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Well-Balanced Positivity Preserving Central-Upwind Scheme on Triangular Grids for the Saint-Venant System

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Abstract

We introduce a new second-order central-upwind scheme for the Saint-Venant system of shallow water equations on triangular grids. We prove that the scheme both preserves stationary steady states (lake at rest) and guarantees the positivity of the computed fluid depth. Moreover, it can be applied to models with discontinuous bottom topography and irregular channel widths. We demonstrate these features of the new scheme, as well as its high resolution and robustness in a number of numerical examples.

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Key Words: Hyperbolic systems of conservation and balance laws, semi-discrete central-upwind schemes, Saint-Venant system of shallow water equations.

1 Introduction

We consider the two-dimensional (2-D) Saint-Venant system of shallow water equations:

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0, \\ (hu)_t + \Big(hu^2 + \dfrac{1}{2}gh^2\Big)_x + (huv)_y = -ghB_x, \\ (hv)_t + (huv)_x + \Big(hv^2 + \dfrac{1}{2}gh^2\Big)_y = -ghB_y, \end{cases} \tag{1.1}$$

where the function $B(x,y)$ represents the bottom elevation, h is the fluid depth above the bottom, $(u,v)^T$ is the velocity vector, and g is the gravitational constant. This system is widely used in many scientific and engineering applications related to modeling of water flows in rivers, lakes and coastal areas. The development of robust and accurate numerical methods for the computation of its solutions is important and challenging problem that has been extensively investigated

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in the recent years. One of the difficulties encountered is the fact that system (1.1) admits nonsmooth solutions: shocks, rarefaction waves and, when the bottom topography function B is discontinuous, contact discontinuities. In the latter case, the solution may not be unique, which makes the design of robust numerical methods more challenging even in the one-dimensional (1-D) case (see, e.g., [2] and the references therein).

A good numerical method for (1.1) should have two major properties, which are crucial for its stability:

(i) The method should be well-balanced, that is, it should exactly preserve the stationary steady-state solutions $h+B \equiv \text{const}$, $u \equiv v \equiv 0$ (lake at rest states). This property diminishes the appearance of unphysical waves of magnitude proportional to the grid size (the so-called "numerical storm"), which are normally present when computing quasi steady-states;

(ii) The method should be positivity preserving, that is, the water depth h should be nonnegative at all times. This property ensures a robust performance of the method on dry ($h = 0$) and almost dry ($h \sim 0$) states.

In the past decade, a number of well-balanced [4, 9, 15, 17, 19, 23, 27, 28, 31, 32, 33, 34, 39, 40] and positivity preserving [4, 19, 23, 32] schemes for (1.1) have been proposed, but only few of them satisfy both major properties (i) and (ii). Among the methods developed are generalizations of the class of accurate, efficient and robust Godunov-type semi-discrete central-upwind schemes, introduced in [20, 21, 24, 25, 22] as universal Riemann-problem-solver-free methods for general multidimensional hyperbolic systems. More precisely, the central-upwind schemes have been extended to compute the solutions of both the 1-D and 2-D Saint-Venant systems. For example, see [19], where well-balanced and positivity preserving central-upwind schemes have been introduced. However, the schemes presented in [19] do not simultaneously satisfy (i) and (ii) over the entire computational domain. In a recent work [23], a new second-order central-upwind scheme, which is well-balanced and positivity preserving at the same time, has been proposed. The key ideas in the development of this scheme are:

- Replacement of the bottom topography function B with its continuous piecewise linear (or bilinear in the 2-D case) approximation;
- Change of conservative variables from $(h, hu, hv)^T$ to $(w := h + B, hu, hv)^T$;
- Special positivity preserving correction of the piecewise linear reconstruction for the water surface w;
- Development of a special finite-volume-type quadrature for the discretization of the cell averages of the geometric source term.

In both [19] and [23], the central-upwind schemes for the 2-D system (1.1) are developed for Cartesian grids. Many real world engineering applications require the use of triangular meshes due to the complicated structure of the computational domains of the problems being investigated. A well-balanced central-upwind scheme on triangular grids has been recently developed in [6], where the presented "triangular" scheme is a (nonconservative) modification of the "triangular" central-upwind scheme from [22] with a special quadrature for the source average over arbitrary triangular cells. The method in [6] is not claimed to be positivity preserving, and is expected to fail on dry states.

In this paper, we present a new second-order semi-discrete central-upwind scheme for computing the solutions of the system (1.1) on triangular grids. Like the central-upwind scheme from [6], our scheme is well-balanced, but the new quadrature for the discretization of the geometric source, presented in §2.2, is much simpler than the one proposed in [6]. In addition, unlike the scheme from [6], the proposed central-upwind scheme is positivity preserving. The latter property is achieved by replacing the (possibly discontinuous) bottom topography function B with its continuous piecewise linear approximation (§2.1) and adjusting the piecewise linear reconstruction for w according to the piecewise linear approximation of B (§2.3). This technique is borrowed from [23] and naturally adopted to triangular meshes.

The new central-upwind scheme is derived in §2 and its positivity preserving property is proved in §2.4. In §3, we demonstrate the high resolution and robustness of the new scheme on a variety of numerical examples.

2 Description of the Scheme

In this section, we describe our new second-order semi-discrete central-upwind scheme for solving the Saint-Venant system of shallow water equations on triangular grids. We first denote the water surface by $w := h + B$ and rewrite (1.1) in terms of the vector $\mathbf{U} := (w, hu, hv)^T$:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U}, B)_x + \mathbf{G}(\mathbf{U}, B)_y = \mathbf{S}(\mathbf{U}, B), \tag{2.1}$$

where the fluxes and the source terms are:

$$\mathbf{F}(\mathbf{U}, B) = \left(hu, \frac{(hu)^2}{w - B} + \frac{1}{2}g(w - B)^2, \frac{(hu)(hv)}{w - B}\right)^T, \tag{2.2}$$

$$\mathbf{G}(\mathbf{U}, B) = \left(hv, \frac{(hu)(hv)}{w - B}, \frac{(hv)^2}{w - B} + \frac{1}{2}g(w - B)^2\right)^T, \tag{2.3}$$

$$\mathbf{S}(\mathbf{U}, B) = \Big(0, -g(w - B)B_x, -g(w - B)B_y\Big)^T. \tag{2.4}$$

We assume that a triangulation $\mathcal{T} := \bigcup\limits_j T_j$ of the computational domain, consisting of triangular cells T_j of size $|T_j|$, is given. We denote by $\vec{n}_{jk} := (\cos(\theta_{jk}), \sin(\theta_{jk}))$ the outer unit normals to the corresponding sides of T_j of length ℓ_{jk}, $k = 1, 2, 3$, see Figure 2.1. Let (x_j, y_j) be the coordinates of the center of mass for T_j and $M_{jk} = (x_{jk}, y_{jk})$ be the midpoint of the k-th side of the triangle T_j, $k = 1, 2, 3$. We denote by T_{j1}, T_{j2} and T_{j3} the neighboring triangles that share a common side with T_j.

A semi-discrete scheme for (2.1) is a system of ODEs for the approximations of the cell averages of the solution:

$$\overline{\mathbf{U}}_j(t) \approx \frac{1}{|T_j|} \int\limits_{T_j} \mathbf{U}(x, y, t)\, dxdy.$$

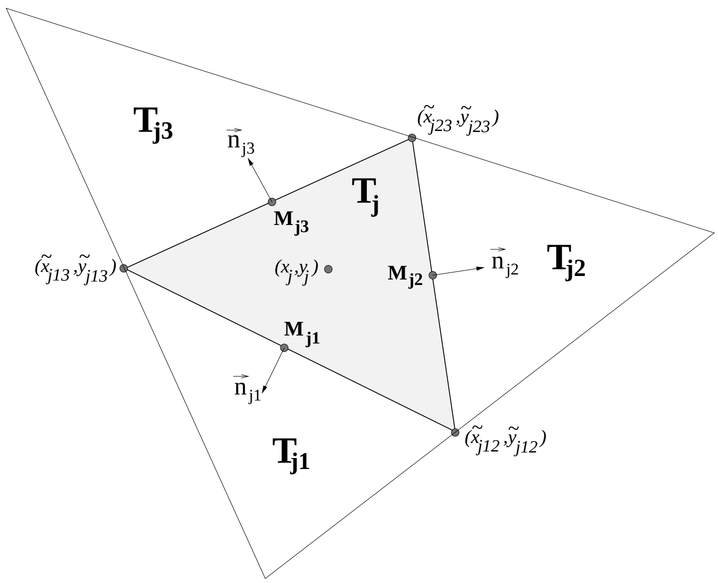

Figure 2.1: A typical triangular cell with three neighbors.

We refer the reader to [22], where a general form of a "triangular" central-upwind scheme for systems of hyperbolic conservation laws is derived. Its second-order version (see [6, 22]) reads:

$$\begin{aligned}\frac{d\overline{\mathbf{U}}_j}{dt} = &-\frac{1}{|T_j|}\sum_{k=1}^{3}\frac{\ell_{jk}\cos(\theta_{jk})}{a_{jk}^{\rm in}+a_{jk}^{\rm out}}\left[a_{jk}^{\rm in}\mathbf{F}(\mathbf{U}_{jk}(M_{jk}),B(M_{jk}))+a_{jk}^{\rm out}\mathbf{F}(\mathbf{U}_j(M_{jk}),B(M_{jk}))\right]\\ &-\frac{1}{|T_j|}\sum_{k=1}^{3}\frac{\ell_{jk}\sin(\theta_{jk})}{a_{jk}^{\rm in}+a_{jk}^{\rm out}}\left[a_{jk}^{\rm in}\mathbf{G}(\mathbf{U}_{jk}(M_{jk}),B(M_{jk}))+a_{jk}^{\rm out}\mathbf{G}(\mathbf{U}_j(M_{jk}),B(M_{jk}))\right]\\ &+\frac{1}{|T_j|}\sum_{k=1}^{3}\ell_{jk}\frac{a_{jk}^{\rm in}a_{jk}^{\rm out}}{a_{jk}^{\rm in}+a_{jk}^{\rm out}}\Big[\mathbf{U}_{jk}(M_{jk})-\mathbf{U}_j(M_{jk})\Big]+\overline{\mathbf{S}}_j,\end{aligned} \tag{2.5}$$

where $\mathbf{U}_j(M_{jk})$ and $\mathbf{U}_{jk}(M_{jk})$ are the corresponding values at M_{jk} of the piecewise linear reconstruction

$$\widetilde{\mathbf{U}}(x,y) := \overline{\mathbf{U}}_j + (\mathbf{U}_x)_j(x-x_j) + (\mathbf{U}_y)_j(y-y_j), \quad (x,y)\in T_j \tag{2.6}$$

of $\mathbf{U}$ at time t, that is:

$$\mathbf{U}_j(M_{jk}) := \lim_{(x,y)\to M_{jk};(x,y)\in T_j}\widetilde{\mathbf{U}}(x,y), \qquad \mathbf{U}_{jk}(M_{jk}) := \lim_{(x,y)\to M_{jk};(x,y)\in T_{jk}}\widetilde{\mathbf{U}}(x,y). \tag{2.7}$$

The numerical derivatives $(\mathbf{U}_x)_j$ and $(\mathbf{U}_y)_j$ are (at least) first-order, componentwise approximations of $\mathbf{U}_x(x_j,y_j,t)$ and $\mathbf{U}_y(x_j,y_j,t)$, respectively, computed via a nonlinear limiter, used to minimize the oscillations of the reconstruction (2.6). One can use any nonlinear limiter. A variety of nonoscillatory reconstructions can be found, for example, in [1, 3, 7, 8, 13, 14, 18, 22, 38]. In our numerical experiments, we have used a componentwise piecewise linear reconstruction, which is a modification of the Cartesian grid minmod reconstruction (see, e.g., [26, 29, 30, 36]). To calculate the numerical derivatives of the ith component of $\mathbf{U}$, $(\mathbf{U}_x^{(i)})_j$ and $(\mathbf{U}_y^{(i)})_j$, we construct three linear interpolations $L_j^{12}(x,y)$, $L_j^{23}(x,y)$ and $L_j^{13}(x,y)$, which are conservative on the triangle T_j and two of the neighboring triangles (T_{j1},T_{j2}), (T_{j2},T_{j3}) and (T_{j1},T_{j3}), respectively.

