

## New Complete Non-compact Spin(7) Manifolds

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### ABSTRACT

We construct new explicit metrics on complete non-compact Riemannian 8-manifolds with holonomy Spin(7). One manifold, which we denote by  $\mathbb{A}_8$ , is topologically  $\mathbb{R}^8$  and another, which we denote by  $\mathbb{B}_8$ , is the bundle of chiral spinors over  $S^4$ . Unlike the previously-known complete non-compact metric of Spin(7) holonomy, which was also defined on the bundle of chiral spinors over  $S^4$ , our new metrics are asymptotically locally conical (ALC): near infinity they approach a circle bundle with fibres of constant length over a cone whose base is the squashed Einstein metric on  $\mathbb{C}\mathbb{P}^3$ . We construct the covariantly-constant spinor and calibrating 4-form. We also obtain an  $L^2$ -normalisable harmonic 4-form for the  $\mathbb{A}_8$  manifold, and two such 4-forms (of opposite dualities) for the  $\mathbb{B}_8$  manifold. We use the metrics to construct new supersymmetric brane solutions in M-theory and string theory. In particular, we construct resolved fractional M2-branes involving the use of the  $L^2$  harmonic 4-forms, and show that for each manifold there is a supersymmetric example. An intriguing feature of the new  $\mathbb{A}_8$  and  $\mathbb{B}_8$  Spin(7) metrics is that they are actually the *same* local solution, with the two different complete manifolds corresponding to taking the radial coordinate to be either positive or negative. We make a comparison with the Taub-NUT and Taub-BOLT metrics, which by contrast do not have special holonomy. In an appendix we construct the general solution of our first-order equations for Spin(7) holonomy, and obtain further regular metrics that are complete on manifolds  $\mathbb{B}_8^+$  and  $\mathbb{B}_8^-$  similar to  $\mathbb{B}_8$ .

# 1 Introduction

There are many explicit examples of Ricci-flat metrics with Kähler or hyper-Kähler special holonomy that are defined on regular non-compact manifolds. There are far fewer analogous examples of Ricci-flat metrics with the exceptional holonomies  $G_2$  in  $D = 7$  or  $\text{Spin}(7)$  in  $D = 8$ . In fact three explicit non-compact  $G_2$  examples and one explicit  $\text{Spin}(7)$  example are known [1, 2]. In this paper we obtain new eight-dimensional metrics of  $\text{Spin}(7)$  holonomy, and show how they can be defined on two topologically inequivalent regular non-compact manifolds. The new metrics are all asymptotically locally conical (ALC), locally approaching  $\mathbb{R} \times S^1 \times \mathbb{CP}^3$ . The radius of the  $S^1$  is asymptotically constant, so the metric approaches an  $S^1$  bundle over a cone with base  $\mathbb{CP}^3$ . However, the Einstein metric on the  $\mathbb{CP}^3$  at the base of the cone is not the Fubini-Study metric, but instead the “squashed” metric described as an  $S^2$  bundle over  $S^4$ . The new solutions can have very different short-distance behaviours, with one approaching flat  $\mathbb{R}^8$  whilst the others approach  $\mathbb{R}^4 \times S^4$  locally. The global topology is that of  $\mathbb{R}^8$  in the first case and the bundle of positive (or negative) chirality spinors over  $S^4$  for the others. An intriguing feature of two of the new metrics, one on each of the inequivalent topologies, is that locally they are actually the same. This metric is complete on a manifold of  $\mathbb{R}^8$  topology if the radial coordinate is taken to be positive, whilst in the region with negative  $r$  it is instead complete on the manifold  $\mathbb{S}(S^4)$  of the bundle of chiral spinors over  $S^4$ . We shall denote the new  $\text{Spin}(7)$  manifold with  $\mathbb{R}^8$  topology by  $\mathbb{A}_8$ , and the new related manifold with  $\mathbb{S}(S^4)$  topology by  $\mathbb{B}_8$ . In appendix A we construct the general solution of the first-order equations that follow by requiring  $\text{Spin}(7)$  holonomy in our metric ansatz, and we show that these lead to further more general classes of regular metrics<sup>1</sup> defined on complete manifolds  $\mathbb{B}_8^\pm$  that are again topologically the bundle of chiral spinors over  $S^4$ .

Our construction is a generalisation of the one that leads to the previously-known metric of  $\text{Spin}(7)$  holonomy. That example is given by [1, 2]

$$ds_8^2 = \left(1 - \frac{\ell^{10/3}}{r^{10/3}}\right)^{-1} dr^2 + \frac{9}{100} r^2 \left(1 - \frac{\ell^{10/3}}{r^{10/3}}\right) h_i^2 + \frac{9}{20} r^2 d\Omega_4^2, \quad (1)$$

where

$$h_i \equiv \sigma_i - A_{(1)}^i, \quad (2)$$

the  $\sigma_i$  are left-invariant 1-forms on  $SU(2)$ ,  $d\Omega_4^2$  is the metric on the unit 4-sphere, and  $A_{(1)}^i$

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<sup>1</sup>This appendix with the general solution and the further complete  $\text{Spin}(7)$  metrics extends the results in an earlier version of this paper.

is the  $SU(2)$  Yang-Mills instanton on  $S^4$ . The  $\sigma_i$  can be written in terms of Euler angles as

$$\sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\varphi, \quad \sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\varphi, \quad \sigma_3 = d\psi + \cos \theta d\varphi. \quad (3)$$

The principal orbits are  $S^7$ , described as an  $S^3$  bundle over  $S^4$ . The solution (1) is asymptotic to a cone over the ‘‘squashed’’ Einstein 7-sphere, and it approaches  $\mathbb{R}^4 \times S^4$  locally at short distance (i.e.  $r \approx \ell$ ). Globally the manifold has the same topology  $\mathbb{S}(S^4)$ , the bundle of chiral spinors over  $S^4$ , as the new Spin(7) manifolds  $\mathbb{B}_8$  and  $\mathbb{B}_8^\pm$  that we obtain in this paper.

## 2 Ansatz, Einstein equation and superpotential for Spin(7) metrics

The generalisation that we shall consider involves allowing the  $S^3$  fibres of the previous construction themselves to be ‘‘squashed.’’ In particular, this encompasses the possibility of having an asymptotic structure of the ‘‘Taub-NUT type,’’ in which the  $U(1)$  fibres in a description of  $S^3$  as a  $U(1)$  bundle over  $S^2$  approach constant length while the radius of the  $S^2$  grows linearly. The appropriate squashing along the  $U(1)$  fibres can be implemented using a description given in [3], where it was observed that if one defines

$$\mu_1 = \sin \theta \sin \psi, \quad \mu_2 = \sin \theta \cos \psi, \quad \mu_3 = \cos \theta, \quad (4)$$

then  $h_i$  can be written (after adapting some conventions) as

$$h_i = -\epsilon_{ijk} \mu^j D\mu^k + \mu^i \sigma, \quad (5)$$

where

$$D\mu^i \equiv d\mu^i + \epsilon_{ijk} A_{(1)}^j \mu^k, \quad \sigma \equiv d\varphi + \mathcal{A}_{(1)}, \quad \mathcal{A}_{(1)} \equiv \cos \theta d\psi - \mu^i A_{(1)}^i. \quad (6)$$

It also follows that

$$\sum_i h_i^2 = \sum_i (D\mu^i)^2 + \sigma^2, \quad (7)$$

In terms of the coordinates  $(\theta, \psi)$  on  $S^2$ , we have

$$\begin{aligned} \sum_i (D\mu^i)^2 &= (d\theta - A_{(1)}^1 \cos \psi + A_{(1)}^2 \sin \psi)^2 \\ &\quad + \sin^2 \theta (d\psi + A_{(1)}^1 \cot \theta \sin \psi + A_{(1)}^2 \cot \theta \cos \psi - A_{(1)}^3)^2. \end{aligned} \quad (8)$$

Finally, one can show that the field strength  $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$ , which follows from (6), is given by

$$\mathcal{F}_{(2)} = \frac{1}{2} \epsilon_{ijk} \mu^k D\mu^i \wedge D\mu^j - \mu^i F_{(2)}^i. \quad (9)$$

Since  $\mu^i \mu^i = 1$ , we see that (7) expresses the metric on the  $S^3$  fibres as a  $U(1)$  bundle over  $S^2$ , with fibre coordinate  $\varphi$ . Note that  $\varphi$  has period  $4\pi$ , while  $\psi$  has period  $2\pi$ . This reversal of the periods by comparison to those for the left-invariant 1-forms (3) is associated with the fact that we effectively transformed from a left-invariant basis to a right-invariant one, in passing to the metric (7) on  $S^3$  [3]. The same transformation, expressed somewhat differently, was used recently in [4].

With these preliminaries, we can now present our more general ansatz for 8-dimensional metrics of Spin(7) holonomy:

$$d\hat{s}_8^2 = dt^2 + a^2 (D\mu^i)^2 + b^2 \sigma^2 + c^2 d\Omega_4^2. \quad (10)$$

Here  $a$ ,  $b$  and  $c$  are functions of the radial variable  $t$ . The metric has cohomogeneity one, with principal orbits that are homogeneously-squashed  $S^7$ . The previous Spin(7) example (1) has  $a = b$ .

A convenient way to obtain the conditions for Ricci-flatness for the ansatz (10) is to perform a Kaluza-Klein dimensional reduction on the  $U(1)$  fibres parameterised by the  $\varphi$  coordinate. This reduction can be written as

$$d\hat{s}_8^2 = e^{-\frac{1}{\sqrt{15}}\phi} ds_7^2 + e^{\sqrt{\frac{3}{5}}\phi} (d\varphi + B_{(1)})^2, \quad (11)$$

where  $ds_7^2$ ,  $B_{(1)}$  and  $\phi$  are all independent of the fibre coordinate  $\varphi$ . The conditions for Ricci flatness of the eight-dimensional metric (11) are then equivalent to the seven-dimensional Einstein-Maxwell-Dilaton equations that follow from the dimensional reduction of the Einstein-Hilbert Lagrangian, which in  $D = 7$  gives

$$e^{-1} \mathcal{L}_7 = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{2\sqrt{\frac{3}{5}}\phi} G_{(2)}^2, \quad (12)$$

where  $G_{(2)} = dB_{(1)}$ . The seven-dimensional equations are

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{2\sqrt{\frac{3}{5}}\phi} (G_{\mu\nu}^2 - \frac{1}{10}G_{(2)}^2 g_{\mu\nu}), \\ \square\phi &= \frac{1}{2}\sqrt{\frac{3}{5}}e^{2\sqrt{\frac{3}{5}}\phi} G_{(2)}^2, \\ d\left(e^{2\sqrt{\frac{3}{5}}\phi} *G_{(2)}\right) &= 0. \end{aligned} \quad (13)$$

Comparing (10) and (11), we see that

$$\begin{aligned} B_{(1)} &= \mathcal{A}_{(1)}, & e^{\sqrt{\frac{5}{3}}\phi} &= b^2, \\ ds_7^2 &= b^{2/5} (dt^2 + a^2 (D\mu^i)^2 + c^2 d\Omega_4^2). \end{aligned} \quad (14)$$

It is easily verified that the field equation for  $G_{(2)}$  given in (13) is automatically satisfied. The metric  $ds_7^2$  lies within the class whose Ricci tensor was calculated in [2], and so using those results it is now a straightforward to obtain the equations for the functions  $a$ ,  $b$  and  $c$  that follow from imposing eight-dimensional Ricci-flatness.

