MATHEMATICAL COHERENCE: CROSS-NATIONAL STUDIES OF TEXTBOOKS AND TEACHERS' KNOWLEDGE OF FRACTIONS

A Dissertation

by

KE WANG

Submitted to the Office of Graduate and Professional Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Chair of Committee,	Yeping Li
Co-Chair of Committee,	Roger Howe
Committee Members,	Dianne Goldsby
	Wen Luo
Head of Department,	Michael de Miranda

May 2019

Major Subject: Curriculum and Instruction

ABSTRACT

Textbooks continue to play an important role in teaching and learning. Coherence is an assumed feature of all textbooks, but it is ill-defined and rarely examined in detail. In the first study, I developed a framework for examining textbook coherence at three levels: macro-logical coherence, meso-lesson-structure coherence, and micro-example-practice coherence. Then I used this framework to analyze textbook coherence on the topic of equivalent fractions in the three series of mathematics textbooks selected from California, Shanghai, and Singapore, respectively. The findings suggest the feasibility of a framework that can contribute to our understanding of textbook coherence and to the possible improvement of specific topic presentation and organization in textbooks.

The second study focused on example-practice coherence by analyzing all problems from the chapters on fractions in the three series of mathematics textbooks. Since problems are important components of school mathematics textbooks, the variation of problems should be in line with students' cognitive levels. To examine example-practice problem coherence, I developed a conceptual framework for problems' cognitive requirement in terms of five aspects of variations: response, operation, reasoning, representation, and connection. After comparing the coherence between example and practice problems in terms of these five aspects, the results show that these textbooks have different emphases. California textbooks emphasize the variations of problems in operation and reasoning. Shanghai textbooks emphasize the variations of problems in connection and representation. Singapore textbooks emphasize the variations of problems in reasoning and response.

The third study aimed to examine U.S. and Chinese teachers' knowledge for teaching equivalent fractions. A conceptual framework for instructional coherence was developed based on the MKT model. This framework includes two aspects: coherence of mathematical content and coherence of mathematical pedagogy. Based on this framework, I designed the interview tasks and interviewed ten U.S. and ten Chinese math teachers, separately. After analyzing and comparing teachers' responses, the findings reveal the different emphases on teaching between selected U.S. and Chinese teachers and also suggest the feasibility of the framework that can contribute to the understanding of the instructional coherence and improve teachers' mathematical knowledge for teaching.

DEDICATION

My parents: Laishu & Lainiao; My parents-in-law: Xinmin & Lijuan,

Who have instilled me the love of life and living and have supported me to pursue my dream;

My lovely wife, Fan,

Who has supported my quest for knowledge throughout our marriage;

My precious daughter, Georgia,

Who makes my life more colorful and happy;

My precious son, Derek,

Who makes my life more meaningful and happy.

ACKNOWLEDGEMENTS

Many thanks go to my super committee members who guided me during my program study and research, to many teachers who joined in my research projects, and to my family members who bring happiness to my study. In particular, I would like to express my deepest gratitude to Dr. Yeping Li, my committee chair, for patiently guiding me in search for research direction, warmly teaching me how to write academic papers, precisely helping me find my own voice as a researcher, and fully supporting me to complete my study.

My highest appreciation also goes to my committee co-chair, Dr. Roger Howe, for his guidance and support throughout this research, which inspired me to explore the topic of coherence deeply. Specifically, I highly appreciate his suggestions on the topic of fractions so that I can understand fractions more completely. I would also like to thank my other committee members, Drs. Dianne Goldsby and Wen Luo, for their valuable time and great suggestions. Dr. Dianne Goldsby encouraged me to integrate my own ideas into teaching undergraduate courses and provided useful materials to support my teaching. Dr. Wen Luo patiently taught me how to use quantitative methods to address the issues in education and provided valuable feedback on the statistical results.

Thanks also go to my friends and colleagues and the department faculty and staff for making my time at Texas A&M University a wonderful experience. Finally, thanks to my parents and parents-in-law for their encouragement and support, to my wife for her patience and love, and to my daughter and son for their support and joy.

CONTRIBUTORS AND FUNDING SOURCES

This dissertation was supported by a dissertation committee consisting of Dr. Yeping Li [chair], Dr. Roger Howe [co-chair], Dr. Dianne Goldsby of the Department of Teaching, Learning, and Culture, and Dr. Wen Luo of the Department of Educational Psychology.

The analyses for Manuscript #1, Manuscript #2, and Manuscript #3 received the support and feedback from Drs. Yeping Li, Roger Howe, Dianne Goldsby, and Wen Luo. The analysis of Manuscript #2 also received feedback from Dr. Wen Luo. The structure, content development, and organization of the whole dissertation received the feedback and support from Drs. Yeping Li, Roger Howe, Dianne Goldsby, and Wen Luo. The editing of the whole dissertation was done with the support from Dr. Yeping Li. All other work conducted for the dissertation was completed by the student independently.

Manuscripts #1 and #2 were partially supported by the CEHD Graduate Research Grant from the College of Education and Human Development and a Graduate Strategic Scholarship from the Department of Teaching, Learning, and Culture at Texas A&M University. Manuscript #3 was partially supported by a Graduate Research Grant from the College of Education and Human Development at Texas A&M University.

NOMENCLATURE

CCCS	California Common Core Standards
CR	Cognitive Requirements
CCSSM	Common Core State Standards for Mathematics
CCSSI	Common Core State Standards Initiative
CK	Content Knowledge
EF	Equivalent Fractions
EFR	Equivalent Fraction Rule
HCK	Horizon Content Knowledge
MKT	Mathematical Knowledge for Teaching
NCTM	National Council of Teachers of Mathematics
NRC	National Research Council
PCK	Pedagogical Content Knowledge
PISA	Programme for International Student Assessment
SEP	Shanghai Education Press
TIMSS	The International Mathematics and Science Study
TEKS	Texas Essential Knowledge and Skills
US	The United States of America
UK	The United Kingdom

TABLE OF CONTENTS

Page

LIST OF FIGURES

LIST OF TABLES

Page

1. INTRODUCTION

Coherence means "consistency in reasoning, or relating, so that one part of the discourse does not destroy or contradict the rest; harmonious connection of the several parts, so that the whole 'hangs together'" (Oxford English Dictionary, 2017). For example, when something has coherence, all its parts fit together well; thus, an argument with coherence is logical and complete, with plenty of supporting facts. Schmidt, Wang, and McKnight (2005) argued that if content standards are to be coherent, they must be articulated as a sequence of topics and performances which reflects the structure of the corresponding discipline. In regard to textbooks, coherence involves different levels, such as operating within a grade level or across grades.

One aspect of a high-quality textbook is readability for as many people as possible. The textbook can start with simple concepts, then gradually introduce more difficult ones so that someone without a relevant background in the topic being discussed can easily understand it. As Socrates said, everyone is teachable; similarly, any textbook should be readable, so coherence is essential for comprehensible textbooks. Furthermore, textbooks are essential tools for a teacher to teach the content effectively; therefore, a coherent textbook plays an important role in promoting the coherence of teachers' instruction (Chen & Li, 2010).

Textbooks are the main bodies of curricula. In this dissertation, I focus on textbooks because textbooks being published by publishing houses or departments of education are in static states while curricula including textbooks, teacher guidebooks, exercises, and other teaching resources are in dynamic states. Therefore, textbooks are more available to be researched objects than curricula. In addition, a close relationship exists between textbooks and curricula. The term coherent curriculum[1] or aligned curriculum refers to an academic program that is (1) well-organized and purposefully designed to facilitate learning, (2) free of academic gaps and needless repetitions, and (3) aligned across lessons, courses, subject areas, and grade levels (Education Reform, 2017). Another important characteristic of a coherent curriculum is that it provides many pedagogical supports (Roseman, Linn, & Koppal, 2008). More specifically, a coherent curriculum should provide one problem with multiple solutions, multiple representations, or transitional language from informal to formal, to support teaching and learning by exploring rather than by merely presenting content. Furthermore, pedagogical support can be incorporated directly into students' materials as well as materials intended specifically for teachers. Roseman et al (2008) posed seven representations of curriculum coherence. They are:

- Taking account of prerequisite knowledge and student misconceptions to enable students to construct new understanding that builds on their prior understanding;
- Helping students appreciate the purpose of classroom activities and the content they are learning to provide an adequate level of motivation to learn;
- Using phenomena and representations to clarify the meaning of abstract ideas and to make the ideas plausible to students;

[1] In this dissertation, we regard curriculum coherence as textbook coherence. Textbooks are the main bodies of curricula. I focused on textbooks rather than curricula. Therefore, I assumed that curriculum coherence can be equally treated as textbook coherence. I then used only textbook coherence and coherent textbooks in this study.

- Helping students interpret phenomena and representations in light of the ideas;
- Promoting student reflection and application of their developing understanding;
- Using continuous assessment and student feedback to inform instruction;
- Enhancing the learning environment to enable students with different abilities and levels of preparation to experience success (pp. 26-27).

Roseman, Linn, and Koppal (2008) also argued that the content of curriculum materials was coherent when it focused on an important set of interrelated ideas and made various kinds of connections explicit. Roseman et al. (2008) claimed that the three important connective aspects contributing to curriculum coherence were: a) alignment with a coherent set of ideas; b) the connections between the ideas of science and phenomena in the natural world; c) the connections to prerequisite and other related ideas. Similarly, Schmidt, Houang, and Cogan (2004) stated that the content characteristics of a coherent curriculum referred to highly non-repetitive, focused, and challenging conceptual organization.

However, it has been documented that some US elementary mathematics teachers believed it was acceptable for children to not understand some ideas the first time (or even the second or third time) because they would learn those ideas again (Watanabe, 2007). This view is incompatible with a focused curriculum or a coherent curriculum. Such repetition (learn them again) can be replaced by standards including a learning trajectory, which links coverage of the topic throughout succeeding grade levels and reduces the repetition occurring during that time period (Houang & Schmidt, 2008). As the National Council of Teachers of Mathematics (NCTM) (2000) states, “a curriculum is more than just a collection of problems and tasks” (p. 14). A school curriculum must present and demonstrate the internal consistency and coherence of curriculum materials. Meanwhile, teachers must pay more attention to knowledge connection within and across grades to realize the internal consistency and coherence of a mathematics curriculum. Teachers are bridges between the content of textbooks and students' learning. It is essential to know teachers' knowledge for teaching. Consequently, based on the perspective of coherence, in this dissertation, I will focus on these two aspects: How do textbooks present equivalent fractions, and what is teachers' mathematical knowledge for teaching equivalent fractions?

1.1. Statement of the Studies

In my dissertation, I will focus on the two above aspects by employing the perspective of coherence: a) How do textbooks present equivalent fractions? and b) What is teachers' knowledge for teaching equivalent fractions? The first aspect includes two topics, and the second aspect includes one topic. Furthermore, I employ this comparative study to address the two issues. I will explore the issue of textbooks and teachers on equivalent fractions in terms of three stages as
Figure 1.1 shows.

In the first stage, I selected three series of featured mathematics textbooks from California, Shanghai, and Singapore. To reduce complexity while retaining the connections between mathematical knowledge, I will focus on equivalent fractions content to analyze, in detail, the characteristics of the three series of mathematics textbooks based on the perspective of coherence. This cross-national study reports the results of analyzing the characteristics of equivalent fractions in elementary mathematics textbooks. This sub-study developed a framework for textbook coherence in terms of three levels: macro-logical coherence, meso-lesson-structure coherence, and micro-example-practice coherence. Then, this framework was used to examine the similarities and differences in the coherent characteristics of equivalent fractions in the three series of mathematics textbooks.

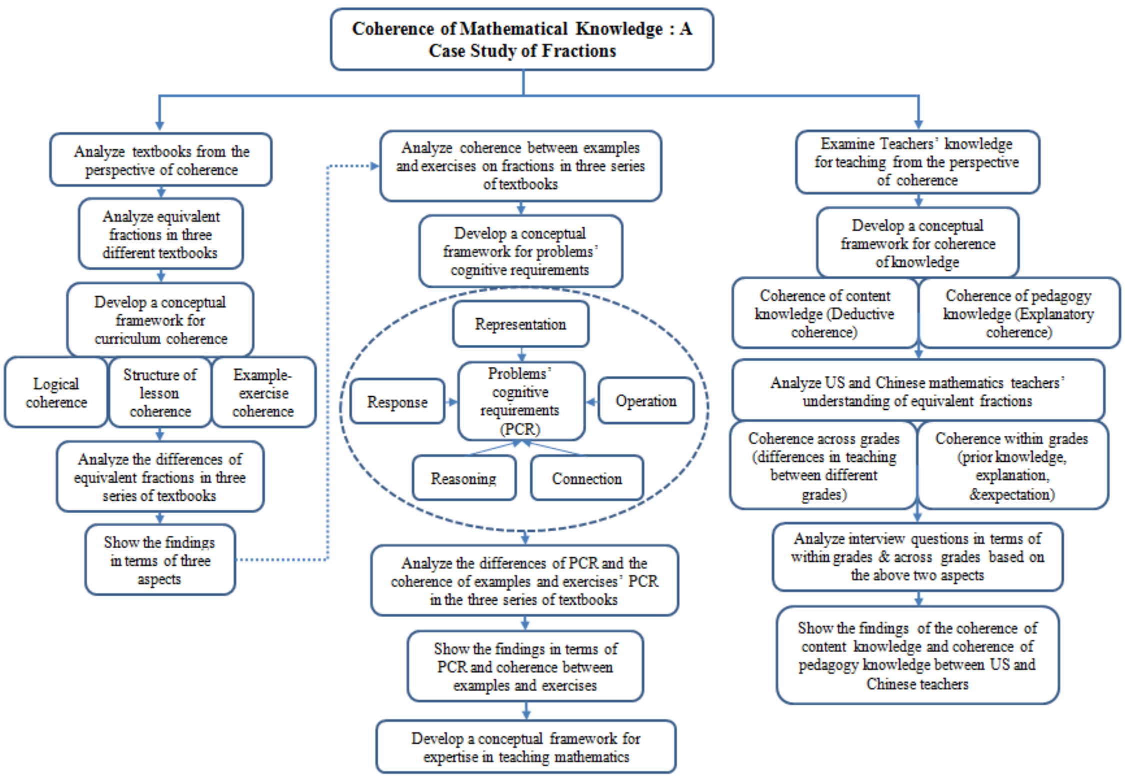


Figure 1.1 The flowchart framework of this dissertation.

