CTP TAMU-05/02 UPR-985-T MCTP-02-18

March 2002 hep-th/0203229

M-theory PP-Waves, Penrose Limits and Supernumerary Supersymmetries

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ABSTRACT

We study supersymmetric pp-waves in M-theory, their dimensional reduction to D0-branes or pp-waves in type IIA, and their T-dualisation to solutions in the type IIB theory. The general class of pp-waves that we consider encompass the Penrose limits of $\mathrm{AdS}_p \times S^q$ with $(p,q) = (4,7)$, $(7,4)$, $(3,3)$, $(3,2)$, $(2,3)$, $(2,2)$, but includes also many other examples that can again lead to exactly-solvable massive strings, but which do not arise from Penrose limits. All the pp-waves in $D = 11$ have 16 "standard" Killing spinors, but in certain cases one finds additional, or "supernumerary," Killing spinors too. These give rise to linearly-realised supersymmetries in the string or matrix models. A focus of our investigation is on the circumstances when the Killing spinors are independent of particular coordinates (x^+ or transverse-space coordinates), since these will survive at the field-theory level in dimensional reduction or T-dualisation.

[†] Research supported in part by DOE grant DE-FG02-95ER40893 and NATO grant 976951.

[⋆] Research supported in full by DOE grant DE-FG02-95ER40899

[‡] Research supported in part by DOE grant DE-FG03-95ER40917.

1 Introduction

The Penrose limit [1] of the $\mathrm{AdS}_5 \times S^5$ solution of type IIB theory is a pp-wave with maximal supersymmetry [2, 3]. This result is of considerable interest within the framework of the AdS/CFT correspondence, since the pp-wave provides a background for which the string theory action in light-cone gauge describes a massive free string, which is exactly solvable [4, 5], thus allowing explicit comparisons with results in the dual gauge theory [5]. The Penrose limit of the $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ solutions of M-theory also gives rise to a maximally-supersymmetric pp-wave, obtained in [6], and this provides a simple background for the DLCQ description of M-theory, and the corresponding matrix-model in this regime [5]. Subsequent papers have explored a variety of consequences and generalisations of these observations [7]-[21].

In a previous paper [17], we studied a wider class of supersymmetric pp-waves in the type IIB theory, generalising the maximally-supersymmetric one that arises as the Penrose limit of $\mathrm{AdS}_5 \times S^5$. In particular, we allowed for a more general structure of the constant self-dual five-form field strength; these structures were motivated by the flat (orbifold) limit of a special holonomy transverse space for the pp waves. In fact any pp-wave within the general class automatically has 16 Killing spinors, which we therefore denoted as "standard" Killing spinors. In special cases one finds that there can be additional Killing spinors, which we denoted as "supernumerary" Killing spinors. The maximum number, 16, of these is achieved for the Penrose limit of $\mathrm{AdS}_5 \times S^5$. (In this class we also found another example of the Penrose limit of $\mathrm{AdS}_3 \times S^3$ arising form an D3/D3-intersection.)

The focus of our study in [17] was to determine the circumstances under which one obtains supernumerary Killing spinors in the type IIB pp-waves. These are important when one considers the exactly-solvable string models in the pp-wave backgrounds; we found that it is the supernumerary Killing spinors that are in one to one correspondence with the associated linearly-realised worldsheet supersymmetries of the corresponding string action. In fact the string theory is solved by going to the light-cone gauge, with the x^+ coordinate in the pp-wave being set equal to the world-sheet time coordinate. In order that the linearly-realised world-sheet supersymmetries be unbroken, it is necessary therefore that the associated supernumerary Killing spinors be independent of the coordinate x^+, which is indeed the case in all the type IIB pp-waves. For instance, all 16 supernumerary Killing spinors in the Penrose limit of $\mathrm{AdS}_5 \times S^5$ have this property [5, 17].

A further significance of having Killing spinors in the type IIB pp-waves that are independent of x^+ is that after performing a T-duality transformation on the x^+ coordinate (which is always a Killing direction), the resulting type IIA solution will also be supersymmetric. It can be lifted to M-theory, where it acquires an interpretation as a supersymmetric deformed M2-brane, i.e. an M2-brane in which an additional 4-form flux is turned on in the transverse space [22]-[28]. An intriguing feature of the deformed M2-branes obtained by this T-dualisation procedure is that if any of the Killing spinors originate from *supernumerary* Killing spinors (which are x^+-independent), then in the M-theory picture they solve the Killing-spinor equations despite violating the criterion that is usually applied [29, 22, 25] for testing whether a supersymmetry survives when the extra 4-form flux is turned on [17].

In this paper, we study supersymmetric pp-waves in M-theory. In particular, we allow for rather general structures for the constant 4-form field strength of M-theory, motivated from the flat (orbifold) limit of special holonomy transverse space for the pp-waves. These possible structures fall into two classes. Focusing on the nature of supersymmetry, we again find that there are always 16 "standard" Killing spinors, and that additional "supernumerary" Killing spinors can arise in special cases. Unlike the case of pp-waves in type IIB, however, it is no longer automatic that supernumerary Killing spinors are independent of x^+. The Penrose limit of $\mathrm{AdS}_4 \times S^7$ or $\mathrm{AdS}_7 \times S^4$ provides the unique example where the number of supernumerary Killing spinors attains its maximum, namely 16 [6]. Unlike the maximally-supersymmetric pp-wave in type IIB, however, here in the M-theory maximally-supersymmetric pp-wave the 16 supernumerary Killing spinors all depend on x^+.

Again our focus is on the occurrence of supernumerary Killing spinors, and also on determining the dependence of all the Killing spinors on the x^+ coordinate and the 9 transverse coordinates z^i. These dependences are of importance when one considers reductions to type IIA, and subsequent T-dualisation to type IIB, since they determine whether there will be supersymmetries in the type IIA and IIB supergravity solutions.

Depending upon whether one reduces from $D = 11$ on x^+ or on one of the transverse coordinates z^i, one either obtains a D0-brane or a pp-wave in the type IIA theory. In the case of a pp-wave, the string theory in this background is again exactly-solvable by going to the light-cone gauge, giving rise to a free massive theory. In the case of a reduction instead on x^+, the D0-brane world-particle action leads to a DLCQ description of the M-theory matrix model in this sector. Thus these backgrounds of M-theory provide for dual descriptions, either in terms of a solvable type IIA string action or in a matrix theory model. In particular, the supernumerary supersymmetry plays a key role in determining the supersymmetry of the string action as well as supersymmetry of the matrix model.

We begin in section 2 by setting up our formalism for the pp-waves in M-theory, and obtaining the criterion for the existence of Killing spinors. We then study the coordinate dependences of the Killing spinors, for the various choices of constant 4-form fluxes in the flat nine-dimensional transverse space that we are considering. This allows us to discuss the supersymmetry of the type IIA D0-branes and pp-waves that we can obtain by dimensional reduction. We also show how some of our pp-wave solutions are related to Penrose limits of M2-brane and M5-brane intersections.

In section 3 we derive the light-cone action for type IIA strings in arbitrary bosonic backgrounds, making use of earlier covariant results for the Green-Schwarz action up to quadratic order in fermions that were obtained in [30]. We also obtain an analogous result for the light-cone action for the type IIB string, again valid for arbitrary bosonic backgrounds, and based on covariant results obtained in [30]. Using these light-cone actions, we study the properties of some of the pp-wave solutions that we obtain by dimensional reduction and T-duality.

In certain cases the M-theory pp-waves have no isometry direction within the nine-dimensional transverse space, and so they cannot be reduced to give pp-waves in ten dimensions. Under these circumstances, where the pp-waves are intrinsically eleven-dimensional, one is led to considering a DLCQ description in this background, leading to a matrix model. In particular, we derive the matrix-model action, whose supersymmetry is in one to one correspondence with the supernumerary supersymmetries of the corresponding pp-wave background.