It is convenient to express the equations for  $a$ ,  $b$  and  $c$  as a Lagrangian system. We find that the equations can be derived from varying  $L \equiv T - V$  where

$$\begin{aligned} T &= 2\alpha'^2 + 12\gamma'^2 + 4\alpha'\beta' + 8\beta'\gamma' + 16\alpha'\gamma', \\ V &= \frac{1}{2}b^2c^4(4a^6 + 2a^4b^2 - 24a^4c^2 - 4a^2c^4 + b^2c^4), \end{aligned} \quad (15)$$

together with the constraint  $T + V = 0$ . Here a prime denotes a derivative with respect to a new radial variable  $\eta$ , defined by  $dt = a^2b^2c^4d\eta$ , and we have also defined  $\alpha = \log a$ ,  $\beta = \log b$ ,  $\gamma = \log c$ .

We find that the potential  $V$  can be derived from a superpotential  $W$ . Writing  $T = \frac{1}{2}g_{ij}(d\alpha^i/d\eta)(d\alpha^j/d\eta)$ , where  $\alpha^i = (\alpha, \beta, \gamma)$ , we have  $V = -\frac{1}{2}g^{ij}(\partial W/\partial\alpha^i)(\partial W/\partial\alpha^j)$ , where

$$W = bc^2(4a^3 + 2a^2b + 4ac^2 - bc^2). \quad (16)$$

From this we can obtain the first-order equations  $d\alpha^i/d\eta = g^{ij}\partial W/\partial\alpha^j$ . Expressed back in terms of the original radial variable  $t$  introduced in (10), these equations are

$$\dot{a} = 1 - \frac{b}{2a} - \frac{a^2}{c^2}, \quad \dot{b} = \frac{b^2}{2a^2} - \frac{b^2}{c^2}, \quad \dot{c} = \frac{a}{c} + \frac{b}{2c}, \quad (17)$$

where a dot denotes a derivative with respect to  $t$ .

Before proceeding to find new solutions to these first-order equations, we can first easily verify that the previous Spin(7) metric (1) is indeed a solution. Also, we may observe that one of the seven-dimensional metrics of  $G_2$  holonomy has principal orbits that are  $\mathbb{C}\mathbb{P}^3$ , written as an  $S^2$  bundle over  $S^4$ , and is given by [1, 2]

$$ds_7^2 = \left(1 - \frac{\ell^4}{r^4}\right)^{-1} dr^2 + \frac{1}{4}r^2 \left(1 - \frac{\ell^4}{r^4}\right) (D\mu^i)^2 + \frac{1}{2}r^2 d\Omega_4^2. \quad (18)$$

This is a solution of the seven-dimensional equations (13) with  $B_{(1)} = 0$  and  $\phi = 0$ , and therefore gives a solution in  $D = 8$  of the form  $d\hat{s}_8^2 = ds_7^2 + d\varphi^2$ . This can be described within the framework of our first-order equations (17) by first rescaling  $b \rightarrow \lambda b$ , and then sending  $\lambda$  to zero, so that the gauge potential  $B_{(1)}$  disappears and  $b = \text{constant}$  is allowed as a solution.<sup>2</sup>

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<sup>2</sup>In appendix A we show how this  $\mathcal{M}_7 \times S^1$  metric arises as a limit of a general class of Spin(7) manifolds.

One can also see the specialisations to the previous results described above at the level of the first-order equations themselves. Setting  $a = b$  gives a consistent truncation of (17), yielding  $\dot{a} = \frac{1}{2}a^2 c^{-2}$ ,  $\dot{c} = \frac{3}{2}a c^{-1}$ , which are indeed the first-order equations for the original Spin(7) metrics. On the other hand, sending  $b \rightarrow 0$  in (17) yields a consistent truncation to  $\dot{a} = 1 - a^2 c^{-2}$ ,  $\dot{c} = a c^{-1}$ , which are the first-order equations for the metrics of  $G_2$  holonomy whose principal orbits are  $S^2$  bundles over  $S^4$ . (The first-order equations for these two cases can be found, for example, in [5].)

Another specialisation of the metric ansatz (10) that makes contact with previous results is to set  $a = c$ , in which case the  $S^2$  bundle over  $S^4$  becomes precisely the usual  $\mathbb{CP}^3$  Einstein manifold, with its  $SU(4)$ -invariant metric. This is incompatible with the first-order equations (17), but it is easily verified that it is consistent with the second-order Einstein equations following from (15). Solutions to these second-order equations then include the 8-dimensional Taub-NUT and Taub-BOLT metrics. The incompatibility with the first-order equations is understandable, since the Taub-NUT and Taub-BOLT 8-metrics do not have special holonomy. Another previously-seen solution of the second-order equations with  $a = c$  is the Ricci-flat Kähler metric on the complex line-bundle over  $\mathbb{CP}^3$ . Although this can arise from a first-order system, it is an inequivalent one that is not related to a specialisation of (17). Its superpotential is  $W = 2a^6 + 6a^4 b^2$  [6], with  $T$ ,  $V$  and  $g_{ij}$  following from setting  $a = c$  in (15). (Other examples of this kind of phenomenon were exhibited recently in [5].)

### 3 Solving the Ricci-flat equations

In order to obtain new solutions of the first-order equations (17) we first introduce a new radial coordinate  $r$ , defined in terms of  $t$  by  $dr = b dt$ . After also defining  $f \equiv c^2$ , we find by taking further derivatives of the first-order equations (17) that  $f$  must satisfy the third-order equation

$$2f^2 f''' + 2f(f' - 3)f'' - (f' + 1)(f' - 1)(f' - 3) = 0. \quad (19)$$

The remaining metric functions are then given by solving

$$a' = \frac{f' - 2}{2a} - \frac{(f' - 1)a}{2f}, \quad b^2 = \frac{4a^2}{(f' - 1)^2}. \quad (20)$$

Naively there now appear to be four constants of integration in total rather than the expected three, but the extra one is eliminated by substituting the solutions back into (17).

We have found two simple independent non-trivial solutions<sup>3</sup> to (19), which can be reduced to  $f = 3r$  and  $f = r + r^2/(2\ell^2)$ . The solution with  $f = 3r$  implies  $a^2 = b^2 = \frac{3}{5}r + kr^{-2/3}$ , and after performing the coordinate transformation  $r \rightarrow 3r^2/20$  this gives precisely the previously-known Spin(7) solution (1), with  $\ell = k^{3/10} (20/3)^{1/5}$ .

Our new simple solutions of Spin(7) holonomy arise from the second solution,  $f = r + r^2/(2\ell^2)$ . After making the coordinate transformation  $r \rightarrow -\ell(r + \ell)$ , this solution leads to the metric

$$ds_8^2 = \frac{(r + \ell)^2 dr^2}{(r + 3\ell)(r - \ell)} + \frac{\ell^2 (r + 3\ell)(r - \ell)}{(r + \ell)^2} \sigma^2 + \frac{1}{4}(r + 3\ell)(r - \ell) (D\mu^i)^2 + \frac{1}{2}(r^2 - \ell^2) d\Omega_4^2, \quad (21)$$

Assuming that the constant  $\ell$  is positive, it is evident that  $r$  should lie in the range  $r \geq \ell$ . We can analyse the behaviour near  $r = \ell$  by defining a new radial coordinate  $\rho$ , where  $\rho^2 = 4\ell(r - \ell)$ . Near  $\rho = 0$  the metric approaches

$$ds_8^2 \approx d\rho^2 + \frac{1}{4}\rho^2 \left[ \sigma^2 + (D\mu^i)^2 + d\Omega_4^2 \right]. \quad (22)$$

The quantity  $\frac{1}{4}(\sigma^2 + (D\mu^i)^2 + d\Omega_4^2)$  is precisely the metric on the unit 7-sphere, and so we see that near  $r = \ell$  the metric  $ds_8^2$  smoothly approaches flat  $\mathbb{R}^8$ . At large  $r$  the function  $b$ , which is the radius in the  $U(1)$  direction  $\sigma$ , approaches a constant, and so the metric approaches an  $S^1$  bundle over a 7-metric. This 7-metric is of the form of a cone over  $\mathbb{C}\mathbb{P}^3$  (described as the  $S^2$  bundle over  $S^4$ ) in this asymptotic region. The manifold of this new Spin(7) metric, which we are denoting by  $\mathbb{A}_8$ , is topologically  $\mathbb{R}^8$ .

We shall use the acronym AC to denote asymptotically conical manifolds. Thus asymptotically our new metrics behave like a circle bundle over an AC manifold in which the length of the  $U(1)$  fibres tends to a constant. The acronym ALF is already in use to describe metrics which tend to a  $U(1)$  bundle over an asymptotically Euclidean or asymptotically locally Euclidean metric with the length of the fibres tending to a constant. We shall therefore adopt the acronym ALC to denote manifolds where the base space of the circle bundle is asymptotically conical.

Ricci-flat ALC metrics, although not with special holonomy, have already been encountered. For example, the higher-dimensional Taub-NUT metric is defined on  $\mathbb{R}^{2n}$  for all  $n$  and it is ALC with the base of the cone being  $\mathbb{C}\mathbb{P}^{n-1}$ . A closely related example is the Taub-BOLT metric which has the same asymptotics but is defined on a line bundle over  $\mathbb{C}\mathbb{P}^{n-1}$ . However, as we shall see later, the metric on the base of the cone differs in this

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<sup>3</sup>The general solution is constructed in appendix A. It gives further inequivalent regular metrics, complete on manifolds  $\mathbb{B}_8^\pm$ . These solutions are more complicated, but still fully explicit (up to quadratures).

case (with  $n = 4$ ) from that in our new metrics. An discussion of ALE Spin(7) manifolds based on the idea of blowing up orbifolds has been given in [7]. As far as we are aware, no explicit examples of this kind have yet been found.