In the second stage, I will focus on the micro-example-practice level to analyze the characteristics of mathematical problems by analyzing all problems in the chapters on fractions from the three series of mathematics textbooks. Because problems are important components of school mathematics textbooks, this stage is intended to examine the coherence between example and practice problems in the three series of textbooks by analyzing the coherence variation of problems. The variation of problems should be in line with students' cognitive levels. Therefore, in this sub-study, based on the prior studies (e.g., Li, 2000; Smith &Stein, 1998; Son & Senk, 2010; Wijaya, van den Heuvel-Panhuizen, & Doorman, 2015), I will develop a conceptual framework for the variation of problems including six aspects: problems' cognitive requirement, response, operation, reasoning, representation, and connection. Then I will build a model for problems' cognitive requirement including other five aspects. This sub-study is intended to explore the coherence of example and practice problems' cognitive requirements in term of the five aspects and to examine the similarities and differences in the coherence of example and practice problems from the three series of textbooks.

In the last stage, I will explore the second aspect of my dissertation based on the perspective of coherence: What is the teachers' knowledge for teaching equivalent fractions? Based on the Mathematical Knowledge for Teaching (MKT) model (Ball, Thames, & Phelps, 2008), to teach this topic, teachers should have subject matter knowledge and pedagogical content knowledge about equivalent fractions. The third sub-study is designed to examine teachers' knowledge for teaching equivalent fractions by employing a new perspective of coherence and comparing US and Chinese mathematics teachers' knowledge of equivalent fractions.

1.2. References

Ball, L. D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Chen, X., & Li, Y. (2010). Instructional coherence in Chinese mathematics classroom—A case study of lessons on fraction division. *International Journal of Science and Mathematics Education, 8*(4), 711-735.

Education Reform (2018, October). *Coherent curriculum*. Retrieved from http://edglossary.org/coherent-curriculum/

Houang, R., & Schmidt, W. (2008, September). TIMSS International Curriculum Analysis and Measuring Educational Opportunities. Paper presented at the 3rd IEA International Research Conference, Taipei, Chinese Taipei. Retrieved from http://www.iea.nl/fileadmin/user_upload/IRC/IRC_2008/Papers/IRC2008_Houang_Schmidt.pdf

Li, Y. (2000). A comparison of problems that follow selected content presentations in American and Chinese mathematics textbooks. *Journal for Research in Mathematics Education, 31*(2), 234-241.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Oxford English Dictionary (2018). *Coherence.* Retrieved from http://www.oed.com/view/Entry/35933?redirectedFrom=coherence#eid

Roseman, J.E., Linn, M.C., & Koppal, M. (2008). Characterizing curriculum coherence. In: Y. Kali, M.C. Linn, & J.E. Roseman (Eds.), Designing coherent science education. NY, NY: Teacher College Press.

Schmidt, W., Houang, R., & Cogan, L. (2004). A coherent curriculum: The case of mathematics. *Journal of Direct Instruction, 4*(1), 13-28.

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of US mathematics and science content standards from an international perspective. *Journal of Curriculum Studies*, *37*(5), 525-559.

Smith, M. S., & Stein, M. K. (1998). Reflections on practice: Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School*, *3*(5), 344-350.

Watanabe, M. (2007). Displaced teacher and state priorities in a high-stakes accountability context. *Educational Policy*, *21*(2), 311-368.

Son, J. W., & Senk, S. L. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics*, *74*(2), 117-142.

Wijaya, A., van den Heuvel-Panhuizen, M., & Doorman, M. (2015). Opportunity-to-learn context-based tasks provided by mathematics textbooks. *Educational Studies in Mathematics*, *89*(1), 41-65.

2. TEXTBOOK COHERENCE: THE CASE OF EQUIVALENT FRACTIONS IN SELECTED MATHEMATICS TEXTBOOKS FROM CALIFORNIA, SHANGHAI, AND SINGAPORE

2.1. Introduction

It is commonly acknowledged that textbooks, as important materials, play a significant role in teaching. Generally, textbooks provide teachers with content; and they also serve as the bridges between students and teachers, as teachers use textbooks to develop their lesson plans (Thompson, Senk, & Johnson, 2012). Mathematics textbooks affect the content that teachers cover in class, the order of the progression of topics, and the ways by which to present mathematical content in teaching (Stein, Remillard, & Smith, 2007). In the United States, two-thirds of middle school mathematics teachers reported that they used one mathematics textbook all or most of the time and taught at least three-fourths of the textbook in a given year (National Research Council, [NRC], 2001); “the correlation between textbook coverage and what teachers teach is 0.95” (Schmidt, Houang, & Cogan, 2002, p. 8). Similarly, in China and Singapore, each student has a student-version mathematics textbook for each school year. Teachers must cover all the content of mathematics textbooks during a certain period. Furthermore, many studies have provided the evidence for the statement that a significant relationship exists between students’ use of textbooks and their achievement (e.g., Agodini et al., 2010; Bhatt & Koedel, 2012). Therefore, it is essential for teachers to use a series of high-quality mathematics textbooks that can be "a powerful catalyst for improving learning for students and teachers alike" (Roseman, Stern, & Koppal, 2010, p.48).

Many studies have reported the characteristics of content presentation in different textbooks across countries in the past 10 years (e.g., Amal, 2011; Charalambous Delaney, Hsu, & Mesa, 2010; Fan & Zhu, 2007; Li, 2000; Son & Senk, 2010; Thompson et al., 2012; Wijaya, van den Heuvel-Panhuizen, & Doorman, 2015). The results from these studies presented the advantages and disadvantages of mathematics textbooks in different countries through textbook comparison. It is commonly acknowledged that coherence is a foundational requirement for textbooks (Roseman et al., 2010). As the Common Core State Standards Initiative (CCSSI) stated, "mathematics is not a list of disconnected topics, tricks, or mnemonics; it is a coherent body of knowledge made up of interconnected concepts" (CCSSI, 2017). Coherence is essential for a set of quality textbooks. However, few studies have employed the perspective of coherence to compare and analyze textbooks (e.g., Schmidt, Houang, & Cogan, 2004; Schmidt, Wang, & McKnight, 2005). One reason may be that, due to the complex nature of mathematical knowledge, no consensus has been reached on the definition of "coherence." This conceptual issue becomes even more complicated when one wants to compare different education systems. Second, mathematics educators still have not developed tools for textbook analysis from the perspective of coherence. Therefore, there is both theoretical and practical significance in analyzing the presentation of textbook content in different education systems from a perspective of coherence.

To simplify the research range, I selected the single topic of equivalent fractions because this topic is usually introduced in the third or fourth grade and reviewed later in both fifth and sixth grades. To be specific, on the one hand, knowledge of equivalent fractions can be used to check students' understanding of fractions in terms of different pictorial models (e.g., area model, length model, set model, and number line) in the elementary stage. Therefore, textbooks can arrange the topic of equivalent fractions after the definition of fractions. On the other hand, equivalent fractions are the foundation of the addition and subtraction of unlike fractions. Students are then expected to master equivalent fractions before learning four operations of fractions. Thus, the topic of equivalent fractions can be arranged before the topic of the addition and subtraction of unlike fractions. According to the Common Core State Standards (Common Core State Standard Initiative, 2010), equivalent fractions instruction begins in third grade and continues through middle school in the US. Meanwhile, equivalent fraction instruction begins in third grade and stops in sixth grade in Shanghai (Shanghai Department of Education, 2017). The Singapore syllabus requires students to learn equivalent fractions in third grade. Therefore, the topic of equivalent fractions is an appropriate medium through which to study the presentations of mathematical content in different textbooks from different education systems.

For this study, based on the definition of curricula coherence in Schmidt et al. (2002) and Cuoco and McCallum (2018), I developed a framework for textbook coherence. Using this framework, I compared the similarities and differences among the presentations of equivalent fractions in California, Shanghai, and Singapore mathematics textbooks. This study provides a whole picture of equivalent fractions in the three series of textbooks. The findings will hopefully not only provide teachers and mathematics educators with the representations of equivalent fractions in three textbook series, but also provide mathematics educators with several perspectives on textbook coherence.

2.2. Literature Review

2.2.1. The Studies of Mathematics Textbooks

Mathematics textbooks received more attention in international mathematics education communities after Schmidt and his colleagues' work was published (e.g., Schmidt & Houang, 2012; Schmidt, McKnight, Calverde, Houang, & Wiley, 1997; Schmidt, Wang, & McKnight, 2005, 2012). Many studies have analyzed the characteristics of textbooks by comparing textbooks from different countries (e.g., Fan & Zhu, 2007; Li, 2000; Wijaya, van den Heuvel-Panhuizen, & Doorman, 2015). These studies used the method of content analysis to analyze two main aspects of textbooks: narrative content and problems. For example, Fan and Zhu (2007) focused on the analysis of problem-solving procedures in selected mathematics textbooks in China, Singapore, and the US. Li (2000) analyzed the presentation of problems in mathematics textbooks from China and the US by developing a three-dimensional framework to illustrate the similarities and differences in expectation related to students' mathematics experiences. Wijaya et al. (2015) employed the perspective of the opportunity-to-learn to analyze four aspects of problems in three Indonesian textbooks. Additionally, Porter (2006) explored the characteristics of high-/low-quality textbooks by using a framework of two-dimensional language presented in a rectangular matrix with topics as rows and cognitive demands (performance goals or performance expectations) as columns to describe mathematical content. However, few studies employed the perspective of coherence to analyze mathematics textbooks (e.g., Schmidt et al., 2004; Schmidt et al., 2005). Therefore, this study will examine the presentation of mathematical content by employing the perspective of coherence.

2.2.2. Textbook Coherence

Mathematics is presented as a coherent, consistent, structured, hierarchical, and organized knowledge system of concepts, laws, and propositions (Suppe, 1977; Koponen & Pehkonen, 2010). The Common Core State Standards Initiative (CCSSI) stated that "mathematics is not a list of disconnected topics, tricks, or mnemonics; it is a coherent body of knowledge made up of interconnected concepts" (CCSSI, 2017). Therefore, coherence is a necessary feature for mathematics textbooks

Coherence can ensure that the system of knowledge is a whole rather than a mixed set of concepts, theorems, propositions, or models (Koponen & Pehkonen, 2010). Coherent knowledge is relatively readable and understandable for learners. The coherentists claim that knowledge is a globally connected system with knowledge of structure. Most importantly, when new knowledge is added to the system, the large parts of the structure are involved, and the structure itself is also affected by the new knowledge (Kosso, 2009). Therefore, coherence between the different content becomes essential for a textbook that includes different topics of content within/across grades.

CCSSI (2017) also defines coherence on mathematics curriculum as "linking topics and thinking across grades." CCSSI further explains:

> The standards are designed around coherent progressions from grade to grade. Learning is carefully connected across grades so that students can build new understanding onto foundations built in previous years..... Each standard is not a new event, but an extension of previous learning. Coherence is also built into the standards in how they reinforce a major topic in a grade by utilizing supporting, complementary topics. For example, instead of presenting the topic of data displays as an end in itself, the topic is used to support grade-level word problems in which students apply mathematical skills to solve problems (CCSSI, 2017).

The coherence that CCSSI describes focuses on the structure of mathematics and the natural pathways through that structure. The natural pathways should follow the logical sequence and cognitive development in designing the sequence of mathematics content. Meanwhile, Koponen and Pehkonen (2010) claimed that the coherence of knowledge is involved in explanatory and deductive coherence. They further said, "The explanatory coherence is quite naturally connected to the methodology of the experiments [in science]. The deductive coherence, on the other hand, is closely related to the deductive use of models and model-type symbolic relations." (p. 262)

Some studies have proposed a framework for curriculum evaluation (e.g., NRC, 2004; Reys, Reys, & Chavez, 2004; Tarr, Reys, Barker, & Billstein, 2006). NRC (2004) proposed a framework for curriculum evaluation including three major components: "(a) the program materials and design principles; (b) the quality, extent, and means of curricular implementation; and (c) the quality, breadth, type, and distribution of outcomes of student learning over time" (p. 4). However, this framework did not include any detailed instructions for each aspect. When employing the framework to assess textbooks, users must connect these components into a research design.

