha\beta}^2 - \frac{1}{6} H^2 \eta_{\alpha\beta}), \\ D^\beta (e^{-\phi} F_{\alpha\beta}^i) &= -\frac{1}{2} e^{-2\phi} H_{\alpha}{}^{\beta\gamma} F_{\beta\gamma}^i, \\ \square \phi &= -\frac{1}{4} e^{-\phi} F_{\alpha\beta}^i F^{i\alpha\beta} - \frac{1}{6} e^{-2\phi} H^2.\end{aligned}\quad (2.16)$$

We find that the system of four-dimensional equations that we have obtained by this 2-sphere reduction can be derived from an action principle, with the Lagrangian given by

$$\mathcal{L} = R * \mathbb{1} - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{-\phi} * F^i \wedge F^i - \frac{1}{2} e^{-2\phi} * H_{(3)} \wedge H_{(3)}, \quad (2.17)$$

where  $F^i = dA^i + g \epsilon_{ijk} A^j \wedge A^k$  and  $H_{(3)} = dB_{(2)} + \frac{1}{2} \omega_{(3)}$ , where  $d\omega_{(3)} = F^i \wedge F^i$ .

It should be emphasised that highly non-trivial cancellations take place when one substitutes the ansatz into the higher-dimensional field equations. In particular, it is only because of the specific details of how the Yang-Mills fields enter not only in the metric ansatz, but also in the ansätze for  $\hat{F}_{(2)}$  and  $\hat{H}_{(3)}$  that one finds that all the dependence on the coordinates  $y^m$  on  $S^2$  eventually cancels out, leading to well-defined four-dimensional equations of motion. This point can be illustrated by examining some of the calculations for the reduction of the Einstein equations in more detail. For the components  $\hat{R}_{\alpha\beta}$  in (2.2), substitution of the ansätze gives

$$\begin{aligned} R_{\alpha\beta} &= \frac{1}{2} \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{4} \square \phi \eta_{\alpha\beta} + \frac{1}{2} e^{-\phi} F_{\alpha\gamma}^i F_\beta^{j\gamma} (\mu_i \mu_j + 4g^2 K_i^a K_{j a} + 4g^2 K_i^a K_{j a}) \\ &\quad - \frac{1}{16} e^{-\phi} F_{\gamma\delta}^i F^{j\gamma\delta} (\mu_i \mu_j + 8g^2 K_i^a K_{j a}) + \frac{1}{4} e^{-2\phi} (H_{\alpha\beta}^2 - \frac{1}{6} H^2 \eta_{\alpha\beta}), \end{aligned} \quad (2.18)$$

where we have kept distinct the contributions in the  $F^i F^j$  terms coming from the different sources. In the  $F_{\alpha\gamma}^i F_\beta^{j\gamma}$  terms, the  $\mu_i \mu_j$  contribution comes from the ansatz for  $\hat{F}_{(2)}$ , while the two  $4g^2 K_i^a K_{j a}$  contributions come from the ansätze for  $\hat{H}_{(3)}$  and the metric. In the  $F_{\gamma\delta}^i F^{j\gamma\delta}$  the two contributions come from the ansätze for  $\hat{F}_{(2)}$  and  $\hat{H}_{(3)}$  respectively. Using (2.8) and (2.10), one finds that  $\mu_i \mu_j + 8g^2 K_i^a K_{j a} = \delta_{ij}$ , and hence all the  $y$ -dependence cancels in (2.18), yielding the first equation in (2.16). The components  $\hat{R}_{ab}$  of the Einstein equations in (2.2) provide another cancellation, with all the  $y$ -dependence associated with the  $F_{\gamma\delta}^i F^{j\gamma\delta}$  terms again conspiring to vanish. A further remarkable feature evident in these components is that the consistent reduction is achieved with only the single “breathing mode scalar”  $\phi$  in the reduction ansatz. Usually, the scalars parameterising deformations of the internal space would be needed in any consistent reduction, since the Yang-Mills fields would act as sources for them. In our present example, these source terms turn out to cancel, and so the normally-expected 5 modulus scalars are consistently set to zero.

The four-dimensional theory that we have arrived at can be cast into a more conventional form by performing a dualisation of  $H_{(3)}$  to an axionic scalar. We do this by the standard procedure of introducing a Lagrange multiplier  $\sigma$  and adding the term  $-\sigma (dH_{(3)} - \frac{1}{2} F^i \wedge F^i)$  to (2.17). This imposes the Bianchi identity for  $H_{(3)}$ . Varying instead with respect to  $H_{(3)}$ , we find that

$$H_{(3)} = e^{2\phi} * d\sigma. \quad (2.19)$$

Substituting this into the modified Lagrangian gives the dualised version

$$\mathcal{L} = R * \mathbb{1} - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} * d\sigma \wedge d\sigma - \frac{1}{2} e^{-\phi} * F^i \wedge F^i + \frac{1}{2} \sigma F^i \wedge F^i. \quad (2.20)$$