We get a different complete manifold, which we are denoting by  $\mathbb{B}_8$ , if we take  $r$  to be negative. It is easier to discuss this by instead setting  $\ell = -\tilde{\ell}$ , where  $\tilde{\ell}$  and  $r$  are taken to be positive. Thus instead of (21) we now have

$$ds_8^2 = \frac{(r - \tilde{\ell})^2 dr^2}{(r - 3\tilde{\ell})(r + \tilde{\ell})} + \frac{\tilde{\ell}^2 (r - 3\tilde{\ell})(r + \tilde{\ell})}{(r - \tilde{\ell})^2} \sigma^2 + \frac{1}{4}(r - 3\tilde{\ell})(r + \tilde{\ell}) (D\mu^i)^2 + \frac{1}{2}(r^2 - \tilde{\ell}^2) d\Omega_4^2, \quad (23)$$

This time, we have  $r \geq 3\tilde{\ell}$ . Defining  $\rho^2 = 4\tilde{\ell}(r - 3\tilde{\ell})$ , we find that near  $r = 3\tilde{\ell}$  the metric has the form

$$ds_8^2 \approx d\rho^2 + \frac{1}{4}\rho^2 [\sigma^2 + (D\mu^i)^2] + 4\tilde{\ell}^2 d\Omega_4^2. \quad (24)$$

The quantity  $\frac{1}{4}[\sigma^2 + (D\mu^i)^2]$  is the metric on the unit 3-sphere, and so in this case we find that the metric smoothly approaches  $\mathbb{R}^4 \times S^4$  locally, at small distance. The large-distance behaviour is the same as for the previous case (21).

Again we have a complete non-compact ALC metric with Spin(7) holonomy with the same base. At short distance, it has the same structure as the previously-known metric of Spin(7) holonomy, obtained in [2]. Thus globally the manifold  $\mathbb{B}_8$  is the bundle of chiral spinors over  $S^4$ .

We can think of the new manifold  $\mathbb{A}_8$  as providing a smooth interpolation between Euclidean 8-space at short distance, and  $\mathcal{M}_7 \times S^1$  at large distance, while  $\mathbb{B}_8$  provides an interpolation between the previous Spin(7) manifold of [1, 2] at short distance and  $\mathcal{M}_7 \times S^1$  at large distance. Here  $\mathcal{M}_7$  denotes the 7-manifold of  $G_2$  holonomy that is the  $\mathbb{R}^3$  bundle over  $S^4$  [1, 2].

In appendix A we construct the general solution of the first-order equations (17). From this, we find additional classes of regular metrics of Spin(7), which are complete on manifolds  $\mathbb{B}_8^\pm$  that are similar to  $\mathbb{B}_8$ . These additional metrics have a non-trivial integration constant  $k$  that parameterises inequivalent solutions.

It is worth remarking that we would obtain identical equations to solve if we were to replace the  $S^4$  metric  $d\Omega_4^2$  in (10) by the Fubini-Study metric on  $\mathbb{C}\mathbb{P}^2$ , scaled so that it has the same cosmological constant as the unit 4-sphere. (In fact the first-order equations in this case are contained within those obtained in [5].) The Yang-Mills connection  $A_{(1)}^i$  would now be the right-handed projection of the spin connection on  $\mathbb{C}\mathbb{P}^2$ . However, the analogue of the  $\mathbb{A}_8$  manifold would now have power-law singularities in the Riemann tensor at  $r = \ell$ , since the principal orbits that collapse to a point would be  $SU(3)/U(1)$  instead of  $S^7$ . The



analogue of the  $\mathbb{B}_8$  manifold would not have power-law curvature singularities at  $r = 3\tilde{\ell}$ , but it would have an orbifold singularity there, approaching  $(\mathbb{R}^4/\mathbb{Z}_2) \times \mathbb{C}\mathbb{P}^2$  locally. The reason for this is that the Yang-Mills connection on  $\mathbb{C}\mathbb{P}^2$  is in  $SO(3)$  rather than  $SU(2)$ , and so the collapsing 3-surfaces at  $r = 3\tilde{\ell}$  will be  $\mathbb{R}\mathbb{P}^3$  rather than  $S^3$ .

## 4 Proof of Spin(7) holonomy

Our procedure for solving the condition of Ricci-flatness for the eight-dimensional metric ansatz (10) involved establishing that there exists a superpotential for the potential in the Lagrangian formulation of the Einstein equations, and hence obtaining the first-order equations (17). The fact that such a first-order system exists provides a strong indication that there is an underlying special holonomy, since such systems of equations typically arise from the conditions for the covariant constancy of a spinor. However, it is still necessary to make a more thorough investigation in order to establish definitively that our new solutions have Spin(7) holonomy.

A convenient way to study this question is by again making use of the Kaluza-Klein reduction (11), so that the equation  $\hat{D}\eta = 0$  for a covariantly-constant spinor in  $D = 8$  can be reformulated in  $D = 7$ . (Here  $\hat{D} \equiv d + \frac{1}{4}\hat{\omega}_{AB}\Gamma_{AB}$  is the Lorentz-covariant exterior derivative that acts on spinors in eight dimensions, where  $\Gamma_{AB} \equiv \frac{1}{2}(\Gamma_A\Gamma_B - \Gamma_B\Gamma_A)$ , and  $\Gamma_A$  are the Dirac matrices that generate the Clifford algebra in eight dimensions.) The advantage of doing this is that we can then make use of results derived in [2] for the spin connection for 7-metrics of the type given in (14). Specifically, we find that under Kaluza-Klein reduction we have

$$\begin{aligned} \hat{D} &= D + \frac{1}{4\sqrt{15}}\partial_a\phi\Gamma_{ab}e^b - \frac{1}{8}\mathcal{F}_{ab}e^2\sqrt{\frac{3}{5}}\phi\Gamma_{ab}(d\varphi + \mathcal{A}_{(1)}) \\ &\quad - \frac{1}{4}\sqrt{\frac{5}{3}}e\sqrt{\frac{3}{5}}\phi\partial_a\phi\Gamma_{a8}(d\varphi + \mathcal{A}_{(1)}) - \frac{1}{4}e^2\sqrt{\frac{3}{5}}\phi\Gamma_{a8}e^b, \end{aligned} \quad (25)$$

where  $D \equiv d + \frac{1}{4}\omega_{ab}\Gamma_{ab}$  is the Lorentz-covariant exterior derivative that acts on spinors in seven dimensions, and  $\omega_{ab}$  can be read off from [2].

Using the results in [2] for the spin connection for 7-metrics of the form appearing in (14), we eventually find that if and only if the metric functions  $a$ ,  $b$  and  $c$  satisfy the the first-order equations (17), then the eight-dimensional equation  $\hat{D}\eta = 0$  has exactly one solution. The solution for the covariantly-constant spinor  $\eta$  can be written as

$$\eta = e^{\frac{1}{2}\theta\Gamma_{71}}e^{\frac{1}{2}\psi\Gamma_{12}}\eta_0, \quad (26)$$

where  $\eta_0$  is independent of  $(r, \theta, \psi, \varphi)$ , and satisfies projection conditions that are all implied by

$$(\Gamma_{12} - \Gamma_{78})\eta_0 = 0, \quad (F_{\alpha\beta}^3 \Gamma_{\alpha\beta} + 4\Gamma_{78})\eta_0 = 0, \quad (F_{\alpha\beta}^1 \Gamma_{\alpha\beta} + 4\Gamma_{71})\eta_0 = 0. \quad (27)$$

Here the tangent-space indices 1 and 2 lie in the  $S^2$  directions,  $(\alpha, \beta)$  lie in the  $S^4$  directions, 7 is in the radial direction, and 8 is in the  $U(1)$  fibre direction. In our conventions, the Yang-Mills instanton fields  $F_{(2)}^i$  on  $S^4$  are given by

$$F_{(2)}^1 = -(e^4 \wedge e^5 + e^3 \wedge e^6), \quad F_{(2)}^2 = -(e^5 \wedge e^3 + e^4 \wedge e^6), \quad F_{(2)}^3 = -(e^3 \wedge e^4 + e^5 \wedge e^6), \quad (28)$$

where  $e^\alpha = (e^3, e^4, e^5, e^6)$  is the basis of tangent-space 1-forms on the unit  $S^4$ . The spinor  $\eta_0$  satisfies the equations for the zero-mode of the Dirac equation on  $S^4$  in the Yang-Mills instanton background.

With these results, we have established that the first-order equations (17) are indeed the integrability conditions for the existence of a single covariantly-constant spinor in the 8-metric (10). This establishes that for any solution of (17), we obtain an 8-metric (10) that has Spin(7) holonomy. The existence of the covariantly-constant spinor  $\eta$  immediately implies the existence of a covariantly-constant self-dual 4-form  $\Phi$ , with components given by  $\Phi_{ABCD} = \bar{\eta} \Gamma_{ABCD} \eta$ . The covariant constancy of  $\eta$  implies that  $\bar{\eta} \eta$  is constant, and so we may choose a normalisation so that  $\bar{\eta} \eta = 1$ . We then find that the 4-form is given by

$$\begin{aligned} \Phi = & -\hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^7 \wedge \hat{e}^8 - \hat{e}^3 \wedge \hat{e}^4 \wedge \hat{e}^5 \wedge \hat{e}^6 + (\hat{e}^1 \wedge \hat{e}^2 + \hat{e}^7 \wedge \hat{e}^8) \wedge \hat{Y}_{(2)} \\ & + (\hat{e}^1 \wedge \hat{e}^8 + \hat{e}^2 \wedge \hat{e}^7) \wedge \frac{\partial \hat{Y}_{(2)}}{\partial \theta} - (\hat{e}^1 \wedge \hat{e}^7 - \hat{e}^2 \wedge \hat{e}^8) \wedge \frac{1}{\sin \theta} \frac{\partial \hat{Y}_{(2)}}{\partial \psi}, \end{aligned} \quad (29)$$

where  $\hat{Y}_{(2)} \equiv c^2 Y_{(2)}$  and  $Y_{(2)} \equiv \mu^i F_{(2)}^i$ , and so

$$\hat{Y}_{(2)} = \frac{1}{2} [\sin \theta (\cos \psi F_{\alpha\beta}^1 + \sin \psi F_{\alpha\beta}^2) + \cos \theta F_{\alpha\beta}^3] \hat{e}^\alpha \wedge \hat{e}^\beta, \quad (30)$$

where as usual  $\hat{e}^\alpha = c e^\alpha$ .

The covariantly-constant self-dual 4-form  $\Phi$ , known as the Cayley form, provides a calibration of the Spin(7) manifold. Thus we have

$$|\Phi(X_1, X_2, X_3, X_4)| \leq 1, \quad (31)$$

where  $(X_1, X_2, X_3, X_4)$  denotes any quadruple of orthonormal vectors. This can be seen from (29), or else from the expression  $\Phi_{ABCD} = \bar{\eta} \Gamma_{ABCD} \eta$ . A calibrated submanifold, or Cayley submanifold,  $\Sigma$ , is one where for each point of  $\Sigma$

$$|\Phi(X_1, X_2, X_3, X_4)| = 1, \quad (32)$$

where the orthonormal vectors  $X_i$  are everywhere tangent to  $\Sigma$ . By inspecting (29) we therefore see that the  $S^4$  zero section of the bundle of chiral spinors is a Cayley submanifold, and hence it is volume minimising in its homology class. Physically, a Cayley submanifold corresponds to a supersymmetric cycle [8, 9].