More precisely, the plane $L_j^{12}(x,y)$, for example, passes through the points whose first two coordinates are the coordinates of the centers of mass of the triangles T_j, T_{j1} and T_{j2}, and the third coordinates are the cell averages of $\mathbf{U}^{(i)}$ over the corresponding triangles. We then select the linear piece with the smallest magnitude of the gradient, say, $L_j^{km}(x,y)$, and set

$$((\mathbf{U}_x^{(i)})_j,(\mathbf{U}_y^{(i)})_j)^T=\nabla L_j^{km}. \tag{2.8}$$

We minimize the oscillations by checking the appearance of local extrema at the points M_{jk}, $1,2,3$. If for some k one of the reconstructed point values $\mathbf{U}_j^{(i)}(M_{jk})$ or $\mathbf{U}_{jk}^{(i)}(M_{jk})$ is not between the cell averages $\overline{\mathbf{U}}_j^{(i)}$ and $\overline{\mathbf{U}}_{jk}^{(i)}$, we replace (2.8) by

$$(\mathbf{U}_x^{(i)})_j=(\mathbf{U}_y^{(i)})_j=0.$$

The quantity $\overline{\mathbf{S}}_j$ in (2.5) is an appropriate discretization of the cell averages of the source term:

$$\overline{\mathbf{S}}_j(t)\approx\frac{1}{|T_j|}\int\limits_{T_j}\mathbf{S}(\mathbf{U}(x,y,t),B(x,y))\,dxdy.$$

In §2.2, we discuss in detail how to compute $\overline{\mathbf{S}}_j$ in a simple way, which guarantees the well-balanced property of the proposed scheme.

Finally, the directional local speeds $a_{jk}^{\rm in}$ and $a_{jk}^{\rm out}$ in (2.5) are defined by

$$\begin{aligned}a_{jk}^{\rm in}(M_{jk})&=-\min\{\lambda_1[V_{jk}(\mathbf{U}_j(M_{jk}))],\lambda_1[V_{jk}(\mathbf{U}_{jk}(M_{jk})],0\},\\ a_{jk}^{\rm out}(M_{jk})&=\max\{\lambda_3[V_{jk}(\mathbf{U}_j(M_{jk}))],\lambda_3[V_{jk}(\mathbf{U}_{jk}(M_{jk})],0\},\end{aligned} \tag{2.9}$$

where $\lambda_1\,[V_{jk}]\le\lambda_2\,[V_{jk}]\le\lambda_3\,[V_{jk}]$ are the eigenvalues of the matrix

$$V_{jk}=\cos(\theta_{jk})\frac{\partial\mathbf{F}}{\partial\mathbf{U}}+\sin(\theta_{jk})\frac{\partial\mathbf{G}}{\partial\mathbf{U}}.$$

If both $a_{jk}^{\rm in}$ and $a_{jk}^{\rm out}$ are zero (or very close to zero), the scheme (2.5) reduces to

$$\begin{aligned}\frac{d\overline{\mathbf{U}}_j}{dt}=&-\frac{1}{|T_j|}\sum_{k=1}^{3}\frac{\ell_{jk}\cos(\theta_{jk})}{2}\left[\mathbf{F}(\mathbf{U}_{jk}(M_{jk}),B(M_{jk}))+\mathbf{F}(\mathbf{U}_j(M_{jk}),B(M_{jk}))\right]\\ &-\frac{1}{|T_j|}\sum_{k=1}^{3}\frac{\ell_{jk}\sin(\theta_{jk})}{2}\left[\mathbf{G}(\mathbf{U}_{jk}(M_{jk}),B(M_{jk}))+\mathbf{G}(\mathbf{U}_j(M_{jk}),B(M_{jk}))\right]+\overline{\mathbf{S}}_j.\end{aligned}$$

A fully discrete scheme is obtained from (2.5) by using a stable ODE solver of an appropriate order. In our numerical experiments, we have used the third-order SSP-RK ODE solver, see [12]. The time step size should satisfy the CFL-condition:

$$\Delta t<\frac{1}{3}\min_{j,k}\left[\frac{r_{jk}}{\max\{a_{jk}^{\rm out},a_{jk}^{\rm in}\}}\right], \tag{2.10}$$

where r_{jk}, $k=1,2,3$ are the three corresponding altitudes of the triangle $T_j\in\mathcal{T}$.

2.1 Piecewise Linear Approximation of the Bottom

We start by replacing the bottom topography function B with its continuous piecewise linear approximation $\widetilde{B}$, which over each cell T_j is given by the formula:

$$\left|\begin{array}{ccc} x-\tilde{x}_{j_{12}} & y-\tilde{y}_{j_{12}} & \widetilde{B}(x,y)-\mathcal{B}_{j_{12}} \\ \tilde{x}_{j_{23}}-\tilde{x}_{j_{12}} & \tilde{y}_{j_{23}}-\tilde{y}_{j_{12}} & \mathcal{B}_{j_{23}}-\mathcal{B}_{j_{12}} \\ \tilde{x}_{j_{13}}-\tilde{x}_{j_{12}} & \tilde{y}_{j_{13}}-\tilde{y}_{j_{12}} & \mathcal{B}_{j_{13}}-\mathcal{B}_{j_{12}} \end{array}\right| = 0, \quad (x,y)\in T_j. \tag{2.11}$$

Here, $\mathcal{B}_{j_\kappa}$ are the values of $\widetilde{B}$ at the vertices $(\tilde{x}_{j_\kappa}, \tilde{y}_{j_\kappa})$, $\kappa = 12, 23, 13$, of the cell T_j (see Figure 2.1), computed according to the following formula:

$$\mathcal{B}_{j_\kappa} := \frac{1}{2}\left(\max_{\xi^2+\eta^2=1}\lim_{h,\ell\to 0} B(\tilde{x}_{j_\kappa}+h\xi, \tilde{y}_{j_\kappa}+\ell\eta) + \min_{\xi^2+\eta^2=1}\lim_{h,\ell\to 0} B(\tilde{x}_{j_\kappa}+h\xi, \tilde{y}_{j_\kappa}+\ell\eta)\right),$$

which reduces to

$$\mathcal{B}_{j_\kappa} = B(\tilde{x}_{j_\kappa}, \tilde{y}_{j_\kappa}),$$

if the function B is continuous at $(\tilde{x}_{j_\kappa}, \tilde{y}_{j_\kappa})$.

Let us denote by B_{jk} the value of the continuous piecewise linear reconstruction at M_{jk}, namely $B_{jk} := \widetilde{B}(M_{jk})$, and by $B_j := \widetilde{B}(x_j, y_j)$ the value of the reconstruction at the center of mass (x_j, y_j) of T_j. Notice that, in general, $B_{jk} \neq B(M_{jk})$ and

$$B_j = \frac{1}{|T_j|}\int\limits_{T_j} \widetilde{B}(x,y)\,dxdy.$$

Moreover, one can easily show that

$$B_j = \frac{1}{3}\left(B_{j1}+B_{j2}+B_{j3}\right) = \frac{1}{3}\left(\mathcal{B}_{j_{12}}+\mathcal{B}_{j_{23}}+\mathcal{B}_{j_{13}}\right). \tag{2.12}$$

Notice that the approach described above is applicable to any bottom topography function, both continuous and discontinuous.

2.2 Well-Balanced Discretization of the Source Term

The well-balanced property of the scheme is guaranteed if the discretized cell average of the source term, $\overline{\mathbf{S}}_j$, exactly balances the numerical fluxes so that the right-hand side (RHS) of (2.5) vanishes for stationary steady states $\mathbf{U} \equiv (C, 0, 0)^T$, where $C =$ const. Notice that for these states $\mathbf{U}_{jk}(M_{jk}) \equiv \mathbf{U}_j(M_{jk}) \equiv (C, 0, 0)^T$, $\forall j, k$. After a substitution of a stationary steady state into (2.5) and taking into account that in this case, $a_{jk}^{\rm in} = a_{jk}^{\rm out}$, see (2.27), the source quadrature should satisfy the following two conditions:

$$-\frac{g}{|T_j|}\sum_{k=1}^{3}\ell_{jk}\cos(\theta_{jk})\frac{(C-B(M_{jk}))^2}{2} + \overline{\mathbf{S}}_j^{(2)} = 0 \tag{2.13}$$

and

$$-\frac{g}{|T_j|}\sum_{k=1}^{3}\ell_{jk}\sin(\theta_{jk})\frac{(C-B(M_{jk}))^2}{2}+\overline{\mathbf{S}}_j^{(3)}=0, \tag{2.14}$$

where $\overline{\mathbf{S}}_j=(0,\overline{\mathbf{S}}_j^{(2)},\overline{\mathbf{S}}_j^{(3)})$,

$$\overline{\mathbf{S}}_j^{(2)}\approx-\frac{g}{|T_j|}\int\limits_{T_j}(C-B(x,y))B_x(x,y)\,dxdy,$$

and

$$\overline{\mathbf{S}}_j^{(3)}\approx-\frac{g}{|T_j|}\int\limits_{T_j}(C-B(x,y))B_y(x,y)\,dxdy.$$