Tarr et al. (2006) proposed a relatively detailed framework for judging the quality of textbooks, which included three aspects: mathematics content emphasis, instructional focus, and teacher support. The first aspect is to examine the degree to which textbooks align with accepted curriculum standards. The second aspect is to examine the degree to which textbooks provide teachers with problems, activities, and investigations that engage students in inquiry-based learning, stimulating classroom discourse, and fostering mathematical reasoning. The last aspect is to examine the extent to which textbooks offer teachers insights and background materials with which to engage students in exploring, creating, and generalizing mathematical ideas, as well as provide support materials for teachers to plan and implement instruction. Although Tarr et al.'s (2006) framework provides detailed instruction on each aspect, this framework does not include our focal aspect: coherence.

Similarly, Reys, Reys, and Chavez (2004) stated that textbooks should present materials coherently, develop ideas in depth, engage students, and motivate learning. They suggested that teachers who assess a textbook should address the following questions:

(a) What key mathematical ideas in each content strand should each grade level address?
(b) How does the content of the textbook align with these key mathematical ideas?
(c) What types of activities does the textbook provide? Are students challenged to think and develop understanding, or are they simply shown how to work some exercises and then asked to practice procedures? Will these activities engage students in mathematical thinking and activities?

> (d) Is there a focus on mathematical thinking and problem solving? Are students expected to explain "why"? Does the textbook encourage students to explore "what if" questions and to offer and text conjectures? (p. 65)

However, Reys et al. (2004) did not state what a coherent approach to present content was, although they mentioned that textbooks should present mathematical content coherently. As NCTM (2000) states, "a curriculum is more than just a collection of problems and tasks" (P. 14). Curriculum editors must pay close attention to the internal consistency and the coherence of curriculum materials.

Schmidt, Wang, and McKnight (2005) stated that content standards were coherent if "they are articulated over time as a sequence of topics and performances consistent with the logical and, if appropriate, hierarchical nature of the disciplinary content from which the subject-matter derives" (p. 528). In other words, textbook coherence must involve different grade levels, such as within a grade or across grades. Therefore, the content characteristics of a coherent textbook must be a highly non-repetitive, focused, and connected conceptual organization (Schmidt, Houang, & Cogan, 2004).

2.2.3. The Conceptual Framework for Textbook Coherence and Research Questions

Schmidt et al. (2005) defined content as coherent "if they are articulated over time as a sequence of topics and performances consistent with the logical and, if appropriate, hierarchical nature of the disciplinary content from which the subject-matter derives" (p. 528). Cuoco and McCallum (2018) further elaborated on this definition from Schmidt et al. (2005) in two directions. One, Cuoco and McCallum distinguished between standards and curriculum; two, they proposed coherence of practice as another aspect of curriculum coherence. That is, this new elaboration includes two aspects: coherence of content and coherence of practice. A coherent sequence of topics might be achieved in three specific ways: logical sequencing, the evolution from particulars to deep structures, and the use of deep structures to make connections. Additionally, the aspect of coherence of practice includes using structure and abstraction. Based on the above elaboration of coherence, I developed a framework including three aspects: the macro-logical coherence, the meso-lesson-structure coherence, and the micro-example-practice coherence. Following, I describe each of the levels.

The first aspect for analysis, the macro-logical coherence corresponds to the logical sequences of Cuoco and McCallum (2018). I not only analyzed the content requirement or learning objectives in the curriculum standards based on the assumption that the curriculum standards were coherent but also used the table of contents to obtain the sequences of topics. In this way, I examined the logical progression of the treatments of equivalent fractions in each textbook series. Next, for the meso-lesson-structure coherence, the detailed treatment of equivalent fractions in each set of textbooks was used to analyze the structure of individual lessons on equivalent fractions. The presentation includes the sequences of topics that make up the components of the lessons about equivalent fractions. The purpose of this meso-level is to examine whether the structure goes through the evolution from particulars to deep structures and uses deep structures to make connections.

Table 2.1 The Conceptual Framework for Curriculum Coherence.

Aspect	Categories
Macro-logical coherence: Assuming that the curriculum standards on mathematical content sequence are logically coherent, check the logical coherence of this topic by comparing it with the curriculum standards.	
Curriculum standards	Compare the content requirement or teaching objectives for the sequence of lessons equivalent fractions to the curriculum standards.
Table of contents	Use the table of contents to obtain the sequences of topics.
Meso-lesson-structure coherence: Examine the structural coherence of equivalent fractions by analyzing the evolution of equivalent fractions in the textbooks and by checking whether the content evolves from particular to deeper structures and uses deep structures to make connections (Schmidt et al, 2005; Cuoco & McCallum, 2018).	
Particular to deeper structures	The method of presenting the concept of equivalent fractions and the Equivalent Fractions Rule* in a lesson
Deep structures to make connections	Use the content of equivalent fractions across the lessons in the textbooks to compare lesson structures on equivalent fractions, which include the components of the fractions lessons and their connections.
Micro-example-practice coherence: Examine the coherence by analyzing and comparing the similarities and differences in the five aspects between examples and practices.	
Representations	Concrete: real world with/without pictures. Pictorial: pictures with information. Abstract: written language or purely mathematics context. Translate among concrete, pictorial, and abstract representation.
Response types	Numerical answers. Numerical expressions. Explanation or solution or justification.
Computation	No computation. Simple computation, fewer than three steps computation. Complicated computation, more than three steps.
Connection	No connection. Connecting to one mathematical concept. Connecting to more than one mathematical concept.
Cognitive requirements	Recall/reproduce. Basic application of skills/concepts. Reflection.

Note. Equivalent Fractions Rule refers to the rule of equivalent fractions, $a/b = (a \times n)/(b \times n) = (a \div m)/(b \div m)$, $b \neq 0$, $m \neq 0$, $n \neq 0$.

Lastly, the micro-example-practice problem coherence corresponds to the coherence of practice in Cuoco and McCallum (2018). To analyze this, I coded all example and practice problems in terms of five aspects: representation, response, computation, connection, and cognition requirements (see Table 2.1). Then I analyzed and compared the coded values of all examples to the values of all practice problems.

The first aspect is *representation.* This refers to how problems are presented, and whether any patterns of problem representations (daily language, pictures, symbol or abstract mathematical language) exist across grade levels (Ding 2016). The second aspect is *response.* The types of responses are classified as numerical answers, numerical expression, and explanation-solution-justification (Li, 2000; Son & Senk, 2010). The third aspect is *computation.* The categories for this are no computation, simple computation, and complicated computations (Li, 2000; Son & Senk, 2010). The fourth aspect is *connection.* The connections of problems refer to whether or not the problems are connected to other knowledge across lessons. For example, a problem may be connected to some situations outside of school or be connected to other mathematical knowledge within a unit or between units/chapters (Charalambous et al., 2010). The categories for this are no connection, connecting to one mathematical concept, and connecting to more than one mathematical concept. The fifth aspect is *cognition*. The cognitive level of a problem is one of the important aspects of the nature of problems. The categories for cognitive level are recall, understanding, and reflection (e.g., Wijaya et al., 2015).

As mentioned above, this framework includes three levels: macro-logic coherence, meso-lesson-structure coherence, and micro-example-practice coherence. The framework is employed to answer the question: What are the similarities and differences between the presentation of equivalent fractions in California, Shanghai, and Singapore mathematics textbooks? To be specific, this study will address the following research questions:

- What is the logical sequence of equivalent fraction concepts in each of the three selected mathematics textbooks?
- How are the lessons on equivalent fractions structured, and how are they introduced and developed?
- What are the characteristics of examples and practice problems presented in these textbooks?

2.3. Method

2.3.1. Textbook Selection

This study selected three series of mathematics textbooks from California, Shanghai, and Singapore. As mentioned previously, this study focused only on the analysis of equivalent fractions. Therefore, the corresponding units of equivalent fractions were selected for analysis in this study. In addition, the three regions have their own specific system contexts. Following, I introduce background information about the three regions and three series of textbooks.

First, California is one of the richest and most populous states in the United States. A number of important education reforms have been implemented in California in recent years (Kirst, 2017[2]). In the US, each state has its own independent education system and each district has the right to recommend several series of textbooks for teachers. Then, teachers have the right to select the appropriate textbooks for their students. The Macmillan/McGraw textbooks are the commonly used in California (Koedel, Li, Polikoff, Hardaway, & Wrabel, 2016). Hence, the series of *California Mathematics* - *Macmillan/McGraw-Hill* (Altieri et al., 2007 G2-6) was selected for use in this study.

Similarly, Shanghai is one of the largest cities in Asia and has its own education system and mathematics curriculum standard, while China has a national education system. Only one series of mathematics textbooks in Shanghai is published by the Shanghai Education Press (SEP) (Huang, Ye, Tong, Song, Ju, & Xu, 2014; Qiu, Huang, Zhang, Ke, Xia, Xu et al. 2014). The selected SEP is a reform textbook series based on the new Shanghai G1-12 Mathematics Curriculum Standard (Shanghai Education Press, 2004). The content of equivalent fractions in the SEP textbook series and their corresponding curriculum standards were selected. The reasons for the selection of Shanghai textbooks rather than other Chinese textbooks are as follows. First, Shanghai is the only city on the Chinese mainland that has participated in international assessments several times in the past (e.g., PISA, 2009, 2012, 2015). Second, Shanghai students use only one set of mathematics textbooks that Shanghai Education Press published. Third, the school minister of the UK, Nick Gibb, announced that training would be provided for

[2] http://iel.org/aera-iel-educational-policy-forums.

8,000 English primary schools--half the country's total--to switch to the Shanghai "mastery' approach. Meanwhile, Shanghai textbooks have been imported into the UK. According to the statement of the School Minister of the UK, this teaching style would become standard in the UK, with the purpose of preventing British youngsters from falling behind their Asian counterparts. Thus, it is essential to analyze the characteristics of the presentation of mathematical content in Shanghai textbooks.

Third, Singapore is a unique country with a very small territorial area and population. English is the first language and instruction language, but it is compulsory for most students to learn Standard Mandarin, Malay, or Tamil--the other official languages of Singapore--in school (Best Singapore Guide, 2017). The US, the UK, and many other nations have imported the Singaporean mathematics curriculum into the elementary stage to improve the quality of their mathematics curricula. Singapore education is strongly influenced by the Confucian culture and integrated includes an integration of Western educational ideas. Therefore, the Singapore education system could be considered a mixed system combining the Confucian culture and Western culture. The series of *My Pals are Here* (2nd Edition) was selected and is one of the most commonly used texts in Singapore (Yan, Reys, & Wu, 2010). Furthermore, the Singapore elementary mathematics syllabus was selected. Table 2.2 shows detailed background information about the differences in educations and textbooks among the three regions.

Table 2.2 Background Information.

Item	California	Shanghai	Singapore
Education system	De-centralized	Centralized	Centralized
Primary education	Grades 1-5	Grades 1-5	Grades 1-6
Academic year	Around 36 weeks	Around 34 weeks	Around 36 weeks
Time allotted to mathematics (weekly)	240 minutes	140 minutes (140 minutes may be allocated to different subjects, including mathematics)	210 minutes to 390 minutes (based on grade level)
Curriculum	Selected	No choice	Selected
Available textbooks	Multiple; published by publishing companies	One; published by Shanghai Education Press	Multiple; published by publishing companies
Price	Provided by schools	Bought by parents (The majority of parents can afford them.)	Bought by parents
Available curriculum materials	Student books; curriculum standards	Student books; teacher manuals; curriculum standards	Student books; curriculum syllabus
Textbooks analyzed in this study	California Mathematics Grade 2-4 (Macmillan McGram-Hill) (Altieri et al., 2007)	Mathematics Grade 3B and Grade 6A (Shanghai Education press) (Qiu et al., 2016)	My Pals are Here (2nd Edition) Grade 3B (Marshall Cavendish Education) (Kheong, Ramakrishnan, & Soon, 2012)
Total number of pages; page size	G3:679; 28cm×22cm; G4:681; 28cm×22cm.	G3B: 77;19cm×26cm; G6A:120;21cm×30cm.	G3B:174; 19cm×26cm.

2.3.2. Code and Analysis

According to the framework for coherence, I analyzed the presentation of the equivalent fractions in the three series of mathematics textbooks from California, Shanghai, and Singapore, then compared the similarities and differences in their presentations. To be specific, first, I analyzed the alignment between students' learning objectives and curriculum standards to answer the first question. Next, to answer the second question, I drew the structure of all content focusing on the introduction of equivalent fractions based on the headings of content in each lesson. Finally, I coded the five aspects of examples and practices, then showed the comparative results in examples and the practices' five aspects. During the process of coding, I coded each problem separately even if several sub-problems focused on the same skill or concept. For example, I coded No. 1-3 in Figure 2.1 as three single problems.

Find equivalent fractions

1. $\frac{1}{2} = \frac{()}{8}$ 2. $\frac{()}{3} = \frac{6}{9}$ 3. $\frac{()}{4} = \frac{()}{12}$

Figure 2.1 Samples of mathematical problems.

To ensure the reliability of the coding for the five aspects of problems, I first coded all the problems, and then recoded problems after two months. Finally, I invited a graduate student to code the same problems using the same coding framework. This coding achieved a 90% agreement between the two coders. The disagreement codes were solved after discussion between the coders.

2.4. Results

In this section, I report the findings following the sequence of three research questions parallel to the three coherence levels in the conceptual framework for textbook coherence. I also use a 3-by-3 manner to show the findings in terms of the coherence of equivalent fractions in the three series of textbooks: California, Shanghai, and Singapore. In each subsection, I show the findings about California, Shanghai, and Singapore textbooks, separately.