### 3 The Fermionic Sector

The easiest way to determine the correct reduction ansatz in the fermionic sector is by looking at the higher-dimensional supersymmetry transformation rules. For the fermions, we have

$$\begin{aligned} \delta \hat{\psi}_M &= \hat{D}_M \hat{\epsilon} + \frac{1}{48} e^{\frac{1}{2}\hat{\phi}} \hat{H}_{NPQ} \hat{\Gamma}^{NPQ} \hat{\Gamma}_M \hat{\epsilon}, \\ \delta \hat{\chi} &= -\frac{1}{4} [\partial_M \hat{\phi} \hat{\Gamma}^M - \frac{1}{6} e^{\frac{1}{2}\hat{\phi}} \hat{H}_{MNP} \hat{\Gamma}^{MNP}] \hat{\epsilon}, \\ \delta \hat{\lambda} &= \frac{1}{4\sqrt{2}} [e^{\frac{1}{4}\hat{\phi}} \hat{F}_{MN} \hat{\Gamma}^{MN} - 8i g e^{-\frac{1}{4}\hat{\phi}}] \hat{\epsilon}, \end{aligned} \quad (3.1)$$

where  $\hat{D}_M$  is the gauge-covariant derivative,  $\hat{D}_M \hat{\epsilon} \equiv (\hat{\nabla}_M - i g \hat{A}_M) \hat{\epsilon}$ . The transformation rules for the bosons are

$$\begin{aligned} \delta \hat{e}_M^A &= -\frac{1}{4} \hat{\epsilon} \hat{\Gamma}^A \hat{\psi}_M + \frac{1}{4} \hat{\psi}_M \hat{\Gamma}^A \hat{\epsilon}, \\ \delta \hat{\phi} &= \frac{1}{2} \hat{\epsilon} \hat{\chi} + \frac{1}{2} \hat{\chi} \hat{\epsilon}, \\ \delta \hat{A}_M &= \frac{1}{2\sqrt{2}} e^{-\frac{1}{2}\hat{\phi}} (\hat{\epsilon} \hat{\Gamma}_M \hat{\lambda} - \hat{\lambda} \hat{\Gamma}_M \hat{\epsilon}), \\ \delta \hat{B}_{MN} &= \hat{A}_{[M} \delta \hat{A}_{N]} + \frac{1}{2} e^{-\hat{\phi}} (\hat{\epsilon} \hat{\Gamma}_{[M} \hat{\psi}_{N]} + \hat{\psi}_{[M} \hat{\Gamma}_{N]} \hat{\epsilon} + \hat{\epsilon} \hat{\Gamma}_{MN} \hat{\chi} - \hat{\chi} \hat{\Gamma}_{MN} \hat{\epsilon}). \end{aligned} \quad (3.2)$$

Since the fermion kinetic terms in the six-dimensional Lagrangian are of the form

$$\hat{e}^{-1} \mathcal{L} = \hat{\psi}_M \hat{\Gamma}^{MNP} \hat{D}_N \hat{\psi}_P + \hat{\chi} \hat{\Gamma}^M \hat{D}_M \hat{\chi} + \hat{\lambda} \hat{\Gamma}^M \hat{D}_M \hat{\lambda}, \quad (3.3)$$

it follows that in order to obtain a four-dimensional theory with canonical fermion kinetic terms having no dilaton exponential scalings, the  $\hat{\chi}$  and  $\hat{\lambda}$  fields and the tangent-frame gravitino components  $\hat{\psi}_A$  should all receive scalings by a factor of  $e^{-\frac{1}{8}\hat{\phi}}$  in their dimensional reductions. It then follows from scaling arguments that in order to obtain a canonical four-dimensional gravitino transformation rule  $\delta \psi_\mu = \nabla_\mu \epsilon + \dots$ , with no dilaton exponential scaling in the derivative term, the six-dimensional supersymmetry parameter  $\hat{\epsilon}$  should receive a scaling by  $e^{\frac{1}{8}\hat{\phi}}$  in its dimensional reduction.

We now introduce the chiral gauge-covariantly constant 2-component spinor  $\eta$  on  $S^2$ , which satisfies the equation

$$(\nabla_a - i A_a^{\text{mono}}) \eta = 0, \quad (3.4)$$

where  $A_a^{\text{mono}}$  is a potential for the Dirac monopole on  $S^2$ . The six-dimensional Dirac matrices  $\hat{\Gamma}_A$  will be decomposed as

$$\hat{\Gamma}_\alpha = \gamma_\alpha \otimes \sigma_3, \quad \hat{\Gamma}_a = \mathbf{1} \otimes \sigma_a, \quad (3.5)$$

where  $1 \leq a \leq 2$ . The six-dimensional supersymmetry parameter is then decomposed as

$$\hat{\epsilon} = e^{\frac{1}{8}\phi} \epsilon \otimes \eta. \quad (3.6)$$

Substituting first into the transformation rule for  $\hat{\lambda}$  in (3.1), and using (2.5), we find that

$$\delta\hat{\lambda} = \sqrt{2} i g e^{\frac{3}{8}\phi} \epsilon \otimes (\sigma_3 - 1) \eta - \frac{1}{4\sqrt{2}} e^{-\frac{5}{8}\phi} \mu_i F_{\alpha\beta}^i \gamma^{\alpha\beta} \epsilon \otimes \eta. \quad (3.7)$$

Thus we deduce that the chirality of  $\eta$  is

$$\sigma_3 \eta = +\eta, \quad (3.8)$$

and that the dimensional reduction of  $\hat{\lambda}$  should be given by

$$\hat{\lambda} = e^{-\frac{1}{8}\phi} \mu_i \lambda^i \otimes \eta. \quad (3.9)$$

We therefore obtain a purely four-dimensional expression for  $\delta\lambda^i$ , with no dependence on the coordinates of  $S^2$ , namely

$$\delta\lambda^i = -\frac{1}{4\sqrt{2}} e^{-\frac{1}{2}\phi} F_{\alpha\beta}^i \gamma^{\alpha\beta} \epsilon. \quad (3.10)$$

The triplet of spin  $\frac{1}{2}$  fields  $\lambda^i$  form the  $N = 1$  superpartners of the  $SU(2)$  Yang-Mills fields  $A_\mu^i$ .