To derive the desired quadrature, we first apply Green's formula,

$$\int\limits_{T_j}\operatorname{div}\vec{\mathcal{G}}\,dxdy=\int\limits_{\partial T_j}\vec{\mathcal{G}}\cdot\vec{n}\,ds,$$

to the vector field $\vec{\mathcal{G}}=(\frac{1}{2}(w(x,y)-B(x,y))^2,0)$ and obtain:

$$\begin{aligned}-\int\limits_{T_j}(w(x,y)-B(x,y))B_x(x,y)\,dxdy &= \sum_{k=1}^{3}\int\limits_{(\partial T_j)_k}\frac{(w(x,y)-B(x,y))^2}{2}\cos(\theta_{jk})\,ds\\ &- \int\limits_{T_j}(w(x,y)-B(x,y))w_x(x,y)\,dxdy,\end{aligned} \tag{2.15}$$

where $(\partial T_j)_k$ is the k-th side of the triangle T_j, $k=1,2,3$. Next, we apply the midpoint rule to the integrals on the RHS of (2.15) and arrive at the following quadrature for the cell average $\overline{\mathbf{S}}_j^{(2)}$:

$$-\frac{g}{|T_j|}\int\limits_{T_j}(w-B)B_x\,dxdy\approx\frac{g}{2|T_j|}\sum_{k=1}^{3}\ell_{jk}(w(M_{jk})-B(M_{jk}))^2\cos(\theta_{jk})-gw_x(x_j,y_j)(\overline{w}_j-\overline{B}_j), \tag{2.16}$$

where

$$\overline{B}_j:=\frac{1}{|T_j|}\int\limits_{T_j}B(x,y)\,dxdy.$$

Likewise, we obtain the quadrature for the cell average $\overline{\mathbf{S}}_j^{(3)}$:

$$-\frac{g}{|T_j|}\int\limits_{T_j}(w-B)B_y\,dxdy\approx\frac{g}{2|T_j|}\sum_{k=1}^{3}\ell_{jk}(w(M_{jk})-B(M_{jk}))^2\sin(\theta_{jk})-gw_y(x_j,y_j)(\overline{w}_j-\overline{B}_j). \tag{2.17}$$

Notice that since $w_x \equiv w_y \equiv 0$ for $w \equiv \text{const}$, the quadratures (2.16)–(2.17) satisfy (2.13)–(2.14) when $\mathbf{U} \equiv (C,0,0)^T$.

We then replace B with its continuous piecewise linear interpolant $\widetilde{B}$. Since the interpolant (2.11) is second order accurate for smooth B, this replacement does not affect the (formal) order of both the central-upwind fluxes in (2.5) and the quadratures (2.16)–(2.17). Finally, the discretization of the source term in (2.5) becomes:

$$\begin{aligned}
\overline{\mathbf{S}}_j^{(2)} &= \frac{g}{2|T_j|}\sum_{k=1}^{3} \ell_{jk}(w_j(M_{jk}) - B_{jk})^2 \cos(\theta_{jk}) - g(w_x)_j(\overline{w}_j - B_j),\\
\overline{\mathbf{S}}_j^{(3)} &= \frac{g}{2|T_j|}\sum_{k=1}^{3} \ell_{jk}(w_j(M_{jk}) - B_{jk})^2 \sin(\theta_{jk}) - g(w_y)_j(\overline{w}_j - B_j),
\end{aligned} \tag{2.18}$$

where $B_{jk} = \widetilde{B}(M_{jk})$ and B_j is given by (2.12).

Remark. The well-balanced quadrature (2.18) is much simpler than the well-balanced source term discretization proposed in [6].

2.3 Positivity Preserving Reconstruction for w

In this section, we describe an algorithm that guarantees positivity of the reconstructed values of the water depth $h_j(M_{jk})$, $k = 1,2,3$, for all j, which are obtained from the corresponding point values of $\widetilde{w}$ (obtained with the help of the minmod-type reconstruction, described in the beginning of §2) and $\widetilde{B}$:

$$h_j(M_{jk}) := w_j(M_{jk}) - B_{jk}, \quad k = 1,2,3. \tag{2.19}$$

Notice that (see the proof of Theorem 2.1) in order to show that the scheme is positivity preserving, one only needs to verify that $h_j(M_{jk}) \ge 0$. None of the aforementioned second-order piecewise linear (or even the first-order piecewise constant) reconstructions for w can guarantee this since it may obviously happen that the cell average $\overline{w}_j > B_j$, but $\overline{w}_j < B_{jk}$ for some k. Therefore, we have to correct the original reconstruction $\widetilde{w}$ so that we ensure the nonnegativity of $h_j(M_{jk})$ for $k = 1,2,3$. There are numerous correction procedures that would guarantee that. The one that we have selected and describe below has less oscillations than some other choices, but we have not conducted a complete study of the dependence of the numerical solution on the the various corrections. Our proposed algorithm modifies $\widetilde{w}$ so that we achieve $\widetilde{w}(x,y) \ge \widetilde{B}(x,y)$ throughout the entire computational domain. The reconstruction $\widetilde{w}$ should be corrected only in those triangles, where $\widetilde{w}(\tilde{x}_{j_\kappa}, \tilde{y}_{j_\kappa}) < \mathcal{B}_{j_\kappa}$ for some κ, $\kappa = 12, 23, 13$. Since $\overline{w}_j \ge B_j$, it is impossible to have $\widetilde{w}(\tilde{x}_{j_\kappa}, \tilde{y}_{j_\kappa}) < \mathcal{B}_{j_\kappa}$ for all three values of κ, that is, at all three vertices of the triangle T_j. Thus, only two cases in which a correction is needed are possible: either there are two indices κ_1 and κ_2, for which $\widetilde{w}(\tilde{x}_{j_{\kappa_1}}, \tilde{y}_{j_{\kappa_1}}) < \mathcal{B}_{j_{\kappa_1}}$ and $\widetilde{w}(\tilde{x}_{j_{\kappa_2}}, \tilde{y}_{j_{\kappa_2}}) < \mathcal{B}_{j_{\kappa_2}}$, or there is only one index κ_1, for which $\widetilde{w}(\tilde{x}_{j_{\kappa_1}}, \tilde{y}_{j_{\kappa_1}}) < \mathcal{B}_{j_{\kappa_1}}$.

In the first case, we will correct the reconstruction $\widetilde{w}$ so that $\widetilde{w}(\tilde{x}_{j_{\kappa_1}}, \tilde{y}_{j_{\kappa_1}}) = \mathcal{B}_{j_{\kappa_1}}$ and $\widetilde{w}(\tilde{x}_{j_{\kappa_2}}, \tilde{y}_{j_{\kappa_2}}) = \mathcal{B}_{j_{\kappa_2}}$. These two conditions, together with the conservation requirement for the corrected reconstruction $\widetilde{w}$, uniquely determine the following correction algorithm:

$$\begin{aligned}
&\text{if} \quad \widetilde{w}(\tilde{x}_{j_{\kappa_1}}, \tilde{y}_{j_{\kappa_1}}) < \mathcal{B}_{j_{\kappa_1}} \quad \text{and} \quad \widetilde{w}(\tilde{x}_{j_{\kappa_2}}, \tilde{y}_{j_{\kappa_2}}) < \mathcal{B}_{j_{\kappa_2}},\\
&\text{set} \quad \widetilde{w}(\tilde{x}_{j_{\kappa_1}}, \tilde{y}_{j_{\kappa_1}}) := \mathcal{B}_{j_{\kappa_1}} \quad \text{and} \quad \widetilde{w}(\tilde{x}_{j_{\kappa_2}}, \tilde{y}_{j_{\kappa_2}}) := \mathcal{B}_{j_{\kappa_2}},
\end{aligned} \tag{2.20}$$

and replace the linear function, originally reconstructed over the triangle T_j, with a new function (still denoted by $\widetilde{w}$) defined by

$$\begin{vmatrix} x-x_j & y-y_j & \widetilde{w}(x,y)-\overline{w}_j \\ \tilde{x}_{j_{\kappa_1}}-x_j & \tilde{y}_{j_{\kappa_1}}-y_j & \mathcal{B}_{j_{\kappa_1}}-\overline{w}_j \\ \tilde{x}_{j_{\kappa_2}}-x_j & \tilde{y}_{j_{\kappa_2}}-y_j & \mathcal{B}_{j_{\kappa_2}}-\overline{w}_j \end{vmatrix}=0, \quad (x,y)\in T_j. \tag{2.21}$$

Note that the corrected reconstruction is the restriction over T_j of the plane that passes through the three points with coordinates $(\tilde{x}_{j_{\kappa_1}},\tilde{y}_{j_{\kappa_1}},\mathcal{B}_{j_{\kappa_1}})$, $(\tilde{x}_{j_{\kappa_2}},\tilde{y}_{j_{\kappa_2}},\mathcal{B}_{j_{\kappa_2}})$, and $(x_j,y_j,\overline{w}_j)$.

In the second case, we will only need to make sure that after the correction $\widetilde{w}(\tilde{x}_{j_{\kappa_1}},\tilde{y}_{j_{\kappa_1}})=\mathcal{B}_{j_{\kappa_1}}$, while we still have $\widetilde{w}(\tilde{x}_{j_{\kappa_2}},\tilde{y}_{j_{\kappa_2}})\ge\mathcal{B}_{j_{\kappa_2}}$ and $\widetilde{w}(\tilde{x}_{j_{\kappa_3}},\tilde{y}_{j_{\kappa_3}})\ge\mathcal{B}_{j_{\kappa_3}}$. This leaves one degree of freedom in the construction of the corrected linear piece over the triangle T_j. To minimize the oscillations, we decide to determine $\widetilde{w}(\tilde{x}_{j_{\kappa_2}},\tilde{y}_{j_{\kappa_2}})$ and $\widetilde{w}(\tilde{x}_{j_{\kappa_3}},\tilde{y}_{j_{\kappa_3}})$ from the relation

$$\widetilde{w}(\tilde{x}_{j_{\kappa_2}},\tilde{y}_{j_{\kappa_2}})-\mathcal{B}_{j_{\kappa_2}}=\widetilde{w}(\tilde{x}_{j_{\kappa_3}},\tilde{y}_{j_{\kappa_3}})-\mathcal{B}_{j_{\kappa_3}}=\frac{3}{2}(\overline{w}_j-B_j), \tag{2.22}$$

and thus replace the original reconstruction over the triangle T_j, with a new one (still denoted by $\widetilde{w}$) defined by

$$\begin{vmatrix} x-x_j & y-y_j & \widetilde{w}(x,y)-\overline{w}_j \\ \tilde{x}_{j_{\kappa_1}}-x_j & \tilde{y}_{j_{\kappa_1}}-y_j & \mathcal{B}_{j_{\kappa_1}}-\overline{w}_j \\ \tilde{x}_{j_{\kappa_2}}-x_j & \tilde{y}_{j_{\kappa_2}}-y_j & W-\overline{w}_j \end{vmatrix}=0, \quad (x,y)\in T_j, \tag{2.23}$$

where $W=\frac{3}{2}(\overline{w}_j-B_j)+\mathcal{B}_{j_{\kappa_2}}$. The corrected reconstruction is the restriction over T_j of the plane that passes through the three points with coordinates $(\tilde{x}_{j_{\kappa_1}},\tilde{y}_{j_{\kappa_1}},\mathcal{B}_{j_{\kappa_1}})$, $(\tilde{x}_{j_{\kappa_2}},\tilde{y}_{j_{\kappa_2}},W)$, and $(x_j,y_j,\overline{w}_j)$.

The correction procedure (2.20)–(2.23) guarantees that the reconstruction of w is conservative and its values are greater or equal to the corresponding values of $\widetilde{B}$ over the whole triangle T_j. Hence the point values of the water height, defined by (2.19), will be nonnegative.