2.4.1. The Logical Coherence of Equivalent Fractions

2.4.1.1. Coherence Between Content and Curriculum Standards

The California Common Core Standard (CCCS) states that the content of equivalent fractions should be taught in grades 3 and 4 (see Table 2.3). CCCS emphasizes the use of visual fraction models (e.g., number line model and area model) to generate equivalent fractions and explain the Equivalent Fractions Rule. For example, CCCS states, “Recognize and generate simple equivalent fractions. Explain why the fractions are equivalent (e.g., by using a visual fraction model).”

Shanghai curriculum standards do not provide any detailed requirements for equivalent fractions. They summarize the key points based on several grade levels rather than on each grade. For example, regarding the requirements of four operations of fractions, the standard states that students must master the addition and subtraction of unlike fractions and the multiplication and division of fractions, as well as have a preliminary understanding of transformation thinking. Therefore, I checked the teacher manual[3] and found learning objectives for equivalent fractions. Shanghai textbooks state that fourth-graders must understand equivalent fractions in terms of observing pictures while sixth-graders must know the mathematical derivation of the Equivalent Fraction Rule (EFR). For example, "preliminary know equivalent fractions by observing pictures" in grade 4; "Understand the principle by understanding the relationship between fractions and division and the principle of division [$a \div b = (a \times k) \div (b \times k) = (a \div n) \div (b \div n)$, ($b \neq 0$, $k \neq 0$, $n \neq 0$)]." Shanghai textbooks also highlight the application of EFR in solving real-world problems in sixth grade (see Table 2.3).

However, the Singapore curriculum syllabus states that students should identify and convert equivalent fractions. To be specific, students must generate the first eight equivalent fractions when giving a fraction, fill out some denominators or numerators (e.g., 2/3= ( )/9), and simplify fractions. The Singapore curriculum syllabus emphasizes the application of EFR in different formats to generate equivalent fractions. The three series of textbooks each use different learning requirements, but all have two emphases: the understanding and the application of equivalent fractions.

---

[3] For Chinese mathematics teachers, the teacher manuals are their most important teaching materials. The editors of the textbooks designed the textbooks based on the curriculum standard. Few teachers check the content requirements in the curriculum standard when they teach. The teacher manuals can help them understand the structure of textbooks and the design ideas of those textbooks. All teachers will design their instruction based on the materials in the teacher manuals. For example, the teaching objectives in the teacher manuals are used to guide teachers in designing key ideas in a lesson. In addition, all mathematics teachers used the same textbooks and must teach all the content in the textbooks based on the teacher manuals. From this perspective, teacher manuals can be used instead of curriculum standards to guide teachers in designing instructions.

Table 2.3 The Requirements of Curriculum Standards on Equivalent Fractions.

	Curriculum standard	G3	G4	G6	Y/N
California	Understand two fractions as equivalent (equal) if they are the same size or on the same point on a number line	Ca			Y*
	Recognize and generate simple equivalent fractions. Explain why the fractions are equivalent (e.g., by using a visual fraction model)	Ca			Y
	Express whole numbers as fractions and recognize fractions that are equivalent to whole numbers (Locate 4/4 at the same point of a number line)	Ca			N
	Explain why a fraction a/b is equivalent to a fraction $((n \times a)/(n \times b))$ by using visual fraction models.		Ca		Y**
	Use EFR to recognize and generate equivalent fractions		Ca		Y
Shanghai	Preliminarily know equivalent fractions by observing pictures		Sh		
	Further know equivalent fractions		Sh		Y
	Understand EFR by understanding the relationship between fractions and division and the principle of division			Sh	Y
	Express a fraction in its simplest form using the principle			Sh	Y
	Understand the concept of a fraction in its simplest form			Sh	Y
	Use EFR to solve sample word problems			Sh	Y
Singapore	Recognize and name equivalent fractions	Si			Y
	List the first eight equivalent fractions of a given fraction	Si			Y
	Write the equivalent fraction of a fraction given the denominator or the numerator	Si			Y
	Express a fraction in its simplest form	Si			Y

Note. G = Grade; Ca = California; Si = Singapore; Sh = Shanghai, Y = Yes; N = No. *California textbooks do not use number line model. There are no opportunities for students to understand two equivalent fractions by identifying the same point on a number line. **California textbooks do not show the abstract expression of equivalent fractions: $(n \times a)/(n \times b)$.

2.4.1.2. Logical Sequence in the Table of Contents

To examine the logical sequence, I listed the table of contents in the chapter containing equivalent fractions (see Table 2.4). Following the different curriculum standards, the equivalent fractions content was arranged in the corresponding grade levels. Finally, I compared the table of contents and the statements of equivalent fractions from curriculum standards.

California textbooks separately introduce equivalent fractions in grades 3 and 4. In addition, two special sections of “Explore” and “Game Time” are included for grades 3 and 4. The content of equivalent fractions cover a total of 16 pages (0.9%, N = 679[4], in grade 3; 1.5%, N = 681, in grade 4, 2.3% total). In grade 3, the content of equivalent fractions is introduced before the topic of the comparison of fractions. Therefore, the third-grade textbook might expect students to understand equivalent fractions based on the understanding of fractions as a part-whole relationship. The textbook in grade 4 has a similar topic sequence to that in grade 3. Therefore, according to only the table of contents, the differences between the two grades are not apparent. The content of equivalent fractions in grade 3 and 4 seem to be repetitive to some degree (see the arrow lines in Table 2.4).

[4] N refers to the total number of the pages for this textbook.

Table 2.4 The Content Structure of a Chapter on Fractions Including Equivalent Fractions in Each Textbook.

	Grade 3	Grade 4
California	Chapter 12 Fractions (6 pages) 1. Parts of a whole 2. Problem-Solving-Investigation: Choose a strategy ***Explore: Math Activity: Equivalent fractions*** ***3. Find equivalent fractions*** ***Game Time: Fraction concentration*** 4. Problem-Solving-Strategy: Draw a picture 5. Compare fractions Explore: Math Activity: Add like fractions[@] 6. Add like fractions Problem-Solving in Science: The buzz on insects Explore: Math Activity: Subtract like fractions 7. Subtract like fractions	Chapter 13 Fractions (10 pages) 1. Parts of a whole 2. Parts of a set 3. Problem-Solving Strategy: Draw a picture ***Explore: Math Activity: Equivalent fractions*** ***4. Equivalent fractions*** ***5. Simplest form*** ***Game Time: Fractions made simple*** 6. Problem-Solving Investigation: Choose a strategy 7. Compare and order fractions Problem Solving in Science: No bones about it 8. Add and subtract fractions 9. Mixed numbers
Shanghai	Grade 4 Fractions (1 page) 1. Compare fractions (1) Compare like fractions (2) Compare fractions with the same numerator (3) ***Equivalent fractions**** 2. Add and subtract fractions 3. Explore: "Fraction wall" (1) Compare two fractions (2) Add and subtract like fractions (3**)** ***Equivalent fractions****	Grade 6 Chapter 2 Fractions (8 pages) 1. Fractions and division ***2. Equivalent Fractions Rules*** 3. Comparing fractions 4. Addition and subtraction of fractions 5. Multiplication of fractions 6. Division of fractions 7. Conversion between fractions and decimals 8. Four operations including fractions and decimals 9. Word problems on fractions
Singapore	Grade 3 Chapter 14 Fractions (6 pages) 1.Numerator and denominator ***2. Understanding equivalent fractions*** ***3. More equivalent fractions: Short cut*** 4. Comparing fractions 5. Adding fractions 6. Subtracting fractions	

Note. [@] "Like fractions" refers to the fractions with the same denominator. The content of equivalent fraction* in Shanghai textbooks is extended to fourth graders.

The content of equivalent fractions in Shanghai textbooks is divided mainly into two sections: two pages (two extended lessons) in grade 4 and eight pages (three lessons in one unit) in grade 6 (4.0%, $N = 120 + 130 = 250$). The content in grade 4 emphasizes that "equal" is one of the categories about the comparison of fractions, while the content in grade 6 emphasizes EFR, that is, equivalent fractions are the results of the application of the EFR. The derivation of EFR must use the relationship between fractions and division. Therefore, the prior knowledge of learning Equivalent Fraction Rule is *Fractions and Division*. The detailed information is shown in Table 2.4. Combined with the requirements of "sixth graders must know the mathematical derivation of the rule of equivalent fractions" (see Table 2.3), Shanghai textbooks put more emphasis on logical coherence in mathematics.

However, Singapore textbooks have only two units that occupied six pages in grade 3 (1.9 %, $N = 174+136 = 310$). Based on the table of contents, the content of equivalent fractions is included only in grade 3. The concept of equivalent fractions is introduced after the concepts of numerator and denominator. EFR appears in the title of *More Equivalent Fractions: Short Cut*. Therefore, I can infer that equivalent fractions are regarded as one of a basic concept in Singapore textbooks. The sequence of topics for equivalent fractions is similar to those in California textbooks. After learning equivalent fractions, students learn to compare, add, and subtract fractions in grade 3. Therefore, Singapore textbooks comprehensively introduce equivalent fractions in grade 3, while California and Shanghai textbooks distribute the content of equivalent fractions over two different grades.

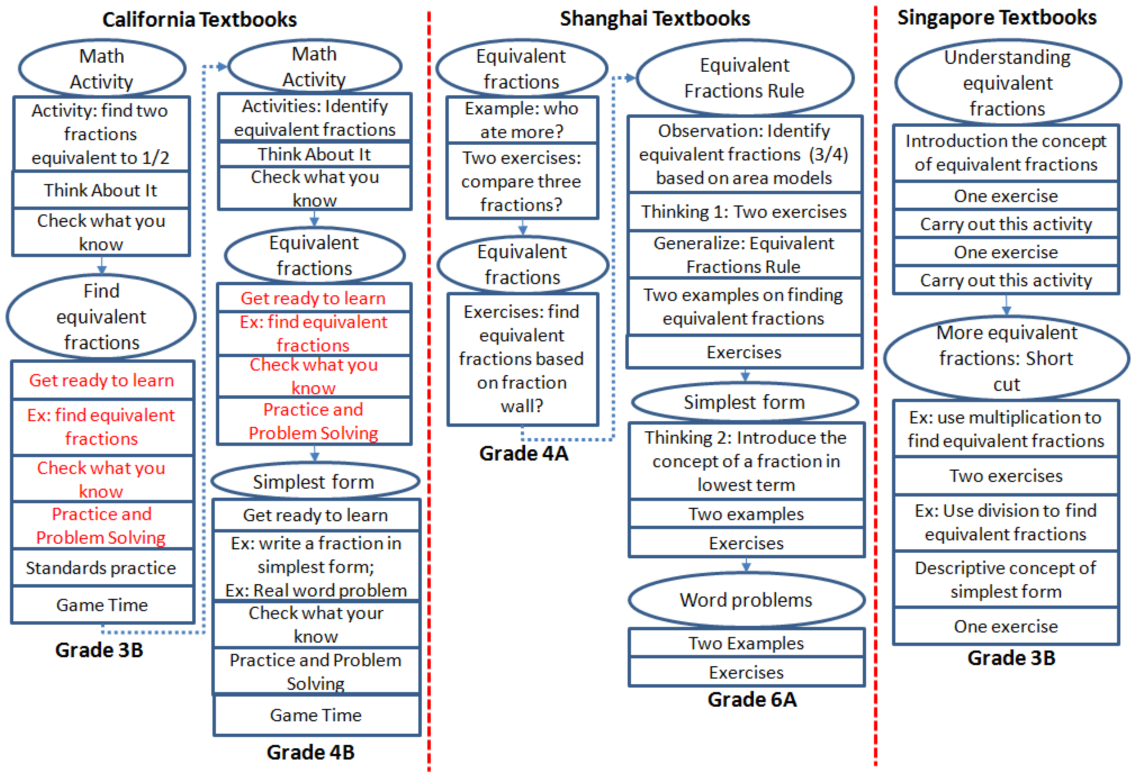


Figure 2.2 The sequence of equivalent fractions in the three series of textbooks.

Note. The elliptical rings show the titles of the lessons while the rectangles show the headings of the sections in those lessons.

2.4.2. The Coherence of Lesson Structure

Based on the content of equivalent fractions, I placed the complete lesson structures in a map as seen in Figure 2.2. The elliptical circles represent the titles of the lessons while the rectangles show the headings of the sections in each lesson. For example, the content of equivalent fractions in California textbooks of grade 3B includes two lessons: *Math Activity* and *Find Equivalent Fractions*. The lesson of *Math Activity* includes three sections: Activity*: Find Two Fractions Equivalent to 1/2, Think About It,* and *Check What You Know* (see Figure 2.2 in detail). I will introduce the lesson structures of equivalent fractions in California, Shanghai, and Singapore textbooks as follows.

2.4.2.1. California Textbooks

Before and after introducing new content, a special activity and game were arranged respectively. The unit of equivalent fractions in California textbooks includes two lessons in grade 3: *Math Activity* and *Find Equivalent Fractions*. The structure of Math Activity consists of *Activity*, *Think About It*, and *Check What You Know*. Generally, the structure of *Math Activity* is similar to the structure of a formal lesson. The activity in *Math Activity* is similar to the example in the lesson of *Find Equivalent Fractions*. The activity is to use length models to generate two equivalent fractions equal to 1/2 by partitioning rectangles in three ways. *Think About It* includes several problems that help students understand the relationship between the unit fractions of two equivalent fractions. *Check What You Know* includes a set of practice exercises that focus on using fraction models to find equivalent fractions. *Math Activity* paves the way for students to learn the lesson of *Equivalent Fractions* from the perspective of drawing fraction models (e.g., length models and area models).

