KRONECKER'S SOLUTION OF PELL'S EQUATION FOR CM FIELDS

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Abstract. We generalize Kronecker's solution of Pell's equation to CM fields K whose Galois group over $\mathbb{Q}$ is an elementary abelian 2-group. This is an identity which relates CM values of a certain Hilbert modular function to products of logarithms of fundamental units. When K is imaginary quadratic, these CM values are algebraic numbers related to elliptic units in the Hilbert class field of K. Assuming Schanuel's conjecture, we show that when K has degree greater than 2 over $\mathbb{Q}$ these CM values are transcendental.

1. Introduction and statement of results

The analytic construction of solutions of certain natural Diophantine equations is a problem of central importance in number theory. One of the most remarkable examples of this is Kronecker's "solution" of Pell's equation

$$x^2 - dy^2 = \pm 1. \tag{1.1}$$

The fundamental unit ε_d in the real quadratic field $\mathbb{Q}(\sqrt{d})$ satisfies (1.1). Kronecker expressed ε_d in terms of values of the Dedekind eta function $\eta(z)$ at CM points on the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ (see the discussion below, and in particular, equation (1.5)).

In this paper we will generalize Kronecker's solution of Pell's equation to CM fields K whose Galois group over $\mathbb{Q}$ is an elementary abelian 2-group (see Theorem 1.3). This is an identity which relates values of a certain Hilbert modular function at CM points on a Hilbert modular variety to products of logarithms of fundamental units. When K is imaginary quadratic, these CM values are algebraic numbers which can be expressed as absolute values of Galois conjugates of elliptic units in the Hilbert class field of K (see [S, p. 103]). In contrast, when K has degree greater than 2 over $\mathbb{Q}$ we will show, assuming Schanuel's conjecture, that these CM values are *transcendental* (see Theorem 1.6). This result is related to interesting recent work of Murty and Murty [MM1, MM2] on transcendental values of class group L–functions for imaginary quadratic fields.

We begin by reviewing Kronecker's solution of Pell's equation. For a quadratic field $\mathbb{Q}(\sqrt{\Delta})$ of discriminant Δ, let χ_Δ be the Kronecker symbol, $L(\chi_\Delta, s)$ be the Dirichlet L–function, $h(\Delta)$ be the class number, ε_Δ be the fundamental unit, and w_Δ be the number of roots of unity. Let $K = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field of discriminant $D < -4$ (so $w_D = 2$). For an ideal class C of K, let $\tau_{\mathfrak{a}} \in \mathbb{H}$ be the CM point of discriminant D on the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ corresponding to $[\mathfrak{a}] = C^{-1}$ (here $\mathbb{H}$ is the complex upper half-plane). More precisely, if $Q(X,Y) = N(\mathfrak{a})X^2 + bXY + cY^2$ is the reduced, primitive, integral binary quadratic form of discriminant $b^2 - 4N(\mathfrak{a})c = D$ corresponding to the class

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C^{-1}, then

$$\tau_{\mathfrak{a}} = \frac{-b+\sqrt{D}}{2N(\mathfrak{a})}$$

is the unique root in $\mathbb{H}$ of the dehomogenized form $Q(X,1)$ (here $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$). Kronecker established the following "limit formula" for the constant term in the Laurent expansion of the partial Dedekind zeta function $\zeta_K(s,C)$ at $s=1$,

$$\lim_{s\to 1}\left[\zeta_K(s,C) - \frac{\pi}{\sqrt{|D|}}\frac{1}{s-1}\right] = \frac{\pi}{\sqrt{|D|}}\left(2\gamma - \log|D| - 2\log g(\tau_{\mathfrak{a}})\right), \tag{1.2}$$

where γ is Euler's constant and $g : \mathbb{H} \to \mathbb{R}^+$ is the $\mathrm{SL}_2(\mathbb{Z})$-invariant function

$$g(z) := \sqrt{(2/\sqrt{|D|})\mathrm{Im}(z)}\,|\eta(z)|^2,$$

where

$$\eta(z) = e(z/24)\prod_{n=1}^{\infty}(1-e(nz)), \quad e(z) := e^{2\pi i z}$$

is Dedekind's weight 1/2 modular form for $\mathrm{SL}_2(\mathbb{Z})$.

Let $D = D_1D_2$ be a nontrivial factorization of D into coprime fundamental discriminants $D_1 > 0$ and $D_2 < 0$. Let χ be the genus character of K corresponding to the decomposition $D = D_1D_2$ and let

$$L_K(\chi,s) = \sum_{C\in\mathrm{CL}(K)} \chi(C)\zeta_K(s,C)$$

be the L–function of χ where $\mathrm{CL}(K)$ is the ideal class group of K. Kronecker established the factorization

$$L_K(\chi,s) = L(\chi_{D_1},s)L(\chi_{D_2},s). \tag{1.3}$$

By orthogonality of group characters, one obtains from (1.2) the formula

$$L_K(\chi,1) = -\frac{2\pi}{\sqrt{|D|}}\sum_{C\in\mathrm{CL}(K)}\chi(C)\log F(\tau_{\mathfrak{a}}).$$

On the other hand, by Dirichlet's class number formula for quadratic fields one has

$$L(\chi_\Delta,1) = \begin{cases} \dfrac{2\log(\varepsilon_\Delta)h(\Delta)}{\sqrt{\Delta}}, & \text{if} \quad \Delta > 0,\\ \dfrac{2\pi h(\Delta)}{w_\Delta\sqrt{|\Delta|}}, & \text{if} \quad \Delta < 0. \end{cases} \tag{1.4}$$

Equating both sides of Kronecker's factorization (1.3) at $s=1$ yields the beautiful identity

$$-\sum_{C\in\mathrm{CL}(K)}\chi(C)\log F(\tau_{\mathfrak{a}}) = \frac{2h(D_1)h(D_2)}{w_{D_2}}\log(\varepsilon_{D_1}),$$

or equivalently

$$\prod_{C\in\mathrm{CL}(K)} F(\tau_{\mathfrak{a}})^{-\chi(C)} = \varepsilon_{D_1}^{2h(D_1)h(D_2)/w_{D_2}}. \tag{1.5}$$

The fundamental unit ε_{D_1} satisfies Pell's equation

$$x^2 - D_1 y^2 = \pm 1,$$

thus one has a "solution" of this equation in terms of the CM values $F(\tau_{\mathfrak{a}})$.