Proceeding in the same vein, we can determine the appropriate ansätze for the dimensional reduction of the remaining fermionic fields, the guiding principle being that one should thereby obtain consistent purely four-dimensional transformation rules, with no dependence on the coordinates of the internal 2-sphere. The summary of our results is the following. The reduction ansätze are

$$\begin{aligned} \hat{\lambda} &= e^{-\frac{1}{8}\phi} \mu_i \lambda^i \otimes \eta, \\ \hat{\chi} &= e^{-\frac{1}{8}\phi} [\chi \otimes \eta + \frac{\sqrt{2}}{3} g K_i^a \lambda^i \otimes \sigma_a \eta], \\ \hat{\psi}_\alpha &= e^{-\frac{1}{8}\phi} [\psi_\alpha \otimes \eta + \frac{1}{\sqrt{2}} g K_i^a \gamma_\alpha \lambda^i \otimes \sigma_a \eta], \\ \hat{\psi}_a &= e^{-\frac{1}{8}\phi} [-\frac{1}{2} \chi \otimes \sigma_a \eta + \frac{3}{\sqrt{2}} g K_i^a \lambda^i \otimes \eta - \frac{i}{\sqrt{2}} g \epsilon_{ab} K_i^b \lambda^i \otimes \eta], \end{aligned} \quad (3.11)$$

leading to the following four-dimensional transformation rules

$$\begin{aligned}
\delta\lambda^i &= -\frac{1}{4\sqrt{2}} e^{-\frac{1}{2}\phi} F_{\alpha\beta}^i \gamma^{\alpha\beta} \epsilon, \\
\delta\chi &= \frac{1}{4} \partial_\alpha \phi \gamma^\alpha \epsilon + \frac{1}{24} e^{-\phi} H_{\alpha\beta\gamma} \gamma^{\alpha\beta\gamma} \epsilon, \\
\delta\psi_\alpha &= \nabla_\alpha \epsilon + \frac{1}{8} \partial_\beta \phi \gamma_\alpha \gamma^\beta \epsilon + \frac{1}{48} e^{-\phi} H_{\beta\gamma\delta} \gamma^{\beta\gamma\delta} \gamma_\alpha \epsilon.
\end{aligned} \tag{3.12}$$

As in the reductions of the bosonic fields discussed earlier, we again find that the way in which the dependences on the internal 2-sphere coordinates  $y^m$  eventually match up, so as to allow four-dimensional transformation rules to be consistently read off, involves some rather non-trivial cancellations between terms.

The four-dimensional transformation rules (3.12) take a more familiar form if we rewrite them using the axion  $\sigma$  dual to  $H_{(3)}$ , given by (2.19). Also, it is advantageous to redefine the gravitino, by setting

$$\psi_\alpha = \psi'_\alpha + \frac{1}{2} \gamma_\alpha \chi. \tag{3.13}$$

The four-dimensional fermionic supersymmetry transformations then become

$$\begin{aligned}
\delta\lambda^i &= -\frac{1}{4\sqrt{2}} e^{-\frac{1}{2}\phi} F_{\alpha\beta}^i \gamma^{\alpha\beta} \epsilon, \\
\delta\chi &= \frac{1}{4} (\partial_\alpha \phi - i e^\phi \partial_\alpha \sigma \gamma_5) \gamma^\alpha \epsilon, \\
\delta\psi'_\alpha &= \nabla_\alpha \epsilon - \frac{i}{4} e^\phi \partial_\alpha \sigma \gamma_5 \epsilon.
\end{aligned} \tag{3.14}$$

If the reduction ansatz (3.11), together with the previous bosonic ansatz, are substituted into the six-dimensional fermionic equations of motion, one will again find that four-dimensional equations of motion emerge in a consistent manner. In this regard we remark that the gravitino  $\psi_\mu$  and the spinor  $\chi$  in the scalar multiplet  $(\phi, \sigma, \chi)$  are uncharged under the  $SU(2)$  Yang-Mills group, while the spinors  $\lambda^i$  in the vector multiplets  $(A_\mu^i, \lambda^i)$  have the expected minimal coupling, with  $D_\mu \lambda^i = \nabla_\mu \lambda^i + 2g \epsilon_{ijk} A_\mu^j \lambda^k$ .

Turning now to the supersymmetry transformation rules for the bosons, we first find by substituting into the variation of  $\hat{\phi}$  in (3.2) that

$$\delta\phi = -\frac{1}{2} (\bar{\epsilon} \chi + \bar{\chi} \epsilon). \tag{3.15}$$

Next, from  $\delta\hat{A}_M$  we find the four-dimensional transformations

$$\delta A_\mu^i = -\frac{1}{2\sqrt{2}} e^{\frac{1}{2}\phi} (\bar{\epsilon} \gamma_\mu \lambda^i - \bar{\lambda}^i \gamma_\mu \epsilon). \tag{3.16}$$

As usual in a dimensional reduction, it is necessary when analysing the supersymmetry transformation of the vielbein to perform also a compensating local Lorentz transformation

in order to maintain the gauge choice  $\hat{e}_m^\alpha = 0$  of the reduction ansatz (2.4), and to allow the six-dimensional supersymmetry transformations to be interpreted in terms of transformations of the four-dimensional fields. Thus we augment  $\delta\hat{e}_M^A$  in (3.2) to

$$\delta\hat{e}_M^A = -\frac{1}{4}\hat{\epsilon}\hat{\Gamma}^A\hat{\psi}_M + \frac{1}{4}\hat{\psi}_M\hat{\Gamma}^A\hat{\epsilon} + \Lambda^A{}_B\hat{e}_M^B, \quad (3.17)$$

where  $\Lambda_{AB} = -\Lambda_{BA}$ , and take

$$\begin{aligned} \Lambda^\alpha{}_b &= \frac{3}{4\sqrt{2}}gK_{ib}(\bar{\epsilon}\gamma^\alpha\lambda^i - \bar{\lambda}^i\gamma^\alpha\epsilon) - \frac{i}{2\sqrt{2}}g\epsilon_{bc}K_i^c(\bar{\epsilon}\gamma^\alpha\lambda^i - \bar{\lambda}^i\gamma^\alpha\epsilon), \\ \Lambda^a{}_b &= -\frac{i}{8}\epsilon^a{}_b(\bar{\epsilon}\chi - \bar{\chi}\epsilon). \end{aligned} \quad (3.18)$$