Equipped with the positivity preserving reconstruction $\widetilde{w}$, we now proceed with the computation of the velocities u and v, and the one-sided local speeds needed in (2.5). Since the obtained values of h may be very small (or even zero), we calculate the velocities the same way as in [23], namely (we omit the j, k, indexes):

$$u=\frac{\sqrt{2}\,h\,(hu)}{\sqrt{h^4+\max(h^4,\varepsilon)}}, \qquad v=\frac{\sqrt{2}\,h\,(hv)}{\sqrt{h^4+\max(h^4,\varepsilon)}}, \tag{2.24}$$

where ε is a prescribed tolerance (we have taken $\varepsilon=\max\limits_j\{|T_j|^2\}$ in all our computations). After evaluating h, u, and v, we recompute the x- and y-discharges and fluxes accordingly, that is, we set:

$$\begin{aligned} &(hu):=h\cdot u, \qquad (hv):=h\cdot v, \\ &\mathbf{F}(\mathbf{U},B):=\left(hu, hu\cdot u+\frac{1}{2}g(w-B)^2, hu\cdot v\right)^T, \\ &\mathbf{G}(\mathbf{U},B):=\left(hv, hv\cdot u, hv\cdot v+\frac{1}{2}g(w-B)^2\right)^T, \end{aligned} \tag{2.25}$$

at the points where these quantities are to be calculated. As noted in [23], this is an important step that allows us to preserve the positivity of the fluid depth h (see Theorem 2.1).

Finally, we denote by $u^\theta_j(M_{jk})$ and $u^\theta_{jk}(M_{jk})$ the normal velocities at the point M_{jk}:

$$u^\theta_j(M_{jk}) := \cos(\theta_{jk})u_j(M_{jk}) + \sin(\theta_{jk})v_j(M_{jk}),\ \ u^\theta_{jk}(M_{jk}) := \cos(\theta_{jk})u_{jk}(M_{jk}) + \sin(\theta_{jk})v_{jk}(M_{jk}), \tag{2.26}$$

and write the formulae for the local one-sided speeds of propagation:

$$\begin{aligned} a^{\rm out}_{jk} &= \max\left\{u^\theta_j(M_{jk}) + \sqrt{gh_j(M_{jk})}\,,\, u^\theta_{jk}(M_{jk}) + \sqrt{gh_{jk}(M_{jk})}\,, 0\right\},\\ a^{\rm in}_{jk} &= -\min\left\{u^\theta_j(M_{jk}) - \sqrt{gh_j(M_{jk})}\,,\, u^\theta_{jk}(M_{jk}) - \sqrt{gh_{jk}(M_{jk})}\,, 0\right\}. \end{aligned} \tag{2.27}$$

2.4 Positivity Preserving Property of the Scheme

In this section, we prove the positivity preserving property of our new well-balanced central-upwind scheme for triangular grids in the case when the system of ODEs (2.5) is discretized in time, using the forward Euler method or a higher-order SSP ODE solver, [12]. The following theorem holds.

Theorem 2.1 *Consider the system* (2.1)–(2.4) *and the central-upwind semi-discrete scheme* (2.5)–(2.7), (2.18), (2.20)–(2.23), (2.27). *Assume that the system of ODEs* (2.5) *is solved by the forward Euler method and that for all* j, $\overline{w}^n_j - B_j \ge 0$ *at time* $t = t^n$. *Then, for all* j, $\overline{w}^{n+1}_j - B_j \ge 0$ *at time* $t = t^{n+1} = t^n + \Delta t$, *provided that* $\Delta t \le \dfrac{1}{6a}\min\limits_{j,k}\{r_{jk}\}$, *where* $a := \max\limits_{j,k}\{a^{\rm out}_{jk}, a^{\rm in}_{jk}\}$ *and* r_{jk}, $k = 1, 2, 3$, *are the altitudes of triangle* T_j.

Proof: We write the first component in equation (2.5) together with the forward Euler temporal discretization as:

$$\begin{aligned} \overline{w}^{n+1}_j = \overline{w}^n_j \; &- \frac{\Delta t}{|T_j|}\sum_{k=1}^{3}\frac{\ell_{jk}\cos(\theta_{jk})}{a^{\rm in}_{jk} + a^{\rm out}_{jk}}\left[a^{\rm in}_{jk}(hu)_{jk}(M_{jk}) + a^{\rm out}_{jk}(hu)_j(M_{jk})\right]\\ &- \frac{\Delta t}{|T_j|}\sum_{k=1}^{3}\frac{\ell_{jk}\sin(\theta_{jk})}{a^{\rm in}_{jk} + a^{\rm out}_{jk}}\left[a^{\rm in}_{jk}(hv)_{jk}(M_{jk}) + a^{\rm out}_{jk}(hv)_j(M_{jk})\right]\\ &+ \frac{\Delta t}{|T_j|}\sum_{k=1}^{3}\ell_{jk}\frac{a^{\rm in}_{jk}a^{\rm out}_{jk}}{a^{\rm in}_{jk} + a^{\rm out}_{jk}}\Big[w_{jk}(M_{jk}) - w_j(M_{jk})\Big], \end{aligned} \tag{2.28}$$

where all quantities on the RHS of (2.28) are evaluated at time level $t = t^n$. Since the piecewise linear interpolant $\widetilde{B}$ of the bottom topography function is continuous, (2.19) implies that

$$w_{jk}(M_{jk}) - w_j(M_{jk}) = h_{jk}(M_{jk}) - h_j(M_{jk}). \tag{2.29}$$

Moreover, (2.12), (2.19), and the fact that $\overline{w}^n_j = \frac{1}{3}\sum\limits_{k=1}^{3} w_j(M_{jk})$ give:

$$\overline{w}^n_j - B_j = \frac{1}{3}\sum_{k=1}^{3} h_j(M_{jk}). \tag{2.30}$$

Using (2.29)–(2.30), subtracting B_j from both sides of (2.28), and using the notation (2.26), we arrive at:

$$\begin{aligned}\overline{h}_j^{n+1} &= \frac{\Delta t}{|T_j|}\sum_{k=1}^{3} h_{jk}(M_{jk})\frac{\ell_{jk}a_{jk}^{\rm in}}{a_{jk}^{\rm in}+a_{jk}^{\rm out}}\left[a_{jk}^{\rm out}-u_{jk}^{\theta}(M_{jk})\right]\\ &+ \sum_{k=1}^{3} h_j(M_{jk})\left(\frac{1}{3}-\frac{\Delta t}{|T_j|}\cdot\frac{\ell_{jk}a_{jk}^{\rm out}}{a_{jk}^{\rm in}+a_{jk}^{\rm out}}\left[a_{jk}^{\rm in}+u_j^{\theta}(M_{jk})\right]\right),\end{aligned}\qquad(2.31)$$

where, as mentioned in (2.25), we have used the fact that $(hu)=h\cdot u$ and $(hv)=h\cdot v$. Next, from the definitions of the local speeds (2.27) we obtain that $a_{jk}^{\rm out}\ge u_{jk}^{\theta}(M_{jk})$ and therefore, all terms in the first sum on the RHS of (2.31) are nonnegative since the corrected reconstruction for w guarantees that $h_{jk}(M_{jk})\ge 0$ for all j and $k=1,2,3$. We also obtain:

$$\frac{\Delta t}{|T_j|}\cdot\frac{\ell_{jk}a_{jk}^{\rm out}}{a_{jk}^{\rm in}+a_{jk}^{\rm out}}\left[a_{jk}^{\rm in}+u_j^{\theta}(M_{jk})\right]\le\frac{\Delta t}{|T_j|}\cdot\frac{\ell_{jk}a_{jk}^{\rm out}}{a_{jk}^{\rm in}+a_{jk}^{\rm out}}\left[a_{jk}^{\rm in}+a_{jk}^{\rm out}\right]\le\frac{1}{3},$$

provided $\Delta t\le\frac{1}{3\alpha}\max_{j,k}\{\frac{|T_j|}{\ell_{jk}}\}$, where $\alpha:=\max_{j,k}\{a_{jk}^{\rm out}\}$. From (2.10) and the fact that $|T_j|=0.5r_{jk}\ell_{jk}$, we conclude that all terms in the second sum on the RHS of (2.31) are also nonnegative as long as $\Delta t<\dfrac{1}{6a}\min_{jk}\{r_{jk}\}$, $a:=\max_{j,k}\{a_{jk}^{\rm out},a_{jk}^{\rm in}\}$, since $h_j(M_{jk})\ge 0$ for all j and $k=1,2,3$. This completes the proof of the theorem. □

Remark. Theorem 2.1 is still valid if one uses a higher-order SSP ODE solver (either the Runge-Kutta or the multistep one), because such solvers can be written as a convex combination of several forward Euler steps.

3 Numerical Experiments

We test our well-balanced positivity preserving central-upwind scheme on several problems in which (almost) dry stationary steady states and/or their small perturbations are present. These examples clearly demonstrate the ability of the proposed scheme to accurately resolve quasi-steady states (small perturbations of stationary steady states) and, at the same time, to preserve the positivity of the fluid depth h (as proved in Theorem 2.1).

In all examples that follow, the gravitational constant is $g=1$. In Examples 1, 2, and 3, the computational domain is a rectangle, and we use the structured triangular mesh outlined in Figure 3.1, while in Example 4, where we simulate a flow in a converging-diverging channel, the mesh is unstructured, see Figure 3.11 (right).

Example 1 — Accuracy Test

The goal of this numerical example is to experimentally check the order of accuracy of the proposed central-upwind scheme. The scheme is applied to the system (2.1)–(2.4) subject to the following initial data and bottom topography:

$$w(x,y,0)=1,\quad u(x,y,0)=0.3,\quad B(x,y)=0.5\exp(-25(x-1)^2-50(y-0.5)^2).\qquad(3.1)$$

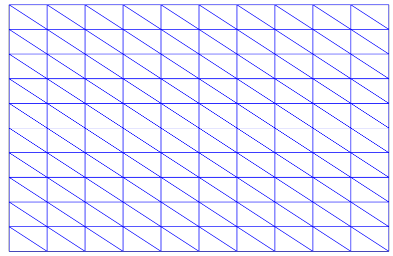

Figure 3.1: Structured triangular mesh. Horizontal to vertical cathetus ratio is 2:1.

For a reference solution, we solve this problem with our method on a 2×400×400 grid. By $t = 0.07$, the solution converges to the steady state, which is in this case nontrivial (nonstationary) but smooth. We show the water surface for the reference solution at time $t = 0.07$ in Figure 3.2. We use this reference solution to test the numerical convergence. The L^1- and L^∞-errors are shown in Table 1. The obtained errors and the rate of convergence are similar to the ones reported in [23, Table 4.1] for the 1-D problem on a uniform grid of the same size. Tests of our method on a finer mesh are prevented by the size of the problem and available computer resources. We believe that further mesh refinement would increase the rates, similar to the behavior shown in [23, Table 4.1].