Recall that a *CM field* is a totally imaginary quadratic extension of a totally real number field. In order to generalize Kronecker's identity (1.5) to CM fields we proceed as follows. First, we evaluate the special value $L_K(\chi, 1)$ where χ is a nontrivial class group character of a CM field K (see Theorem 1.1). To do this we establish a suitable version of the Kronecker limit formula for CM fields, which relates the constant term in the Laurent expansion at $s = 1$ of $\zeta_K(s, C)$ to values of a Hilbert modular function at CM points on a Hilbert modular variety (see Theorem 4.1). Second, we identify the CM fields which possess a genus character χ whose L–function $L_K(\chi, s)$ factors as a product of quadratic Dirichlet L–functions. These are the CM fields whose Galois group over $\mathbb{Q}$ is an elementary abelian 2-group. Given such a factorization, we can evaluate $L_K(\chi, 1)$ using Dirichlet's class number formula for quadratic fields. By equating the two different evaluations of $L_K(\chi, 1)$ we will generalize (1.5).

Note that a limit formula for CM fields was established by Konno in [K]. See also the work of Asai [A], who calculated the constant term in the Laurent expansion at $s = 1$ of the real-analytic Eisenstein series associated to any number field of class number 1. Our approach to the limit formula for CM fields differs from [K]. In particular, we proceed via the Fourier expansion of the Hilbert modular Eisenstein series, which enables us to use periods of this Eisenstein series to explicitly determine the CM zero-cycles along which we evaluate the Hilbert modular function.

In order to state our results we fix the following notation. Let F be a totally real number field of degree n over $\mathbb{Q}$ with embeddings $\sigma_1, \dots, \sigma_n$ and ring of integers $\mathcal{O}_F$. Let K be a CM extension of F with a CM type Φ, and let

$$\mathcal{CM}(K, \Phi, \mathcal{O}_F) = \{z_{\mathfrak{a}} \in \mathbb{H}^n : \ [\mathfrak{a}] \in \mathrm{CL}(K)\}$$

be the zero-cycle of CM points on the Hilbert modular variety $X_F = \mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{H}^n$ (see Section 3). Let R_K, w_K and d_K be the regulator, number of roots of unity, and absolute discriminant of K, respectively.

In the following theorem we give a formula for the special value $L_K(\chi, 1)$.

Theorem 1.1. *Let F be a totally real number field of degree n over $\mathbb{Q}$ with narrow class number 1. Let K be a CM extension of F with a CM type Φ. For each class $C \in \mathrm{CL}(K)$, let $z_{\mathfrak{a}}$ be the CM point in $\mathcal{CM}(K, \Phi, \mathcal{O}_F)$ corresponding to C^{-1}. Then for each nontrivial class group character χ of K,*

$$L_K(\chi, 1) = -\frac{2^{n+1}\pi^n R_K}{w_K\sqrt{d_K}} \sum_{C \in \mathrm{CL}(K)} \chi(C) \log G(z_{\mathfrak{a}}),$$

where $G : \mathbb{H}^n \to \mathbb{R}^+$ is the $\mathrm{SL}_2(\mathcal{O}_F)$-invariant function

$$G(z) := \sqrt{\left(2^n d_F/\sqrt{d_K}\right) \prod_{i=1}^{n} \mathrm{Im}(z_i) \cdot \phi(z)^2}, \quad z = (z_1, \dots, z_n) \in \mathbb{H}^n$$

and $\phi(z)$ is the positive, real-analytic function generalizing $|\eta(z)|$ defined by (1.6).

Remark 1.2. The narrow class number 1 assumption in Theorem 1.1 can be removed by working adelically. We have worked classically throughout the paper to emphasize the parallels with Kronecker's original work.

The function $\phi(z)$ in Theorem 1.1 is defined by

$$\phi(z) := f(z)^{-\sqrt{d_F}/2\pi^n r_F}, \tag{1.6}$$

where r_F is the residue of $\zeta_F(2s-1)$ at $s = 1/2$ and

$$f(z) := \exp\left(\zeta_F(2)\prod_{i=1}^{n} y_i + \frac{\pi^n}{\sqrt{d_F}}\sum_{\tilde{a}\in\mathcal{O}_F^*}{}' \sum_{\substack{\tilde{a}=ab\\ a\in\mathcal{O}_F^*\\ b\in\mathcal{O}_F/\mathcal{O}_F^\times}} \frac{e^{-2\pi i S(aby)}}{N_{F/\mathbb{Q}}((b))} e^{2\pi i T(abx)}\right),$$

where $z = x + iy \in \mathbb{H}^n$, $\mathcal{O}_F^*$ is the dual lattice, $\mathcal{O}_F^\times$ is the unit group,

$$S(aby) = \sum_{i=1}^{n} |\sigma_i(ab)|\, y_i,$$

$$T(abx) = \sum_{i=1}^{n} \sigma_i(ab) x_i,$$

and the prime means the sum is over nonzero elements. In Proposition 4.3 we will show that $\phi(z)$ transforms like

$$\phi(Mz) = \left|\prod_{i=1}^{n}(\sigma_i(\gamma)z_i + \sigma_i(\delta))\right|^{\frac{1}{2}} \phi(z)$$

for $M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F)$.

Our main result is the following theorem generalizing Kronecker's identity (1.5).

Theorem 1.3. *Let F be a totally real number field with narrow class number 1. Let K be a CM extension of F with Gal$(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^r$ for some integer $r \geq 2$, and let E be an unramified quadratic extension of K with Gal$(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^{r+1}$. Let χ be the genus character of K arising from the extension E/K. Let Δ_i for $1 \leq i \leq 2^r$ be the discriminants of the quadratic subfields $\mathbb{Q}(\sqrt{\Delta_i})$ of E which are not contained in K and define $S_R := \{\Delta_i : \Delta_i > 0\}$ and $S_I := \{\Delta_i : \Delta_i < 0\}$. Then*

$$\prod_{C\in\mathrm{CL}(K)} G(z_{\mathfrak{a}})^{-\chi(C)} = \exp\left(\frac{\alpha}{\prod_{i=1}^{2^r}\sqrt{|\Delta_i|}}\frac{\sqrt{d_K}}{R_F}\prod_{\Delta_i\in S_R}\log(\varepsilon_{\Delta_i})\right),$$

where

$$\alpha := \frac{w_K \prod_{i=1}^{2^r} h(\Delta_i)}{\prod_{\Delta_i\in S_I} w_{\Delta_i}} \in \mathbb{Q}.$$

In the following theorem we give an explicit example of Theorem 1.3 for CM biquadratic fields.