2 Supersymmetry of pp-waves in M-theory

2.1 General formalism

We shall consider pp-wave solutions of $D=11$ supergravity, where the metric and 4-form are given by

$$ds_{11}^2 = -4dx^+\,dx^- + H\,{dx^+}^2 + dz_i^2\,, \qquad (1)$$

$$F_{(4)} = \mu\,dx^+\wedge\Phi_{(3)}\,, \qquad (2)$$

where $\Phi_{(3)}$ is a harmonic 3-form in the flat nine-dimensional transverse space whose metric is dz_i^2, μ is a constant, and we are taking H here to depend only on z^i. In the vielbein basis $e^+=dx^+$, $e^-=-2dx^-+\frac{1}{2}H\,dx^+$, $e^i=dz^i$, for which the metric is $ds_{11}^2=2e^+\,e^-+e^i\,e^i$, the spin connection is given by

$$\omega_{+i}=\tfrac{1}{2}\partial_i H\,e^+\,,\qquad \omega_{-i}=\omega_{+-}=\omega_{ij}=0\,, \qquad (3)$$

and the only non-vanishing Riemann tensor components, in the vielbein basis, are

$$R_{+i+j} = -\tfrac{1}{2}\partial_i\partial_j H\,. \tag{4}$$

This implies that the only non-vanishing Ricci-tensor component is $R_{++} = -\frac{1}{2}\Box H$. The $D = 11$ supergravity equations are therefore satisfied if H obeys the equation

$$\Box\, H = -\tfrac{1}{6}\,\mu^2\,|\Phi_{(3)}|^2\,. \tag{5}$$

In this paper we shall focus on the cases where $\Phi_{(3)}$ is a covariantly-constant 3-form. It is sufficient for our purposes to take the solution for H to be

$$H = c_0 + \frac{Q}{r^7} - \sum_i \mu_i^2\, z_i^2\,, \tag{6}$$

where c_0, Q and μ_i are constants, and $r^2 \equiv z_i\, z_i$. It follows from (5) that the μ_i are subject to the condition

$$\sum_i \mu_i^2 = \tfrac{1}{12}\,\mu^2\,|\Phi_{(3)}|^2\,. \tag{7}$$

When $\mu = 0$ (and hence $\mu_i = 0$), the solution becomes a standard pp-wave in $D = 11$, whose dimensional reduction gives a D0-brane in the type IIA theory, and the pp-wave charge Q becomes the charge of the D0-brane.

The supercovariant derivative appearing in the supersymmetry transformation rule $\delta\psi_M = D_M\,\epsilon$ is given by

$$D_M = \nabla_M - \tfrac{1}{288}\,(\Gamma_M{}^{N_1\cdots N_4}\,F_{N_1\cdots N_4} - 8F_{MN_1\cdots N_3}\,\Gamma^{N_1\cdots N_3})\,. \tag{8}$$

Defining $D_M = \nabla_M + \Omega_M$, we therefore find

$$\begin{aligned} &\nabla_+ = \partial_+ + \tfrac{1}{4}\partial_i H\,\Gamma_-\,\Gamma_i\,, \qquad \nabla_- = \partial_-\,, \qquad \nabla_i = \partial_i\,,\\ &\Omega_+ = -\tfrac{\mathrm{i}}{12}\mu\,(1 + \Gamma_-\,\Gamma_+)\,W\,, \qquad \Omega_- = 0\,, \qquad \Omega_i = \tfrac{\mathrm{i}}{24}\,\mu\,\Gamma_-\,(\Gamma_i\,W + 3W\,\Gamma_i)\,, \end{aligned} \tag{9}$$

where we have defined

$$W \equiv \tfrac{\mathrm{i}}{6}\Phi_{ijk}\,\Gamma_{ijk}\,. \tag{10}$$

It follows immediately from (9) that Killing spinors ϵ, satisfying $D_M\,\epsilon = 0$, are independent of x^-. Since $\Omega_i\,\Omega_j = 0$ we have $\partial_i\,\partial_j\,\epsilon = 0$ and hence it follows that

$$\epsilon = (1 - z^i\,\Omega_i)\,\chi\,, \tag{11}$$

where χ depends only on x^+. Finally, from $D_+\,\epsilon = 0$ one deduces that

$$\partial_+\,\chi - \tfrac{\mathrm{i}}{12}\mu\,(\Gamma_-\,\Gamma_+ + 1)\,W\,\chi = 0 \tag{12}$$

and

$$\Big[\mu^2\, z^i\, (\Gamma_i\, W^2 + 9W^2\, \Gamma_i + 6W\, \Gamma_i\, W) + 72\partial_i H\, \Gamma_i\Big]\, \Gamma_-\, \chi = 0\,. \qquad (13)$$

Thus (13) determines the number of Killing spinors, while (11) and (12) determine their z^i and x^+ dependence.

Since $\Gamma_+\, \Gamma_- + \Gamma_-\, \Gamma_+ = 2$ and $\Gamma_+{}^2 = \Gamma_-{}^2 = 0$, we have a unique decomposition $\chi = \chi_+ + \chi_-$ for any χ, where $\chi_+ \equiv \frac{1}{2}\Gamma_+\, \Gamma_-\, \chi$ and $\chi_- \equiv \frac{1}{2}\Gamma_-\, \Gamma_+\, \chi$ have the defining properties $\Gamma_+\, \chi_+ = 0$ and $\Gamma_-\, \chi_- = 0$.

It is evident from (13) that there will always be 16 Killing spinors corresponding to $\chi = \chi_-$. Accordingly, we refer to these as "standard Killing spinors," since they exist for any choice of the function H that satisfies the field equation (5). In particular, the pp-wave charge Q can be non-zero. In certain cases there can also be additional Killing spinors corresponding to $\chi = \chi_+$. We refer to these as "supernumerary Killing spinors." We shall construct the two categories of Killing spinors, and discuss their coordinate dependences, in the following two subsections.

Before discussing the two categories of Killing spinors, let us be a little more specific in our choice of constant 3-forms $\Phi_{(3)}$. It turns out to be natural to restrict attention to cases where the associated matrix W, defined by (10), is a sum of individual terms W_α that all commute with each other. This can be seen to lead to two inequivalent maximal sets of terms, which we shall refer to as Case 1 and Case 2. For the two cases we have

Case1 :

$$\Phi_{(3)} = m_1\, dz^{129} + m_2\, dz^{349} + m_3\, dz^{569} + m_4\, dz^{789}\,, \qquad (14)$$

Case2 :

$$\begin{aligned}\Phi_{(3)} = m_1\, dz^{123} + m_2\, dz^{145} + m_3\, dz^{167} + m_4\, dz^{246} \\ +m_5\, dz^{257} + m_6\, dz^{347} + m_7\, dz^{356}\,,\end{aligned} \qquad (15)$$

where we have defined $dz^{ijk} \equiv dz^i \wedge dz^j \wedge dz^k$. It should be noted that unless all four of the m_α coefficients are non-zero in Case 1, it is in fact encompassed (after a relabelling of coordinates) within Case 2.

It is straightforward to verify that if we construct W as in (10), and write it as

$$W = \sum_\alpha m_\alpha\, W_\alpha\,, \qquad (16)$$

where W_α denotes the individual Γ_{ijk} structures (for example $W_1 = \mathrm{i}\, \Gamma_{129}$ is one of the four structures in Case 1), then we shall have $[W_\alpha, W_\beta] = 0$. In consequence, for either Case 1 or Case 2 we can choose a basis for the gamma matrices in which the W_α are all diagonal. It is useful to have in mind such a diagonal choice of basis in the subsequent discussion.

For our canonical choices, one can see that if the m_α are taken equal then $\Phi_{(3)}$ in Case 1 can be expressed as $m\, dz_9 \wedge J$, where J is the Kähler form for the eight-dimensional flat space with metric $\sum_{i=1}^8 dz_i^2$. Likewise, if the m_α are set equal in Case 2, $\Phi_{(3)}$ can be expressed as $m\,\Psi_{(3)}$, where $\Psi_{(3)}$ is a G_2-invariant associative 3-form in the flat seven-dimensional space with metric $\sum_{i=1}^7 dz_i^2$.