The second lesson of *Find Equivalent Fractions* consists of *Get Ready to Learn, Example, Check What You Know, Practice and Problem Solving*, and *Game Time* (see Figure 2.3). *Get Ready to Learn* shows a real-world problem. Then the definition of equivalent fractions was described without any explanation. *Example* shows a solution to the problem from *Get Ready to Learn* by using length models and area models. *Check*

What You Know shows several practices that are very similar to the example. *Practice and Problem Solving* includes various kinds of problems. To be specific, the first half of the problems is similar to the exercises from *Check What You Know*. The second half of the problems called *H.O.T. Problem* includes open-ended problems. Finally, *Game Time* describes a game focused on newly learned content.

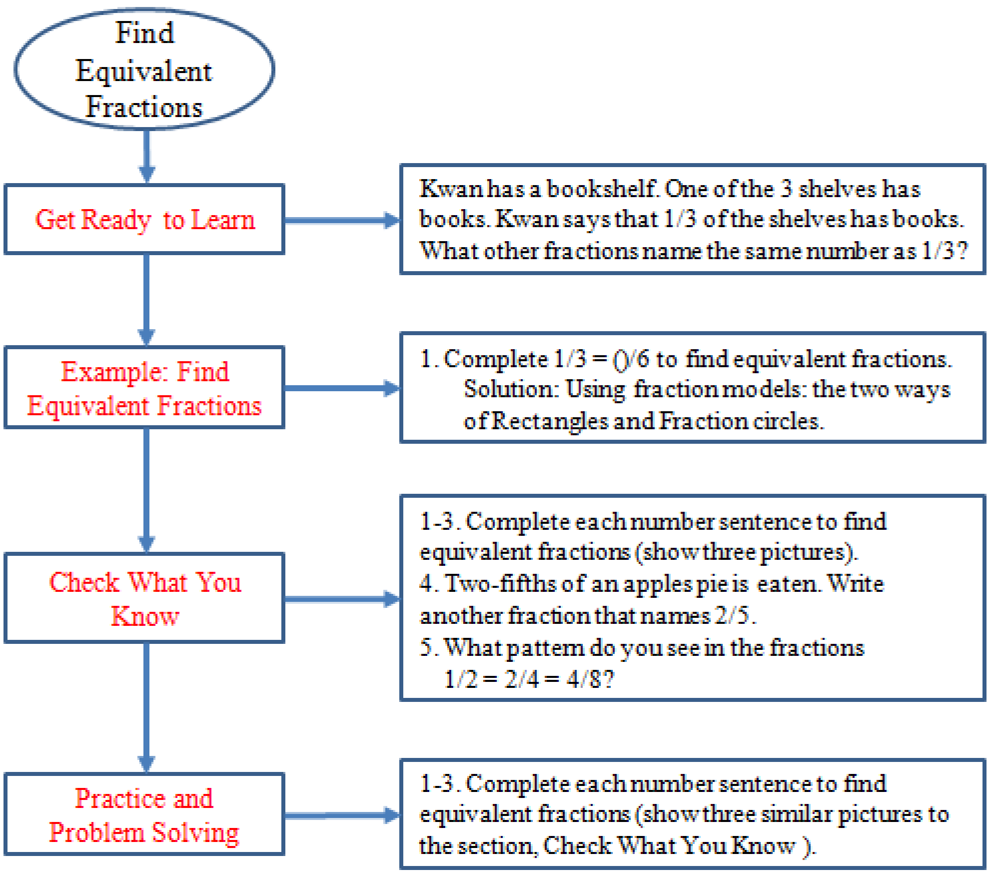


Figure 2.3 The content of equivalent fractions adapted from California mathematics grade 3 (Altieri et al. 2007, pp. 516-518).

The content on equivalent fractions includes three lessons in grade 4: *Math Activity*, *Equivalent Fractions*, and *Simplest Form*. The structure of the first lessons is similar to that of the *Math Activity* while the structures of two other lessons are the same as that of the second lesson of *Find Equivalent Fractions* in grade 3. The only difference between the two first lessons in different grades is that the first lesson in grade 4 added an example of using number line models to identify two equivalent fractions. The differences between the second two lessons are that grade 3 emphasized using fraction models to find and identify equivalent fractions while grade 4 emphasized using the methods of multiplication and division to find equivalent fractions. However, grade 4 does not show the rule of equivalent fractions. As Figure 2.4 shows, the example gives the solutions directly rather than provide an opportunity for students to explore the method of "multiplication and division." (see Altieri et al. 2007, California mathematics G4, p. 518)

Example

1. Find three fractions that are equivalent to 4/8.

To find equivalent fractions, you can use multiplication or division.

One way:

$$\frac{4}{8} \times \frac{2}{2} = \frac{8}{16}$$

Multiply the numerator and the denominator by the same number, 2.

Another way:

$$\frac{4}{8} \div \frac{2}{2} = \frac{2}{4}$$

$$\frac{2}{4} \div \frac{2}{2} = \frac{1}{2}$$

Divide the numerator and the denominator by the same number, 2.

Figure 2.4 The content adapted from California Mathematics grade 4 (Altieri et al. 2007, p. 518).

The third lesson of *Simplest Form* has the same lesson structure as *Find Equivalent fraction* in grade 3. The practice problems focused on the application of the procedural rule of equivalent fractions. The types of problems include simplifying fractions in a numerical expression, word problems, and open-ended problems. Generally, grade 3 focused on using fraction models to find equivalent fractions and emphasized understanding modeling equivalent fractions. Grade 4 focused on using the property of fractions to find and identify equivalent fractions.

2.4.2.2. Shanghai Textbooks

Shanghai textbooks introduce intense knowledge of equivalent fractions in grade 6A. However, two extended lessons in grade 4B are the main introduction to the method of using fraction models to explain that fractions are equal. One lesson focuses on area models and length models, while the other introduces the application of the fraction wall/strips in finding equivalent fractions. Therefore, in grade 4, Shanghai textbooks expect that students know that several fractions are equal by drawing fraction models.

In grade 6, three lessons are involved in the unit on the *Equivalent Fraction Rule*. The first lesson introduces and derives EFR, which includes five sections: *Observation, Thinking*, *Generalization, Example*, and *Exercise* (see Figure 2.5). The *Observation* section is used to guide students in observing the pattern of the equivalent fractions of 3/4 based on area models and shows the pattern of multiplying and dividing the numerator and denominator by a whole number. *Thinking* consists of two exercises corresponding to the *observation* section. One exercise involves labeling the equivalent fractions of 1/2 based on four shaded-area pictures. The other exercise requires that students use the pattern to fill out a number. *Generalization* shows an abstract format of the property of fractions: $\frac{a}{b}=\frac{a\times k}{b\times k}=\frac{a\div n}{b\div n}, b\neq 0, k\neq 0, n\neq 0$. *Examples* includes two examples of the application of EFR. *Exercises* and consists of a set of exercises.

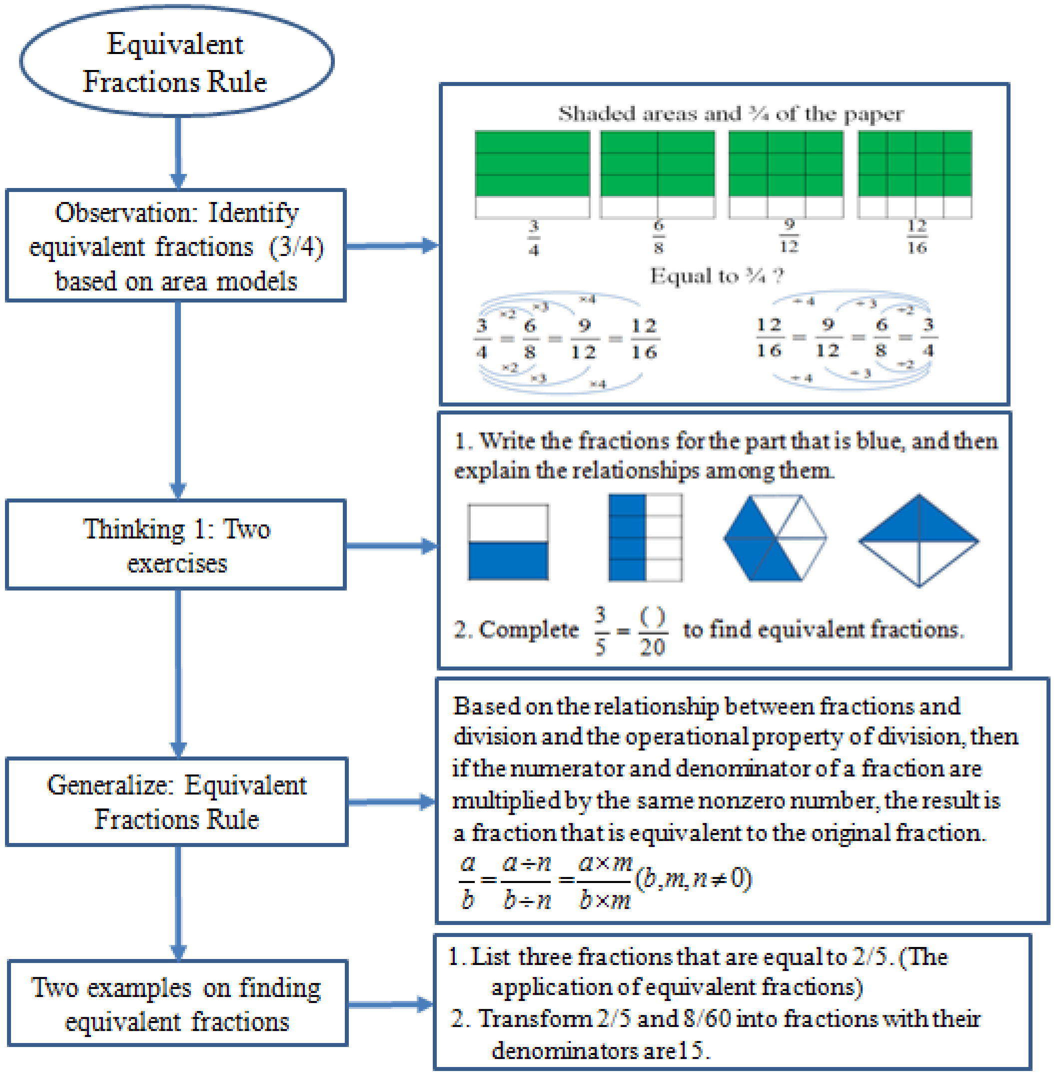


Figure 2.5 The content of equivalent fractions adapted from Shanghai textbook grade 6A (Qiu et al. 2014, pp. 31-32).

The second lesson consists of three sections: *Thinking*, *Example*, and *Exercise*. *Thinking* introduces the concept of simplest form by solving a problem on finding equivalent fractions: A fraction is in its simplest form when its numerator and denominator have no common factor other than 1. *Example* shows two examples: One is how to simplify a fraction and the other is how to solve word problems with measurement. *Exercise* includes a set of problems about simplifying fractions and word problems. The third lesson is to extend the application of equivalent fractions to real-world problems with tables. It includes two sections: *Examples* and *Exercises*. Two worked-out examples with tables are shown. *Exercises* include two real-world problems with tables and one open-ended problem without tables.

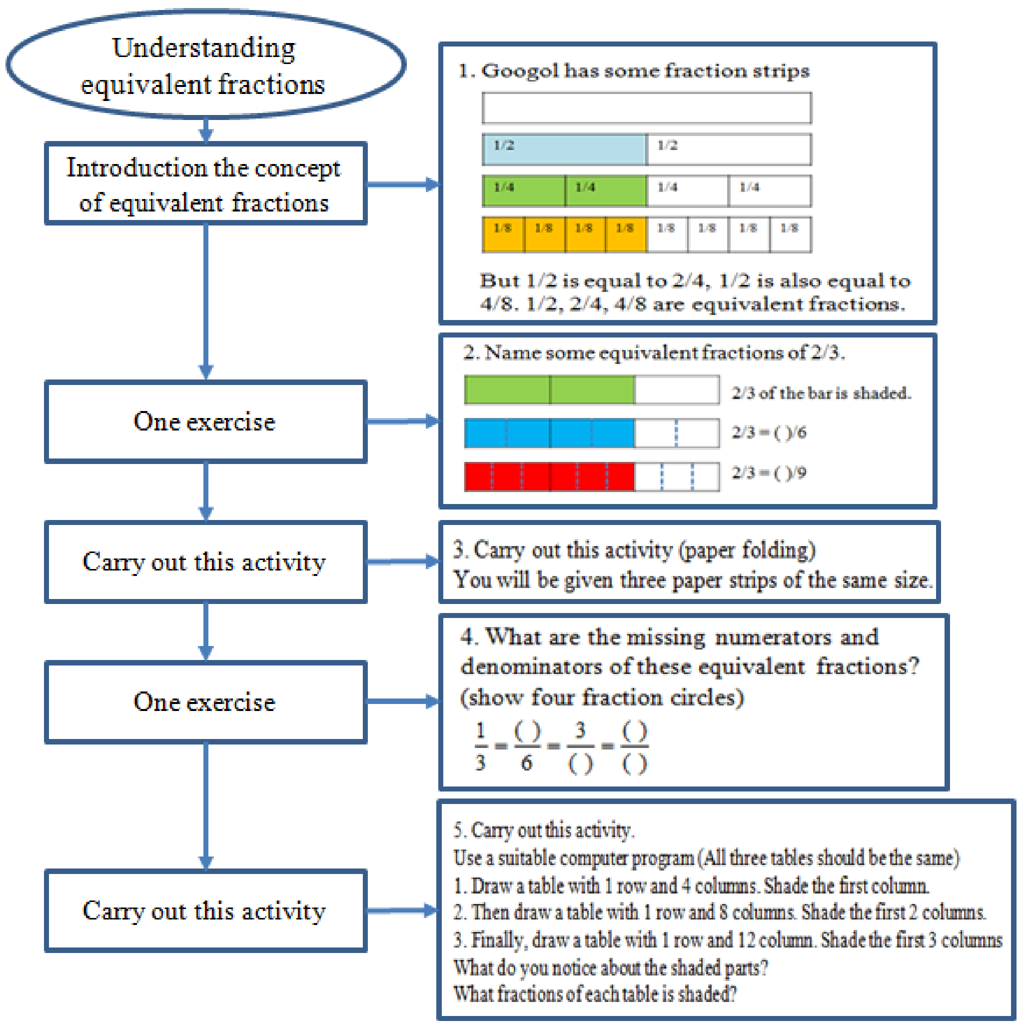


Figure 2.6 The content of equivalent fractions adapted from Singapore textbooks grade 3B (Kheong, Ramakrishnan, & Choo, 2007, pp. 69-71).