Additionally, it is advantageous to perform a further purely four-dimensional local Lorentz transformation, whose purpose is simply to neaten up and simplify the results:

$$\Lambda^\alpha{}_\beta = \frac{1}{8}(\bar{\epsilon}\gamma^\alpha{}_\beta\chi - \bar{\chi}\gamma^\alpha{}_\beta\epsilon). \quad (3.19)$$

The various four-dimensional sectors of the six-dimensional vielbein transformation rules then reproduce previously-derived results for  $\delta\phi$  and  $\delta A_\mu$ , and give the four-dimensional vierbein transformation

$$\delta e_\mu^\alpha = -\frac{1}{4}(\bar{\epsilon}\gamma^\alpha\psi'_\mu - \bar{\psi}'_\mu\gamma^\alpha\epsilon). \quad (3.20)$$

## 4 Uplifting of Four-Dimensional Black Holes

Since we have obtained a consistent Pauli reduction of the six-dimensional theory, it follows that any solution in four dimensions can be lifted back to give a solution in six dimensions. A case of interest is an extremal black hole in four dimensions, carrying a magnetic charge supported by a field in the abelian  $U(1)$  subgroup of the  $SU(2)$  Yang-Mills. Thus taking  $F^i$  to be zero except for  $F^3 = F$ , in four dimensions we shall have

$$\begin{aligned} ds_4^2 &= -\mathcal{H}^{-1}dt^2 + \mathcal{H}[d\rho^2 + \rho^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\varphi}^2)], \\ e^\phi &= \mathcal{H}, \quad F = -Q\sin\tilde{\theta}d\tilde{\theta} \wedge d\tilde{\varphi}, \quad \mathcal{H} = 1 + \frac{Q}{\rho}. \end{aligned} \quad (4.1)$$

Note that although this solution is extremal it is not supersymmetric within the  $N = 1$  theory, since the charge is carried by a gauge field in a matter multiplet. (See (3.12).)

Lifting to six dimensions using the reduction formulae (2.3) and (2.5), we find that the metric is given by

$$d\hat{s}_6^2 = -\mathcal{H}^{-\frac{1}{2}}dt^2 + \mathcal{H}^{\frac{3}{2}}[d\rho^2 + \rho^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\varphi}^2)] + \frac{1}{8g^2}\mathcal{H}^{-\frac{1}{2}}[d\theta^2 + \sin^2\theta(d\varphi + 2gQ\cos\tilde{\theta}d\tilde{\varphi})^2], \quad (4.2)$$

while the other six-dimensional fields are given by

$$\begin{aligned}
\hat{F}_{(2)} &= \frac{1}{2g} \sin \theta d\theta \wedge (d\varphi + 2gQ \cos \tilde{\theta} d\tilde{\varphi}) + Q \cos \theta \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\varphi}, \\
\hat{H}_{(3)} &= \frac{Q}{4g} \sin \tilde{\theta} \sin^2 \theta d\tilde{\theta} \wedge d\tilde{\varphi} \wedge d\varphi, \\
e^{\hat{\phi}} &= \mathcal{H}^{-1}.
\end{aligned} \tag{4.3}$$

Since the azimuthal coordinate  $\varphi$  on the Pauli reduction 2-sphere has period  $2\pi$ , it follows that the 1-form  $(d\varphi + 2gQ \cos \tilde{\theta} d\tilde{\varphi})$  appearing in (4.2) and (4.3) will be globally defined if

$$2gQ = \frac{1}{2}n, \tag{4.4}$$

where  $n$  is an integer. However, the six-dimensional metric will still become singular on the horizon at  $\rho = 0$ . We can study the near-horizon limit by taking  $\mathcal{H} \rightarrow Q/\rho$ . Defining a new radial coordinate by  $r = 4\rho^{\frac{1}{4}} Q^{\frac{3}{4}}$ , we find that in the near-horizon limit (4.2) approaches

$$d\hat{s}_6^2 = dr^2 + r^2 \left\{ -\frac{dt^2}{16Q^2} + \frac{1}{8n^2} [d\theta^2 + \sin^2 \theta (d\varphi + \frac{1}{2}n \cos \tilde{\theta} d\tilde{\varphi})^2 + \frac{1}{2}n^2 (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2)] \right\}. \tag{4.5}$$

Thus in this limit the metric approaches a cone over  $\mathbb{R}$  times an  $S^2$  bundle over  $S^2$ . If  $n$  is odd the bundle is non-trivial, while if  $n$  is even it is trivial.

## 5 Four-dimensional Physics and Compactification Scale

Having shown in this paper that there exists an exact consistent Pauli reduction of the six-dimensional gauged  $N = (1, 0)$  supergravity, it is of interest to examine in further detail the resulting four-dimensional  $N = 1$  theory, and its relation to the six-dimensional starting point.