2.2 The 16 standard Killing spinors

The 16 standard Killing spinors correspond to taking $\chi = \chi_-$, i.e. they are defined by $\Gamma_-\,\chi = 0$. It is evident from (9) and (11) that they are all independent of all of the z^i coordinates. It is also evident from (12) that they will have x^+ dependence given by

$$\chi = e^{\frac{i}{4}\,\mu\, x^+\, W}\,\chi_0\,, \tag{17}$$

where χ_0 is any constant spinor satisfying $\Gamma_-\,\chi_0 = 0$. If W annihilates any of these spinors, then the associated "standard" Killing spinor will be independent of x^+ (and so, in fact, it will be independent of all the coordinates). The discussion now divides into two, according to whether we take $\Phi_{(3)}$ to be given by (14) or (15):

Case 1:

For $\Phi_{(3)}$ given by (14), the eigenvalues of W are

$$\lambda_i = \pm m_1 \pm m_2 \pm m_3 \pm m_4\,, \tag{18}$$

where the $\pm$ choices are all independent. Each eigenvalue occurs twice, making the 32 in total. In the subspace of eigenspinors annihilated by Γ_- one gets each eigenvalue once, and likewise in the subspace annihilated by Γ_+. For the standard Killing spinors arising in Case 1, it therefore follows that the possible numbers that are independent of x^+ can be $N_{\rm stan} =$ 0, 2, 4 or 8.

The number $N_{\rm stan} = 0$ is achieved for generic choices of the m_α; $N_{\rm stan} = 2$ is achieved for choices where $m_1 + m_2 + m_3 + m_4 = 0$; $N_{\rm stan} = 4$ is achieved for choices with $m_4 = 0$ and $m_1 + m_2 + m_3 = 0$ (or permutations); and $N_{\rm stan} = 8$ is achieved for choices with $m_3 = m_4 = 0$ and $m_1 + m_2 = 0$ (or permutations).

Case 2:

When $\Phi_{(3)}$ is given by (15), we find that again W generically has sixteen different eigenvalues, each occurring twice. One copy of the sixteen again occurs in each of the Γ_- and Γ_+ subspaces. The eigenvalues are given by $\pm\lambda_i$, where

$$\lambda_8 = m_1 + m_2 + m_3 - m_4 + m_5 + m_6 + m_7\,, \tag{19}$$

and λ_i for $1 \le i \le 7$ is given by reversing the sign of each m_α that occurs as a coefficient of any term containing the gamma matrix Γ_i. The numbers of standard Killing spinors that are independent of x^+ that can be achieved for these Case 2 examples are therefore $N_{\rm stan} = 2n$, where $0 \le n \le 6$ is the number of $\lambda_i = 0$ that are arranged to vanish by choosing the m_α appropriately. Thus for Case 2 we can have $N_{\rm stan} = 0$, 2, 4, 6, 8, 10 or 12 standard Killing spinors that are independent of x^+.

2.3 Supernumerary Killing spinors

We now turn to the discussion of supernumerary Killing spinors, for which $\Gamma_+\,\chi = 0$. In a generic pp-wave solution there will be none of these, but they can arise in special cases when H given in (6) is quadratic in z^i (i.e. the pp-wave charge $Q = 0$), and the distribution of μ_i coefficients (which must in any case satisfy (7)) is chosen appropriately. The numbers of supernumerary Killing spinors that can be achieved depends also on the choice of $\Phi_{(3)}$.

The equation (13) that determines the number of Killing spinors admits solutions for supernumerary Killing spinors ($\Gamma_+\,\chi = 0$) if H in (6) depends on z^i only quadratically, and

$$\Big[\mu^2\,(\Gamma_i\,W^2 + 9W^2\,\Gamma_i + 6W\,\Gamma_i\,W) - 144\mu_i^2\,\Gamma_i\Big]\,\chi = 0\,, \tag{20}$$

where $\Gamma_+\,\chi = 0$.

As we discussed when we made our choices (14) or (15) for $\Phi_{(3)}$, the resulting terms W_α defined by (10) and (16) can all be simultaneously diagonalised (for each of Case 1 or Case 2 separately), by means of an appropriate similarity transformation of the gamma matrices. It is convenient to assume that such a basis for the gamma matrices has been chosen.

With respect to a diagonal basis, it is evident that

$$X_i \equiv \Gamma_i\,W\,\Gamma_i + 3W \tag{21}$$

is also diagonal, and therefore that (20) can be rewritten as

$$(\mu^2\,X_i^2 - 144\mu_i^2)\,\Gamma_i\,\chi = 0\,, \qquad \text{for each } i\,. \tag{22}$$

From (11), it now follows that the solutions of (22) will give the supernumerary Killing spinors, with z^i dependence given by

$$\epsilon = (1 - \tfrac{\mathrm{i}}{2}\,\mu_i\, z^i\, \Gamma_-\, \Gamma_i)\, \chi\,. \tag{23}$$

In particular, this means that a supernumerary Killing spinor is independent of a given coordinate z^i if and only if the associated coefficient μ_i in (6) is zero.

The discussion of the supernumerary Killing spinors now divides into the two possibilities for $\Phi_{(3)}$, given by (14) or (15).

Case 1:

In this case $\Phi_{(3)}$ is given by (14). In the direction $i = 9$ we have $X_9 = 4W$, and so $\mu_9^2 = \frac{1}{9}\,\mu^2\,\lambda^2$, where λ is one of the eigenvalues of W given in (18). Without loss of generality, since the other eigenvalues differ only in sign permutations of the m_α, we can take

$$\mu_9^2 = \tfrac{1}{9}\mu^2\,(m_1 + m_2 + m_3 + m_4)^2\,. \tag{24}$$

The remaining μ_i for $1 \le i \le 8$ are then given by

$$\begin{aligned}
\mu_1^2 = \mu_2^2 &= \tfrac{1}{36}\,\mu^2\,(-2m_1 + m_2 + m_3 + m_4)^2\,,\\
\mu_3^2 = \mu_4^2 &= \tfrac{1}{36}\,\mu^2\,(m_1 - 2m_2 + m_3 + m_4)^2\,,\\
\mu_5^2 = \mu_6^2 &= \tfrac{1}{36}\,\mu^2\,(m_1 + m_2 - 2m_3 + m_4)^2\,,\\
\mu_7^2 = \mu_8^2 &= \tfrac{1}{36}\,\mu^2\,(m_1 + m_2 + m_3 - 2m_4)^2\,.
\end{aligned} \tag{25}$$

To see this, we note that if $(X_9 - \kappa_9)\,\Gamma_9\,\chi = 0$ then $4W\,\chi = \kappa_9\,\chi$. We are taking $\kappa_9 = 4(m_1 + m_2 + m_3 + m_4)$. Substituting into $(X_1 - \kappa_1)\,\Gamma_1\,\chi = 0$ we therefore find $\frac{1}{4}\kappa_9\,\chi + 3\widetilde{W}\,\chi - \kappa_1\,\chi = 0$, where $\widetilde{W} = \Gamma_1\,W\,\Gamma_1$. From (14) and (10) it follows that the diagonal matrix $\widetilde{W}$ has eigenvalues that are just those of W but with m_2, m_3 and m_4 reversed in sign, and so $\widetilde{W}\,\chi = (m_1 - m_2 - m_3 - m_4)\,\chi$. Thus we deduce that $\kappa_1 = 2(2m_1 - m_2 - m_3 - m_4)$. Applying an analogous argument for each direction i, we arrive at (25).