Theorem 1.4. *Let $F = \mathbb{Q}(\sqrt{p})$ where $p \equiv 1 \mod 4$ is a prime such that F has narrow class number 1. Let $D = D_1 D_2 < 0$ be a composite fundamental discriminant with $D_1 > 0$ and $D_2 < 0$ fundamental discriminants. Let $K = \mathbb{Q}(\sqrt{D}, \sqrt{pD})$ and $E = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}, \sqrt{p})$. Let χ be the genus character of K arising from the extension E/K. Then*

$$\prod_{C \in \mathrm{CL}(K)} G(z_{\mathfrak{a}})^{-\chi(C)} = \exp\left(\beta \sqrt{d_K} \frac{\log(\varepsilon_{D_1}) \log(\varepsilon_{pD_1})}{\log(\varepsilon_p)}\right),$$

where

$$\beta := \frac{w_K h(D_1) h(D_2) h(pD_1) h(pD_2)}{p D_1 D_2 w_{D_2} w_{pD_2}} \in \mathbb{Q}.$$

Kronecker's identity (1.5) implies that the product of CM values

$$\prod_{C \in \mathrm{CL}(K)} g(\tau_{\mathfrak{a}})^{-\chi(C)}$$

is an algebraic number. This product is also related to elliptic units in the Hilbert class field H of $K = \mathbb{Q}(\sqrt{D})$. Namely, using quotients of powers of $\eta(\tau_{\mathfrak{a}})$ and the theory of complex multiplication, one can construct a sequence ζ_ℓ, $\ell = 1, \dots, h(D) - 1$, of independent units in H (see [S, p. 103]). If σ_k is the automorphism of H/K corresponding to the ideal class C_k under the isomorphism

$$\mathrm{Gal}(H/K) \to \mathrm{CL}(K),$$

one can show that

$$\frac{g(\tau_{\mathfrak{a}_k})}{g(\tau_{\mathfrak{a}_k \mathfrak{a}_\ell^{-1}})} = |\zeta_\ell^{(k)}|^{1/12h(D)}, \quad k, \ell = 1, \dots, h(D) - 1,$$

where $\zeta_\ell^{(k)} := \sigma_k(\zeta_\ell)$. In particular, the quotients $g(\tau_{\mathfrak{a}_k})/g(\tau_{\mathfrak{a}_k \mathfrak{a}_\ell^{-1}})$ are algebraic.

More generally, let H_K be the Hilbert class field of a CM field K as in Theorem 1.1 and let h_K be the class number of K. In light of the preceding facts, it is natural to ask whether the products of CM values

$$\prod_{C \in \mathrm{CL}(K)} G(z_{\mathfrak{a}})^{-\chi(C)}$$

are algebraic, and if so, whether they are related to analogs of elliptic units in H_K. We will show, assuming Schanuel's conjecture, that these products are *transcendental*.

Recall the following well-known conjecture of Schanuel from transcendental number theory (see e.g. [Wa, Conjecture 1.14]).

Conjecture 1.5 (Schanuel)**.** *Given complex numbers $x_1, ..., x_n$ that are linearly independent over $\mathbb{Q}$, the field*

$$\overline{\mathbb{Q}}(x_1, ..., x_n, \exp(x_1), ..., \exp(x_n))$$

has transcendence degree at least n over $\overline{\mathbb{Q}}$.

We will prove the following theorem.

Theorem 1.6. *Let notation and assumptions be as in Theorem 1.3. Then assuming Schanuel's conjecture, the numbers*

$$\prod_{C\in \mathrm{CL}(K)} G(z_{\mathfrak{a}})^{-\chi(C)}$$

are transcendental.

Theorem 1.6 indicates that one cannot in general expect the quotients

$$\frac{G(z_{\mathfrak{a}_k})}{G(z_{\mathfrak{a}_k\mathfrak{a}_\ell^{-1}})}, \quad k,\ell = 1,\dots,h_K-1,$$

to be related to analogs of elliptic units in H_K. For example, if we assume in Theorem 1.6 that K has class number 2, then Schanuel's conjecture implies that the quotients $G(z_{\mathfrak{a}})/G(z_{\mathcal{O}_K})$ are transcendental. Note that there are more than 150 CM biquadratic fields with class number 2 (see [BWW]).

Organization. The paper is organized as follows. In Section 2 we calculate the Laurent expansion at $s=1$ of the Hilbert modular Eisenstein series. In Section 3 we review some facts regarding CM zero-cycles on Hilbert modular varieties. Finally, in Sections 4, 5, 6, and 7, we prove Theorems 1.1, 1.3, 1.4, and 1.6, respectively.

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2. Laurent expansion of the Hilbert modular Eisenstein series

Let F be a totally real number field with class number 1. Let F have degree n over $\mathbb{Q}$ with embeddings $\sigma_1,\dots,\sigma_n$ and let

$$z = x+iy = (z_1,\dots,z_n)\in\mathbb{H}^n.$$

Let $\mathcal{O}_F$ be the ring of integers of F and $\mathrm{SL}_2(\mathcal{O}_F)$ be the Hilbert modular group. Then $\mathrm{SL}_2(\mathcal{O}_F)$ acts componentwise on $\mathbb{H}^n$ by linear fractional transformations,

$$Mz = (\sigma_1(M)z_1,\dots,\sigma_n(M)z_n)\,, \quad M\in\mathrm{SL}_2(\mathcal{O}_F).$$

Let

$$N(y(z)) = \prod_{j=1}^n \mathrm{Im}(z_j) = \prod_{j=1}^n y_j$$

denote the product of the imaginary parts of the components of $z\in\mathbb{H}^n$. Define the real-analytic Hilbert modular Eisenstein series

$$\mathcal{E}(z,s) := \sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathcal{O}_F)} N(y(Mz))^s, \quad z\in\mathbb{H}^n, \quad \mathrm{Re}(s)>1,$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F)\right\}.$$

Furthermore, let

$$N(a+bz)=\prod_{j=1}^{n}(\sigma_j(a)+\sigma_j(b)z_j)$$

for $(a,b)\in\mathcal{O}_F\times\mathcal{O}_F$ and define the Eisenstein series

$$E(z,s):=\sum_{(a,b)\in\mathcal{O}_F\times\mathcal{O}_F/\mathcal{O}_F^\times}{}' \frac{N(y(z))^s}{|N(a+bz)|^{2s}},\quad z\in\mathbb{H}^n,\quad \mathrm{Re}(s)>1,$$

where the sum is over a complete set of nonzero, nonassociated representatives of $\mathcal{O}_F\times\mathcal{O}_F$ (recall that (a,b) and (a',b') are *associated* if there exists a unit $\epsilon\in\mathcal{O}_F^\times$ such that $(a,b)=(\epsilon a',\epsilon b')$). The two Eisenstein series are related by

$$E(z,s)=\zeta_F(2s)\mathcal{E}(z,s), \tag{2.1}$$

where $\zeta_F(s)$ is the Dedekind zeta function of F.