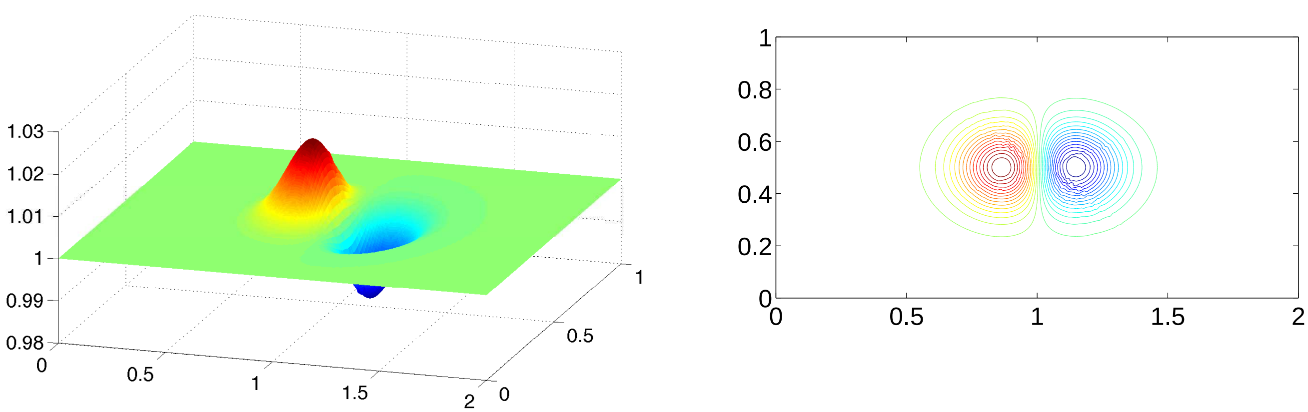

Figure 3.2: w component of the solution of the IVP (2.1)–(2.4), (3.1) on a $2\times 400 \times 400$ grid: the 3-D view (left) and the contour plot (right).

Example 2 — Small Perturbation of a Stationary Steady-State Solution

Here, we first solve the initial value problem (IVP) proposed in [27]. The computational domain is $[0, 2] \times [0, 1]$ and the bottom consists of an elliptical shaped hump:

$$B(x, y) = 0.8 \exp(-5(x - 0.9)^2 - 50(y - 0.5)^2). \tag{3.2}$$

Table 1: Example 1: L^1- and L^∞-errors and numerical orders of accuracy.

Number of cells	L^1-error	Order	L^∞-error	Order
$2 \times 50 \times 50$	6.59e-04	–	8.02e-03	–
$2 \times 100 \times 100$	2.87e-04	1.20	3.59e-03	1.16
$2 \times 200 \times 200$	1.00e-04	1.52	1.21e-03	1.57

Initially, the water is at rest and its surface is flat everywhere except for $0.05 < x < 0.15$:

$$w(x,y,0) = \begin{cases} 1+\varepsilon, & 0.05 < x < 0.15, \\ 1, & \text{otherwise}, \end{cases} \qquad u(x,y,0) \equiv v(x,y,0) \equiv 0, \tag{3.3}$$

where the perturbation height is $\varepsilon = 0.01$. Figure 3.3 displays the right-going disturbance as it propagates past the hump. The water surface, $w(x,y,t)$, is presented at times $t = 0.6, 0.9, 1.2, 1.5$ and 1.8. One can observe the high resolution of complex small features of the flow (compare with [6, 19, 27]).

We then modify the initial data (3.3) by taking a very small perturbation height $\varepsilon = 10^{-14}$, which is comparable with the machine error, and numerically verify property (i) of the proposed central-upwind scheme. In Figure 3.4, we plot $\max\limits_{x,y}(w-1)$ as a function of t, computed on a very coarse $2 \times 10 \times 10$ mesh, on time. As one can clearly see, no instabilities are developed and the balance between the fluxes and the geometric source terms is preserved numerically.

Next, we modify the IVP (2.1)–(2.4), (3.2), (3.3) to numerically study the case of a submerged flat plateau, see Figure 3.5 (left). Notice that the plateau is very close to the water surface, and that the initial water depth over the plateau is equal to the perturbation height $\varepsilon = 0.01$. The computational domain is now $[-0.5, 0.5] \times [-0.5, 0.5]$, the radially symmetric bottom topography is given by

$$B(r) = \begin{cases} 0.99, & r \le 0.1, \\ 9.9(0.2 - r), & 0.1 < r < 0.2, \\ 0, & \text{otherwise}, \end{cases} \tag{3.4}$$

where $r := \sqrt{x^2 + y^2}$, and the initial data are:

$$w(x,y,0) = \begin{cases} 1+\varepsilon, & -0.4 < x < -0.3, \\ 1, & \text{otherwise}, \end{cases} \qquad u(x,y,0) \equiv v(x,y,0) \equiv 0. \tag{3.5}$$

The solution, computed on two different grids, is shown at both early stages of the disturbance propagation, see Figure 3.6, and at later times, see Figure 3.7. Since the area over the plateau is almost dry, the right-going disturbance mostly bends around that area, while only small portion of the wave propagates over the area. As one can see from Figures 3.6 and 3.7, the general structure of the solution is well resolved on the coarser mesh, while a finer mesh is clearly needed to achieve high resolution of the solution over the plateau area. Notice that the positivity of h is preserved and no instabilities are developed at the almost dry area.

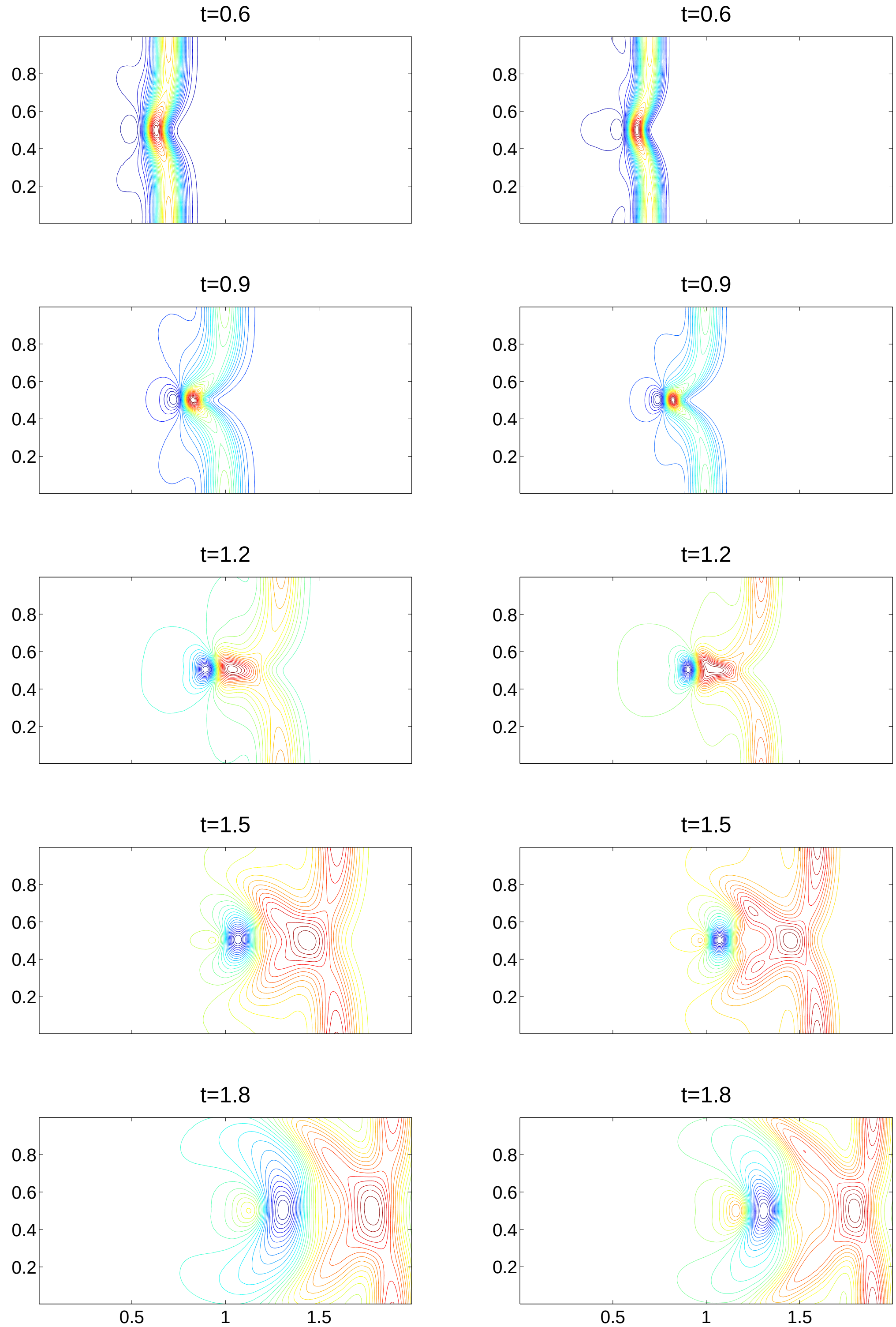

Figure 3.3: w component of the solution of the IVP (2.1)–(2.4), (3.2), (3.3) with $\varepsilon = 0.01$ on $2\times 200\times 200$ (left column) and $2\times 400\times 400$ (right column) grids.

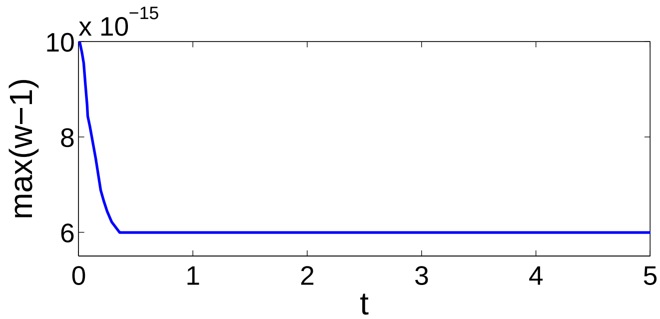

Figure 3.4: $\max_{x,y}(w-1)$ as a function of t, where w is the solution of the IVP (2.1)–(2.4), (3.2), (3.3) with $\varepsilon = 10^{-14}$.

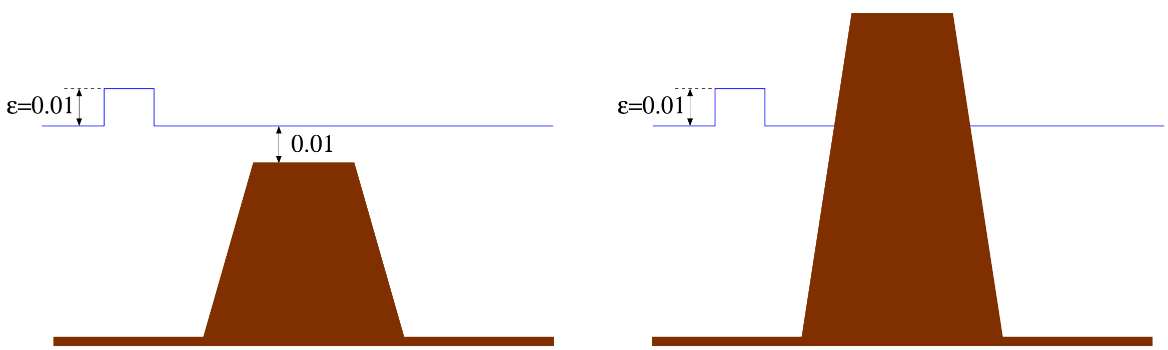

Figure 3.5: 1-D slice of the bottom topographies (3.4), left, and (3.6), right. These plots are not to scale.