For a generic choice of the constants m_α, there are precisely two supernumerary Killing spinors. This is because a given bosonic solution has fixed values for the coefficients μ_α, and so there are two solutions to (20) since there is a twofold degeneracy in (25). In special cases, where the m_α are chosen so that two or more of the expressions in (25) are equal, there can therefore be more solutions of (20). It is an elementary exercise to enumerate all the possible numbers of supernumerary supersymmetries that can be achieved for specific choices of m_α.

As in the case of type IIB pp-wave solutions, the supernumerary supersymmetries can lead to a variety of "non-standard" fractions of total supersymmetry that exceed $\frac{1}{2}$ [17].

It is worth remarking that for Case 1, as a consequence of the equation $X_9 = 4W$, it follows that supernumerary Killing spinors are independent of x^+ if and only if they are independent of z_9.

Case 2:

When the 3-form $\Phi_{(3)}$ is given by (15), it follows that we shall have $X_8 = X_9 = 2W$, and so from (22) we shall have $\mu_8^2 = \mu_9^2 = \frac{1}{36}\,\mu^2\,\lambda^2$, where λ is one of the eigenvalues of W. These are now given by λ_8 in (19), together with λ_i for $1 \le i \le 7$ as described below (19). Without loss of generality, since the μ_α have not yet been specified, we may choose

$$\mu_8^2 = \mu_9^2 = \tfrac{1}{36}\mu^2\,\lambda_8^2 = \tfrac{1}{36}\mu^2\,(m_1 + m_2 + m_3 - m_4 + m_5 + m_6 + m_7)^2\,. \qquad (26)$$

For the other constants μ_i, we shall therefore have

$$\mu_i^2 = \tfrac{1}{144}\,\mu^2\,(\lambda_8 - 3\lambda_i)^2\,, \qquad 1 \le i \le 7\,. \qquad (27)$$

where λ_8 and λ_i are given by (19) and below. For example, we shall have $\mu_1^2 = \frac{1}{36}\mu^2\,(2m_1 + 2m_2 + 2m_3 + m_4 - m_5 - m_6 - m_7)^2$.

2.4 Supersymmetry of type IIA pp-waves and D0-branes

Having obtained the M-theory pp-waves, we can dimensionally reduce the solutions to $D = 10$, giving rise to D0-branes if we reduce on the x^+ coordinate, or to type IIA pp-waves if we reduce instead on any of the z^i coordinates. Of course a reduction on a particular z^i coordinate is possible only if it is a Killing direction, which means that the associated coefficient μ_i in the metric function H must vanish.[1]

First let us consider a reduction on x^+. This is a Killing direction for all pp-wave solutions. However, some or all of the Killing spinors in a given solution may be dependent on the x^+ coordinate, in which case they will not survive in the reduction of the solution to type IIA supergravity. As we saw from (12), the criterion for a Killing spinor to be independent of x^+ is that it should be annihilated by W. For the standard Killing spinors, the fraction of the 16 Killing spinors that will survive the reduction on x^+ depends on

[1] There are, of course, many other Killing vectors in the pp-wave metric, which could be used for Kaluza-Klein reduction, but we are not considering these here (see, however, section 2.5). Some examples are discussed in [20]; these typically give rise to an extra (constant) flux in the lower dimension, coming from a non-vanishing Kaluza-Klein vector potential.

the detailed structure of W. The 16 standard Killing spinors exist for any solution for H, subject to (5). In particular, they exist when the D0-brane charge Q is turned on. The supernumerary Killing spinors, on the other hand, are all eigenvectors of W with the same eigenvalue, and hence they are all x^+-independent if $W\chi = 0$, but x^+-dependent if $W\,\chi \neq 0$. The supernumerary Killing spinors exist only if $Q = 0$ and the μ_i are distributed appropriately.

For a reduction on one of the transverse coordinates z^i, the the corresponding constant μ_i in the expression for H in the metric (1) must vanish, in order that $\partial/\partial z^i$ be a Killing vector. As we saw in (23), the Killing spinors are then also independent of the coordinate z^i, and hence they will all survive in the reduction. In section 2.2, it was observed that the 16 standard Killing spinors are all independent of z^i.

When $\Phi_{(3)}$ is contained within the Case 1 in (14), the direction z_9 is singled out. It was observed in the discussion of Case 1 in section 2.3 that if $\mu_9 = 0$ then we have also $W\chi = 0$, and so this implies that if z_9 is a Killing direction then the supernumerary Killing spinors will not only be independent of z_9, but also of x^+.

In Case 2, where $\Phi_{(3)}$ is given by (15), the directions z_8 and z_9 are singled out. If we arrange for $\mu_8 = \mu_9 = 0$ (they are always equal), then as shown in 2.3 we also have $W\chi = 0$, and so the supernumerary Killing spinors will be independent of x^+ as well as the reduction coordinate z_8 or z_9.

In fact these various reductions to type IIA can be related to the general type IIB pp-waves obtained in [17], by means of T-duality. If we dimensionally reduce on z_9 in Case 1, or on z_8 or z_9 in Case 2, which is possible if parameters are chosen so that the corresponding μ_i coefficient vanishes, the resulting supernumerary Killing spinors are all independent x^+. This implies that the type IIA string action then has linearly-realised supersymmetries.

We can also obtain large classes of type IIA pp-wave solutions in which other μ_i parameters are instead zero. (That is to say, μ_i other than μ_9 in Case 1, or $\mu_8 = \mu_9$ in Case 2.) In these circumstances, there can exist supernumerary Killing spinors that are dependent on x^+. One would then obtain a type IIA solution where some, or all, of the world-sheet supersymmetries were non-linearly realised.

2.5 An explicit example

Here we consider an explicit example where only m_1 and m_2 are non-vanishing. (This can equally well be for either Case 1 in (14) or Case 2 in (15), since the two are then equivalent after coordinate relabellings.) Taking m_1 and m_2 as given, we can then arrange for two choices for the μ_i coefficients that will give rise to supernumerary Killing spinors. These are summarised in the following table:

$\mu_1^2 = \mu_2^2$	$\mu_3^2 = \mu_4^2$	$\mu_5^2 = \mu_6^2$	$\mu_7^2 = \mu_8^2$	μ_9^2
$\frac{1}{36}(-2m_1 + m_2)^2$	$\frac{1}{36}(m_1 - 2m_2)^2$	$\frac{1}{36}(m_1 + m_2)^2$	$\frac{1}{36}(m_1 + m_2)^2$	$\frac{1}{9}(m_1 + m_2)^2$
$\frac{1}{36}(2m_1 + m_2)^2$	$\frac{1}{36}(m_1 + 2m_2)^2$	$\frac{1}{36}(m_1 - m_2)^2$	$\frac{1}{36}(m_1 - m_2)^2$	$\frac{1}{9}(m_1 - m_2)^2$

Table 1: Choices for μ_i with fixed m_1 and m_2 that give supernumerary Killing spinors

The second choice is nothing but the first, with one of the two m_i reversed in sign. However, if the m_α are given, fixed parameters, then these two choices for the μ_α correspond to two independent solutions.[2]

For generic but fixed values of m_α, each choice in Table 1 gives rise to 8 supernumerary Killing spinors. When $m_1 = m_2$ in the second choice, the 8 supernumerary Killing spinors are all independent of x^+. This then corresponds to the Penrose limit of the M2/M5-brane system ($AdS_3 \times S^3 \times T^4$). By contrast, in the first choice in Table 1 the 8 supernumerary Killing spinors depend on the x^+ coordinate. For both choices, 8 of the 16 standard Killing spinors are independent of x^+. Another special case arises when either $m_1 = \pm 2m_2$ or $m_2 = \pm 2m_1$, which implies that $\mu_1 = \mu_2 = 0$ or $\mu_3 = \mu_4 = 0$. Interestingly enough, this case is T-dual to the maximally supersymmetric pp-wave arising from the Penrose limit of $AdS_5 \times S^5$. Suppose, for example, we have the choice giving $\mu_1 = \mu_2 = 0$. We can then reduce on z_1 and T-dualise on z_2. The resulting type IIB solution is the maximally-supersymmetric pp-wave, in a slightly non-standard coordinate system that was introduced in [20] to make certain Killing directions in the transverse space manifest. The reverse procedure of T-dualisation and lifting was performed in [20], to give the pp-wave in $D = 11$.