The four-dimensional theory admits a Minkowski vacuum, in which the dilaton field  $\phi$  is a constant. As can be seen from (2.17), the value of this constant, say  $\phi = \phi_0$ , can be arbitrary. It is evident from the metric reduction ansatz (2.3) that the geometric radius of the compactifying 2-sphere, as measured in the six-dimensional metric, will then be given by

$$\mathcal{R} = \frac{1}{2\sqrt{2}g} e^{-\frac{1}{4}\phi_0}. \tag{5.1}$$

From the form of the four-dimensional Lagrangian (2.17), we see that we should rescale the  $SU(2)$  Yang-Mills fields  $A^i$ , and hence also the gauge-coupling constant  $g$ , by appropriate powers of  $e^{\phi_0}$  in order to work with canonically-normalised fields. Thus, we define

$$\tilde{A}^i = e^{-\frac{1}{2}\phi_0} A^i, \quad \tilde{g} = e^{\frac{1}{2}\phi_0} g, \tag{5.2}$$

in terms of which the Lagrangian expanded around the  $\phi = \phi_0$  background will take the form

$$\mathcal{L} = R * \mathbf{1} - \frac{1}{2} * \tilde{F}^i \wedge \tilde{F}^i + \dots, \quad (5.3)$$

where  $\tilde{F}^i = d\tilde{A}^i + \tilde{g} \epsilon_{ijk} \tilde{A}^j \wedge \tilde{A}^k$ . Thus the radius  $\mathcal{R}$  and the rescaled Yang-Mills coupling constant  $\tilde{g}$  are related by

$$\mathcal{R} = \frac{1}{2\sqrt{2}\tilde{g}} e^{+\frac{1}{4}\phi_0}, \quad (5.4)$$

which should be contrasted with the expression (5.1) involving the gauge-coupling  $g$  in the non-canonical normalisation. Note that the coupling constant  $\tilde{g}$  has dimensions of inverse length. When the action (5.3) is converted to standard units, one obtains a classical Yang-Mills coupling constant  $g_{\text{YM}} = \tilde{g}(\sqrt{4\pi G})/c$  which again is dimensionful. Quantum perturbation theory is an expansion in powers of the dimensionless constant  $\hbar g_{\text{YM}}^2/(4\pi c) = (\tilde{g} L_{\text{planck}})^2$ . Thus if we wanted the  $SU(2)$  gauge-coupling constant  $g_{\text{YM}}$  to be of order unity, we should take  $\tilde{g} \sim 1/L_{\text{planck}}$ .

Note that one can include a Yang-Mills sector in the original  $D = 6$  theory, implying additional terms

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2} e^{\frac{1}{2}\hat{\phi}} * \hat{F}^I \wedge \hat{F}^I \quad (5.5)$$

that are added to (2.1), where  $\hat{F}^I = d\hat{A}^I + g' f_{JK}^I \hat{A}^J \wedge \hat{A}^K$ . These fields reduce to four dimensions simply by setting  $\hat{A}^I = A^I$ , implying additional terms

$$\mathcal{L} = -\frac{1}{2} e^{-\phi} * F^I \wedge F^I \quad (5.6)$$

in (2.17). These fields might be taken to have the gauge group  $E_6 \times E_7$ , with the fields of the standard model being embedded in them. Conversion to canonically-normalised fields in the  $\phi = \phi_0$  vacuum would again require rescalings

$$\tilde{A}^I = e^{-\frac{1}{2}\phi_0} A^I, \quad \tilde{g}' = e^{\frac{1}{2}\phi_0} g', \quad (5.7)$$

just as for the  $SU(2)$  fields in (5.2). In the absence of fine-tuning of the ratio of coupling constants we would expect  $g' \sim g$  in  $D = 6$ , and hence  $\tilde{g}' \sim \tilde{g}$  in  $D = 4$ .

Our discussion in this paper has focussed on the massless sector of the dimensionally-reduced theory, but one can also study the massive ‘‘Kaluza-Klein modes,’’ at least in a linearised expansion around the Minkowski vacuum. It is easily seen that the mass scale  $M$  for these modes is given by

$$\frac{M c}{\hbar} \sim \frac{1}{\mathcal{R}} e^{\frac{1}{4}\phi_0} \sim g e^{\frac{1}{2}\phi_0} = \tilde{g}, \quad (5.8)$$

which would therefore be of Planck scale if the physical Yang-Mills coupling constant were of order unity.

For example, if we set  $\hat{A}_{(1)} = (2g)^{-1} \cos \theta d\varphi + A$ , then the contribution to the four-dimensional Lagrangian involving  $A$ , the massive boson associated with the (broken)  $U(1)$  gauge symmetry of the six-dimensional theory, is found to be

$$\mathcal{L}_A = -\frac{1}{2}e^{-\phi} *F \wedge F - 8g^2 *A \wedge A, \quad (5.9)$$

showing that  $A$  is a Proca field with mass  $4g e^{\frac{1}{2}\phi_0} (\hbar/c) = 4\tilde{g} (\hbar/c)$ .

As another example, the tower of massive spin-2 fields will have a mass spectrum given by the spectrum of the scalar Laplacian on the compactifying 2-sphere. Expanding around the Minkowski vacuum with  $\phi = \phi_0$  we find that the masses are given by

$$\frac{M_\ell c}{\hbar} = \frac{\sqrt{\ell(\ell+1)}}{\mathcal{R}} e^{\frac{1}{4}\phi_0} = \sqrt{8\ell(\ell+1)} e^{\frac{1}{2}\phi_0} g = \sqrt{8\ell(\ell+1)} \tilde{g}. \quad (5.10)$$

Our findings for the spectrum of massive modes differ in one respect from those in [12], where it was argued that there were two distinct mass scales for Kaluza-Klein modes; a “standard” set of states with scale  $1/\mathcal{R}$ , and a “systematically light” set with scale  $(1/\mathcal{R}) e^{\frac{1}{4}\phi_0}$ . In fact we find that *all* massive modes have the latter scale, as is easily seen when one expands the fields in harmonics around the vacuum solution: The small fluctuations are governed, to leading order, by the six-dimensional d’Alembertian  $\hat{\square}_6$ . In the  $\phi = \phi_0$  vacuum this takes the form  $\hat{\square}_6 \sim e^{-\frac{1}{2}\phi_0} \square_4 + e^{\frac{1}{2}\phi_0} \square_2$ , and so with four-dimensional (mass)<sup>2</sup>  $\sim \square_4$ , this implies a mass scale  $m \sim e^{\frac{1}{2}\phi_0} g \sim \tilde{g}$ , since  $\square_2$  is the Laplacian on the  $R_{mn} = 8g^2 g_{mn}$  2-sphere. Thus there is a *universal* mass-scale  $\tilde{g} \sim (1/\mathcal{R}) e^{\frac{1}{4}\phi_0}$  for all massive modes in the Kaluza-Klein spectrum. Furthermore, all the Kaluza-Klein massive modes will be of Planck scale and above, if the Yang-Mills coupling is taken to be of order of magnitude unity.