In the final part of Example 2, we consider the situation in which the bottom hump is above the water surface so that there is a disk-shaped island at the origin, see Figure 3.5 (right):

$$B(r) = \begin{cases} 1.1, & r \le 0.1, \\ 11(0.2-r), & 0.1 < r < 0.2, \\ 0, & \text{otherwise}. \end{cases} \tag{3.6}$$

The computational domain is $[-0.5, 0.5] \times [-0.5, 0.5]$ and the initial data are given by

$$w(x,y,0) = \begin{cases} 1+\varepsilon, & -0.4 < x < -0.3, \\ \max\left\{1, B(\sqrt{x^2+y^2})\right\}, & \text{otherwise}, \end{cases} \qquad u(x,y,0) \equiv v(x,y,0) \equiv 0. \tag{3.7}$$

As one can see from Figure 3.8, where the computed water surface w is shown, the right-going disturbance bends around the island while the general solution structure is quite similar to the one obtained in the submerged plateau case. The major difference is that completely dry states and states that change their status back and forth between dry and almost dry (at the areas around the island) are now present. Nevertheless, the solution obtained by the proposed

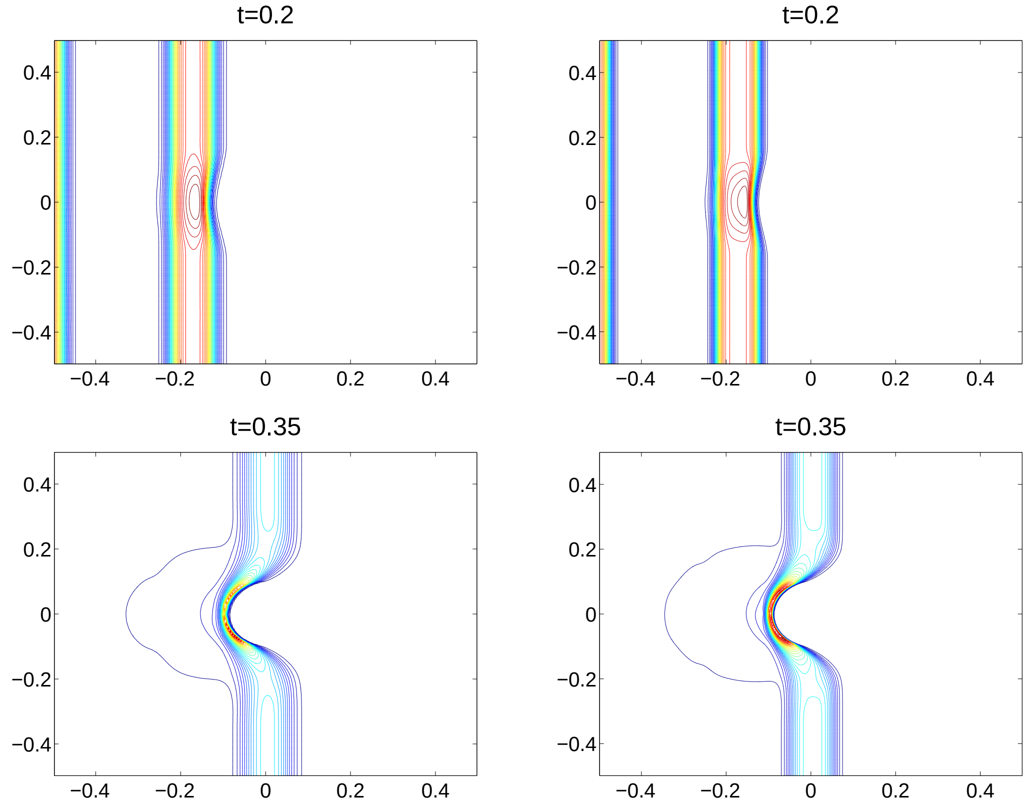

Figure 3.6: w component of the solution of the IVP (2.1)–(2.4), (3.4), (3.5) at small times on $2\times200\times200$ (left column) and $2\times400\times400$ (right column) grids.

positivity preserving central-upwind scheme remains consistent and stable. This demonstrates the robustness of our method.

Example 3 — Saint-Venant System with Friction and Discontinuous Bottom

It is a well-known fact that more realistic shallow water models based on the Saint-Venant system (1.1) should include additional friction and/or viscosity terms. In [10], such models were derived from the Navier-Stokes equations for incompressible flows with a free moving boundary. Presence of friction and viscosity terms guarantees uniqueness of the steady state solution, especially in the case when the fluid propagates into a certain region and gradually occupies parts of initially dry areas, as, for example, in Figure 3.10.

We consider the simplest model in which only friction terms, $-\kappa(h)u$ and $-\kappa(h)v$, are added to the RHS of the second and third equations in (1.1):

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0, \\ (hu)_t + \left(hu^2 + \dfrac{1}{2}gh^2\right)_x + (huv)_y = -ghB_x - \kappa(h)u, \\ (hv)_t + (huv)_x + \left(hv^2 + \dfrac{1}{2}gh^2\right)_y = -ghB_y - \kappa(h)v. \end{cases} \tag{3.8}$$

We numerically solve the system (3.8) on the domain $[-0.25, 1.75] \times [-0.5, 0.5]$, assuming that

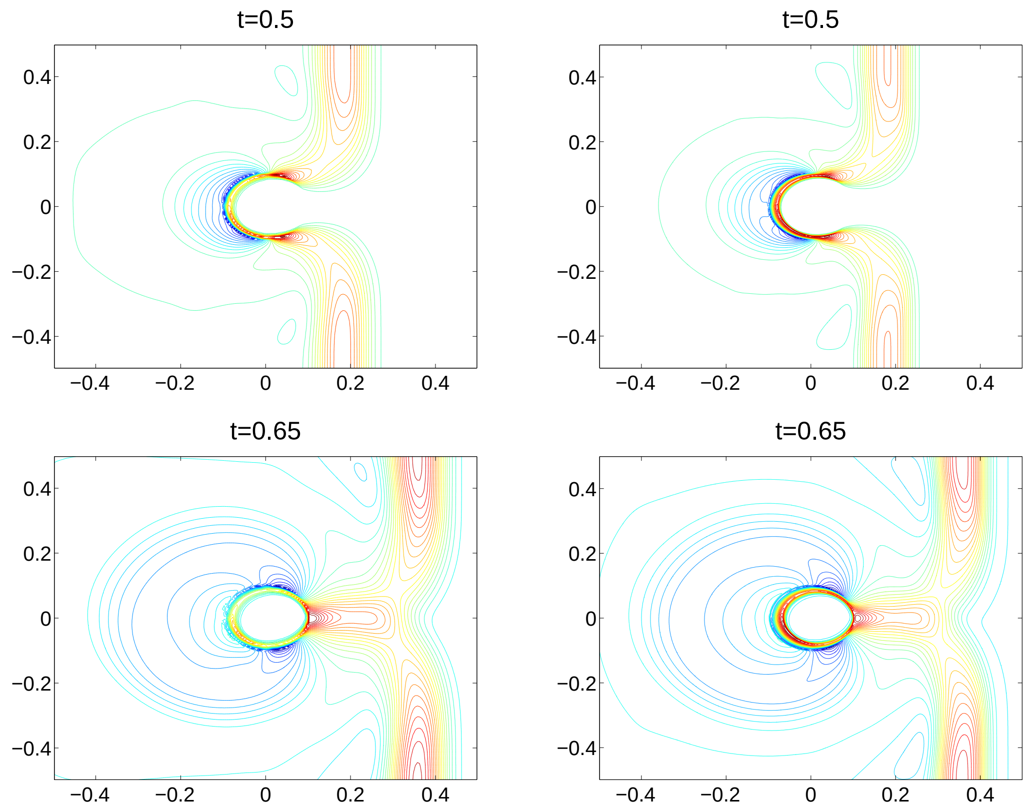

Figure 3.7: w component of the solution of the IVP (2.1)–(2.4), (3.4), (3.5) at larger times on $2\times 200\times 200$ (left column) and $2\times 400\times 400$ (right column) grids.

the friction coefficient is $\kappa(h) = 0.001(1 + 10h)^{-1}$, and the bottom topography function has a discontinuity along the vertical line $x = 1$ and is given by:

$$B(x,y) = 100y^4 + \begin{cases} 1, & x < 0, \\ \cos^2(\pi x), & 0 \le x \le 0.4, \\ \cos^2(\pi x) + 0.28(\cos(10\pi(x-0.5)) + 1), & 0.4 \le x \le 0.5, \\ 0.5\cos^4(\pi x) + 0.28(\cos(10\pi(x-0.5)) + 1), & 0.5 \le x \le 0.6, \\ 0.5\cos^4(\pi x), & 0.5 \le x < 1, \\ 0.28\sin(2\pi(x-1)), & 1 < x \le 1.5, \\ 0, & x > 1.5. \end{cases} \tag{3.9}$$

This topography B mimics a mountain river valley, which, together with the surrounding mountains, is shown in Figure 3.9. We take the following initial data:

$$w(x,y,0) = \begin{cases} \max\Big\{1.8, B(x,y)\Big\}, & x < 0, \\ B(x,y), & x > 0, \end{cases} \qquad u(x,y,0) \equiv v(x,y,0) \equiv 0, \tag{3.10}$$

and implement solid wall boundary conditions. These data correspond to the situation when the second of the three dams, initially located at the vertical lines $x = -0.25$ (the left boundary