2.6 AdS Penrose limits

Various AdS spacetimes can arise in M-theory as near-horizon limits of M-brane intersections. The M2-brane and M5-brane themselves have near-horizon limits $AdS_4 \times S^7$ and

[2]In our previous discussion, we adopted an "active" viewpoint when discussing the possible occurrences of supernumerary Killing spinors, rather than the "passive" viewpoint we are adopting here. Namely, we previously took a fixed choice for how the eigenvalues were to be expressed in terms of the m_α, and then covered the spectrum of possibilities by allowing the m_α to be chosen freely. The two viewpoints are clearly equivalent, if appropriate care is taken.

$\mathrm{AdS}_7 \times S^4$ respectively; these both have the same Penrose limit. The resulting pp-wave is maximally supersymmetric, and is the one obtained in [6], with

$$\begin{aligned} H &= c_0 - \tfrac{1}{9}\mu^2\,(z_1^2+z_2^2+z_3^2) - \tfrac{1}{36}\mu^2\,(z_4^2+\cdots z_9^2)\,,\\ \Phi_{(3)} &= \mu\, dz^{123}\,. \end{aligned} \tag{28}$$

(The constant c_0 was not included in the Penrose limit taken in [6], but it can easily be included, as was shown in [17] for any Penrose limit.) There is no isometry along any of the z^i directions. All the Killing spinors depend on x^+, and so a dimensional reduction on the x^+ coordinate will result in a type IIA supergravity solution that breaks all the supersymmetry.

The Penrose limits of intersecting M-branes that give rise to AdS structure were discussed in [8]. An M2/M5 brane intersection gives rise to $\mathrm{AdS}_3 \times S^3 \times T^4$ in the near-horizon limit. The Penrose limit is then given by

$$\begin{aligned} H &= c_0 - \tfrac{1}{4}\mu^2(z_1^2+z_2^2+z_3^2+z_4^2)\,,\\ \Phi_{(3)} &= \mu\,(dz^{129}+dz^{349})\,. \end{aligned} \tag{29}$$

There are in total 24 Killing spinors, of which 16 are the standard Killing spinors, together with 8 supernumerary Killing spinors. Out of the 16 standard Killing spinors, 8 are independent of x^+. All 8 supernumerary Killing spinors are independent of x^+, giving a total of 16 Killing spinors independent of x^+. Thus the D0-brane brane after reduction to type IIA will have 16 Killing spinors, but reduced to 8 if the D0-brane charge Q is turned on (since then the supernumerary Killing spinors will already be lost in $D=11$). The solution has also isometries along the five z^i coordinates with $i=5,6,7,8,9$. All 24 Killing spinors are independent of these coordinate, and so they will all survive if the M-theory solution is instead reduced on any of these five z^i coordinates, giving a pp-wave in type IIA. In particular, if we reduce the solution on the z^9 coordinate the pp-wave will have a constant (NS-NS) 3-form $F_{(3)}$, and in fact this pp-wave is itself the Penrose limit of the NS1/NS5 system in type IIA. If instead we reduce the $D=11$ solution on z_5, z_6, z_7 or z_8, the ten-dimensional pp-wave is supported by a constant R-R 4-form $F_{(4)}$.

M2/M2/M2 and M5/M5/M5 brane intersections give rise to $\mathrm{AdS}_2 \times S^3$ and $\mathrm{AdS}_3 \times S^2$ respectively. Both have the same Penrose limit, which can be written as

$$\begin{aligned} H &= c_0 - \tfrac{1}{4}\mu^2\,(z_7^2+z_8^2) - \mu^2\,z_9^2\,,\\ \Phi_{(3)} &= \mu(dz^{129}+dz^{349}+dz^{569})\,, \end{aligned} \tag{30}$$

In this case there are the 16 standard Killing spinors plus 4 supernumerary Killing spinors, giving a total of 20 in all. All the Killing spinors depend on x^+, and hence after reduction on x^+ to type IIA the resulting D0-brane will have no supersymmetry. There are also isometries in the z^i coordinates with $i = 1, \dots 6$. Since none of the Killing spinors depends on any of these coordinates, the type IIA pp-wave that results from reducing instead on one of these will have all 20 Killing spinors.

The M2/M2/M5/M5 brane intersection system gives an $\mathrm{AdS}_2 \times S^2$ in its near-horizon limit. The Penrose limit is given by

$$\begin{aligned} H &= c_0 - \mu^2 \, (z_1^2 + z_6^2) \, , \\ \Phi_{(3)} &= \mu \, (dz^{123} + dz^{145} + dz^{246} + dz^{356}) \, . \end{aligned} \tag{31}$$

In this case, 4 out of the 16 standard Killing spinors are independent of x^+. Additionally, there are 4 supernumerary Killing spinors, which are all x^+-independent.

3 Matrix model and type II string actions

3.1 Type IIA string action

Whenever there is an isometry in any of the z^i directions, the $D = 11$ pp-wave can be dimensionally reduced on such a coordinate, to give a pp-wave in the type IIA theory. In general, the resulting pp-wave can have non-vanishing constant backgrounds both for the NS-NS 3-form and R-R 4-form. The precise details depend in the usual way on the structure of $\Phi_{(3)}$ in $D = 11$.

The Green-Schwarz action for the type IIA string in an arbitrary bosonic background was derived, in component form up to and including second-order in the fermionic coordinates θ, in [30] (the form of the R-R couplings had previously been obtained schematically in [31]):

$$\begin{aligned} \mathcal{L}_2 &= -\tfrac{1}{2}\sqrt{-h}\, h^{ij}\, \partial_i X^\mu\, \partial_j\, X^\nu\, g_{\mu\nu} + \tfrac{1}{2}\epsilon^{ij}\, \partial_i\, X^\mu\, \partial_j\, X^\nu\, A_{\mu\nu} \\ &\quad -\mathrm{i}\, \bar\theta\, \beta^{ij}\, \Gamma_\mu\, D_j\theta\, \partial_i X^\mu + \tfrac{\mathrm{i}}{8}\partial_i X^\mu\, \partial_j X^\nu\, \bar\theta\, \beta^{ij}\, \Gamma_{11}\, \Gamma_\mu{}^{\rho\sigma}\, \theta\, F_{\nu\rho\sigma} \\ &\quad -\tfrac{\mathrm{i}}{16}\partial_i X^\mu \partial_j X^\nu e^\phi\, \bar\theta\, \beta^{ij} \Big(\Gamma_{11}\, \Gamma_\mu\, \Gamma^{\rho\sigma}\, \Gamma_\nu\, F_{\rho\sigma} + \tfrac{1}{12}\Gamma_\mu\, \Gamma^{\rho\sigma\lambda\tau}\, \Gamma_\nu\, F_{\rho\sigma\lambda\tau}\Big)\, \theta \, , \end{aligned} \tag{32}$$

where

$$\beta^{ij} \equiv \sqrt{-h}\, h^{ij} - \epsilon^{ij}\, \Gamma_{11} \, , \qquad D_i\theta \equiv \partial_i\theta + \tfrac{1}{4}\partial_i X^\mu\, \omega_\mu{}^{mn}\, \Gamma_{mn}\, \theta \, . \tag{33}$$

The field strengths are given by

$$F_{(4)} = dA_{(3)} - A_{(1)} \wedge dA_{(2)} \, , \qquad F_{(3)} = dA_{(2)} \, , \qquad F_{(2)} = dA_{(1)} \, . \tag{34}$$