## 5 $L^2$ -normalisable harmonic 4-forms

In this section, we obtain  $L^2$  normalisable harmonic 4-forms for each of the new Spin(7) 8-manifolds  $\mathbb{A}_8$  and  $\mathbb{B}_8$ . Specifically, we obtain one such 4-form, which is anti-self-dual, for the manifold  $\mathbb{A}_8$  that is topologically  $\mathbb{R}^8$ , and two such 4-forms, one of each duality, for the manifold  $\mathbb{B}_8$  of the chiral spin bundle over  $S^4$ .

We start from the following ansatz for the harmonic 4-forms,

$$\begin{aligned} G_{(4)} = & u_1 (h a^2 b dr \wedge \sigma \wedge X_{(2)} \pm c^4 \Omega_{(4)}) + u_2 (h b c^2 dr \wedge \sigma \wedge Y_{(2)} \pm a^2 c^2 X_{(2)} \wedge Y_{(2)}) \\ & + u_3 (h a c^2 dr \wedge Y_{(3)} \mp b a c^2 \sigma \wedge X_{(3)}), \end{aligned} \quad (33)$$

where  $\Omega_{(4)}$  is the volume form of the unit  $S^4$ , and

$$\begin{aligned} X_{(2)} &\equiv \frac{1}{2} \epsilon_{ijk} \mu^i D\mu^j \wedge D\mu^k, & X_{(3)} &\equiv D\mu^i \wedge F_{(2)}^i, \\ Y_{(2)} &\equiv \mu^i F_{(2)}^i, & Y_{(3)} &\equiv \epsilon_{ijk} \mu^i D\mu^j \wedge F_{(2)}^k. \end{aligned} \quad (34)$$

The upper and lower sign choices in (33) correspond to self-dual and anti-self-dual 4-forms respectively. The various 2-forms and 3-forms defined in (34) satisfy

$$d\sigma = X_{(2)} - Y_{(2)}, \quad dX_{(2)} = X_{(3)} = dY_{(2)}, \quad dY_{(3)} = 2X_{(2)} \wedge Y_{(2)} + 4\Omega_{(4)}. \quad (35)$$

Note that in (33) we have introduced a radial coordinate  $r$  that is related to  $t$  by  $dt = h dr$ .

$G_{(4)}$  will be harmonic if  $dG_{(4)} = 0$ . This implies that

$$\begin{aligned} (c^4 u_1)' &= \pm 2(-h b c^2 u_2 + 2h a c^2 u_3), \\ (a^2 c^2 u_2)' &= \pm (-h a^2 b u_1 + h b c^2 u_2 + 2h a c^2 u_3), \\ (a b c^2 u_3)' &= \pm (h a^2 b u_1 + h b c^2 u_2). \end{aligned} \quad (36)$$

The  $\pm$  signs correspond to self-dual and anti-self-dual respectively, and a prime denotes a derivative with respect to  $r$ . In the remainder of this section, we shall for convenience set the scalar parameters  $\ell$  and  $\tilde{\ell}$  in the metrics (21) and (23) to unity.<sup>4</sup>

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<sup>4</sup>Care must be exercised when taking the square roots of  $a^2$ ,  $b^2$  and  $c^2$  in the metrics (21) and (23), if

For the metric (21) on the manifold  $\mathbb{A}_8$  that is topologically  $\mathbb{R}^8$ , we find that there is a normalisable harmonic 4-form that is anti-self-dual, *i.e.*, the lower choice of the sign is used in (33) and (36). The solution is given by

$$u_1 = \frac{2}{(r+1)^3(r+3)}, \quad u_2 = -\frac{r^2+10r+13}{(r+1)^3(r+3)^3}, \quad u_3 = -\frac{2}{(r+1)^2(r+3)^3}. \quad (37)$$

The norm of the harmonic anti-self-dual 4-form is then given by

$$\begin{aligned} |G_{(4)}|^2 &= 48(u_1^2 + 2u_2^2 + 4u_3^2) \\ &= \frac{96(3r^4 + 44r^3 + 242r^2 + 492r + 339)}{(r+1)^6(r+3)^6}. \end{aligned} \quad (38)$$

Clearly  $G_{(4)}$  is  $L^2$ -normalisable, and in fact we have  $\int_1^\infty \sqrt{g} |G_{(4)}|^2 dr = 9/4$ . We have chosen the integration constants from (36) appropriately in order to select the solution in  $L^2$ . (There also exists a solution for a self-dual harmonic 4-form. It can be made square integrable at small distance, but there is no choice of integration constants for which it is  $L^2$  normalisable, owing to its large distance behaviour.)

For the metric (23) on  $\mathbb{B}_8$ , the bundle of chiral spinors over  $S^4$ , we find that there exists a normalisable harmonic 4-form that is anti-self-dual, *i.e.*, the lower choice of sign is used in (33) and (36). The solution is given by

$$\begin{aligned} u_1 &= \frac{2(r^4 + 8r^3 + 34r^2 - 48r + 21)}{(r-1)^3(r+1)^5} & u_2 &= -\frac{r^4 + 4r^3 - 18r^2 + 52r - 23}{(r-1)^3(r+1)^5}, \\ u_3 &= \frac{2(r^2 + 14r - 11)}{(r-1)^2(r+1)^5}. \end{aligned} \quad (39)$$

The square of the anti-self-dual 4-form is given by

$$|G_{(4)}|^2 = \frac{96(3r^8 + 40r^7 + 252r^6 + 1064r^5 + 2506r^4 - 12936r^3 + 18284r^2 - 10824r + 2379)}{(r-1)^6(r+1)^{10}}, \quad (40)$$

and its  $L^2$ -normalisability can be seen by noting that  $\int_3^\infty \sqrt{g} |G_{(4)}|^2 dr = 189/16$ .

Both of the above harmonic anti-self-dual 4-forms (37) and (39) on  $\mathbb{A}_8$  and  $\mathbb{B}_8$  satisfy the linear relation

$$u_1 + 2u_2 - 4u_3 = 0. \quad (41)$$

This observation will prove useful later, for showing the supersymmetry of resolved brane solutions.

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one wants the functions  $a$ ,  $b$  and  $c$  to solve precisely the first-order equations (17), since these equations are sensitive to the signs of  $a$ ,  $b$  and  $c$ . (Of course there are equivalent first-order equations that differ by precisely these sign factors, and which also imply solutions of the Einstein equations.) We are assuming here that the signs are chosen so that precisely (17) are satisfied. This can be achieved by taking all square roots to be positive, except for  $b$  in the case of (21) on  $\mathbb{A}_8$ .

We also find a second  $L^2$ -normalisable harmonic 4-form in the new Spin(7) manifold  $\mathbb{B}_8$ . This 4-form is self-dual, and is given by

$$\begin{aligned} u_1 &= -\frac{2(5r^3 - 9r^2 + 15r - 3)}{(r-1)^3(r+1)^4}, \\ u_2 &= \frac{(r-3)(5r^2 - 2r + 1)}{(r-1)^3(r+1)^4}, \quad u_3 = -\frac{2(r-3)}{(r-1)^2(r+1)^4}. \end{aligned} \quad (42)$$

In contrast to the previous harmonic 4-forms, there is no linear relation between the functions  $u_1$ ,  $u_2$  and  $u_3$  here. The magnitude of  $G_{(4)}$  is given by

$$|G_{(4)}|^2 = \frac{96(75r^6 - 350r^5 + 829r^4 - 932r^3 + 885r^2 - 414r + 99)}{(r-1)^6(r+1)^8}. \quad (43)$$

It integrates to give  $\int_3^\infty \sqrt{g} |G_{(4)}|^2 dr = 189/4$ .

It is interesting to note that for the anti-self-dual harmonic 4-form on  $\mathbb{A}_8$ , given by (37), we can write it in terms of a globally-defined potential,  $G_{(4)} = dB_{(3)}$ . Specifically, we find that  $B_{(3)}$  can be written as

$$B_{(3)} = (r-1)^2 \left[ -\frac{1}{8(r+1)^2} \sigma \wedge X_{(2)} + \frac{(r+5)}{8(r+1)(r+3)^2} \sigma \wedge Y_{(2)} - \frac{1}{16(r+3)^2} Y_{(3)} \right]. \quad (44)$$

One can see from (21) that this has a vanishing magnitude  $|B_{(3)}|^2$  at  $r = 1$ . On the other hand the analogous expressions for the potential  $B_{(3)}$  for the two harmonic 4-forms (39) and (42), which are similarly expressible as functions of  $r$  times the three 3-form structures in (44), turn out to have a diverging magnitude at  $r = 3$ . In all three cases the  $r$ -dependent prefactors tend to constants at infinity.

## 6 Comparison with Taub-NUT and Taub-BOLT metrics

As mentioned above, the new 8-metrics of Spin(7) holonomy that we have obtained in this paper have an asymptotic large-distance behaviour that is similar to the one seen in the 8-dimensional Taub-NUT and Taub-BOLT metrics. Unlike those metrics, however, ours admit a covariantly-constant spinor, and so they have special holonomy Spin(7).

It is worthwhile looking at the comparison with the Taub-NUT and Taub-BOLT 8-metrics in a little more detail. The 8-dimensional Taub-NUT metric can be written as (see, for example, [10, 11, 5])

$$ds_8^2 = \frac{5(r+\ell)^3}{8(r-\ell)(r^2+4\ell r+5\ell^2)} dr^2 + \frac{8(r-\ell)(r^2+4\ell r+5\ell^2)}{5(r+\ell)^3} \sigma^2 + (r^2-\ell^2) d\Sigma_3^2, \quad (45)$$

where  $d\Sigma_3^2$  is the Fubini-Study metric on the ‘‘unit’’  $\mathbb{C}\mathbb{P}^3$ , and  $\sigma = d\varphi + A_{(1)}$ , where  $dA_{(1)} = 2J$  and  $J$  is the Kähler form on  $\mathbb{C}\mathbb{P}^3$ . The Taub-BOLT metric can be written as [11]

$$ds_8^2 = \frac{dr^2}{F(r)} + F(r) \sigma^2 + (r^2 - \ell^2) d\Sigma_3^2, \quad (46)$$

where

$$F = \frac{8(r^6 - 5\ell^2 r^4 + 15\ell^4 r^2 - 10m r + 5\ell^6)}{5(r^2 - \ell^2)^3}, \quad (47)$$

and we choose the integration constant  $m = \frac{8}{5}\ell^5$ . This choice means that  $F$  vanishes at  $r = 4\ell$ , which is a smooth 6-dimensional fixed point set (geometrically a  $\mathbb{CP}^3$ ) of the  $U(1)$  action.