2.4.2.3. Singapore Textbooks

Singapore textbooks introduce equivalent fractions in grade3B and use only two lessons to introduce all content on equivalent fractions. The structure of the lessons in Singapore textbooks is completely different from that of California textbooks. The first lesson of *Understanding Equivalent Fractions* includes five problems (see Figure 2.6). The second lesson of *More Equivalent Fractions: Short Cut* includes six problems. The textbooks seem to expect students to learn the concept of equivalent fractions and the property of equivalent fractions in solving problems.

Based on Figure 2.2, the content of equivalent fractions in *California textbooks* was repeated to some degree. Singapore did not show any indications of repetition in the table of contents. Shanghai textbooks bridge the content of equivalent fractions in grade 4 and grade 6 by identifying equivalent fractions based on area models (see Figure 2.2). Also, the sequence of equivalent fractions in Shanghai textbooks briefly shows the sequence of all related fraction content from the concept of fractions to four operations to application in word problems.

2.4.3. The Coherence of Example-Practice Problems

The ratio of the number of examples to practice problems in California textbooks is very low (.006) compared to Shanghai textbooks (.180) and Singapore textbooks (.185). This means California textbooks have many more exercises than worked-out examples. California textbooks include only seven examples and 117 practice problems, while Shanghai textbooks have nine examples and 50 practice problems and Singapore textbooks have five examples and 27 practice problems (see Table 2.5 for more information). Table 5 shows the percentage of each subcategory for five aspects of example and practice problems in the three series of textbooks. For example, in the aspect of representation, (86, 65) means that the percentages of the third categories of example problems' and practice problems' representation (abstract representation) in California textbooks are 86% and 65%, separately. This means that the statements of 65% of practice problems use abstract representation like written language or purely mathematical context (see Table 2.1 for more information about the categories of each aspect). I analyzed the coherence characteristics of mathematical problems based on five aspects: problem representations, response types, computation, connections, and cognitive requirements and used radar maps to show the results as follows.

Table 2.5 Detailed Information About Worked Problems in the Three Series of Textbooks.

A	**Example & Practice (%, %)**								
	California N_{ex} = 7, N_{pr} = 117			**Shanghai** N_{ex} = 9, N_{pr} = 50			**Singapore** N_{ex} = 5, N_{pr} = 27		
C	1st	2nd	3rd	1st	2nd	3rd	1st	2nd	3rd
Pre	(14,16)	(0,19)	(86,65)	(33,14)	(22, 6)	(45,80)	(20,15)	(60,26)	(20,59)
Res	(71,63)	(29,25)	(0,12)	(22,36)	(33,58)	(45, 6)	(40, 94)	(40, 0)	(20, 4)
Com	(57,35)	(43,59)	(0,6)	(11,8)	(67,82)	(22,10)	(40,41)	(60,59)	(0, 0)
Con	(86,80)	(14,20)	(0,0)	(56,84)	(33,16)	(11, 0)	(100,100)	(0, 0)	(0, 0)
Cog	(29,74)	(71,24)	(0,3)	(22,78)	(45,18)	(33, 4)	(20, 96)	(80, 4)	(0, 0)

Note. A = Aspect; C = Categories; N_{ex} = Number of examples, N_{pr} = Number of practices; Pre = Representation; Res = Response; Com = Computation; Con= Connection; Cog = Cognition.

2.4.3.1. The Coherent Characteristics of Examples and Practices

As Figure 2.7 shows, in California textbooks, the five aspects of examples and practice problems do not have any obvious patterns. Compared to practices, examples have bigger values of cognition and representation aspects and have smaller values of response and computation aspects. However, as shown in Figure 2.8, the levels of problems in grade 4 are higher than those of problems in grade 3 except for the aspect of representation. This means that the coherence of problems in grades 3 and 4 is strong. Shanghai and Singapore textbooks have a similar pattern. To be specific, compared to the five aspects of practices in Shanghai and Singapore textbooks, the other four aspects of examples have bigger values, except for the representations aspect.

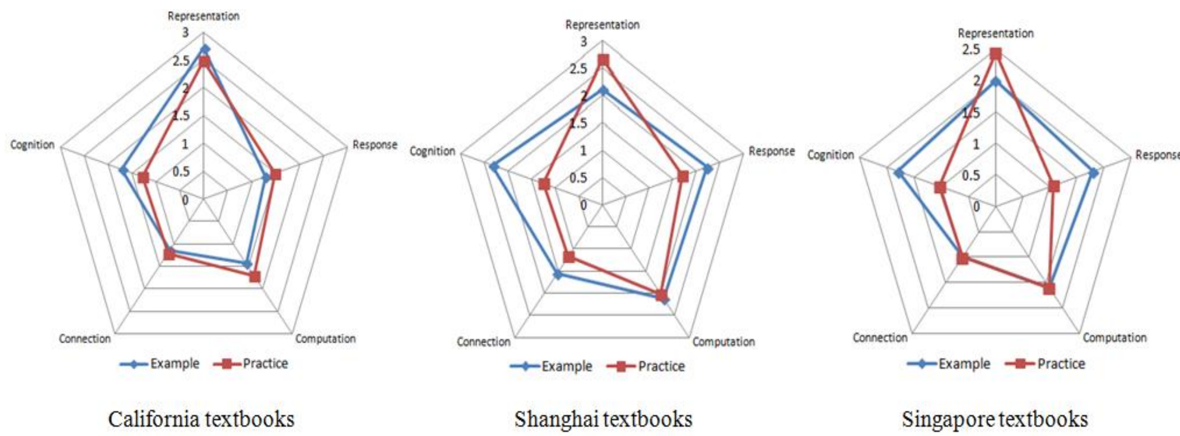


Figure 2.7 The comparison between examples and practices problems in the three series of textbooks.

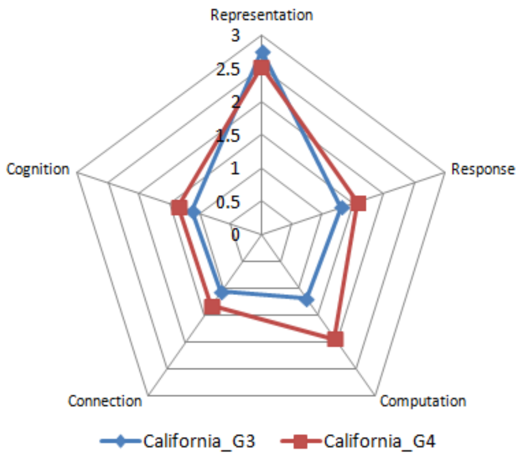
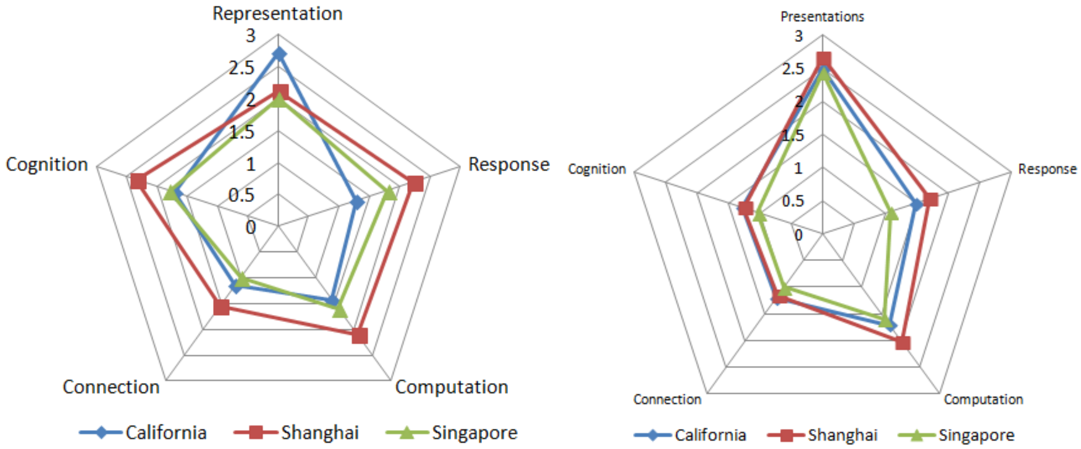


Figure 2.8 The comparison of problems on equivalent fractions between grade 3 and grade 4 textbooks.

2.4.3.2. The Coherence of Examples and Practice Across Regions

Compared to California and Singapore textbooks, the values of all aspects on examples in Shanghai textbooks are larger except for representation. Shanghai textbooks tend to use concrete and pictorial representations to show problems. All worked-out examples in California textbooks used similar written mathematical language to show all problems except for one real-world problem. The values of all aspects on examples in Singapore textbooks are the middle of those in Shanghai and California textbooks (see Figure 2.9a).

On the other hand, the values of the five aspects of practices in Singapore textbooks are relatively lower, especially in the aspects of response, computation, and cognition. Compared to California textbooks, I found that Shanghai textbooks have high values of computation and response; the values of the others are almost the same. The practices in Shanghai textbooks have relatively higher values in all five aspects. This suggests that the problems in Shanghai textbooks are the most difficult (see Figure 2.9b).



a. Examples b. Practices

Figure 2.9 The comparison of examples/practices in the three series of textbooks.

2.5. Discussion

This study examined the characteristics of equivalent fractions in California, Shanghai, and Singapore mathematics textbooks based on three coherent aspects: logical coherence, lesson structures, and the coherence of examples and practices.

2.5.1. The Coherence of Logical Sequences of Equivalent Fractions

In the three different curriculum requirements for equivalent fractions, students are required to recognize and generate equivalent fractions and simplify fractions. However, the emphases on the understanding and application of EFR are a slightly different. The California Common Core Standard highlights using all kinds of models to understand, recognize, generate, and explain equivalent fractions. As far as EFR are concerned, the California Common Core Standard requires the use of visual different fraction models to explain the rule of $\frac{a}{b}=\frac{n\times a}{n\times b}$ rather than $\frac{a}{b}=\frac{n\times a}{n\times b}=\frac{a\div n}{b\div n}$. Perhaps it is relatively easier to explain the method of multiplication than the method of division using fraction models.

Shanghai teaching requirements on equivalent fractions focus on understanding EFR based on fractions as division and the operational principle of division ($a\div b=(a\times k)\div(b\div k)=(a\div n)\div(b\div n), b\neq 0, k\neq 0, n\neq 0$). This suggests that the Shanghai curriculum underlines the conceptual understanding of equivalent fractions from the perspective of mathematical content' coherence and assumes that students are familiar with the operational principle of division. To be specific, students have learned the principle of division: $a\div b=(a\times k)\div(b\div k)=(a\div n)\div(b\div n), b\neq 0, k\neq 0, n\neq 0$, in Shanghai textbooks. In addition, students learn fractions as division, $a/b = a \div b$, before they learn equivalent fractions (see Table 4). Based on the prior knowledge of the operational property of division and fractions as division, students can derive the property of fractions. Additionally, besides highlighting the understanding of concepts, Shanghai textbooks emphasize the application of equivalent fractions to solve word problems.

In the Singapore curriculum syllabus, the content requirements of equivalent fractions emphasize the application of equivalent fractions rather than understanding. The requirements use the words of recognizing, naming, listing, writing, and expressing, except understanding. Thus, based on these words, the Singapore syllabus emphasized the application of equivalent fractions.

Generally, a mathematics curriculum standard is a region or national guiding document that has significant effects on the coherent presentation of the content in mathematics textbooks. It states that students must achieve the level of the requirements after learning the content. The presentation of mathematical content includes the sequence of topics, the structure of lessons, and the mathematical problems (examples and practices). Each textbook can have its own organization of content on the condition that it meets the requirements in the curriculum standards. For example, NCTM (2000) mentioned that by using parallel number lines, each showing a unit fraction and its multiples, students can see fractions as numbers, note their relationship to 1, and see relationships among fractions, including equivalence (p. 150). The number line model is essential for teachers to teach fractions (Carmer et al., 2002). In terms of textbooks, Singapore textbooks do not use number lines to find equivalent fractions. Shanghai textbooks use the fractions wall (early number line model), while California textbooks use the number line to find equivalent fractions.