The Eisenstein series $E(z,s)$ has the Fourier expansion

$$\begin{aligned} E(z,s)&=N(y(z))^s\zeta_F(2s)+\frac{N(y(z))^{1-s}}{\sqrt{d_F}}\left[\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right]^n\zeta_F(2s-1) \qquad (2.2)\\ &+\frac{2^nN(y(z))^{\frac{1}{2}}}{\sqrt{d_F}}\left[\frac{\pi^s}{\Gamma(s)}\right]^n\times\\ &\sum_{\tilde{a}\in\mathcal{O}_F^*}{}'\sum_{\substack{\tilde{a}=ab\\ a\in\mathcal{O}_F^*\\ b\in\mathcal{O}_F/\mathcal{O}_F^\times}}\left(\frac{N_{F/\mathbb{Q}}((a))}{N_{F/\mathbb{Q}}((b))}\right)^{s-\frac{1}{2}}e^{2\pi iT(abx)}\prod_{j=1}^{n}K_{s-\frac{1}{2}}(2\pi\left|\sigma_j(ab)\right|y_j)\\ &=:A(s)+B(s)+C(s), \end{aligned}$$

where $\mathcal{O}_F^*$ is the dual lattice, d_F is the absolute discriminant, $T(ax)=\sum_{j=1}^n\sigma_j(a)x_j$ is the trace, $K_s(v)$ is the usual K-Bessel function of order s, and $A(s),B(s),C(s)$ are the three functions on the right hand side of (2.2), respectively.

The Fourier expansion provides a meromorphic continuation of $E(z,s)$ to $\mathbb{C}$ with a simple pole at $s=1$. We now use this to compute the Laurent expansion at $s=1$.

The Laurent expansion of $A(s)$ at $s=1$ is

$$A(s)=N(y(z))\zeta_F(2)+O(s-1).$$

Next, observe that

$$\frac{N(y(z))^{1-s}}{\sqrt{d_F}}=\frac{1}{\sqrt{d_F}}-\frac{\log N(y(z))}{\sqrt{d_F}}(s-1)+O(s-1)^2,$$

$$\left[\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right]^n=\pi^n-2n\pi^n\log(2)(s-1)+O(s-1)^2,$$

and

$$\zeta_F(2s-1)=\frac{r_F}{2(s-1)}+A_F+O(s-1).$$

After a calculation, we find that the Laurent expansion of $B(s)$ at $s = 1$ is

$$B(s) = \frac{\pi^n r_F}{2\sqrt{d_F}}\frac{1}{(s-1)} + \frac{\pi^n}{\sqrt{d_F}}A_F - \frac{\pi^n r_F}{2\sqrt{d_F}}\left[\log N(y(z)) + 2n\log(2)\right] + O(s-1).$$

Using

$$K_{1/2}(v) = \sqrt{\pi/2v}e^{-v}$$

we compute

$$\prod_{j=1}^{n} K_{1/2}(2\pi\left|\sigma_j(ab)\right| y_j) = \frac{N(y(z))^{-1/2}}{2^n}N_{F/\mathbb{Q}}((ab))^{-1/2}e^{-2\pi S(aby)},$$

where

$$S(aby) = \sum_{j=1}^{n}\left|\sigma_j(ab)\right| y_j.$$

Thus the Laurent expansion of $C(s)$ at $s = 1$ is

$$C(s) = \frac{\pi^n}{\sqrt{d_F}}\sum_{\tilde{a}\in\mathcal{O}_F^*}{}' \sum_{\substack{\tilde{a}=ab \\ a\in\mathcal{O}_F^* \\ b\in\mathcal{O}_F/\mathcal{O}_F^\times}} \frac{e^{-2\pi i S(aby)}}{N_{F/\mathbb{Q}}((b))}e^{2\pi i T(abx)} + O(s-1).$$

Putting things together, we find that the Laurent expansion of $E(z, s)$ at $s = 1$ is

$$E(z, s) = \frac{E_{-1}}{s-1} + E_0(z) + O(s-1), \tag{2.3}$$

where the residue

$$E_{-1} = \frac{\pi^n r_F}{2\sqrt{d_F}},$$

and

$$E_0(z) = \frac{\pi^n}{\sqrt{d_F}}A_F - E_{-1}2n\log(2) + \log\left(N(y(z))^{-E_{-1}}f(z)\right), \tag{2.4}$$

where

$$\log f(z) = N(y(z))\zeta_F(2) + \frac{\pi^n}{\sqrt{d_F}}\sum_{\tilde{a}\in\mathcal{O}_F^*}{}' \sum_{\substack{\tilde{a}=ab \\ a\in\mathcal{O}_F^* \\ b\in\mathcal{O}_F/\mathcal{O}_F^\times}} \frac{e^{-2\pi i S(aby)}}{N_{F/\mathbb{Q}}((b))}e^{2\pi i T(abx)}.$$

3. CM zero-cycles on Hilbert modular varieties

In this section we review some facts we will need regarding CM zero-cycles on Hilbert modular varieties following Bruinier and Yang [BY, Section 3]. See also the recent book of Howard and Yang [HY]. Let F be a totally real number field of degree n over $\mathbb{Q}$. For $S \subset F$, let S^+ be the subset of S consisting of totally positive elements. For a fractional ideal $\mathfrak{f}_0$ of F, let

$$\Gamma(\mathfrak{f}_0) = \mathrm{SL}(\mathcal{O}_F \oplus \mathfrak{f}_0) = \{M = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}_2(F) : \ \alpha,\delta\in\mathcal{O}_F,\ \beta\in\mathfrak{f}_0,\ \gamma\in\mathfrak{f}_0^{-1}\}.$$

Recall that $\Gamma(\mathfrak{f}_0)$ acts on $\mathbb{H}^n$ by

$$Mz = (\sigma_1(M)z_1, \dots, \sigma_n(M)z_n).$$

The quotient space

$$X(\mathfrak{f}_0) = \Gamma(\mathfrak{f}_0)\backslash\mathbb{H}^n$$

is the (open) Hilbert modular variety associated to $\mathfrak{f}_0$. The variety $X(\mathfrak{f}_0)$ parameterizes isomorphism classes of triples (A, i, m) where (A, i) is an abelian variety with real multiplication $i : \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$ and

$$m : \left(\mathfrak{M}_A, \mathfrak{M}_A^+\right) \to \left((\partial_F \mathfrak{f}_0)^{-1}, (\partial_F \mathfrak{f}_0)^{-1,+}\right)$$

is an $\mathcal{O}_F$-isomorphism from $\mathfrak{M}_A$ to $(\partial_F \mathfrak{f}_0)^{-1}$ which maps $\mathfrak{M}_A^+$ to $(\partial_F \mathfrak{f}_0)^{-1,+}$. Here $\mathfrak{M}_A$ is the polarization module of A and $\mathfrak{M}_A^+$ is its positive cone.