We have seen that owing to the presence of the dilaton field, the vacuum energy scale is not set, as one might have supposed, by  $1/\mathcal{R}^4$ , but rather by  $M_K^4$ , which is  $(1/\mathcal{R}^4) e^{\phi_0}$ . The fact that it is not  $\mathcal{R}$  alone that occurs in the expression for the vacuum energy is a reflection of the fact that  $\mathcal{R}$  and  $\phi_0$  separately have no intrinsic four-dimensional physical significance; as we noted in section 2,  $\phi_0$  can be absorbed into a rescaling of the gauge coupling constant  $g$ . However, an invariant statement about the four-dimensional bulk Yang-Mills coupling constant for the Salam-Sezgin model is that

$$g_{\text{YM}} \sim \frac{M_K}{M_{\text{planck}}}. \quad (5.11)$$

If the dimensionless Yang-Mills coupling constants are of order unity, then the only scale in the problem is the Planck scale.

In summary, in order to have a vacuum energy of the observed (Hubble) magnitude, rather than  $1/L_{\text{planck}}^4$ , one must choose the  $U(1)$  coupling constant to be about  $10^{-31}$  in dimensionless units.

The model in [13] introduced 3-branes at the north and south poles of the 2-sphere, with tensions  $T_3 = M_{\text{weak}}^4$ , where  $M_{\text{weak}} \sim 100$  GeV, the weak scale. The associated deficit angles, in units of  $2\pi$ , are  $\epsilon = T_3/(4G_6)$ , where  $G_6$  is the six-dimensional Newton constant, related to the four-dimensional Newton constant  $G$  by

$$G_6 = 4\pi G \mathcal{R}^2 e^{\frac{1}{2}\phi_0}. \quad (5.12)$$

Thus

$$\epsilon = \frac{\pi M_{\text{weak}}^4 \mathcal{R}^2 e^{\frac{1}{2}\phi_0}}{M_{\text{planck}}^2}. \quad (5.13)$$

By its very nature,  $\epsilon$  cannot exceed 1.

The presence of the 3-branes alters the Dirac quantisation condition. Suppose that the solution is now supported by a Dirac monopole configuration on  $S^2$  that lies only partly in the original  $U(1)$  field  $\hat{F}_{(2)}$ , with the remainder of the total contribution required by the field equations supported within the additional six-dimensional Yang-Mills sector  $\hat{F}^I$ ;

$$\hat{F}_{(2)} = \frac{\cos \beta}{2g} \Omega_{(2)}, \quad T_I \hat{F}^I = T_0 \frac{\sin \beta}{2g} \Omega_{(2)}, \quad (5.14)$$

where  $\beta$  is the ‘‘mixing angle,’’ and  $T_0$  denotes the  $U(1)$  generator within the Yang-Mills sector. Of course if  $\cos \beta \neq 1$ , supersymmetry is broken. Because the azimuthal coordinate  $\varphi$  on  $S^2$  now has period  $2\pi(1-\epsilon)$ , the Dirac quantisation cannot in general be simultaneously satisfied for both the  $U(1)$  and Yang-Mills groups. Single-valuedness of the gravitino and the gauginos imply

$$\cos \beta = \frac{N}{1-\epsilon}, \quad \frac{g' \sin \beta}{g} = \frac{N'}{1-\epsilon} \quad (5.15)$$

respectively, where  $N$  and  $N'$  are integers. (The cases  $\beta = 0$  and  $\beta = \frac{1}{2}\pi$  were given in [13], with the integers  $N$  or  $N'$  taken to be 1.) In fact the first condition cannot be satisfied for a positive-tension brane. The second allows a moderate-sized  $g'$  for small  $g$ , if either the mixing angle  $\beta$  is chosen to be very small, or else if the deficit-angle parameter  $\epsilon$  is chosen to be very close to its maximum value of unity. Thus because

$$\frac{g'}{g} = \frac{\tilde{g}'}{\tilde{g}}, \quad (5.16)$$

one could obtain a very small value for  $\tilde{g}$ , needed for the suppression of the vacuum energy, whilst having  $\tilde{g}'$  of order unity, if  $\epsilon$  were taken to be equal to its maximum value 1 minus a quantity of order  $10^{-31}$ . The resulting internal space, achieved by this choice, now resembles a needle rather than a rugby ball, and has much in common with the  $S^1/Z_2$  orbifold of Horava and Witten [14]. This suggests the intriguing possibility that in this *maximal-tension limit*, the six-dimensional Salam-Sezgin model approaches the five-dimensional Horava-Witten models considered in [15]. The alternative possibility is to choose  $\beta \sim 10^{-31}$ . Further investigation of these possibilities is clearly worthwhile.