Therefore, the content requirements from curriculum standards determine the range and depth of content in each grade. The California CCCS emphasizes logical coherence in concrete fraction models, while the Shanghai curriculum emphasize logical coherence in abstract mathematical knowledge. The Singapore syllabus emphasizes logical coherence in problem-solving.

2.5.2. The Coherence of Lesson Structures

2.5.2.1. The Evaluation from Particular to Deep Structures

In California textbooks, the structure of equivalent fractions' lessons has a pattern: presentation of concepts, examples, and practices. However, there are two types of lessons on equivalent fractions. First is an exploration lesson (Math Activity) before the students learn equivalent fractions. The lessons in California textbooks use three fraction models (area models, length models, and number line) in the two lessons of *Math Activity* and emphasize that students must draw pictures to find equivalent fractions. Second, the two lessons of equivalent fractions include the presentation of concepts, examples, and practice. They directly show the concepts and EFR. Generally, in California textbooks, the two lessons in grade 3 emphasized the use of fraction models to find equivalent fractions. The lessons in grade 4 focused on the application of multiplication and division to find equivalent fractions. Perhaps this sequence of topics is related to the grade levels because third- and fourth-graders are not able to comprehend the abstract EFR.

In Shanghai textbooks, two lessons on equivalent fractions in grade 4 are under the heading of fractions comparison. As this stage, fraction models are employed to teach about fractions. Next, fraction models are used to teach about equivalent fractions. The content of equivalent fractions in grade 4 is extended. From the perspective of coherence, this stage is to preliminarily learn equivalent fractions based on fraction models. In grade 6, the learning of equivalent fractions focuses on mathematical structure. The first lesson on equivalent fractions follows the sequence of finding a conjecture, checking, proving, and applying. More specifically, the first lesson shows an open-ended problem and guides students in finding the patterns of the numerators and denominators of equivalent fractions. Then the lesson presents two exercises to check the pattern. However, the pattern is a conjecture that must be justified. Next, the lesson shows that the abstract of the principle of equivalent fractions can be derived based on fractions as division and the principle of fractions. Following the first lesson, the second lesson applies the property of fractions in solving a problem. In solving the problem, a newly posed question pushes students into a situation in which they must produce a new concept of simplest form. Then two examples and several practices follow. The third lesson is on the application of equivalent fractions to two real-world problems. The three lessons in grade 6 of equivalent fractions are introduction, application, and problem-solving. The first lesson introduces of the basic property of fractions. The second lesson introduces the concept of simplest form based on the application of the basic property of fractions. The third lesson emphasizes real-world problem-solving. The three lessons consist of a whole unit on equivalent fractions. Therefore, Shanghai textbooks show a tracking that evolves from particulars (find the method of multiplication and division) to deep structures (derivation of the property of fractions) and tend to emphasize that mathematics is logical and deductive rather than inductive.

In Singapore textbooks, the structure of the single lesson is not very clear. Singapore textbooks introduce a concept by solving problems. Then they present exercises. The examples and practices are shown alternatively. There are no clear and strict concepts on the property of equivalent fractions. Singapore textbooks directly provide the ways of multiplying and dividing to find equivalent fractions rather than provide opportunities for students to explore the principle of equivalent fractions. However, the lesson structure of the whole unit on equivalent fractions is clear and obvious. The first lesson is to find equivalent fractions based on fraction models (e.g., drawing pictures). The second lesson is to find a shortcut for using multiplication and division to generate equivalent fractions. It also arranges the topic of simplest form into this lesson, as simplifying fractions is possibly the application of EFR (the means of dividing to find equivalent fractions).

Generally, the Shanghai lessons' structures follow the strategy of concrete fading in the introduction of equivalent fractions and emphasize upward spiral learning. That is, each lesson is developed based on the prior lesson. Shanghai textbooks show a clear, coherent lesson structure that includes an introduction, examples, and practices and that emphasizes the understanding of concepts from mathematical content development. California textbooks have little repetition between using fractions models and applying EFR. Singapore textbooks emphasize the problem solving rather than the understanding of the concept of equivalent fractions and EFR.

2.5.2.2. Using deep structure to make connections

In California textbooks, fraction models, as a method, are used to find equivalent fractions in grade 3, while multiplication-division, as another method, is applied to generate equivalent fractions. However, California textbooks do not provide opportunities for students to explore the relationship between these two methods. Most importantly, California textbook showed a confusing expression in introducing the property of equivalent fractions: $\frac{4}{8}\times\frac{2}{2}=\frac{8}{16}$ and $\frac{4}{8}\div\frac{2}{2}=\frac{2}{4}$, although the expressions are correct. The expression implies that $\frac{4}{8}\times\frac{2}{2}=\frac{4\times 2}{8\times 2}$ and $\frac{4}{8}\div\frac{2}{2}=\frac{4\div 2}{8\div 2}$. The principle of equivalent fractions is that the value of a/b is the same when both *a* and *b* multiply or divide a non-zero whole number. The two expressions are used to inappropriately multiply/divide the numerator and the denominator by the same non-zero whole number. Fourth-graders did not learn how to multiply a fraction by a fraction and how to divide a fraction by a fraction at the same time they learned equivalent fractions. This might lead students to mistakenly multiply a fraction by a whole number to generate an equivalent fraction, for example, $\frac{2}{3}\times 2=\frac{4}{6}$.

However, Shanghai textbooks use fraction models to teach students how to find equivalent fractions as a means of teaching EFR. This is different from other methods that emphasize using models to find equivalent fractions. Shanghai textbooks emphasize using the property of fractions to find equivalent fractions. This is consistent with the nature of Chinese textbooks that emphasize a spiral understanding of mathematical knowledge. Students should use prior mathematical knowledge to learn new mathematical knowledge. For example, when fractions as divisions were introduced, EFR could be derived based on fractions as division and the principle of division. That is, EFR should be derived strictly from prior knowledge rather than shown directly without explanation. When students find the pattern of the numerators and denominators of equivalent fractions, the found pattern is only a conjecture rather than a principle.

Shanghai textbooks try to emphasize the reasoning of mathematical knowledge to help students understand the property of equivalent fractions.

Similarly, in Singapore textbooks, the length model is a tool used to show the relationship between numerators/denominators from some equivalent fractions (e.g., 2/3 = 4/6 = 6/9 = 8/12). The purpose is to find the pattern in the numerators and denominators of equivalent fractions. Singapore textbooks provide this method only to find equivalent fractions rather than to provide opportunities to prove this property. In other words, Singapore textbooks employ inductive reasoning (particulars) to find the pattern of several equivalent fractions. Therefore, I can conclude that Singapore textbooks do not make a connection to other mathematical knowledge.

In summary, first, the concept and basic property of equivalent fractions and the concept of simplest form in California textbooks are presented directly rather than explored. Shanghai and Singapore textbooks use worked-out examples to explore the concept and property of equivalent fractions by solving problems. In terms of the differences between Shanghai and Singapore textbooks with respect to the presentation of the concepts, Shanghai presents the property of fractions by using rigorous mathematical language rather than the descriptive language that Singapore textbooks use. Furthermore, only Shanghai textbooks emphasized that the multiplier in the principle of equivalent fractions is not zero: the two other textbooks do not provide a mathematical abstract statement of the principle of equivalent fractions. For the content of the simplest form, California and Shanghai textbooks used a lesson to introduce the concept and application of simplest form, while Singapore textbooks introduced the concept of simplest form only by the way of solving a work-out example at the end of the lesson of *More Equivalent Fractions: Short Cut*.

In addition, California textbooks have two special sections: *Math Activity* and *Game Time*. The Math Activity is used before the lesson on equivalent fractions. *Game Time* is arranged after the exercise section. Singapore textbooks arranged the activities in the teaching. However, the Shanghai textbook did not include a section of activity and named several sections such as *Observation*, *Thinking*, and *Generalization* that focus on mathematical thinking.

2.5.3. The Comparison of Example-Practice Problems in the Three Series of Textbooks

First, the percentages of problems on equivalent fractions in the three textbooks are different. The content percentage of equivalent fractions in California textbooks is low in the three series of textbooks. The percentage of equivalent fractions in Shanghai textbooks is highest. Repetitions of equivalent fractions in California textbooks also provide evidence of consistent criticism of repetitive content in US mathematics textbooks (Alajmi, 2012; Schmidt et al., 2002; Schmidt et al., 2005).

Furthermore, the purposes of worked-out examples in the three series of textbooks are different. When the introduction sections do not explain the concept of equivalent fractions in California textbooks, the examples play two roles: One is to explain equivalent fractions and another is to show how to find equivalent fractions. When the introduction uses examples to explain the concept of simplest form, the examples play only one role which shows the detailed procedural solution. However, the introduction examples in Shanghai textbooks focus on the explanation and exploration of EFR. The examples are to practice the application of EFR. The practices and examples use different varieties to deepen the understanding of the concepts and EFR, such as one problem with multiple sub-goals and one problem with multiple solutions. Singapore textbooks do not include any formal examples. All examples are to introduce the ways by which to find equivalent fractions. All practices without extension completely practice the methods shown in the examples.

Finally, the presentations of problems in terms of the five aspects are different. First, abstract representations of problems are the predominant type of presentation in the three series. After learning the concepts and principles of equivalent fractions, the procedural practices are the main problem format. Second, California textbooks have different percentages of response types, though the numerical answer accounts for the majority of response types, while the type of numerical expression of problems occupies over half in Shanghai textbooks. The problems in Singapore textbooks have predominantly numerical-answer response types. Therefore, Shanghai textbooks emphasize operation. Third, all three series of textbooks emphasize computation, especially Shanghai textbooks. California and Singapore have similar distributions of computation. Fourth, for the problems' connections, California textbooks present various types of problems that connect to other mathematical knowledge. The problems in Singapore textbooks are only in the field of just learned content that the examples show. Shanghai textbooks emphasized in-depth understanding and application of learned knowledge. Finally, the cognitive levels of practice problems in the three series of textbooks are mainly at the first level.

2.6. Implications and Conclusions

Textbook coherence is a complicated issue and no completed evaluation system exists for researchers to evaluate textbook coherence. Although Schmidt, Wang, and McKnight (2005) studied curriculum coherence, the study focused only on the macro level, including the sequences of mathematical topics, rather than the micro level, such as the structure of lessons and the nature of problems. Therefore, this study seeks to explore textbook coherence from both the macro and micro levels by building up three aspects: logical sequence, lesson structure, and example-practice problems.

The purpose of cross-cultural comparison is to reflect on one’s practices and to learn from others (Ding, 2014; Shimizu & Kaur, 2013). In this study, the findings based on California, Shanghai, and Singapore textbooks contribute insights to improve curriculum coherence in terms of logical sequence, lesson structures, and example-practice problems. Shanghai textbooks’ stressing the underlying structural relations and mathematical logical coherence is consistent with prior findings about Chinese textbook presentations of mathematical content (e.g., Ding & Li, 2010, 2014). These aspects of developing curriculum coherence may be learned by textbook designers in the United States and other countries. Likewise, U.S. textbooks’ unique real problems and emphasis on modeling to understand mathematical concepts (e.g., area models, length models, and number lines) may be learned by Chinese and Singapore textbook designers and others. Singapore textbooks’ emphasis on problem-solving by translating from concrete representations (activities), pictorial representations to abstract representations also contributes to students' learning of problem-solving.

Indeed, findings about Singapore textbooks' preference in solving problems are consistent with the conclusion in Fan and Zhu (2007) that Singapore textbooks seemed to stress more problem-solving. Although all three series of textbooks have employed the representation of translation in learning mathematical knowledge, they have various emphases. Given that examples with greater variability in five aspects can better facilitate the connection to knowledge and the translation of problems among different formats, representations, or cognitive requirements, they may promote students' effective learning and help teachers build a more coherent system of knowledge.

The findings together raise important questions about not only how to design coherent textbooks but also how to use them successfully in classrooms to support coherent instruction and learning. Findings about Shanghai textbooks in this study appear to parallel prior findings on Chinese teachers' knowledge structures and classroom teaching (e.g., Ding et al., 2013; Zhou, Peverly, & Xin, 2006). As such, future studies may include more mathematical content, across more regions and explore detailed connections between coherent textbooks, coherent instruction, and student learning.

However, this study has two limitations. First, I focused only on the topic of equivalent fractions in three series of textbooks from three regions. Therefore, the conclusion cannot be easily generalized. Future studies may examine more topics and more versions of textbook series from different publishers. Second, I analyzed textbooks only through the perspective of textbook coherence. Various perspectives could result in completely different findings. Therefore, I must emphasize that the purpose of this study is to provide a perspective from which teachers, mathematics educators, and textbook editors can learn from the coherence characteristics of textbooks rather than to evaluate the quality of the textbooks.

2.7. References

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and Instruction*, *16*(3), 183-198.

Alajmi, A. H. (2012). How do elementary textbooks address fractions? A review of mathematics textbooks in the USA, Japan, and Kuwait. *Educational Studies in Mathematics*, *79*(2), 239-261.