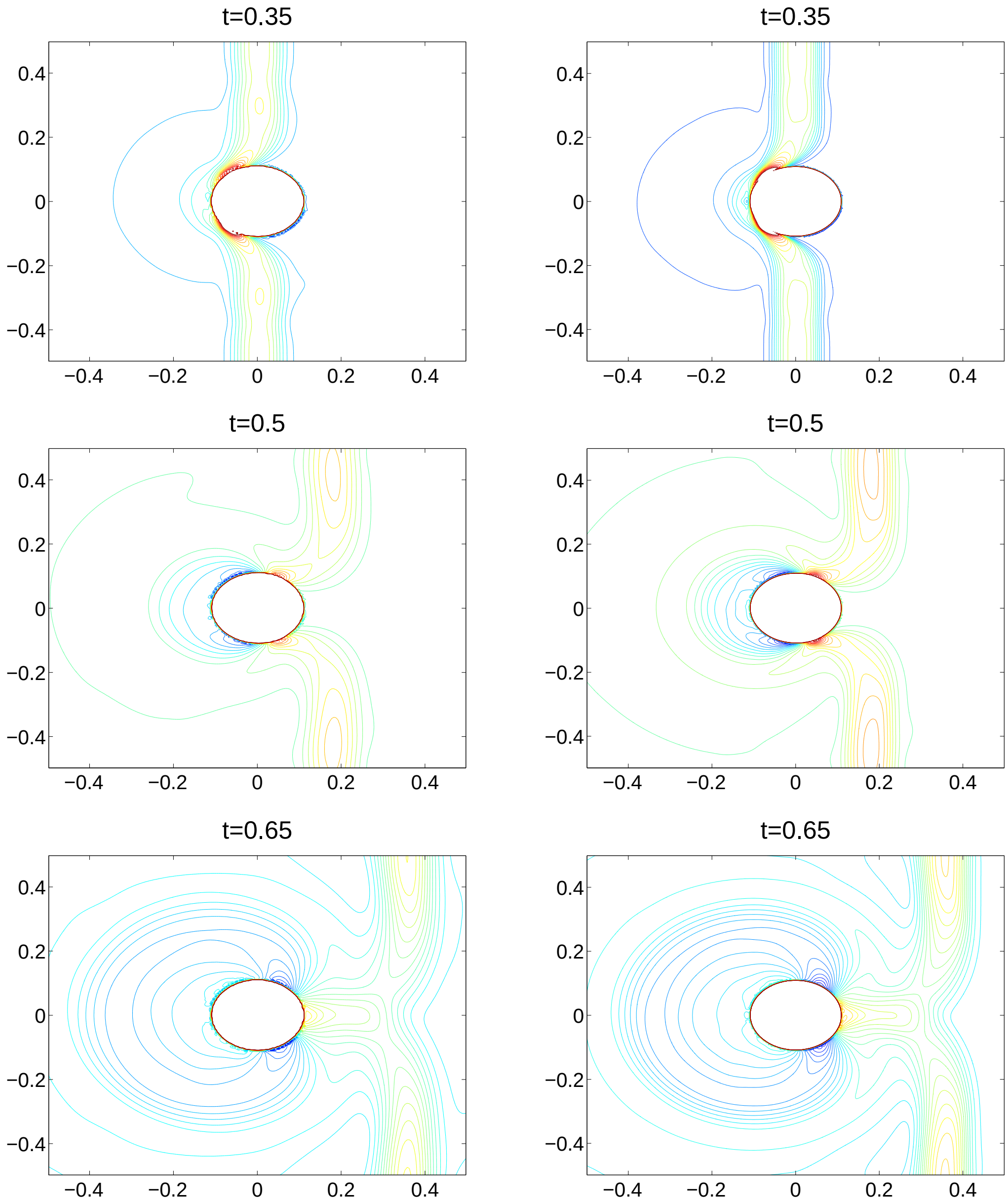

Figure 3.8: w component of the solution of the IVP (2.1)–(2.4), (3.6), (3.7) on $2\times200\times200$ (left column) and $2\times400\times400$ (right column) grids.

of the computational domain), $x = 0$, and $x = 1.75$ (the right boundary of the computational domain), breaks down at time $t = 0$, the water propagates into the initially dry area $x > 0$, and a stationary steady state is achieved after a certain period of time (this problem is a modification of the 1-D test problem from [23]).

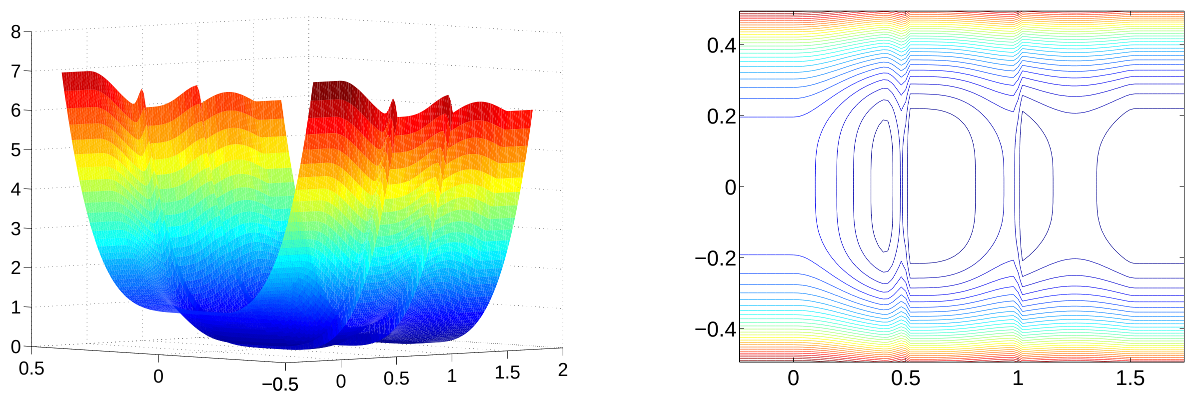

Figure 3.9: River valley topography — three-dimensional view (left) and contour plot (right).

We apply the well-balanced positivity preserving central-upwind scheme to this initial-boundary value problem (IBVP). The friction term in (3.8) is discretized in a straightforward manner. Notice that this affects neither the well-balanced (since $u \equiv v \equiv 0$ at stationary steady states) nor the positivity preserving (since the first equation has not been changed) properties of our scheme. The solution of the IBVP (3.8)–(3.10) at times $t = 0.3$, 1, 4 and 7 is computed using two different grids. At later times, the computed solution almost does not change, thus the solution at time $t = 7$ can be viewed as a numerical stationary steady state. The solution (the water depth h) is shown in Figure 3.10, where one can clearly see the dynamics of the fluid flow as it moves from the region $x < 0$ into the initially dry area $x > 0$ and gradually settles down into a stationary steady state. Notice that this state includes dry areas and therefore its computation requires a method that is both well-balanced and positivity preserving on the entire computational domain.

Example 4 — Flows in Converging-Diverging Channels

In the last example, borrowed from [14] (see also [6]), we study water flow in open converging-diverging channel of length 3 with symmetric constrictions of length 1 at the center. The exact geometry of each channel is determined by its breadth, which is equal to $2y_{\rm b}(x)$, where

$$y_{\rm b}(x) = \begin{cases} 0.5 - 0.5(1-d)\cos^2(\pi(x-1.5)), & |x-1.5| \le 0.5, \\ 0.5, & \text{otherwise}, \end{cases}$$

and d is the minimum channel breadth (in our numerical experiments we take $d = 0.9$ and $d = 0.6$). Thus, the computational domain is $[0,3] \times [-y_{\rm b}(x), y_{\rm b}(x)]$, see Figure 3.11 (left). We take the following initial data:

$$w(x,y,0) = \max\Big\{1, B(x,y)\Big\}, \quad u(x,y,0) = 2, \quad v(x,y,0) = 0. \tag{3.11}$$

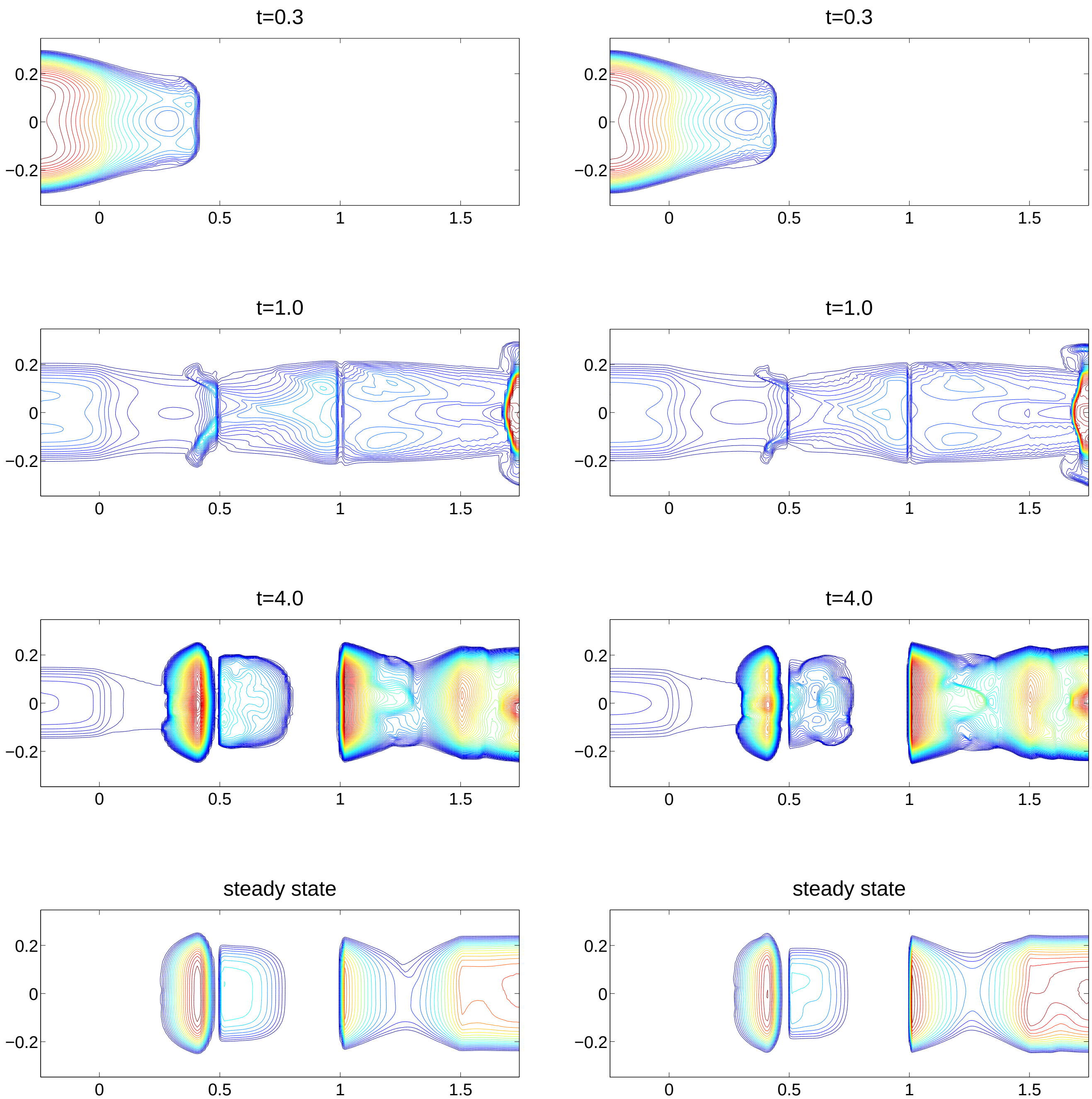

Figure 3.10: Solution (h) of the IBVP (2.1)–(2.4), (3.6), (3.7) on $2\times200\times200$ (left column) and $2\times400\times400$ (right column) grids.