Let K be a CM extension of F and $\Phi = (\sigma_1, \dots, \sigma_n)$ be a CM type of K. A point $z = (A, i, m) \in X(\mathfrak{f}_0)$ is a *CM point* of type (K, Φ) if one of the following equivalent definitions holds:

(1) As a point $z \in \mathbb{H}^n$, there is a point $\tau \in K$ such that

$$\Phi(\tau) = (\sigma_1(\tau), \dots, \sigma_n(\tau)) = z$$

and

$$\Lambda_\tau = \mathfrak{f}_0 + \mathcal{O}_F \tau$$

is a fractional ideal of K.

(2) (A, i') is a CM abelian variety of type (K, Φ) with complex multiplication $i : \mathcal{O}_K \hookrightarrow \mathrm{End}(A)$ such that $i = i'_{|\mathcal{O}_F}$.

Fix $\varepsilon_0 \in K^\times$ such that $\overline{\varepsilon_0} = -\varepsilon_0$ and $\Phi(\varepsilon_0) = (\sigma_1(\varepsilon_0), \dots, \sigma_n(\varepsilon_0)) \in \mathbb{H}^n$. Let $\mathfrak{a}$ be a fractional ideal of K and $\mathfrak{f}_\mathfrak{a} = \varepsilon_0 \partial_{K/F} \mathfrak{a}\bar{\mathfrak{a}} \cap F$. By [BY, Lemma 3.1], the CM abelian variety $(A_\mathfrak{a} = \mathbb{C}^n/\Phi(\mathfrak{a}), i)$ defines a CM point on $X(\mathfrak{f}_0)$ if there exists an $r \in F^\times$ such that $\mathfrak{f}_\mathfrak{a} = r\mathfrak{f}_0$. Thus any pair $(\mathfrak{a}, r)$ with $\mathfrak{a}$ a fractional ideal of K and $r \in F^\times$ with $\mathfrak{f}_\mathfrak{a} = r\mathfrak{f}_0$ defines a CM point $(A_\mathfrak{a}, i, m) \in X(\mathfrak{f}_0)$ (we refer the reader to [BY] for a discussion of how the $\mathcal{O}_F$-isomorphism m depends on r). Two such pairs $(\mathfrak{a}_1, r_1)$ and $(\mathfrak{a}_2, r_2)$ are equivalent if there exists an $\alpha \in K^\times$ such that $\mathfrak{a}_2 = \alpha\mathfrak{a}_1$ and $r_2 = r_1\alpha\bar{\alpha}$. Write $[\mathfrak{a}, r]$ for the class of $(\mathfrak{a}, r)$ and identify it with its associated CM point $(A_\mathfrak{a}, i, m) \in X(\mathfrak{f}_0)$.

By [BY, Lemma 3.2], given a CM point $[\mathfrak{a}, r] \in X(\mathfrak{f}_0)$ there is a decomposition

$$\mathfrak{a} = \mathcal{O}_F \alpha + \mathfrak{f}_0 \beta$$

with $z = \alpha/\beta \in K^\times \cap \mathbb{H}^n = \{z \in K^\times : \ \Phi(z) \in \mathbb{H}^n\}$. Moreover, z represents the CM point $[\mathfrak{a}, r] \in X(\mathfrak{f}_0)$.

Let $\mathcal{CM}(K, \Phi, \mathfrak{f}_0)$ be the set of CM points $[\mathfrak{a}, r] \in X(\mathfrak{f}_0)$, which we view as a CM zero-cycle in $X(\mathfrak{f}_0)$. Let

$$\mathcal{CM}(K, \Phi) = \sum_{[\mathfrak{f}_0] \in \mathrm{CL}(F)^+} \mathcal{CM}(K, \Phi, \mathfrak{f}_0),$$

where $\mathrm{CL}(F)^+$ is the narrow ideal class group of F. The forgetful map

$$\mathcal{CM}(K, \Phi) \to \mathrm{CL}(K),$$
$$[\mathfrak{a}, r] \mapsto [\mathfrak{a}]$$

is surjective. Each fiber is indexed by $\epsilon \in \mathcal{O}_F^{\times,+}/N_{K/F}\mathcal{O}_K^\times$. Here $\#(\mathcal{O}_F^{\times,+}/N_{K/F}\mathcal{O}_K^\times)$ equals 1 or 2; in particular, it equals 1 if $\epsilon \in N_{K/F}\mathcal{O}_K^\times$.

Assume now that F has narrow class number 1. Then

$$\mathcal{CM}(K,\Phi) = \mathcal{CM}(K,\Phi,\mathcal{O}_F),$$

and the forgetful map

$$\mathcal{CM}(K,\Phi) \to \mathrm{CL}(K)$$

is injective (hence bijective) since $N_{K/F}\mathcal{O}_K^\times = \mathcal{O}_F^\times$. We will repeatedly use this bijection to identify the zero-cycle of CM points $\mathcal{CM}(K,\Phi,\mathcal{O}_F) \subset X_F := X(\mathcal{O}_F)$ with the set

$$\{z_{\mathfrak{a}} \in K^\times \cap \mathbb{H}^n : [\mathfrak{a}] \in \mathrm{CL}(K)\},$$

where $z_{\mathfrak{a}}$ represents $[\mathfrak{a}, r] \in X_F$ as above. The reader should keep in mind that the later set depends on Φ.

4. Proof of Theorem 1.1

We first establish the following version of the Kronecker limit formula for CM fields.