The general Taub-BOLT and Taub-NUT metrics are thus constructed as metrics of cohomogeneity one with principal orbits that are  $\mathbb{CP}^3$  with its standard  $SU(4)$ -invariant Fubini-Study metric. By contrast, although our new metrics are again of cohomogeneity one with  $\mathbb{CP}^3$  principal orbits, the metric on  $\mathbb{CP}^3$  is “squashed,” and is constructed as an  $S^2$  bundle over  $S^4$ , with isometry group  $SO(3) \times SO(5)$ . At large distance our new solutions are ALC, i.e. of the form of an  $S^1$  bundle over the cone with base the “squashed” Einstein metric on  $\mathbb{CP}^3$ . By contrast, the  $D = 8$  Taub-NUT and Taub-BOLT metrics are asymptotically cylindrical, having the form of an  $S^1$  bundle over the cone with base the “round” Fubini-Study metric on  $\mathbb{CP}^3$ .

At short distance, our solution approaches either  $\mathbb{R}^8$  or  $\mathbb{R}^4 \times S^4$  locally, depending on whether the parameter  $\ell$  in (21) is taken to be positive or negative. The manifold  $\mathbb{A}_8$  that approaches  $\mathbb{R}^8$  is very similar in its short-distance behaviour to the 8-dimensional Taub-NUT metric, and indeed both metrics are defined on  $\mathbb{R}^8$ . On the other hand, the 8-dimensional Taub-BOLT locally approaches  $\mathbb{R}^2 \times \mathbb{CP}^3$  at short distance. Thus while our solution on  $\mathbb{B}_8$ , given in (23), which approaches  $\mathbb{R}^4 \times S^4$  locally at short distance, could be thought of as somewhat analogous to  $D = 8$  Taub-BOLT, it is of a quite different structure. In four dimensions the terms NUT and BOLT were originally defined [12] as zero-dimensional and two-dimensional fixed point sets of a  $U(1)$  action. They are not infrequently extended to cover the more general case of the degenerate orbits of a higher-dimensional isometry group  $G$ , say. These are the orbits which are smaller in dimension than the generic or principal orbits. However a subtlety now arises because (even in four dimensions) such degenerate orbits may or may not be the fixed point sets of a  $U(1)$  subgroup of the isometry group  $G$ . In the present cases the degenerate orbit is also the fixed point set of a circle action, and so the original and the extended meaning both apply. In all the cases we consider, the circle action is generated by the Killing field  $\frac{\partial}{\partial\varphi}$ , and since its length squared is  $g(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}) = b^2 = F$ , it has a fixed point set when  $b$  or  $F$  vanishes.

A further point of interest concerns the feature of our new Spin(7) metric obtained in section 3 that it can be defined on two inequivalent regular non-compact manifolds, depending on whether  $r$  is positive or negative. (We equivalently presented the choice in

terms of an  $r$  that is always positive, but with opposite signs for the scale parameter  $\ell$ .) In fact although this feature is somewhat unusual, it does also occur in at least one other previously-known metric. Specifically, the 6-dimensional Taub-NUT metric can be written as

$$ds_6^2 = \frac{(r + \ell)^2}{2(r - \ell)(r + 3\ell)} dr^2 + \frac{2(r - \ell)(r + 3\ell)}{(r + \ell)^2} \sigma^2 + (r^2 - \ell^2) d\Sigma_2^2, \quad (48)$$

where  $d\Sigma_2^2$  is the Fubini-Study metric on the unit  $CP^2$ , and  $\sigma = d\varphi + A$  with  $dA = 2J$  (see, for example, [10, 11, 5]). If one takes  $\ell$  to be positive, then this metric is defined for  $r \geq \ell$ , and near  $r = \ell$  it approaches  $\mathbb{R}^6$ . This can be seen by letting  $\rho^2 = 2\ell(r - \ell)$ , so that near  $\rho = 0$  we have  $ds_6^2 \approx d\rho^2 + \rho^2(\sigma^2 + d\Sigma_2^2)$ , and  $\sigma^2 + d\Sigma_2^2$  can be recognised as the metric on the unit  $S^5$  described as a  $U(1)$  bundle over  $\mathbb{C}P^2$ , provided that  $\varphi$  has period  $4\pi$ .

If, on the other hand, we set  $\ell = -\tilde{\ell}$ , so that the metric becomes

$$ds_6^2 = \frac{(r - \tilde{\ell})^2}{2(r + \tilde{\ell})(r - 3\tilde{\ell})} dr^2 + \frac{2\tilde{\ell}^2(r + \tilde{\ell})(r - 3\tilde{\ell})}{(r - \tilde{\ell})^2} \sigma^2 + (r^2 - \tilde{\ell}^2) d\Sigma_2^2, \quad (49)$$

then we now have  $r \geq 3\tilde{\ell}$ . Near to  $r = 3\tilde{\ell}$  we can introduce a new radial coordinate such that  $\rho^2 = 2\tilde{\ell}(r - 3\tilde{\ell})$ , and so the metric approaches

$$d\hat{s}_6^2 \approx d\rho^2 + \rho^2 \sigma^2 + 8\tilde{\ell}^2 d\Sigma_2^2. \quad (50)$$

Regularity at  $\rho = 0$  requires that the coordinate  $\varphi$  should have period  $\Delta\varphi = 2\pi$ . The period that would be needed for the  $U(1)$  bundle over  $CP^2$  to be  $S^5$  is  $\Delta\varphi = 4\pi$ . Therefore the level surfaces of the principal orbits are  $S^5/\mathbb{Z}_2$ . The metric smoothly approaches  $\mathbb{R}^2 \times \mathbb{C}P^2$  locally at short distance, and has the usual cylindrical Taub-NUT form at large  $r$ . Somewhat surprisingly, we find that this is in fact precisely the  $D = 6$  Taub-BOLT solution. Thus we have the remarkable result that in  $D = 6$  the Taub-BOLT metric is nothing but the Taub-NUT metric, seen from the other side of  $r = 0$ .

This feature of the 6-dimensional Taub-NUT solution, of admitting a different global interpretation for the opposite sign of  $r$  or the scale parameter  $\ell$ , does not appear to extend to the Taub-NUT metrics in  $D \geq 8$ . For example, in (45) there is no additional real root of the radial function multiplying  $\sigma^2$ , analogous to the  $(r + 3\ell)$  factor in (48). Although a Taub-BOLT solution exists in  $D = 8$ , it is given by the quite different metric (46). In  $D = 10$  there does exist another real root, but it corresponds to an un-removable conical singularity since it would require that the fibre coordinate  $\varphi$  have an irrational period. As with all the higher-dimensional cases there does also exist a Taub-BOLT solution, but it has a quite different form. It seems likely that the feature of a Taub-NUT metric having a second regular manifold corresponding to the negative- $r$  region is peculiar to the six-dimensional

case, and it happens neither in  $D = 4$  nor in  $D \geq 8$ . Only for  $D = 6$  is Taub-NUT its own Taub-BOLT.

## 7 Applications in M-theory and string theory

The new Spin(7) manifolds have a variety of applications in M-theory and string theory. For the present purposes, these can be discussed as the level of the classical low-energy effective supergravity field theories. The bosonic Lagrangian for the  $D = 11$  supergravity limit of M-theory is

$$\mathcal{L}_{11} = R * \mathbf{1} - \frac{1}{2} * F_{(4)} \wedge F_{(4)} + \frac{1}{6} F_{(4)} \wedge F_{(4)} \wedge A_{(3)}, \quad (51)$$

where  $F_{(4)} = dA_{(3)}$ . The low-energy limit for type IIA string theory follows by performing a Kaluza-Klein dimensional reduction of (51) on a circle.

### 7.1 D6-branes as Spin(7) manifolds

The new Spin(7) manifolds  $\mathbb{A}_8$ ,  $\mathbb{B}_8$  and  $\mathbb{B}_8^\pm$  provide new supersymmetric vacua in  $D = 11$  M-theory, simply by taking the direct product with a three-dimensional Minkowski spacetime  $M_3$ , and setting  $F_{(4)} = 0$ . We can then dimensionally reduce the solution on the  $\varphi$  fibre coordinate, using the  $11 \rightarrow 10$  analogue of (11), to give a wrapped D6-brane in type IIA string theory:

$$\begin{aligned} ds_{\text{str}}^2 &= \frac{b}{N} \left( -dt^2 + dx_1^2 + dx_2^2 + c^2 d\Omega_4^2 + h^2 dr^2 + a^2 D\mu^i D\mu^i \right), \\ e^{\frac{4}{3}\phi} &= \frac{b^2}{N^2}, \quad \mathcal{F}_{(2)} = N \left( \frac{1}{2} \epsilon_{ijk} \mu^k D\mu^i \wedge D\mu^j - \mu^i F_{(2)}^i \right). \end{aligned} \quad (52)$$

Here  $ds_{\text{str}}^2$  is the string-frame metric in  $D = 10$ , related to the Einstein-frame metric  $ds_{10}^2$  by  $ds_{\text{str}}^2 = e^{\frac{1}{2}\phi} ds_{10}^2$ . The string coupling constant is given by  $g = e^{\phi_0}$ , where  $\phi_0$  is the asymptotic value of  $\phi$  at large distance;  $g = (\ell/N)^{3/2}$ . We have introduced an integer  $N$  which is the number of D6-branes. This corresponds to the  $D = 11$  solution with the  $\varphi$  fibre coordinate having a period of  $4\pi/N$ . (There will be an orbifold singularity at the origin if  $N \neq 1$ .) The solution can be viewed as D6-branes wrapped around the 4-sphere. At small distance, the wrapping 4-sphere either collapses or stabilises to a fixed radius, depending on which of our two manifolds is used. Using the  $\mathbb{A}_8$  manifold we have an interpolation from  $M_{11}$  at short distance to  $M_3 \times S^1 \times \mathcal{M}_4$  at large distance, while for  $\mathbb{B}_8$  or  $\mathbb{B}_8^\pm$  the interpolation is from  $M_3 \times \mathcal{M}_8$  at short distance to  $M_3 \times S^1 \times \mathcal{M}_7$  at large distance. Here  $M_n$  denotes  $n$ -dimensional Minkowski spacetime,  $\mathcal{M}_7$  is the manifold of  $G_2$  holonomy on the  $\mathbb{R}^3$  bundle



over  $S^4$ , and  $\mathcal{M}_8$  is the previously-known manifold of Spin(7) holonomy on the  $\mathbb{R}^4$  bundle over  $S^4$ . The world volume at large distance becomes

$$M_3 \times S^1, \tag{53}$$

with the string coupling constant being  $g_{\text{str}} = R^{3/2}$ , where  $R$  is the radius of  $S^1$ . Taking  $g_{\text{str}}$  large implies a decompactification of  $S^1$ , thus rendering the world-volume theory to be effectively a Poincaré invariant  $M_4$ . Therefore, this limit may provide an M-theory realisation of a four-dimensional field theory with a zero cosmological constant and infinite Bose-Fermi mass splitting [13], i.e. these properties are a consequence [13] of the underlying  $\mathcal{N} = 1$  supersymmetry of the three-dimensional field theory on  $M_3$ . (Note however, that in the limit of large radius for the  $S^1$ , the size of the non-compact manifold  $\mathcal{M}_7$  of  $G_2$  holonomy also grows, and so the decoupling of the degrees of freedom associated with  $\mathcal{M}_7$  from those on the effective  $M_4$  has to be addressed. In fact for the more general Spin(7) metrics (85) obtained in appendix A, the presence of the additional non-trivial parameter  $k$  allows us to find a limit (95) where the  $S^1$  and the  $\mathcal{M}_7$  do fully decouple. The manifold  $\mathcal{M}_7$  can then be viewed as a blow-up of an orbifold point in a compact manifold of  $G_2$  holonomy, while the  $S^1$  effectively decompactifies.)