The fermionic coordinates θ are non-chiral, and can be written as $\theta = \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}$, where $\Gamma_{11}\,\theta = \begin{pmatrix} \theta^1 \\ -\theta^2 \end{pmatrix}$. In a notation adapted to the passage to light-cone gauge, we can introduce world-sheet Dirac matrices ϱ_i, with $\varrho_0 = -\mathrm{i}\,\tau_2$, $\varrho_1 = \tau_1$ and $\varrho_2 = \tau_3$, where τ_i are the Pauli matrices. The ϱ_i act on the upper and lower 16 components θ^1 and θ^2 of the column vector $\Psi = \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}$, and ϱ_2 is the chirality operator. The conjugate spinor in this notation is then $\bar\Psi = \Psi^\dagger\,\Gamma_0\,\varrho_0$, and we therefore have the "dictionary" $\bar\theta\,\mathcal{O}\,\theta \longrightarrow -\bar\Psi\,\varrho_0\,\mathcal{O}\,\Psi$ and $\bar\theta\,\Gamma_{11}\,\mathcal{O}\,\theta \longrightarrow \bar\Psi\,\varrho_1\,\mathcal{O}\,\Psi$, where $\mathcal{O}$ is any matrix or operator constructed from the Γ_i matrices. In the light-cone gauge, where $X^+ = \tau$, $\Gamma_-\,\theta = 0$ and $\sqrt{-h}\,h^{ij} = \eta^{ij}$, the fermionic part of the type IIA Green-Schwarz action (32) therefore becomes

$$\mathcal{L}_F = \mathrm{i}\,\bar\Psi\,\Gamma_+\,\not{D}\,\Psi - \tfrac{\mathrm{i}}{4}\bar\Psi\,\varrho_1\,\Gamma_+\,\not{F}_{(3)}\,\Psi - \tfrac{\mathrm{i}}{4}e^{\phi}\,\bar\Psi\,\Gamma_+\,(\varrho_1\,\not{F}_{(2)} - \varrho_0\,\not{F}_{(4)})\,\Psi\,, \tag{35}$$

where we have defined

$$\not{F}_{(2)} \equiv \Gamma^i\,F_{+i}\,,\qquad \not{F}_{(3)} \equiv \tfrac12\Gamma^{ij}\,F_{+ij}\,,\qquad \not{F}_{(4)} \equiv \tfrac16\Gamma^{ijk}\,F_{+ijk}\,. \tag{36}$$

In the pp-wave backgrounds we are considering here, the world-sheet Dirac operator $\not{D}$ just reduces to $\not\partial$ in the light-cone gauge.

If the dimensional reduction from the $D=11$ pp-wave to $D=10$ is performed on a Killing direction z^i whose differential dz^i does not appear in the expression for $\Phi_{(3)}$ in $D=11$, then the solution in $D=10$ is a pp-wave with only the R-R 4-form $F_{(4)}$ as a source. This situation can be achieved for any $\Phi_{(3)}$ contained within Case 2, provided that one reduces on the z_8 or z_9 coordinate. Since Case 1 is encompassed by Case 2 (after appropriate coordinate relabellings) in all situations except where all four m_α are non-vanishing, it is only in this last circumstance that one is forced into a dimensional reduction in which the differential dz^i of the reduction coordinate is present in $\Phi_{(3)}$ in $D=11$. Of course in other cases too, one may choose to perform the dimensional reduction on such a coordinate z^i, provided that the associated coefficient μ_i in the quadratic metric function H vanishes, implying that $\partial/\partial z^i$ is a Killing vector.

Let us first consider the case where the differential dz^i of the reduction coordinate z^i does not appear in $\Phi_{(3)}$ in $D=11$. It then follows from (32) and (35) that after choosing the light-cone gauge, the associated type IIA string action will be[3]

$$\mathcal{L} = \sum_{i=1}^{8} (\tfrac{1}{2}\dot{z}_i^2 - \tfrac{1}{2}z_i'^2 - \tfrac{1}{2}\mu_i^2 z_i^2) + \bar{\Psi}(\mathrm{i}\,\partial\!\!\!/ + \tfrac{1}{4}\,\mu\,\varrho_0\,W)\,\Gamma_+\,\Psi\,. \tag{37}$$

Thus the masses of the fermions will be given by the eigenvalues of W.

Suppose now that we instead perform a reduction on a coordinate z^i whose differential dz^i does appear in $\Phi_{(4)}$ in $D = 11$. We now find that $F_{(4)}$ in $D = 11$ reduces as

$$F_{(4)} \longrightarrow F_4 + dz \wedge F_{(3)} \tag{38}$$

in $D = 10$, where z is the reduction coordinate. We now have the non-vanishing NS-NS field $F_{(3)}$ in the type IIA background, together, possibly, with a non-vanishing R-R 4-form $F_{(4)}$. If the reduction of the 3-form $\Phi_{(3)}$ is written as $\Phi_{(3)} \longrightarrow \Phi_3 + dz \wedge \Phi_2$, and if we define

$$Y \equiv \tfrac{\mathrm{i}}{2}\Phi_{ij}\,\Gamma^{ij}\,, \tag{39}$$

then it follows from (32) and (35) that after choosing the light-cone gauge, we shall obtain the string action

$$\mathcal{L} = \sum_{i=1}^{8} (\tfrac{1}{2}\dot{z}_i^2 - \tfrac{1}{2}z_i'^2 + \mu\,z_i'\,B_i - \tfrac{1}{2}\mu_i^2 z_i^2) + \bar{\Psi}(\mathrm{i}\,\partial\!\!\!/ + \tfrac{1}{4}\,\mu\,\varrho_0\,W + \tfrac{1}{4}\,\mu\,\varrho_0\,Y)\,\Gamma_+\,\Psi\,. \tag{40}$$

Here, B_i denotes the components of the 1-form $B_{(1)}$ whose exterior derivative gives $\Phi_{(2)} = dB_{(1)}$. Since $\Phi_{(2)} = \frac{1}{2}\Phi_{ij}\,dz^i \wedge dz^j$ where Φ_{ij} are constants, we may therefore take B_i to be given by

$$B_i = \tfrac{1}{2}\Phi_{ji}\,z^j\,. \tag{41}$$

Thus the string action in this case is given by

$$\begin{aligned}\mathcal{L} &= \sum_{i=1}^{8} (\tfrac{1}{2}\dot{z}_i^2 - \tfrac{1}{2}(z_i' - \tfrac{1}{2}\mu\,\Phi_{ij}\,z_j)^2 - \tfrac{1}{2}\mu_i^2 z_i^2) + \tfrac{1}{8}\mu^2\Phi_{ik}\,\Phi_{jk}\,z^j\,z^k \\ &\quad + \bar{\Psi}(\mathrm{i}\,\partial\!\!\!/ + \tfrac{1}{4}\,\mu\,\varrho_0\,W + \tfrac{1}{4}\,\mu\,\varrho_0\,Y)\,\Gamma_+\,\Psi\,. \end{aligned}\tag{42}$$

Thus the boson masses, as well as the fermion masses, are modified by the presence of the NS-NS 3-form field. This is a generalisation of a result obtained in [5].

As an example, let us consider the pp-wave in $D = 11$ resulting from taking $\Phi_{(3)}$ to be given by Case 1, as in (14). After dimensional reduction on the coordinate z_9, which will be a Killing direction provided that

$$m_1 + m_2 + m_3 + m_4 = 0 \tag{43}$$

[3]See also [32] for a related discussion of the type IIA Green-Schwarz action in a gravitational wave background.

(see (24)), the type IIA light-cone action will be given by

$$\mathcal{L} = \sum_{i=1}^{8}(\tfrac12 \dot z_i^2 - \tfrac12 (z_i' - \tfrac12 \mu\, \Phi_{ij}\, z_j)^2 - \tfrac12 \mu_i^2 z_i^2) + \tfrac18 \mu^2 [m_1^2\,(z_1^2+z_2^2) + \cdots + m_4^2\,(z_7^2+z_8^2)]$$
$$+\bar\Psi(\mathrm{i}\,\partial\!\!\!/ + \tfrac14\,\mu\,\varrho_0\, Y)\,\Gamma_+\,\Psi\,, \qquad (44)$$

with Y given by

$$Y = \mathrm{i}\, m_1\,\Gamma_{12} + \mathrm{i}\, m_2\,\Gamma_{34} + \mathrm{i}\, m_3\,\Gamma_{56} + \mathrm{i}\, m_4\,\Gamma_{78}\,. \qquad (45)$$

(The matrix W is absent here, since all terms in $\Phi_{(3)}$ in $D=11$ involved a factor dz_9. Thus the $D=10$ background is purely NS-NS in this example.)