Because the Kaluza-Klein scale is so small, the only viable possibility would seem to be to get all the observed four-dimensional gauge and matter fields from the 3-branes themselves, with all the bulk gauge fields constituting a “hidden sector” [13]. In this scenario, there would be no direct phenomenological reason for requiring the bulk gauge-field coupling constants  $\tilde{g}$  and  $\tilde{g}'$  to be of order unity, and so one might aim to achieve the appropriately small value for the bulk vacuum energy  $(1/\mathcal{R}^4)e^{\phi_0}$  by choosing  $\tilde{g} \sim 10^{-31}/L_{\text{planck}}$ .

Issues of fine tuning or “what is natural” are complex, and there seems to be no universally agreed notion or convention about what they mean. They are clearly dependent on the theory one considers, and the choice of parameters specifying that theory. In the present case, the Salam-Sezgin model is completely specified by giving the quantity  $\tilde{g} = g e^{\frac{1}{2}\phi_0}$ , or equivalently, the four-dimensional Yang-Mills coupling constant and the Planck mass. In terms of these, the choice  $g_{\text{YM}} \sim 10^{-31}$  could be said to be “fine tuned” and “unnatural.”

This appearance of fine tuning might well change if the theory were modified, for example by providing a potential that determined the value of  $\phi_0$ , or if one were able to calculate quantities such as the six-dimensional Newton constant, or six-dimensional gauge-coupling constants, in terms of four-dimensional physical quantities. In the absence of a precise model that goes beyond the level of supergravity, little more can be said at present.

## 6 Conclusions

In this paper, we have shown that six-dimensional chiral  $N = (1, 0)$  gauged Einstein-Maxwell supergravity provides one of the rare examples where a consistent dimensional reduction on a coset space is possible. Specifically, we have shown that it admits a consistent Pauli reduction on  $S^2$ , giving a four-dimensional theory, namely  $N = 1$  supergravity coupled to an  $SU(2)$  Yang-Mills vector multiplet and a scalar multiplet. Since, by definition, a consistent reduction has the property that all solutions of the lower-dimensional equations of motion

provide solutions of the higher-dimensional equations of motion, this reduction allows one to lift any solution of the four-dimensional  $N = 1$  theory back to six dimensions.

This dimensional reduction is of considerable interest in its own right, since examples of consistent Pauli reductions are so few and far between. A further unusual feature, not seen in any of the other known examples, is that only one scalar field (the breathing mode) appears in the metric reduction ansatz. In all other examples of consistent Pauli reductions, such as the  $S^7$  and  $S^4$  reductions of eleven-dimensional supergravity, the  $S^5$  reduction of type IIB supergravity, the  $S^3$  reduction of the bosonic string, and the  $S^2$  reduction of a certain Einstein-Maxwell-dilaton system, the inclusion of  $\frac{1}{2}n(n+3)$  scalars that parameterise inhomogeneous distortions of the  $S^n$  is necessary for consistency, since the  $SO(n+1)$  Yang-Mills fields act as sources for these scalars. By contrast, in the consistent  $S^2$  reduction of the chiral six-dimensional theory studied in this paper, the analogous source terms turn out to arise with zero coefficient.<sup>1</sup>

It is interesting also to note that had we been concerned only with the reduction of the bosonic sector of the six-dimensional theory, a broader class of consistent  $S^2$  reductions would have been possible. The reason for this is that in the purely bosonic sector, the only place where the gauge-coupling constant  $g$  appears in the six-dimensional theory is in the scalar potential term in (2.1). One can then choose this to be distinct from the constant  $g$  appearing in the bosonic reduction ansätze (2.3) and (2.5), without upsetting the consistency of the reduction. Thus if  $g$  is relabelled as  $g_0$  in the original Lagrangian (2.1), while keeping the constant  $g$  in (2.3) and (2.5), we obtain a consistent reduction that yields four-dimensional equations of motion derivable now from

$$\mathcal{L} = R * \mathbf{1} - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{-\phi} * F^i \wedge F^i - \frac{1}{2} e^{-2\phi} * H_{(3)} \wedge H_{(3)} - 8(g_0^2 - g^2) e^\phi * \mathbf{1}, \quad (6.1)$$

where  $F^i = dA^i + g \epsilon_{ijk} A^j \wedge A^k$  and  $H_{(3)} = dB_{(2)} + \frac{1}{2} \omega_{(3)}$ , where  $d\omega_{(3)} = F^i \wedge F^i$ . This freedom to construct a more general class of consistent reductions, yielding a scalar potential in four dimensions, does not, however, extend to the reduction of the full supergravity theory. The gauge coupling constant of the six-dimensional theory, which we have temporarily relabelled as  $g_0$  here, appears now in the gauge covariant derivatives of all the fermionic fields;  $(\hat{\nabla}_M - i g_0 \hat{A}_M) \hat{\chi}$ , *etc.* One finds that the consistent reduction of these fields, discussed in section 3, is possible only if  $g_0 = g$ . (For example, one obtains extra terms of the

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<sup>1</sup>In fact according to a linearised analysis of the entire spectrum of massless and massive fields given in [10], these 5 scalars are members of a massive supermultiplet, and so their omission from the consistent Pauli reduction ansatz is likely to be obligatory rather than optional.

form  $i(g_0 - g) \mu_i A_\alpha^i \hat{e}$  in the transformation rule for  $\hat{\psi}_\alpha$ , which would leave uncanceled  $y$ -dependence unless  $g_0 = g$ .)

One question that is left unanswered by this work is whether the six-dimensional chiral gauged supergravity can itself be obtained by any consistent dimensional reduction from a yet higher dimension. Various attempts have been made, but to date none has been successful. Thus a string origin for the six-dimensional theory remains elusive.

## Note Added

After the first version of this paper was submitted to the archive, an M/string-theory origin for the Salam-Sezgin theory has been found [16]. Additionally, a new class of 3-brane solutions in the Salam-Sezgin theory [17] has been found, which has some bearing on the issues of Dirac quantisation raised in section 5.