There are several differences between this wrapped D6-brane and the D6-brane that comes from the  $S^1$  reduction of the manifold  $G_2$  holonomy with  $S^3 \times S^3$  principal orbits, which was discussed in [4].<sup>5</sup> Since in our case the radius of the  $U(1)$  fibres becomes constant at infinity, the D6-brane solution asymptotically approaches a product of  $M_3$  and the cone metric of the  $S^2$  bundle over  $S^4$ . The value of the dilaton stabilises at large distance. This situation is analogous to the unwrapped D6-brane in the maximally-supersymmetric theory, where it lifts to  $D = 11$  to become a product of  $M_7$  with a four-dimensional Taub-NUT. On the other hand the dilaton becomes singular at short distance, where the  $U(1)$  fibres shrink to zero. The D6-brane in [4] has the opposite behaviour: the radius diverges at large distance but stabilises to a fixed value at small distance.

## 7.2 M2-branes

Another application of metrics with Spin(7) holonomy is in the construction of eleven-dimensional Lorentzian metrics that solve the equations of eleven-dimensional supergravity theory, with the 4-form  $F_{(4)}$  in (51) non-zero. Metrics representing M2-branes are given

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<sup>5</sup>Some related ideas have also been discussed in [14, 15]. See also the talk by E. Witten at the Santa Barbara “David Fest,” and forthcoming work by Atiyah and Witten.

locally by

$$ds^2 = H^{-\frac{2}{3}}(-dt^2 + dx_1^2 + dx_2^2) + H^{\frac{1}{3}}ds_8^2, \quad (54)$$

where  $H$  is a harmonic function on the 8-manifold with metric  $ds_8^2$ . Taking the metric  $ds_8^2$  to be of holonomy  $\text{Spin}(7)$  guarantees that the eleven-dimensional solution (including the 4-form  $F_{(4)} = dt \wedge dx_1 \wedge dx_2 \wedge dH^{-1}$ ) admits at least one Killing spinor. In the present case the simplest example to consider is when  $H$  depends only on the radial variable  $r$ . For the case of  $\mathbb{A}_8$ , with metric (21), one then has

$$\begin{aligned} H &= 1 + Q \int_r^\infty \frac{dx}{(x-\ell)^4(x+3\ell)^2} \\ &= 1 + \frac{Q(3r^3 - 3\ell r^2 - 11\ell^2 r + 27\ell^3)}{192\ell^4(r-\ell)^3(r+3\ell)} + \frac{Q}{256\ell^5} \log\left(\frac{r-\ell}{r+3\ell}\right). \end{aligned} \quad (55)$$

In the case that  $\ell$  and the constant are both positive,  $H$  will be bounded and positive for  $r > \ell$ . Near  $r = \ell$  we have

$$H \propto (r - \ell)^{-3}. \quad (56)$$

This corresponds to the horizon of the M2-brane, which becomes  $\text{AdS}_{(4)} \times S^7$ . Thus we see that the M2-brane interpolates between  $\text{M}_3 \times \text{ALC}_8$  at infinity and  $\text{AdS}_4 \times S^7$ . This solution represents an  $\mathcal{N} = 1$  dual supersymmetric gauge theory in three dimensions that flows from the UV region (large distance) to the maximally supersymmetric conformal IR region (small distance).

It is convenient to write

$$H = \frac{G}{(r - \ell)^3}, \quad (57)$$

where  $G$  is a positive smooth function for  $r > a$ , and  $a$  is a constant that is less than  $\ell$ . Substitution in (54) shows that the apparent singularity at  $r = \ell$  is a coordinate singularity and represents a degenerate event horizon. Near  $r = \ell$ , the metric tends to the direct product  $\text{AdS}_4$  times  $\mathbb{CP}^3$ , where  $\text{AdS}_4$  is four-dimensional anti-de-Sitter spacetime, i.e.  $SO(3,2)/SO(3,1)$  with its standard Lorentzian metric. It is possible to extend the Lorentzian metric to  $r < \ell$ , but  $r = -\ell$  represents a spacetime singularity.

In the case that  $\ell = -\tilde{\ell}$  is negative, the harmonic function blows up near  $r = 3\tilde{\ell}$ , and this appears to represent a spacetime singularity. In particular it does not seem to be possible to construct a spacetime in which one pass between the positive and negative  $r$  regions along a smooth timelike (or indeed, as far as we can see, spacelike) curve. A similar analysis can be given for the more general manifolds  $\mathbb{B}_8^\pm$  found in appendix A.

### 7.3 Resolved M2-branes

Since both the Spin(7) manifolds  $\mathbb{A}_8$  and  $\mathbb{B}_8$  admit  $L^2$ -normalisable harmonic 4-forms, we can construct resolved M2-branes, whose metrics take the identical form as the regular M2-brane (54), but with the 4-form  $F_{(4)}$  having an additional contribution

$$F_{(4)} = dt \wedge dx_1 \wedge dx_2 \wedge dH^{-1} + m G_{(4)}. \quad (58)$$

Instead of being harmonic, as in section 7.2, the function  $H$  now satisfies

$$\square H = -\frac{1}{48}m^2 |G_{(4)}|^2 \quad (59)$$

on the Ricci-flat 8-dimensional space.

For the manifold  $\mathbb{A}_8$ , we have one harmonic normalisable harmonic 4-form, and its magnitude is given in (38). It follows that

$$H = 1 + \frac{m^2(3r^2 + 26r + 63)}{20(r+1)^2(r+3)^5}. \quad (60)$$

The solution is smooth everywhere; it interpolates between eleven-dimensional Minkowski spacetime at small distance and  $M_3 \times S^1 \times \mathcal{M}_7$  at large distance. Here  $\mathcal{M}_7$  is the 7-manifold of  $G_2$  holonomy that is the  $\mathbb{R}^3$  bundle over  $S^4$ .

For the manifold  $\mathbb{B}_8$ , we have two harmonic normalisable 4-forms, whose magnitudes are given by (40) and (43). It follows that the function  $H$  is given by

$$H = 1 + \frac{m^2(1323r^6 + 9786r^5 + 32937r^4 + 64428r^3 + 52237r^2 - 136934r + 29983)}{1680(r+1)^9(r-1)^2}, \quad (61)$$

and

$$H = 1 + \frac{m^2(63r^4 - 80r^3 + 114r^2 + 63)}{20(r+1)^7(r-1)^2}, \quad (62)$$

respectively.

The additional  $G_{(4)}$  term added to the 4-form field strength (58) has the possibility of breaking the supersymmetry of the original unresolved brane solution. The criterion for preserving the supersymmetry is that the covariantly-constant spinor  $\eta$  in the Ricci-flat 8-manifold should be such that [16, 17, 18]

$$G_{abcd} \Gamma^{bcd} \eta = 0. \quad (63)$$

Using our results for the covariantly-constant spinor in  $\mathbb{A}_8$  or  $\mathbb{B}_8$ , we find that the supersymmetry will remain unbroken provided that the functions  $u_i$  in the harmonic 4-form (33)

satisfy precisely the linear relation given in (41). Thus our resolved M2-branes with anti-self-dual harmonic 4-forms in both the  $\mathbb{A}_8$  and  $\mathbb{B}_8$  manifolds are supersymmetric. By contrast, the resolved M2-brane using the self-dual harmonic 4-form in  $\mathbb{B}_8$  is not supersymmetric.

Resolved M2-branes in various manifolds were also constructed in the previous papers [18, 6, 5]. One important difference is that in all three of the new resolved M2-brane solutions obtained above, the 4-form  $F_{(4)}$  carries a magnetic M5-brane charge in addition to the electric M2-brane charge. The magnetic charge is given by

$$Q_m = \frac{1}{\omega_4} \int F_{(4)} = q m, \quad (64)$$

where  $\omega_4$  is the volume of the unit 4-sphere, and  $q = \frac{1}{2}, \frac{1}{2}$  and  $\frac{5}{2}$  for the three solutions respectively. Thus our resolved M2-brane solutions describe fractional magnetic M2-branes as wrapped M5-branes, together with the usual electric M2-brane. This generalises the fractional D3-branes [19, 20] of type IIB theory to the case of M-theory. It was argued in [21] that there should be no supersymmetric fractional M2-branes in asymptotically conical manifolds. Thus our fractional M2-branes do not contradict the no-go theorem, since the  $\mathbb{A}_8$  and  $\mathbb{B}_8$  Spin(7) manifolds are not asymptotically conical, but instead have the ALC structure with an  $S^1$  whose radius tends to a constant at infinity.

In [22], a supergravity solution of an ordinary D2-brane together with a fractional D2-brane from the wrapping of a D4-brane around the  $S^2$  in a manifold of  $G_2$  holonomy was obtained. It was conjectured [5] that this D2-brane should be related to the resolved M2-brane with a transverse 8-space of Spin(7) holonomy. Here we have provided a concrete realisation. In our 2-brane solution, in addition to the regular D2-brane (coming from the double dimensional reduction of the M2-brane) and fractional D2-branes as wrapped D4-branes (coming from the vertical reduction of the M5-brane), we also have wrapped D6-brane charges. This connection between D2-branes and M2-branes is rather different from the one in a flat transverse space that was discussed in [23], where the D2-brane was viewed as a periodic array of M2-branes in the eleventh direction.