Note that the choice of gauge for writing $B_{(1)} \equiv B_i\, dz^i$ is not unique. In this example we could, for instance, choose, instead of writing it in the "symmetrical" gauge

$$B_{(1)} = \tfrac12 \Phi_{ij}\, z^i\, dz^j = \tfrac12 m_1\,(z_1\, dz_2 - z_2\, dz_1) + \cdots + \tfrac12 m_4\,(z_7\, dz_8 - z_8\, dz_7)\,, \qquad (46)$$

to write it in the "asymmetrical" form

$$B_{(1)} = m_1\, z_1\, dz_2 + m_2\, z_3\, dz_4 + m_3\, z_5\, dz_6 + m_4\, z_7\, dz_8 \qquad (47)$$

In this choice of gauge we would instead obtain the string action

$$\mathcal{L} = \sum_{i=1}^{8}(\tfrac12 \dot z_i^2 - \tfrac12 (z_i' - \tfrac12 \mu\, \Phi_{ij}\, z_j)^2 - \tfrac12 \mu_i^2 z_i^2) + \tfrac12 \mu^2 [m_1^2\, z_1^2 + m_2^2 z_3^2 + m_3^2\, z_5^2 + m_4^2\, z_7^2]$$
$$+\bar\Psi(\mathrm{i}\,\partial\!\!\!/ + \tfrac14\,\mu\,\varrho_0\, Y)\,\Gamma_+\,\Psi\,. \qquad (48)$$

Of course the different gauge choices just change the action by a total derivative, and so they are equivalent in the closed string sector.

3.2 Type IIB string action

Many of our examples can be T-dualised to pp-waves in type IIB theory when there are two μ_i that vanish. In some cases, when the type IIA pp-wave is supported only by the $F_{(3)}$, the solution is also valid in type IIB theory supported by the NS-NS $F_{(3)}$ or the R-R $F_{(3)}$, or both using S-duality rotation. Thus in this section, we consider the light-cone type IIB string action in such a background.

In [30], the Green-Schwarz action for the type IIB string in an arbitrary bosonic background was derived, giving all terms up to and including quadratic order in the fermionic coordinates. In the notation of [30], the two Majorana-Weyl fermions were denoted by θ^1 and θ^2. If we put these in a column vector $\theta \equiv \begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix}$, and define world-sheet Dirac matrices ϱ_i by $\varrho_0 = -\mathrm{i}\,\tau_2$, $\varrho_1 = \tau_1$ and $\varrho_2 = \tau_3$, where τ_i are the Pauli matrices, then we find the following "dictionary" for converting the notation in [30] to the one we wish to use here. For any matrix or operator $\mathcal{O}$ constructed from the target-space Dirac matrices, we shall have

$$\begin{aligned} &\bar\theta\,\varrho_0\,\mathcal{O}\,\theta = -\bar\theta^1\,\mathcal{O}\,\theta^1 - \bar\theta^2\,\mathcal{O}\,\theta^2\,, \qquad \bar\theta\,\varrho_1\,\mathcal{O}\,\theta = \bar\theta^1\,\mathcal{O}\,\theta^1 - \bar\theta^2\,\mathcal{O}\,\theta^2\,,\\ &\bar\theta\,\mathcal{O}\,\theta = 2\bar\theta^{[1}\,\mathcal{O}\,\theta^{2]}\,, \qquad \bar\theta\,\varrho_2\,\mathcal{O}\,\theta = -2\bar\theta^{(1}\,\mathcal{O}\,\theta^{2)}\,, \end{aligned} \tag{49}$$

where the conjugate of θ is defined by $\bar\theta = \theta^\dagger\,\Gamma_0\,\varrho_0 = (-\bar\theta^2, \bar\theta^1)$. Substituting into equation (3.29) of [30] (in the updated v2. where a minor typographical error has been corrected and conventions adjusted), the type IIB Green-Schwarz action up to $O(\theta^2)$ is given by

$$\begin{aligned} \mathcal{L} \;=\;& -\tfrac12\sqrt{-h}\,h^{ij}\,\partial_i X^\mu\,\partial_j X^\nu\,g_{\mu\nu} + \tfrac12\epsilon^{ij}\,\partial_i X^\mu\,\partial_j X^\nu\,B_{\mu\nu}\\ &+\mathrm{i}\,\partial_i X^\mu\,\bar\theta\,\gamma^{ij}\,\varrho_0\,\Gamma_\mu\,D_j\,\theta - \tfrac{\mathrm{i}}{8}\partial_i X^\mu\,\partial_j X^\nu\,\bar\theta\,\gamma^{ij}\,\varrho_1\,\Gamma_\mu{}^{\rho\sigma}\,\theta\,G_{\nu\rho\sigma}\\ &+\tfrac{\mathrm{i}}{8}e^\phi\,\partial_i X^\mu\,\partial_j X^\nu\,\bar\theta\,\gamma^{ij}\,\Gamma_\mu\,[\Gamma^\rho\,\partial_\rho\chi + \tfrac16\varrho_2\,\Gamma^{\rho_1\rho_2\rho_3}\,F_{\rho_1\rho_2\rho_3} + \tfrac{1}{240}\Gamma^{\rho_1\cdots\rho_5}\,F_{\rho_1\cdots\rho_5}]\,\Gamma_\nu\,\theta\,, \end{aligned} \tag{50}$$

where $G_{(3)} = dB_{(2)}$ is the NS-NS 3-form, ϕ and χ are the dilaton and axion, and $F_{(3)}$ and $F_{(5)}$ are the R-R 3-form and self-dual 5-form, and we have defined

$$\gamma^{ij} \equiv \sqrt{-h}\,h^{ij} - \epsilon^{ij}\,\varrho_2\,. \tag{51}$$

In the light-cone gauge, $X^+ = \tau$, $\theta = \Psi$ with $\Gamma_-\,\Psi = 0$, and $\sqrt{-h}\,h^{ij} = \eta^{ij}$, we therefore find that the fermionic part of the action becomes

$$\mathcal{L}_F = \mathrm{i}\,\bar\Psi\,\Gamma_+\,\not{D}\,\Psi + \tfrac{\mathrm{i}}{4}\bar\Psi\,\varrho_1\,\Gamma_+\,\not{G}_3\,\Psi - \tfrac{\mathrm{i}}{4}\,e^\phi\,\bar\Psi\,\Gamma_+\,(\partial_+\chi\; + \varrho_2\;\not{F}_{(3)} + \tfrac12\,\not{F}_{(5)})\,\Psi\,, \tag{52}$$

where we have defined

$$\not{G}_{(3)} \equiv \tfrac12\Gamma^{ij}\,G_{+ij}\,, \qquad \not{F}_{(3)} \equiv \tfrac12\Gamma^{ij}\,F_{+ij}\,, \qquad \not{F}_{(5)} \equiv \tfrac{1}{24}\Gamma^{ijk\ell}\,F_{+ijk\ell}\,. \tag{53}$$

Note that for all the pp-waves, it follows from (3) that $\not{D}$ is just $\not{\partial}$.

To apply this action to our examples, let us first consider Case 2, with $\mu_8 = \mu_9 = 0$. This example can be T-dualised to type IIB, where it gives rise to pp-waves of the sort described in [17]. In these cases the pp-wave is supported only by the self-dual R-R 5-form. The action was obtained in [17].