Theorem 4.1. *Let F be a totally real number field of degree n over $\mathbb{Q}$ with narrow class number 1. Let K be a CM extension of F with a CM type Φ. For each class $C \in \mathrm{CL}(K)$, let $z_{\mathfrak{a}}$ be the CM point in $\mathcal{CM}(K,\Phi,\mathcal{O}_F)$ corresponding to C^{-1}. Then we have*

$$\lim_{s\to 1}\left[\zeta_K(s,C) - \frac{(2\pi)^n R_K}{w_K\sqrt{d_K}}\frac{1}{s-1}\right] =$$
$$\frac{(2\pi)^n R_K}{w_K\sqrt{d_K}}\left(\frac{\pi^n A_F}{E_{-1}\sqrt{d_F}} + 2\log(d_F) - \log(d_K) - 2\log G(z_{\mathfrak{a}})\right),$$

where

$$G(z) := \sqrt{\left(2^n d_F/\sqrt{d_K}\right) N(y(z)) \cdot \phi(z)^2} \tag{4.1}$$

and

$$\phi(z) := f(z)^{-1/4E_{-1}}.$$

Proof. Fix a CM type Φ for K. Let $C \in \mathrm{CL}(K)$, and fix an integral ideal $\mathfrak{a} \in C^{-1}$. Then the partial Dedekind zeta function equals

$$\begin{aligned}\zeta_K(s,C) &= \sum_{\mathfrak{b}\in C}{}' N_{K/\mathbb{Q}}(\mathfrak{b})^{-s}\\ &= \sum_{(\omega)\subset\mathfrak{a}}{}' N_{K/\mathbb{Q}}(\mathfrak{a}^{-1}(\omega))^{-s}\\ &= N_{K/\mathbb{Q}}(\mathfrak{a})^s \sum_{\omega\in\mathfrak{a}/\mathcal{O}_K^\times}{}' N_{K/\mathbb{Q}}((\omega))^{-s}.\end{aligned}$$

Notice that

$$\sum_{\omega\in\mathfrak{a}/\mathcal{O}_K^\times}{}' N_{K/\mathbb{Q}}((\omega))^{-s} = \frac{1}{\left|\mathcal{O}_K^\times : \mathcal{O}_F^\times\right|}\sum_{\omega\in\mathfrak{a}/\mathcal{O}_F^\times}{}' N_{K/\mathbb{Q}}((\omega))^{-s}.$$

Thus we have

$$\zeta_K(s,C) = \frac{N_{K/\mathbb{Q}}(\mathfrak{a})^s}{\left|\mathcal{O}_K^\times : \mathcal{O}_F^\times\right|} \sum_{\omega \in \mathfrak{a}/\mathcal{O}_F^\times}{}' N_{K/\mathbb{Q}}((\omega))^{-s}.$$

By the facts in Section 3 there exists a decomposition

$$\mathfrak{a} = \mathcal{O}_F \alpha + \mathcal{O}_F \beta,$$

where $z_\mathfrak{a} = \beta/\alpha \in K^\times \cap \mathbb{H}^n$ and $z_\mathfrak{a}$ represents the CM point $[\mathfrak{a}, r] \in X_F$ (here $\mathfrak{f}_0 = \mathcal{O}_F$ since $\#\mathrm{CL}(F)^+ = 1$). Then

$$\begin{aligned}\sum_{\omega \in \mathfrak{a}/\mathcal{O}_F^\times}{}' N_{K/\mathbb{Q}}((\omega))^{-s} &= \sum_{(a,b)\in \mathcal{O}_F \times \mathcal{O}_F/\mathcal{O}_F^\times}{}' N_{K/\mathbb{Q}}((a\alpha + b\beta))^{-s} \\ &= N_{K/\mathbb{Q}}((\alpha))^{-s} \sum_{(a,b)\in \mathcal{O}_F \times \mathcal{O}_F/\mathcal{O}_F^\times}{}' N_{K/\mathbb{Q}}((a + bz_\mathfrak{a})).\end{aligned}$$

By a calculation with the CM type Φ we obtain

$$N_{K/\mathbb{Q}}((a + bz_\mathfrak{a})) = \left|N\left(a + bz_\mathfrak{a}\right)\right|^2,$$

where we have identified $z_\mathfrak{a}$ with $\Phi(z_\mathfrak{a}) \in \mathbb{H}^n$. Moreover, one has

$$N_{K/\mathbb{Q}}(\mathfrak{a}/(\alpha)) = N(y(z_\mathfrak{a})) \frac{2^n d_F}{\sqrt{d_K}}.$$

By combining the preceding calculations, we obtain

$$\begin{aligned}\zeta_K(s,C) &= \left(\frac{2^n d_F}{\sqrt{d_K}}\right)^s \frac{1}{\left|\mathcal{O}_K^\times : \mathcal{O}_F^\times\right|} \sum_{(a,b)\in \mathcal{O}_F \times \mathcal{O}_F/\mathcal{O}_F^\times}{}' \frac{N(y(z_\mathfrak{a}))^s}{\left|N(a + bz_\mathfrak{a})\right|^{2s}} \\ &= \left(\frac{2^n d_F}{\sqrt{d_K}}\right)^s \frac{1}{\left|\mathcal{O}_K^\times : \mathcal{O}_F^\times\right|} E(z_\mathfrak{a}, s).\end{aligned}$$

Observe that

$$\left(\frac{2^n d_F}{\sqrt{d_K}}\right)^{s-1} = 1 + \log\left(\frac{2^n d_F}{\sqrt{d_K}}\right)(s-1) + O(s-1)^2.$$

Then after a calculation using the Laurent expansion

$$E(z_\mathfrak{a}, s) = \frac{E_{-1}}{s-1} + E_0(z_\mathfrak{a}) + O(s-1)$$

given by (2.3), we obtain the limit formula in the theorem. □

Remark 4.2. If $F = \mathbb{Q}$ in Theorem 4.1, we recover the Kronecker limit formula (1.2).

The function $\phi(z)$ is positive and real-analytic. In the following proposition, we identify how $\phi(z)$ transforms with respect to $\mathrm{SL}_2(\mathcal{O}_F)$ (see also [S, pp. 108-109]).

Proposition 4.3. *For all* $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F)$, *we have*

$$\phi(Mz) = \left|N(\gamma z + \delta)\right|^{\frac{1}{2}} \phi(z).$$

Proof. From the relation (2.1) we see that $E(z,s)$ has weight 0 with respect to $\mathrm{SL}_2(\mathcal{O}_F)$. Then the Laurent expansion (2.3) implies that $E_0(Mz) = E_0(z)$, which by (2.4) implies that

$$\log f(Mz) = \log f(z) + E_{-1} \log \left(\frac{N(\mathrm{Im}(Mz))}{N(\mathrm{Im}(z))} \right).$$

A straightforward calculation shows that

$$\frac{N(\mathrm{Im}(Mz))}{N(\mathrm{Im}(z))} = |N(\gamma z + \delta)|^{-2},$$

and thus

$$f(Mz) = |N(\gamma z + \delta)|^{-2E_{-1}} f(z).$$

The result now follows from the definition of $\phi(z)$. □

Remark 4.4. By Proposition 4.3, the function $G : \mathbb{H}^n \to \mathbb{R}^+$ defined by (4.1) has weight 0 with respect to $\mathrm{SL}_2(\mathcal{O}_F)$ and thus is well-defined on CM points.