All the M-theory solutions we discussed in this section can be reduced on the principal orbits to give rise to four-dimensional domain walls, given by

$$ds_4^2 = a^4 b^2 c^8 (H^{\frac{5}{3}} (-dt^2 + dx_1^2 + dx_2^2) + H^{\frac{8}{3}} h^2 dr^2). \quad (65)$$

## 8 Conclusions

In this paper, we have constructed new explicit complete non-compact 8-metrics of Spin(7) holonomy. Our procedure involved writing down the ansatz (10) for metrics of cohomology

generality one, for which the principal orbits are  $S^7$ , described as a homogeneous manifold with  $S^4$  base and  $S^3$  fibres that are themselves Hopf fibred over  $S^2$  and squashed along the  $U(1)$ . This provides a more general ansatz than the one that led to the previous complete non-compact metric of Spin(7) holonomy obtained in [1, 2]. We then showed that there exists a first-order system of equations whose solutions yield Ricci-flat metrics. We first found simple solutions that give rise to two new complete non-compact metrics.<sup>6</sup> One is complete on the manifold that we denote by  $\mathbb{A}_8$ , which is topologically  $\mathbb{R}^8$ . The other is complete on a manifold that we denote by  $\mathbb{B}_8$ , which is topologically the bundle of chiral spinors over  $S^4$ . Both the new metrics are asymptotically locally conical (ALC), approaching  $S^1 \times \mathcal{M}_7$  locally at infinity, where  $\mathcal{M}_7$  is the manifold of  $G_2$  holonomy defined on the  $\mathbb{R}^3$  bundle over  $S^4$ . Thus the new manifolds have an asymptotically cylindrical structure that is rather like Taub-NUT or Taub-BOLT. This is quite different from the asymptotically conical structure of the previously-known Spin(7) example found in [1, 2]. At short distance  $\mathbb{A}_8$  approaches Euclidean  $\mathbb{R}^8$  locally, while  $\mathbb{B}_8$  approaches  $\mathbb{R}^4 \times S^4$  locally. We also obtained the general solution to the first order equations (17) in Appendix A, and showed that there exist further classes of regular metrics of Spin(7) holonomy, complete on manifolds which we denote by  $\mathbb{B}_8^\pm$ , with the same topology as  $\mathbb{B}_8$ . These have a non-trivial parameter  $k$ , with the earlier simple solutions corresponding to the limit where  $k = 0$ .

We exhibited the Spin(7) holonomy of the new metrics by constructing the covariantly-constant spinor associated with the special holonomy. From this, we also constructed the calibrating covariantly-constant self-dual 4-form  $\Phi$ . We then showed that the manifolds  $\mathbb{A}_8$  and  $\mathbb{B}_8$  both admit an  $L^2$ -normalisable anti-self-dual harmonic 4-form, and that  $\mathbb{B}_8$  also admits a second  $L^2$ -normalisable harmonic 4-form, which is self-dual.

The new Spin(7) manifolds have a variety of applications in M-theory and string theory. We discussed the eleven-dimensional solutions obtained by taking the product of the Spin(7) metrics with 3-dimensional Minkowski spacetime, and the ten-dimensional solutions obtained by reducing on the  $U(1)$  fibres in the Spin(7) manifolds. These give higher-dimensional analogues of the relation between charged black holes in  $D = 4$  (Kaluza-Klein monopoles) and a product of time and Taub-NUT in  $D = 5$ . In the limit where the string coupling is strong, the world volume geometry  $M_3 \times S^1$  corresponds to the large-radius limit of  $S^1$ , and thus it is effectively the Poincaré invariant  $M_4$ . This feature of M-theory compactified on these Spin(7) manifolds may provide a concrete realisation of the proposal in [13] for explaining the vanishing cosmological constant, the absence of Fermi-Bose mass

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<sup>6</sup>It would be interesting also to study these solutions using the methods developed in [24].

degeneracy, and the absence of a massless dilaton in four-dimensional field theory.

We also discussed M2-brane solutions, in which the  $\mathbb{A}_8$ ,  $\mathbb{B}_8$  or  $\mathbb{B}_8^\pm$  Spin(7) manifold replaces the usual flat 8-space transverse to the membrane. These solutions can be “resolved” by adding an extra contribution to the 4-form in  $D = 11$ , proportional to a harmonic 4-form on the 8-manifold. We showed that for each of the  $\mathbb{A}_8$  and  $\mathbb{B}_8$  manifolds there is a resolved M2-brane solution that preserves the single supersymmetry of the unresolved solution. The second  $L^2$ -normalisable harmonic 4-form in the  $\mathbb{B}_8$  manifold gives a resolved M2-brane that breaks supersymmetry. The additional contributions to  $F_{(4)}$  in the resolved solutions give rise to non-vanishing magnetic M5-brane fluxes in the system, and hence our solutions are the supergravity duals of fractional M2-branes.

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## A General solution of the first-order equations

Here, we obtain expressions that yield the general solution of the first-order equations (17). Specifically, we show how the third-order equation for  $f$  given in (19) may be solved. From this, one can then solve for  $a$  and  $b$  as in (20), and eliminate the spurious fourth constant of integration resulting from this procedure by substituting the results back into the first-order equations (17). Before presenting the general solution of (19), we may note that it can be written in the “factorised” form

$$f Q' - (f' + 1) Q = 0, \quad (66)$$

where  $Q \equiv 2f W' + (f' - 3) W$  and  $W \equiv f' - 1$ . In fact for a generic solution, where  $Q$  itself is non-zero, the solutions for  $a$  and  $b$  can be written entirely algebraically in terms of  $f$ , as

$$a^2 = \frac{(f' - 1)(f' - 3)f}{Q}, \quad b^2 = \frac{2a^2}{(f' - 1)^2}. \quad (67)$$

Thus for a solution where  $Q \neq 0$  the three integration constants for the first-order system (17) are simply the three integration constants for the third-order equation (19), and no further substitution back into (19) is necessary. As we shall see below,  $Q$  is non-vanishing for all but one degenerate solution of (19).

The first stage in solving (19) is to let

$$f(r) = e^{\int^x g(s) ds}, \quad (68)$$

where the new radial variable  $x$  is defined implicitly in terms of  $r$  by

$$\frac{df}{dr} = \frac{1}{x}. \quad (69)$$

Using  $f'$  to denote  $df/dr$ , we therefore have

$$f' = \frac{1}{x}, \quad f'' = -\frac{1}{f g x^3}, \quad f''' = \frac{1}{f^2} \left[ \frac{3}{g^2 x^5} + \frac{1}{g x^4} + \frac{1}{g^3 x^4} \frac{dg}{dx} \right]. \quad (70)$$

Substituting into the original 3'rd-order equation (19) gives the first-order equation

$$2x \frac{dg}{dx} + 6g + 6x^2 g^2 - x^2 (x^2 - 1)(3x - 1) g^3 = 0. \quad (71)$$

Note that  $f$  no longer appears explicitly; this is a consequence of the scaling symmetry  $f \rightarrow \lambda f$ ,  $r \rightarrow \lambda r$  of the original equation (19). A further simplification can be achieved by setting  $g = 1/(\gamma(x) x^3)$ , and also defining  $x = 1/\rho$ . Then, we find

$$2\gamma \frac{d\gamma}{d\rho} + 6\gamma = (1 - \rho^2)(3 - \rho). \quad (72)$$

We may first note that two specific solutions are

$$\gamma = \frac{1}{2}(1 - \rho^2), \quad \gamma = \frac{1}{2}(\rho - 1)(\rho - 3). \quad (73)$$

The first of these leads back to our new solution in this paper. The second also gives a solution for  $f$  (with an arbitrary multiplicative constant of integration). However, in this latter case it turns out that after solving for  $a$  and  $b$  and plugging back into the original first-order equations, this arbitrary constant of integration has to be zero and so the second solution in (73) is trivial. In fact it corresponds to solutions of (19) which, in the factorised form (66), have  $Q = 0$ .

The next step in obtaining the general solution is to change variable once again, from  $\gamma$  to  $z$ , defined by

$$z \equiv \frac{(1 - \rho)^2}{2(1 - \rho - \gamma)}. \quad (74)$$

In terms of this new variable, (72) becomes

$$\frac{dz}{d\rho} = \frac{2z(1 - z^2)}{\rho + 2z - 1}. \quad (75)$$

It turns out that the solution to this equation cannot be given explicitly in the form of  $z$  expressed as a function of  $\rho$ , but it can be explicitly solved in the form of  $\rho$  expressed as a function of  $z$ . To do this, it is convenient to characterise this relation in the equivalent form

$$u(z^2) + \frac{1 - \rho}{2z} (1 - z^2)^{1/4} = 0, \quad (76)$$

for some function  $u$  to be determined.

Differentiating (76) with respect to  $\rho$ , using (75) to substitute for  $dz/d\rho$ , and using (76) itself to substitute for  $\rho$ , we find that  $u(y)$  satisfies

$$4y \frac{du(y)}{dy} + u(y) = (1 - y)^{-3/4}, \quad (77)$$

where  $y = z^2$ . The solution to this equation is

$$u(y) = -\tilde{k} y^{-1/4} + {}_2F_1\left[\frac{1}{4}, \frac{3}{4}; \frac{5}{4}; y\right], \quad (78)$$

where  $\tilde{k}$  is an arbitrary constant. Thus we conclude that the general solution of (72) for  $\gamma(\rho)$  is given by

$${}_2F_1\left[\frac{1}{4}, \frac{3}{4}; \frac{5}{4}; z^2\right] = \frac{\tilde{k}}{\sqrt{z}} - \frac{1 - \rho}{2z} (1 - z^2)^{1/4}, \quad (79)$$

where  $z$  is given by (74) and  $\tilde{k}$  is an arbitrary constant.



The first special solution  $\gamma = \frac{1}{2}(1 - \rho^2)$  in (73) corresponds to  $z = 1$ . It is easily seen that this is indeed a special case of (79), with  $\tilde{k} = [\Gamma(\frac{1}{4})]^2 / (4\sqrt{\pi})$ . The second special solution  $\gamma = \frac{1}{2}(\rho - 1)(\rho - 3)$  in (73) corresponds to  $z = -1$ . Again, this is seen to be a special case of (79), now with  $\tilde{k} = i[\Gamma(\frac{1}{4})]^2 / (4\sqrt{\pi})$ .

Using identities for hypergeometric functions, another way to write the general solution (79) is

$$(1 - z^2)^{1/4} {}_2F_1[1, \frac{1}{2}; \frac{5}{4}; 1 - z^2] = \frac{k}{\sqrt{z}} + \frac{(1 - \rho)}{2z} (1 - z^2)^{1/4}. \quad (80)$$

(Here, the arbitrary constant  $k$  is zero for the solutions with  $z = \pm 1$ .) We may write the general solution as  $\rho - 1 = v(z)$ , where

$$v(z) = \frac{2k\sqrt{z}}{(1 - z^2)^{1/4}} - 2z {}_2F_1[1, \frac{1}{2}; \frac{5}{4}; 1 - z^2]. \quad (81)$$

Note that from (77) it follows that

$$2z(1 - z^2) \frac{dv}{dz} = v + 2z. \quad (82)$$

The general solution can now be presented explicitly, in the sense that it is reduced to quadratures. It is convenient in general to take  $z$  to be the radial coordinate in the metric. Retracing the steps of the various redefinitions, we eventually obtain

$$c^2 = f = \exp \left[ \int^z \frac{[v(z') + 1] dz'}{v(z')(1 - z'^2)} \right], \quad a^2 = \frac{[v(z) - 2] z f}{v(z)(1 + z)}, \quad b^2 = \frac{4a^2}{v(z)^2}. \quad (83)$$

The coordinate  $r$  is given in terms of  $z$  by

$$dr = \frac{f dz}{v(z)(1 - z^2)}. \quad (84)$$

Thus the general solution for the metric can be written as

$$ds_8^2 = \frac{v f dz^2}{4z(1 - z^2)(1 - z)(v - 2)} + \frac{(v - 2) z f}{(1 + z)v} (D\mu^i)^2 + \frac{4(v - 2) z f}{(1 + z)v^3} \sigma^2 + f d\Omega_4^2. \quad (85)$$

Note that from (83) we may express  $f$  as

$$f = \left( \frac{1 + z}{1 - z} \right)^{1/2} \exp \left[ \int^z \frac{dz'}{v(z')(1 - z'^2)} \right]. \quad (86)$$

Of the three expected constants of integration for the first-order system (17) two are “trivial,” in the sense that they correspond to a constant shift and rescaling of the radial coordinate. The non-trivial third constant of integration is associated with  $k$  in (81).