For another example consider Case 1, with all four of the m_α non-vanishing, but chosen so that $\mu_9 = 0$, thus permitting a reduction in the z_9 direction to give a type IIA solution. The solution is then supported purely by the NS-NS 3-form. There is in general no further isometry direction among the remaining z_i coordinates that could allow us to perform a T-duality transformation. However, since only the NS-NS 3-form field is involved, we can clearly take the identical configuration and view it as a solution instead of the type IIB theory, again supported by the NS-NS 3-form. Having done so, we can then choose to perform an S-duality transformation, thereby introducing a non-vanishing R-R 3-form as well (or instead). It is then straightforward to see from (52) and (53), together with imposing the light-cone gauge in the bosonic part of (50), that these type IIB pp-wave backgrounds will have a light-cone string action given by

$$\mathcal{L} = \sum_{i=1}^{8} (\tfrac{1}{2}\dot{z}_i^2 - \tfrac{1}{2}(z_i' - \tfrac{1}{2}B_{ij}\, z^j)^2 - \tfrac{1}{2}\mu_i^2 z_i^2) + \bar{\Psi}(\mathrm{i}\, \not{\partial} + \tfrac{\mathrm{i}}{4}\varrho_1\, \not{G}_{(3)} - \tfrac{\mathrm{i}}{4}\varrho_2\, \not{F}_{(3)})\,\Gamma_+\,\Psi\,, \tag{54}$$

where $\not{G}_{(3)}$ and $\not{F}_{(3)}$ are the NS-NS and R-R 3-form contributions respectively.

3.3 Matrix model action

There are many examples in our general discussion where all the coefficients μ_i in the metric function H are non-vanishing, implying that the pp-wave is intrinsically eleven-dimensional. In these cases, the system is best described by a D0-brane action. Namely, one can perform a DLCQ compactification [33]-[36] along the light-cone coordinate $x^- \equiv x^- + 2\pi R$, and consider the sector with momentum $2p^+ = -p^- = N/R$. The dynamics of this sector is then described by a $U(N)$ matrix model with the strength of interactions governed by $g \sim 2R$. The procedure as it applies to the case of the Penrose limit of $\mathrm{AdS}_4 \times S^7$ or $\mathrm{AdS}_7 \times S^4$ was given in [5]. The form of the action for the general, constant W, as studied in this paper, can be derived along the same lines, and is structurally of the same form. This is due to the fact that the 4-form field strength enters the D0-brane particle action in the light-cone gauge (i.e. the $U(N)$ matrix model) only through $W = \frac{\mathrm{i}}{6}\Phi_{ijk}\,\Gamma^{ijk}$. The form of the action is thus given by

$$\begin{aligned} L &= \sum_{i=1}^{9}[(\dot{X}^i)^2 - \mu_i^2 (X^i)^2] + \Psi^T\,\dot{\Psi} + \tfrac{\mathrm{i}}{4}\,\mu\,\Psi^T\,W\,\Psi \\ &\quad -\tfrac{2}{3}\mu\, g \sum_{i,j,k=1}^{9} \mathrm{Tr}(X^i\,X^j\,X^k)\,\Phi_{ijk} + 2g^2\,\mathrm{Tr}([X^i, X^j]^2) + 2\mathrm{i}g\,\mathrm{Tr}(\Psi^T\,\Gamma^i\,[\Psi, X^i])\,. \end{aligned} \tag{55}$$

Note that in addition to the standard matrix-model interactions there are also the fermionic and bosonic mass terms, and additionally the term tri-linear in X^i that is related to the Myers effect [37].

The supersymmetry of this quantum mechanical matrix model fixes the coefficients in front of the fermionic mass terms and the interaction terms in the same way as it was derived for the special case of $W = \Phi_{123}\,\Gamma^{123}$ in [5]. Indeed, the existence of supersymmetry is dictated by the existence of the supernumerary Killing spinors. In fact, the supersymmetry transformation parameter is exactly the supernumerary Killing spinor:

$$\begin{aligned} \delta X^i &= \Psi\,\Gamma^i\,\epsilon\,, \\ \delta\Psi &= \Big(\dot X^i\,\Gamma_i + \mu\,X^i\,(-\tfrac14 W\,\Gamma_i + \tfrac1{12}\Gamma_i\,W) + {\rm i}\,g\,[X^i,X^j]\,\Gamma_{ij}\Big)\,\epsilon\,, \\ \epsilon &= e^{\mu\,W\,t}\,\epsilon_0\,. \end{aligned} \tag{56}$$

The case where $W=\Gamma_{123}$ was given in [5]. In that case, the system is fully supersymmetric, and hence ϵ_0 is an arbitrary constant spinor. Furthermore, since W has no zero eigenvalues in that example, all the supersymmetry parameters ϵ are time-dependent.

For the more general W's that we have considered in this paper, ϵ_0 is subject to further projection constraints, in accordance with the supernumerary Killing spinors. In our more general cases W can annihilate ϵ_0, implying that ϵ is then time-independent. In such an example, the pp-wave can also be reduced to give rise to a pp-wave in type IIA, thus giving an exactly-solvable string action. The existence of two routes, one corresponding to the matrix model of the D0-particle action, and the other corresponding to the free massive Type IIA string action, therefore suggests that these are dual descriptions of the theory when the background is of this particular type.

4 Conclusions

In this paper, we studied a general class of supersymmetric pp-waves in M-theory, by turning on constant 3-forms, motivated by the orbifold (flat) limit of a special holonomy transverse space for the pp-wave. These 3-forms fall into two classes, one motivated by the Kähler form of the eight-dimensional special holonomy transverse space, and the other motivated by the associative 3-form of a seven-dimensional transverse space of G_2 holonomy.

This general class of pp-waves encompass the Penrose limits of $\mathrm{AdS}_p\times S^q$ with $(p,q)=(4,7)$, $(7,4)$, $(3,3)$, $(3,2)$, $(2,3)$, $(2,2)$ which are associated with the near horizon limits of the M2-brane, M5-brane, and M2/M5, M5/M5/M5, M2/M2/M2 and M2/M2/M5/M5 intersections, respectively. In addition this general class contains many additional examples of pp-waves that are do not correspond to any known Penrose limit.

We focused on the study of the target space supersymmetry. In addition to 16 "standard" Killing spinors that always arise, we determined the conditions under which additional "supernumerary" Killing spinors appear. We also analysed the conditions under which the Killing spinors are independent of the light cone x^+ coordinate, or of one or more of the nine transverse coordinates. These conditions determine whether the reduction of the M-theory pp-waves to type IIA supergravity, and subsequent T-dualisation to type IIB, remain supersymmetric.

Since x^+ is always a Killing direction the M-theory pp-wave can always be reduced on this coordinate, leading to a D0-brane configuration of the type IIA theory. Its world-particle action corresponds to a DLCQ description of a matrix-theory action for M-theory, with unbroken supersymmetry governed by x^+-independent supernumerary supersymmetries of the M-theory pp-wave background.

On the other hand the independence of a Killing spinor on a transverse coordinate allows for a reduction on this coordinate down to a supersymmetric type IIA pp-wave. The light cone string actions in these backgrounds correspond to exactly-solvable free massive string theories, and again the supernumerary supersymmetries play a key role in determining the supersymmetry of the string action.

Note Added

We have updated the discussion of the type IIA and type IIB string actions in this version of the paper, taking into account the corrections and improvements in v2. of [30]. There was only one minor typographical in the type IIB action in the earlier version of [30], but there were various inelegant notations and conventions, all of which have now been changed, and these changes are incorporated in this version of the present paper. A detailed discussion of the changes is given in the Addendum section in v2. of [30]. We are grateful to Kelly Stelle for discussions leading to these improvements.

Acknowledgements

We are grateful to Gary Gibbons, Jim Liu and Justin Vázquez-Poritz for conversations. H.L. and C.N.P. are grateful to UPenn for hospitality and financial support during the course of this work.

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