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## A Curvature Calculations

For convenience, we present the results here for the spin connection and the Ricci curvature for the following class of metrics:

$$d\hat{s}^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi} g_{mn} (dy^m + \tilde{g} A^i K_i^m)(dy^n + \tilde{g} A^j K_j^n), \quad (\text{A.1})$$

where  $\alpha$  and  $\beta$  are constants,  $ds^2$  is the lower-dimensional metric of dimension  $d_x$ , and  $g_{mn} dy^m dy^n$  is the undistorted “internal” metric, of dimension  $d_y$ . We use the orthonormal basis

$$\hat{e}^\alpha = e^{\alpha\phi} e^\alpha, \quad \hat{e}^a = e^{\beta\phi} (e^a + \tilde{g} A^i K_i^a). \quad (\text{A.2})$$

The Killing vectors  $K_i^m$  on the internal space will be assumed to satisfy the algebra  $[K_i, K_j] = -f_{ij}{}^k K_k$ .

The torsion-free spin connection is given by

$$\begin{aligned}
\hat{\omega}_{\alpha\beta} &= \omega_{\alpha\beta} + \alpha e^{-\alpha\phi} (\partial_\beta\phi \hat{e}^\alpha - \partial_\alpha\phi \hat{e}^\beta) - \frac{1}{2}\tilde{g} e^{(\beta-2\alpha)\phi} F_{\alpha\beta}^i K_i^a \hat{e}^a, \\
\hat{\omega}_{\alpha b} &= -\frac{1}{2}\tilde{g} e^{(\beta-2\alpha)\phi} F_{\alpha\beta}^i K_i^b \hat{e}^\beta - \beta e^{-\alpha\phi} \partial_\alpha\phi \hat{e}^b, \\
\hat{\omega}_{ab} &= \omega_{ab} - \tilde{g} e^{-\alpha\phi} A^i \nabla_a K_b^i \hat{e}^\alpha,
\end{aligned} \tag{A.3}$$

where  $\omega_{\alpha\beta}$  and  $\omega_{ab}$  are the spin connections for the lower-dimensional spacetime and the undistorted internal space respectively. The Yang-Mills field strengths are defined by

$$F^i = dA^i + \frac{1}{2}\tilde{g} f_{jk}{}^i A^j \wedge A^k. \tag{A.4}$$

The orthonormal components of the Ricci tensor are given by

$$\begin{aligned}
\hat{R}_{\alpha\beta} &= e^{-2\alpha\phi} \left[ R_{\alpha\beta} - \alpha \square\phi \eta_{\alpha\beta} - (\alpha(d_x - 2) + \beta d_y)(\nabla_\alpha \nabla_\beta \phi + \alpha(\nabla\phi)^2 \eta_{\alpha\beta}) \right. \\
&\quad \left. + (\alpha^2(d_x - 2) - \beta d_y(\beta - 2\alpha))\nabla_\alpha\phi \nabla_\beta\phi \right] - \frac{1}{2}\tilde{g}^2 e^{(2\beta-4\alpha)\phi} F_{\alpha\gamma}^i F_\beta^{j\gamma} K_i^a K_{ja}, \\
\hat{R}_{\alpha b} &= \frac{1}{2}\tilde{g} e^{(\beta-3\alpha)\phi} [D^\beta F_{\alpha\beta}^i K_{ib} + [\alpha(d_x - 4) + \beta(d_y + 2)] F_{\alpha\beta}^i K_{ib} \nabla^\beta\phi], \\
\hat{R}_{ab} &= e^{-2\beta\phi} R_{ab} - \beta e^{-2\alpha\phi} [\square\phi + (\alpha(d_x - 2) + \beta d_y)(\nabla\phi)^2] \delta_{ab} \\
&\quad + \frac{1}{4}\tilde{g}^2 e^{(2\beta-4\alpha)\phi} F_{\alpha\beta}^i F^{i\alpha\beta} K_{ia} K_{jb},
\end{aligned} \tag{A.5}$$

where  $R_{\alpha\beta}$  and  $R_{ab}$  are the Ricci tensors of the lower-dimensional spacetime and the undistorted internal space respectively, and the derivative  $D_\alpha$  is both spacetime and Yang-Mills covariant.

A canonical choice for the constants  $\alpha$  and  $\beta$  is to take  $\alpha(d_x - 2) + \beta d_y = 0$ , since this ensures that the higher-dimensional Einstein-Hilbert action will yield a normal Einstein-Hilbert term in the lower dimension, with no power of  $e^\phi$  multiplying it. It can be seen from (A.5) that this choice leads to considerable simplifications.

Our specific case in this paper has  $d_x = 4$ ,  $d_y = 2$ ,  $\alpha = -\beta = \frac{1}{4}$  and  $\tilde{g} = 2g$ , and so we shall have

$$\begin{aligned}
\hat{R}_{\alpha\beta} &= e^{-\frac{1}{2}\phi} \left[ R_{\alpha\beta} - \frac{1}{4}\square\phi \eta_{\alpha\beta} - \frac{1}{4}\nabla_\alpha\phi \nabla_\beta\phi \right] - 2g^2 e^{-\frac{3}{2}\phi} F_{\alpha\gamma}^i F_\beta^{j\gamma} K_i^a K_{ja}, \\
\hat{R}_{\alpha b} &= g D^\beta (e^{-\phi} F_{\alpha\beta}^i) K_{ib}, \\
\hat{R}_{ab} &= e^{\frac{1}{2}\phi} R_{ab} + \frac{1}{4} e^{-\frac{1}{2}\phi} \square\phi \delta_{ab} + g^2 e^{-\frac{3}{2}\phi} F_{\alpha\beta}^i F^{i\alpha\beta} K_{ia} K_{jb},
\end{aligned} \tag{A.6}$$

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