In order to recognise the solutions that give rise to regular metrics on complete manifolds, it is helpful to study the phase-plane diagram for the first-order equation (82), which can be expressed as

$$\frac{dz}{d\tau} = 2z(1 - z^2), \quad \frac{dv}{d\tau} = v + 2z, \quad (87)$$

where  $\tau$  is an auxiliary “time” parameter. The solutions can be studied by looking at the flows generated by the 2-vector field  $\{dz/d\tau, dv/d\tau\} = \{2z(1-z^2), v+2z\}$  in the  $(z, v)$  plane. For any such flow, it is then necessary to investigate the global structure of the associated metric (85) for regularity. We find that regular solutions can arise in the following four cases, namely

$$\begin{aligned}
(1) : \quad & z = 1 \text{ (fixed);} \quad v = -2 \text{ to } v = -\infty, \\
(2) : \quad & z = 1 \text{ (fixed);} \quad v = +2 \text{ to } v = +\infty, \\
(3) : \quad & z_0 \leq z \leq 1; \quad v = +2 \text{ to } v = +\infty, \quad (0 < z_0 < 1), \\
(4) : \quad & 1 \leq z \leq z_0; \quad v = +2 \text{ to } v = +\infty, \quad (1 < z_0 < \infty). \quad (88)
\end{aligned}$$

Note that  $v = \pm\infty$  corresponds to the asymptotic large-distance region, and in all four cases the metrics have similar asymptotic structures, precisely as we have already seen in the  $\mathbb{A}_8$  and  $\mathbb{B}_8$  cases.  $v = -2$  corresponds to the short-distance behaviour of the  $\mathbb{A}_8$  metric, approaching Euclidean  $\mathbb{R}^8$  at the origin where the  $S^7$  principal orbits degenerate to a point.  $v = 2$  on the other hand corresponds to the short-distance behaviour seen in the  $\mathbb{B}_8$  metric, approaching  $\mathbb{R}^4 \times S^4$  locally. In fact solution (1) is the metric (21) on  $\mathbb{A}_8$  found in section 3, and solution (2) is the metric (23) on  $\mathbb{B}_8$  found there also. These both have  $k = 0$  in (81).

Solution (3) arises when  $k$  is any positive number, with  $z_0$  being the corresponding value of  $z$ , with  $0 < z_0 < 1$ , for which  $v(z_0) = 2$ . The value of  $z_0$  is correlated with the value of  $k$ , ranging from  $z_0 = 0$  for  $k = \infty$ , to  $z_0 = 1$  for  $k = 0$ .<sup>7</sup> Near  $z = 1$  it follows from (81) that we shall have

$$v = 2^{3/4} k (1 - z)^{-1/4} - 2 + \dots, \quad f = c_0 (1 - z)^{-1/2} + \dots, \quad (89)$$

where  $c_0$  is an arbitrary constant of integration. Defining  $y \equiv (2c_0)^{-1/2} (1 - z)^{-1/4}$ , we see that as  $z \rightarrow 1$  we shall have  $y \rightarrow \infty$  and

$$ds_8^2 \approx dy^2 + \frac{1}{4}y^2 (D\mu^i)^2 + \frac{1}{2}y^2 d\Omega_4^2 + \frac{c_0}{k^2 \sqrt{2}} \sigma^2, \quad (90)$$

and so this more general metric has the same large-distance asymptotic form as do  $\mathbb{A}_8$  and  $\mathbb{B}_8$ . Near  $z = z_0$  we shall have  $v(z) = 2 + v'(z_0)(z - z_0) + \dots$ , and defining a new radial

<sup>7</sup>For the case  $k = 0$ , for which the regular solution is  $\gamma = \frac{1}{2}(1 - \rho^2)$  in (73), and which leads to the metrics (21) and (23), the quantity  $z$  is not a good choice for the radial coordinate, since it is fixed at  $z = 1$ . This case can be regarded as a singular limit within the general formalism we are using here. Specifically, if we let  $z = 1 - 16\epsilon^4 \tilde{\ell}^4 (r + \tilde{\ell})^{-4}$ ,  $k = 2^{1/4} \epsilon$ , and choose the integration constant in (86) so that  $f = \frac{1}{2}(r^2 - \tilde{\ell}^2)$ , then upon sending  $\epsilon$  to zero we recover the metric (23).

coordinate  $R$  by  $(z - z_0) = \frac{1}{4}R^2$  near  $z = z_0$ , we shall have

$$ds_8^2 \approx \frac{f_0}{2z_0(1-z_0^2)(1-z_0)v'(z_0)} \left[ dR^2 + \frac{1}{4}v'(z_0)^2 z_0^2 (1-z_0)^2 R^2 [(D\mu^i)^2 + \sigma^2] \right] + f_0 d\Omega_4^2, \quad (91)$$

where  $f_0$  is the value of  $f$  at  $z = z_0$ . From (82) we have that  $z_0(1-z_0)v'(z_0) = 1$ , and so we see from (91) that at short distance the metric (91) approaches  $\mathbb{R}^4 \times S^4$  locally. Thus these more general solution (3) in (88) with  $k > 0$  is complete on a manifold that is very similar to the manifold  $\mathbb{B}_8$  of the solution (23), with an  $S^4$  BOLT at  $z = z_0$ . We shall denote it by  $\mathbb{B}_8^-$ , where the superscript indicates that  $z$  starts from a value  $z_0 < 1$  at short distance, flowing to  $z = 1$  asymptotically.

Solution (4) arises in the region where  $z \geq 1$ , and again the flow runs from an  $S^4$  BOLT at  $z_0$  (now  $> 1$ ) at which  $v(z_0) = 2$ , to the asymptotic region as  $z$  approaches 1. It follows from (81) that in this case we should first introduce a new constant  $\kappa$  such that

$$v(z) = \frac{2\kappa\sqrt{z}}{(z^2-1)^{1/4}} - 2z {}_2F_1\left[1, \frac{1}{2}; \frac{5}{4}; 1-z^2\right]. \quad (92)$$

Since  $v$  has the asymptotic form

$$v \sim 2\kappa - \frac{2\sqrt{\pi}\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} + \frac{1}{z} + \mathcal{O}(z^{-2}) \quad (93)$$

at large  $z$ , one can show that we shall only be able to find the required regular starting-point with  $v(z_0) = 2$  if  $\kappa$  is bounded by

$$0 \leq \kappa \leq 1 + \frac{\sqrt{\pi}\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}. \quad (94)$$

(The lower limit corresponds to  $z_0 = 1$ , while the upper limit corresponds to  $z_0 = \infty$ .) Under these circumstances we can find the necessary  $z_0$  which corresponds to an  $S^4$  BOLT at short distance. We shall denote this solution by  $\mathbb{B}_8^+$ . Note that the simple solution  $\mathbb{B}_8$  in (23) can be viewed as a  $k \rightarrow 0$  or  $\kappa \rightarrow 0$  limit of the more complicated  $\mathbb{B}_8^-$  or  $\mathbb{B}_8^+$  solutions.

The arguments in section 4 show that in common with  $\mathbb{A}_8$  and  $\mathbb{B}_8$  of section 3, the additional solutions  $\mathbb{B}_8^-$  and  $\mathbb{B}_8^+$  also have Spin(7) holonomy.

We observed at the end of section 2 that a particular example of a solution of the first-order equations (17) is the direct product metric  $ds_8^2 = ds_7^2 + d\varphi^2$ , where  $ds_7^2$  is the Ricci-flat 7-metric of  $G_2$  holonomy on the  $\mathbb{R}^3$  bundle over  $S^4$  [1, 2], and  $\varphi$  is a coordinate on a circle. We are now in a position to see how this solution can arise as a limit of our new Spin(7) metrics. Specifically, it arises as the  $k \rightarrow \infty$  limit of Solution (3) listed in (88). This is

the limit where the constant  $z_0$ , which sets the lower limit for the range  $z_0 \leq z \leq 1$  for  $z$ , becomes zero.

At the same time as sending  $k$  to infinity, we can rescale the fibre coordinate  $\varphi$  appearing in definition (6) for  $\sigma = d\varphi + \mathcal{A}_{(1)}$ , according to  $\varphi \rightarrow k\varphi$ . From (81) and (86) we see that when  $k$  becomes very large we shall have

$$v \rightarrow \frac{2k\sqrt{z}}{(1-z^2)^{1/4}}, \quad f \rightarrow \left(\frac{1+z}{1-z}\right)^{1/2}, \quad (95)$$

and so in the limit of infinite  $k$  the metric (85) becomes

$$ds_8^2 = \frac{dz^2}{4z(1-z)^2(1-z^2)^{1/2}} + \frac{z}{(1-z^2)^{1/2}}(D\mu^i)^2 + \left(\frac{1+z}{1-z}\right)^{1/2}d\Omega_4^2 + d\varphi^2. \quad (96)$$

Defining a new radial coordinate  $r$  by  $r^4 = (1+z)(1-z)^{-1}$ , we see that this becomes  $ds_8^2 = ds_7^2 + d\varphi^2$ , where

$$ds_7^2 = \frac{2dr^2}{1-r^{-4}} + \frac{1}{2}r^2(1-r^{-4})(D\mu^i)^2 + r^2d\Omega_4^2. \quad (97)$$

This can be recognised as the metric of  $G_2$  holonomy on the manifold  $\mathcal{M}_7$  of the  $\mathbb{R}^3$  bundle over  $S^4$ , which was constructed in [1, 2].<sup>8</sup> Thus the family of new Spin(7) manifolds that we are denoting by  $\mathbb{B}_8^-$  has a non-trivial parameter  $k$  such that the  $k = \infty$  limit degenerates to  $\mathcal{M}_7 \times S^1$ , while the  $k = 0$  limit reduces to the case  $\mathbb{B}_8$  given by (23).

Finally, we should stress that the analysis in this appendix assumes that  $f$  is not solely linearly dependent on  $r$ , since if it is, we see from (69) that  $x$  is then a constant. This case is easily analysed separately, and the conclusion is that the only additional solution is the previous metric of Spin(7) holonomy found in [1, 2] (corresponding to  $f = 3r$ , as we saw in section 3).

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<sup>8</sup>If we had not rescaled the fibre coordinate  $\varphi$  by a factor of  $k$  before taking the limit  $k \rightarrow \infty$ , the radius of the  $S^1$  would have tended to zero. This limit is known as the Gromov-Hausdorff convergence [25].

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