In the case of a flat bottom $B(x,y) \equiv 0$, these initial data correspond to the data considered in [14] with the Froude number equals to 2. Both the upper and lower y-boundaries are reflecting (solid wall), the left x-boundary is an inflow boundary with $u = 2$ and the right x-boundary is a zero-order outflow boundary. Finally, the bottom topography, shown in Figure 3.12, is given by

$$B(x,y) = B_{\max}\left(e^{-10(x-1.9)^2-50(y-0.2)^2} + e^{-20(x-2.2)^2-50(y+0.2)^2}\right), \tag{3.12}$$

where $B_{\max}$ is a parameter.

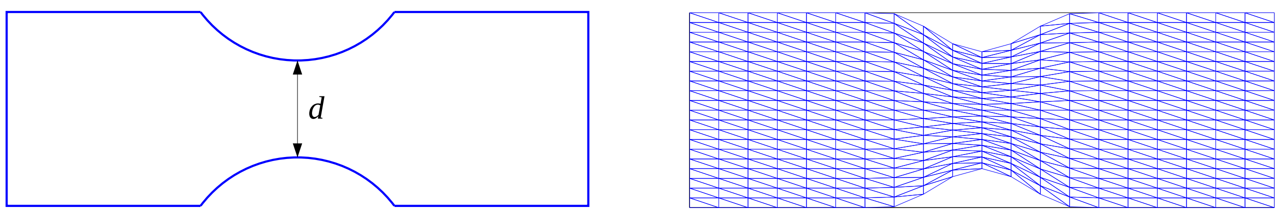

Figure 3.11: Example 4: computational domain (left) and its unstructured triangulation (right).

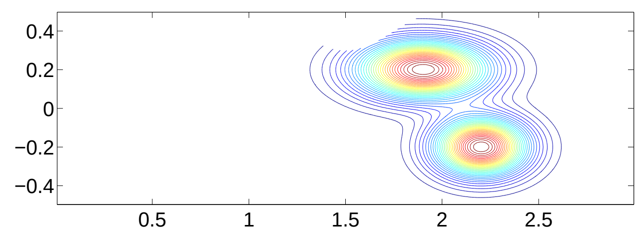

Figure 3.12: Example 4: bottom topography for $(d, B_{\max}) = (0.6, 1)$.

We apply the proposed central-upwind scheme on an unstructured triangular mesh obtained from the structured one, outlined in Figure 3.1, using the mapping

$$(x,y) \to \begin{cases} (x, (1-(1-d)\cos^2(\pi(x-1.5)))y), & |x-1.5| \le 0.5, \\ (x,y), & \text{otherwise}. \end{cases}$$

The resulting triangulation is shown in Figure 3.11 (right). We test our method on the following 4 sets of parameters: $(d, B_{\max}) = (0.9, 0)$, $(0.9, 1)$, $(0.9, 2)$ and $(0.6, 1)$. In every test, we run the simulations on $2\times 200 \times 200$ and $2\times 400 \times 400$ grids until the steady state (which is, in this case, not a stationary one) is reached at about $t = 2$. The obtained results are shown in Figures 3.13–3.16.

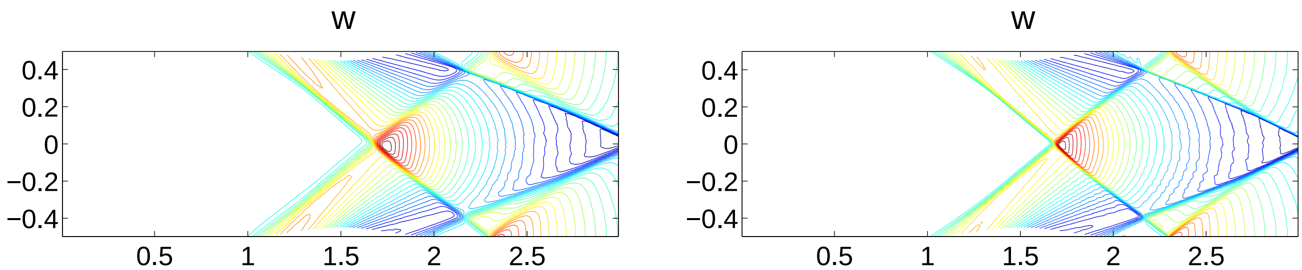

Figure 3.13: Example 4: steady-state solution (w) for $(d, B_{\max}) = (0.9, 0)$ on $2\times 200 \times 200$ (left) and $2\times 400 \times 400$ (right) grids.

We first verify that in the flat bottom case, $(d, B_{\max}) = (0.9, 0)$, the results obtained by the central-upwind scheme are in good agreement with the solution computed by an alternative finite-volume scheme, see Figure 3.13, and compare it with Figure 10 (e) in [14].

We then modify the IBVP by including two asymmetric elliptical Gaussian mounds in the bottom topography, that is, by taking $(d, B_{\max}) = (0.9, 1)$. This bottom function is similar to the one used in [6], but in our case the water depth at the top of both mounds is zero. The proposed central-upwind scheme successfully captures the complicated steady-state solution emerging in this case, see Figure 3.14.

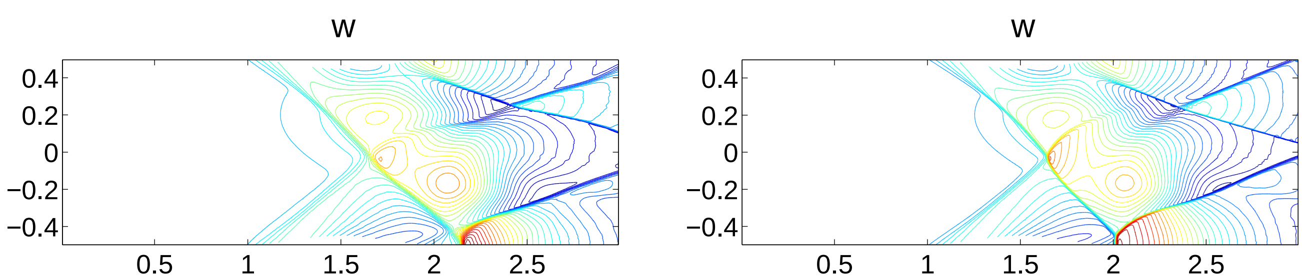

Figure 3.14: Example 4: steady-state solution (w) for $(d, B_{\max}) = (0.9, 1)$ on $2\times 200 \times 200$ (left) and $2\times 400 \times 400$ (right) grids.

Next, we increase $B_{\max}$ to 2, which models the presence of two Gaussian-shaped islands. Our scheme still exhibits a superb performance in this case, as one can see in Figure 3.15, where we show both the water surface (w) and depth (h) to better illustrate the structure of the computed steady-state solution at/near the islands.

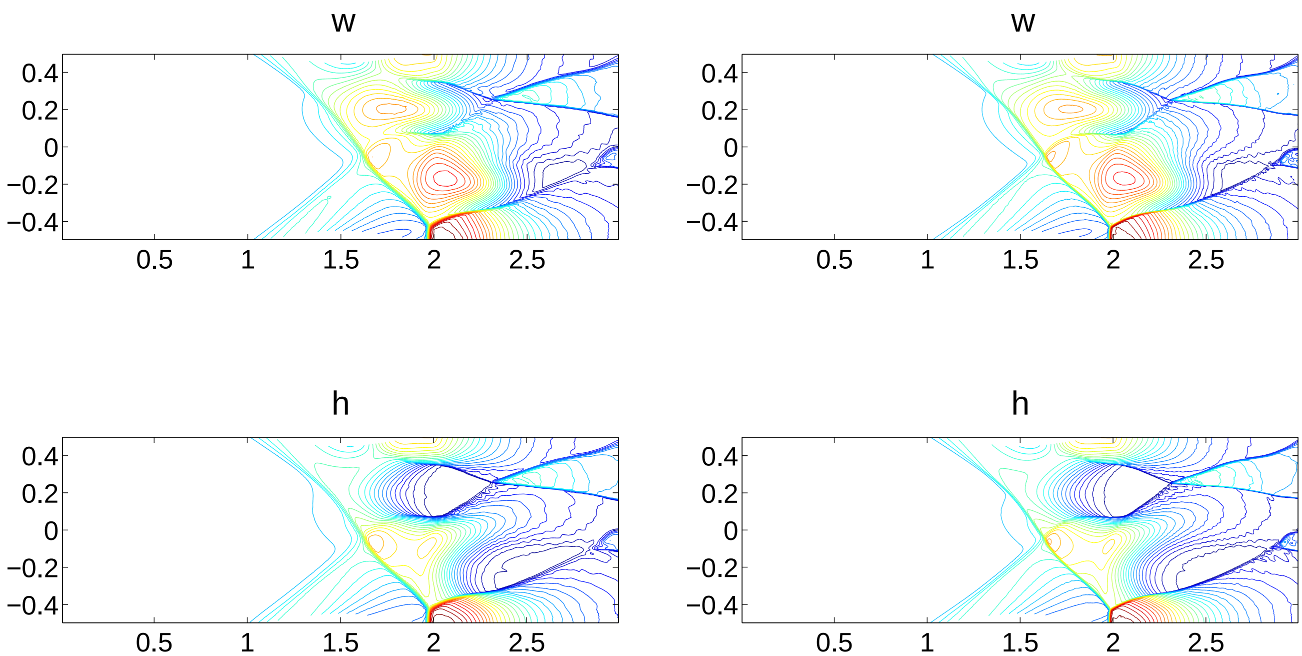

Figure 3.15: Example 4: steady-state solution (w – top, h – bottom) for $(d, B_{\max}) = (0.9, 2)$ on $2 \times 200 \times 200$ (left) and $2 \times 400 \times 400$ (right) grids.

Finally, we modify the shape of the channel by taking $d = 0.6$, and compute the steady-state solution for $B_{\max} = 1$. The results, presented in Figure 3.16, are of the same high quality as in the case of a wider channel studied above.

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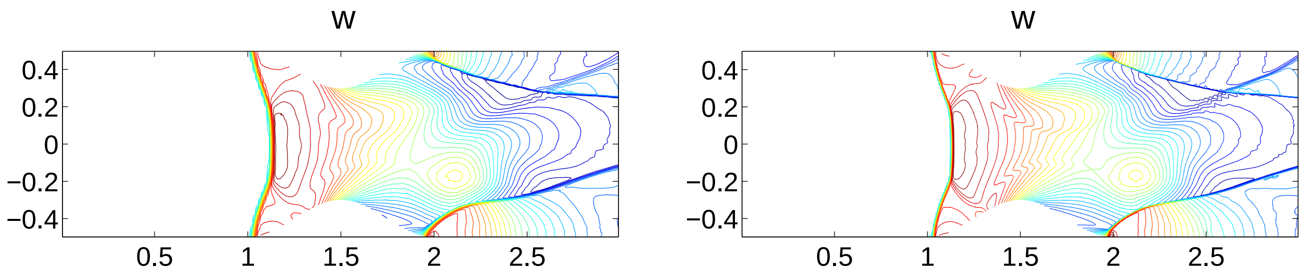

Figure 3.16: Example 4: steady-state solution (w) for $(d, B_{\max}) = (0.6, 1)$ on $2\times 200\times 200$ (left) and $2\times 400\times 400$ (right) grids.

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