We can now deduce Theorem 1.1.

Proof of Theorem 1.1: For a class group character χ of K, let

$$L_K(\chi, s) = \sum_{C \in \mathrm{CL}(K)} \chi(C) \zeta_K(s, C)$$

be its associated L–function. By orthogonality for group characters, if χ is nontrivial we have

$$\sum_{C \in \mathrm{CL}(K)} \chi(C) = 0.$$

The theorem now follows from Theorem 4.1. □

5. Proof of Theorem 1.3

Let K be a CM field with $\mathrm{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^r$ for some integer $r \geq 2$, and let E be an unramified quadratic extension of K with $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^{r+1}$. Then the zeta function $\zeta_E(s)$ (resp. $\zeta_K(s)$) factors as $\zeta(s)$ times the product of the quadratic Dirichlet L–functions associated to the quadratic subfields of E (resp. K). Note that there are $2^r - 1$ quadratic subfields of K, $2^{r+1} - 1$ quadratic subfields of E, and 2^r quadratic subfields of E that are not contained in K. By class field theory, the unramified extension E/K gives rise to a real class group character χ of K (a genus character) whose L-function factors as

$$L_K(\chi, s) = \frac{\zeta_E(s)}{\zeta_K(s)}.$$

Then by the preceding facts we obtain the factorization

$$L_K(\chi, s) = \prod_{i=1}^{2^r} L(\chi_{\Delta_i}, s),$$

where χ_{Δ_i} for $1 \leq i \leq 2^r$ are the Kronecker symbols associated to the quadratic subfields $\mathbb{Q}(\sqrt{\Delta_i})$ of E which are not contained in K.

Divide the discriminants Δ_i into two disjoint sets, $S_R := \{\Delta_i : \ \Delta_i > 0\}$ and $S_I := \{\Delta_i : \ \Delta_i < 0\}$. Then we obtain the following formula for $L_K(\chi, 1)$ using Dirichlet's class number formula (1.4) for quadratic fields,

$$L_K(\chi, 1) = \frac{2^{2^r} \pi^{\# S_I} \prod_{i=1}^{2^r} h(\Delta_i) \prod_{\Delta_i \in S_R} \log(\varepsilon_{\Delta_i})}{\prod_{i=1}^{2^r} \sqrt{|\Delta_i|} \prod_{\Delta_i \in S_I} w_{\Delta_i}}. \tag{5.1}$$

On the other hand, by Theorem 1.1 we have

$$L_K(\chi, 1) = \frac{2^{n+1} \pi^n R_K}{w_K \sqrt{d_K}} \left(- \sum_{C \in \mathrm{CL}(K)} \chi(C) \log G(z_{\mathfrak{a}}) \right). \tag{5.2}$$

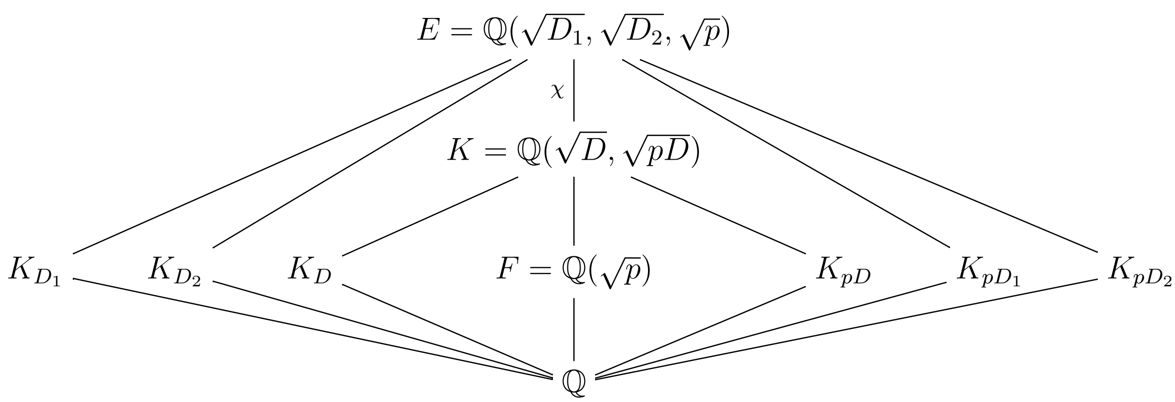
Observe that $\# S_R = \# S_I = 2^{r-1} = [F : \mathbb{Q}] = n$, and the regulators of K and F satisfy the relation

$$R_K = 2^{n-1} R_F$$

(see [W, p. 41]). The theorem now follows by equating (5.1) and (5.2) and simplifying the resulting expression. ☐

6. Proof Theorem 1.4

Let $F = \mathbb{Q}(\sqrt{p})$ where $p \equiv 1 \mod 4$ is a prime such that F has narrow class number 1. Let $D = D_1 D_2 < 0$ be a composite fundamental discriminant with $D_1 > 0$ and $D_2 < 0$ fundamental discriminants. Let $K = \mathbb{Q}(\sqrt{D}, \sqrt{pD})$, which is a CM biquadratic extension of $\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let $E = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}, \sqrt{p})$, which is an unramified quadratic extension of K with $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$. Let χ be the genus character of K arising from the extension E/K, and let K_Δ denote $\mathbb{Q}(\sqrt{\Delta})$ for a fundamental discriminant Δ. Then we have the following diagram:



We have

$$L_K(\chi, s) = \frac{\zeta_E(s)}{\zeta_K(s)}.$$

Then the factorizations

$$\zeta_E(s) = \zeta(s) L(\chi_p, s) L(\chi_D, s) L(\chi_{pD}, s) L(\chi_{D_1}, s) L(\chi_{D_2}, s) L(\chi_{pD_1}, s) L(\chi_{pD_2}, s)$$

and

$$\zeta_K(s) = \zeta(s)L(\chi_p, s)L(\chi_D, s)L(\chi_{pD}, s)$$

yield

$$L_K(\chi, s) = L(\chi_{D_1}, s)L(\chi_{D_2}, s)L(\chi_{pD_1}, s)L(\chi_{pD_2}, s).$$

By Dirichlet's class number formula (1.4) for quadratic fields, we have

$$L_K(\chi, 1) = \frac{2\log(\varepsilon_{D_1})h(D_1)}{\sqrt{D_1}} \frac{2\pi h(D_2)}{w_{D_2}\sqrt{|D_2|}} \frac{2\log(\varepsilon_{pD_1})h(pD_1)}{\sqrt{pD_1}} \frac{2\pi h(pD_2)}{w_{pD_2}\sqrt{|pD_2|}}. \tag{6.1}$$

On the other hand, by Theorem 1.1 we have

$$L_K(\chi, 1) = \frac{16\pi^2 \log(\varepsilon_p)}{w_K\sqrt{d_K}} \left(- \sum_{C \in \mathrm{CL}(K)} \chi(C) \log G(z_{\mathfrak{a}}) \right), \tag{6.2}$$

where we used $R_K = 2\log(\varepsilon_p)$ (see [W, Proposition 4.16]). The theorem now follows by equating (6.1) and (6.2) and simplifying the resulting expression. □

7. Proof of Theorem 1.6

Assume first that $r = 2$. Then $K \cong (\mathbb{Z}/2\mathbb{Z})^2$, $E \cong (\mathbb{Z}/2\mathbb{Z})^3$, and the maximal totally real subfield F of K is real quadratic. Let $\mathbb{Q}(\sqrt{D_1})$ and $\mathbb{Q}(\sqrt{D_2})$ be the real quadratic subfields of E which are not contained in K, and let $F = \mathbb{Q}(\sqrt{D_3})$. Then because $R_K = 2\log(\varepsilon_{D_3})$, it suffices to show that $A := \exp(B)$ is transcendental, where

$$B := Q_1\sqrt{Q_2}\frac{\log(\varepsilon_{D_1})\log(\varepsilon_{D_2})}{\log(\varepsilon_{D_3})}$$

for rational numbers $Q_1, Q_2 \in \mathbb{Q}$.

Let $x_1 := \log(\varepsilon_{D_1}), x_2 := \log(\varepsilon_{D_2})$ and $x_3 := \log(\varepsilon_{D_3})$. Then

$$\overline{\mathbb{Q}}(x_1, x_2, x_3, \exp(x_1), \exp(x_2), \exp(x_3)) = \overline{\mathbb{Q}}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3})).$$

Because $\varepsilon_{D_1}, \varepsilon_{D_2}$ and ε_{D_3} are multiplicatively independent, x_1, x_2 and x_3 are linearly independent over $\mathbb{Q}$. Then by Schanuel's conjecture (see Conjecture 1.5), the field

$$\overline{\mathbb{Q}}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3}))$$

has transcendence degree at least 3 over $\overline{\mathbb{Q}}$, and hence exactly 3 as it is generated by 3 elements. In particular, x_1, x_2 and x_3 are algebraically independent over $\overline{\mathbb{Q}}$.

We claim that because x_1, x_2 and x_3 are algebraically independent over $\overline{\mathbb{Q}}$, the numbers x_1, x_2, x_3 and $x_4 := B$ are linearly independent over $\mathbb{Q}$. To see this, suppose to the contrary that there exist rational numbers $\alpha_i \in \mathbb{Q}$, not all zero, such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 B = 0. \tag{7.1}$$

Define the polynomial

$$q(t_1, t_2, t_3) := \alpha_1 t_1 t_3 + \alpha_2 t_2 t_3 + \alpha_3 t_3^2 + \alpha_4 Q_1 \sqrt{Q_2} t_1 t_2.$$

Then (7.1) implies that $q(x_1, x_2, x_3) = 0$, which contradicts the algebraic independence of x_1, x_2 and x_3 over $\overline{\mathbb{Q}}$. Thus x_1, x_2, x_3 and x_4 are linearly independent over $\mathbb{Q}$. By another application of Schanuel's conjecture, the field

$$\begin{aligned}&\overline{\mathbb{Q}}(x_1, x_2, x_3, x_4, \exp(x_1), \exp(x_2), \exp(x_3), \exp(x_4))\\ &= \overline{\mathbb{Q}}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3}), B, A)\\ &= \overline{\mathbb{Q}}(\log(\varepsilon_{D_1}), \log(\varepsilon_{D_2}), \log(\varepsilon_{D_3}), A)\end{aligned}$$

has transcendence degree at least 4 over $\overline{\mathbb{Q}}$, hence A must be transcendental. This completes the proof when $r = 2$.

Next assume that $r \geq 2$. Then $K \cong (\mathbb{Z}/2\mathbb{Z})^r$, $E \cong (\mathbb{Z}/2\mathbb{Z})^{r+1}$, and $F \cong (\mathbb{Z}/2\mathbb{Z})^{r-1}$. The rank of the unit group $\mathcal{O}_F^\times$ is $n-1$, where $n = [F : \mathbb{Q}]$, and recall that the regulators of K and F satisfy the relation

$$R_K = 2^{n-1} R_F.$$

Let $\varepsilon_1, \ldots, \varepsilon_{n-1}$ be fundamental units for the $n-1$ real quadratic subfields of F. These units form a set of multiplicatively independent units in F which are a basis for $\mathcal{O}_F^\times/\{\pm 1\}$, and thus

$$R_F = |\det(\log|\sigma_i(\varepsilon_j)|)_{1 \leq i,j \leq n-1}|$$

where the σ_i run through any $n-1$ embeddings of F. The conjugate of a unit in a real quadratic field is, up to a sign, its inverse. Thus for $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, either $\sigma(\varepsilon_j) = \varepsilon_j$ or $\sigma(\varepsilon_j) = \pm\varepsilon_j^{-1}$. It follows that the regulator R_F is a positive integer multiple of the product $\log(\varepsilon_1) \cdots \log(\varepsilon_{n-1})$. Therefore it suffices to show that $\exp(C)$ is transcendental, where

$$C := Q_3 \sqrt{Q_4} \frac{\prod_{\Delta_i \in S_R} \log(\varepsilon_{\Delta_i})}{\log(\varepsilon_1) \cdots \log(\varepsilon_{n-1})}$$

for rational numbers $Q_3, Q_4 \in \mathbb{Q}$. Because the units $\{\varepsilon_1, \ldots, \varepsilon_{n-1}\} \cup \{\varepsilon_{\Delta_i} : \Delta_i \in S_R\}$ are multiplicatively independent, a straightforward modification of the argument for $r = 2$ shows that $\exp(C)$ is transcendental.

$\square$

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