Altieri, M. B., Balka, D. S., Day, R. D., Gonsalves, P. D., Grace, E. C., Krulik, S. et al., (2007). *California mathematics: Concepts, skills, and problem solving* (grade 3). Columbus, OH: McGraw-Hill.

Altieri, M. B., Balka, D. S., Day, R. D., Gonsalves, P. D., Grace, E. C., Krulik, S. et al., (2007). *California mathematics: Concepts, skills, and problem solving* (grade 4). Columbus, OH: McGraw-Hill.

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D. A. Grows (Ed.), *Handbook on research on mathematics teaching and learning* (pp. 296–333). New York: Macmillan.

California State Board of Education (2013) *California common core state standards: Mathematics* (Electronic Edition). California State Board of Education.

Chan, W., Leu, Y., & Chan, C. (2007). Exploring group-wise conceptual deficiencies of fractions for fifth and sixth graders in Taiwan. *Journal of Experimental Education, 76*, 26-57.

Chen, X., & Li, Y. (2010). Instructional coherence in Chinese mathematics classroom—A case study of lessons on fraction division. *International Journal of Science and Mathematics Education*, *8*(4), 711-735.

Cramer, K. A., Post, T. R., & delMas, R. C. (2002). Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the rational number project curriculum. *Journal for Research in Mathematics Education*, *33*(2), 111-144.

Curriculum Planning and Development Division (2006). *Mathematics syllabus primary*. Singapore: Ministry of Education.

Cuoco, A., & McCallum, W. (2018). Curricular coherence in mathematics. In Y. Li, W. J. Lewis, & J. J., Madden (Eds.), *Mathematics matters in education* (pp. 245-256). Cham, Switzerland: Springer.

Ding, M. (2016). Opportunities to learn: Inverse relations in US and Chinese textbooks. *Mathematical Thinking and Learning, 18*(1), 45-68.

Ding, M., Li, X., Capraro, M. M., & Kulm, G. (2012) A case study of teacher responses to a doubling error and difficulty in learning equivalent fractions, *Investigations in Mathematics Learning*, *4*(2), 42-73.

Fan, L., & Zhu, Y. (2007). Representation of problem-solving procedures: A comparative look at China, Singapore, and US mathematics textbooks. *Educational Studies in Mathematics, 66*(1), 61-75.

Fyfe, E. R., McNeil, N. M., Son, J. Y., & Goldstone, R. L. (2014). Concreteness fading in mathematics and science instruction: A systematic review. *Educational Psychology Review, 26*(1), 9-25.

Garner, R. (1992). Learning from school texts. *Educational Psychologist*, *27*(1), 53-63.

Huang, J., Ye, W., Tong, H., Song, Y., Hu, P., & Xu, P. (2014). *Mathematics textbook* (4B). Shanghai, China: Shanghai Education Press.

Huang, J., Ye, W., Tong, H., Song, Y., Hu, P., & Xu, P. (2014). *Reference materials for mathematics teaching* (4B). Shanghai, China: Shanghai Education Press.

Kamii, C., & Clark, R. B. (1995). Equivalent fractions: Their difficulty and educational implications. *Journal of Mathematics Behavior, 14,* 365-378.

Kheong, F. H. Ramakrishnan, C., & Choo, M. (2007). *My pals are here Math* (2nd Edition) (3B). Singapore: Marshall Cavendish Education.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49-84). Hillsdale, NJ: Erlbaum.

Koedel, C., Li, D., Polikoff, M. S., Hardaway, T., & Wrabel, S., L. (2017). Mathematics curriculum effects on student achievement in California. *AERA Open 3*(1), 1-22.

Li, Y. (2000). A comparison of problems that follow selected content presentations in American and Chinese mathematics textbooks. *Journal for Research in Mathematics Education, 31*(2), 234-241.

McKnight, C., Crosswhite, J., Dossey, J., Kifer, L., Swafford, J., Travers, K., and Cooney, T. (1987). *The underachieving curriculum: Assessing U.S. school mathematics from an international perspective*. A national report on the Second International Mathematics Study. Champaign, IL: Stipes.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

National Research Council. (2001). *Knowing what students know: The science and design of educational assessment.* Washington, DC: The National Academy Press.

National Research Council. (2004). *On evaluating curricular effectiveness: Judging the quality of K–12 mathematics evaluations*. Washington, DC: National Academy Press.

Porter, A. C. (2006). Curriculum assessment. In J. L. Green, G. Camilli, & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 141–159). Washington, DC: American Educational Research Association.

Qiu, W., Huang, H., Sun, H., Song, J., Wu, H., Li, Y., Xu, X. et al. (2014). *Mathematics textbook* (6A). Shanghai, China: Shanghai Education Press.

Qiu, W., Huang, H., Sun, H., Song, J., Wu, H., Li, Y., Xu, X. et al. (2014). *Reference materials for mathematics teaching* (6A). Shanghai, China: Shanghai Education Press.

Renkl, A., Atkinson, R. K., Maier, U., & Staley, R. (2002). From example study to problem solving: Smooth transitions help learning. *The Journal of Experimental Education*, *70*, 293-315.

Reys, B. J., Reys, R. E., & Chavez, O. (2004). Why mathematics textbooks matter. *Educational Leadership*, *61*(5), 61-66.

Roseman, J. E., Stern, L., & Koppal, M. (2010). A method for analyzing the coherence of high school biology textbooks. *Journal of Research in Science Teaching*, *47*(1), 47-70.

Schmidt, W., Houang, R., & Cogan, L. (2002). A coherent curriculum. *American Education*. Summer, 1-17.

Schmidt, W., Houang, R., & Cogan, L. (2004). A coherent curriculum: The case of mathematics. *Journal of Direct Instruction*, *4*(1), 13-28.

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of US mathematics and science content standards from an international perspective. *Journal of Curriculum Studies*, *37*(5), 525-559.

Schmidt, W. H., & Houang, R. T. (2007). Lack of focus in the mathematics curriculum: Symptom or cause? In T. Loveless (Ed.), *Lessons learned: What international assessments tell us about math achievement* (pp. 65–84). Washington, DC: Brookings Press.

Sun, X. (2011). “Variation problems” and their roles in the topic of fraction division in Chinese mathematics textbook examples. *Educational Studies in Mathematics*, *76*(1), 65-85.

Tarr, J. E., Reys, R. E., Reys, B. J., Chavez, O., Shih, J., & Osterlind, S. J. (2008). The impact of middle-grades mathematics curricula and the classroom learning environment on student achievement. *Journal for Research in Mathematics Education*, *39*(3), 247-280.

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. *Journal for Research in Mathematics Education*, *43*(3), 253-295.

Tarr, J. E., Reys, B. J., Barker, D. D., & Billstein, R. (2006). Selecting high-quality mathematics textbooks. *Mathematics Teaching in the Middle School*, *12*(1), 50-54.

Van de Walle, J. A. (2004). *Elementary and middle school mathematics: Teaching developmentally.* Boston, MA: Pearson.

Wijaya, A., van den Heuvel-Panhuizen, M., & Doorman, M. (2015). Opportunity-to-learn context-based tasks provided by mathematics textbooks. *Educational Studies in Mathematics*, *89*(1), 41-65.

Yang, D. C., Reys, R. E., & Wu, L. L. (2010). Comparing the development of fractions in the fifth-and sixth-graders’ textbooks of Singapore, Taiwan, and the USA. *School Science and Mathematics*, *110*(3), 118-127.

3. OPPORTUNITIES TO DEVELOP COGNITIVE SKILLS: PROBLEMS' COGNITIVE REQUIREMENTS SPECIFIED IN CALIFORNIA, SHANGHAI, AND SINGAPORE TEXTBOOKS

3.1. Introduction

School textbooks serve as critical vehicles for knowledge acquisition in school and are capable of replacing teacher instruction as the primary source of information in the upper grades (e.g., Drake, Land, & Tyminski, 2014; Garner, 1992). Concerns about the role of textbooks in mathematics teaching and learning have grown in the international mathematics education community, especially with the publication of analysis results about curricula and students' achievements from the International Mathematics and Science Study (TIMSS) (e.g., Schmidt & Houang, 2012; Schmidt, Wang, & McKnight, 2005). Many studies have claimed that textbooks were one of the key factors contributing to cross-national differences in students' mathematical achievement (e.g., Li, 2000; Tornroos, 2005; Schmidt, Wang, & McKnight, 2005; Xin, 2007; Zhu & Fan, 2006). To search for the possible reasons behind the differences in performance on international tests, researchers have examined the characteristics of textbooks based on the assumption that textbooks played an important role in the process of teaching and learning (e.g., Tornroos, 2005; Xin, 2007; Zhu & Fan, 2006).

Mathematical problems are important components of school mathematics textbooks. The issue of how problems and problem-solving are treated in school mathematics textbooks has received considerable attention from researchers (e.g., Fan &

Zhu, 2007). Much attention has been paid to the representation of problem types (e.g., Ding & Li, 2014), the procedure of problem solving (e.g., Kolovou, Van den Heuvel-Panhuizen, & Bakker, 2009), and the presentation of problems (e.g., Li, 2000, Van den Heuvel-Panhuizen, 2005). These studies have revealed the characteristics of problem presentation by comparing textbooks from different educational systems. These characteristics include the contexts of problems, cognitive requirements of problems, problem information, and response requirements of problems provided. These characteristics provide more perspectives through which students can understand the nature of problems (e.g., Wijaya, van den Heuvel-Panhuizen, & Doorman, 2015).

Furthermore, in the studies of textbook analysis, many focused on the variation of problems in textbooks. As Zhang, Wang, Huang, and Kimmins (2017) claimed, in the past two decades Chinese editors of mathematics textbooks have emphasized the use of the variation of problems in textbooks and the way to employ the variation of problems in textbooks is a popular topic. Sun (2011) discussed the roles of the variation of example problems in mathematics textbooks by analyzing "one problem multiple solutions (varying solutions)," "one problem multiple changes (varying conditions and conclusion)," and "multiple problems one solution (varying presentations)" (p. 67). Sun (2011) also reviewed the feature of Chinese variation-practice pedagogy and employed this feature to analyze the variation of examples of fractions in the Chinese and US mathematics textbooks. Additionally, Zhang et al. (2017) analyzed the functions of the use of the variation of problems in a Chinese popular mathematics textbook series. The functions include mathematical concepts, principles, skills, and mathematical thinking.

To be specific, the use of the variation of problems is intended to help students understand mathematical concepts/principles as well as to help them practice specific skills/mathematical thinking. They stated that “the consistent use of the variation of problems in textbooks and in classroom instruction provides strong support for students’ learning that may provide further explanation of Chinese students’ excellent performance of mathematics on international comparative assessment” (p. 237). Furthermore, there are various classifications of the variation of problems, such as conceptual variation and procedural variation (Gu, Huang, & Gu, 2018); developmental variation and multi-faceted variation (Park & Leung, 2004); explicit variation and implicit variation (Huang, Mok, & Leung, 2006); and form variation, solution variation, and content variation (Xiao, 2000).

Few studies, however, have employed the perspective of textbook coherence to compare and analyze the variation of problems in textbooks. One reason for this may be that, due to the complex variation of mathematical problems, various explanations exist for the definitions of variation. This conceptual issue becomes even more complicated when one wants to compare education systems. Second, mathematics educators have not developed tools for evaluating the variation of problems from the perspective of coherence. Therefore, there is both theoretical and practical significance in analyzing the variation of problems from textbooks’ content in different education systems from a perspective of coherence.

In the present study, I selected a series of school mathematics textbooks from three regions: California, Shanghai, and Singapore. For the variation of problems, I defined variation as the alteration of problems' cognitive requirements (hereafter referred to as CR). Problems' CR could be modified by altering one or more aspects: response, representation, operation, reasoning, and connection. Then I built a model for problems' CR. To simplify the research range, I selected problems from chapters about fractions in the three series of textbooks for two reasons. First, the topic of fractions is usually introduced from second grade to sixth grade. Second, this topic is considered as one of the most problematic areas for students.

This study examines the coherence of variation of fraction problems' CR in the three series of textbooks. By employing quantitative analysis to compare the model for problems' CR between example and practice problems, this study not only could provide useful documentation and knowledge of how the fraction problems from the three series were shown in a coherent variation of problems in terms of the five aspects (i.e., response, representation, operation, reasoning, and connection) but also could provide possible ways to improve the coherence of example-practice problems in mathematics textbooks.

3.2. Literature Review and Research Questions

3.2.1. Analysis of Problems

Many studies have focused on the aspects of problems: problem representations, responses, connections, operations, reasoning, and cognitive requirements (e.g., Fan & Zhu, 2007; Li, 2000; Son & Senk, 2010; Wijaya et al., 2015). For example, Li (2000) developed a three-dimensional framework (for mathematical features, contextual features, and performance requirements) for analyzing the problems presented in textbooks. This framework helps analyze the similarities and differences of problems that followed the content presentation of the addition and subtraction of integers in several American and Chinese mathematics textbooks. The first dimension concerned mathematical features, such as whether only a computational procedure is required, or whether multiple computational procedures are required. The second dimension concerned contextual features, such as whether there was a purely mathematical context in numerical or word forms, or an illustrative context with pictorial representations or stories. The third dimension of performance requirements involved two aspects. The first aspect concerned response types and distinguished three categories (numerical answer only, numerical expression only, and explanation or solution required). The second aspect concerned the cognitive requirements and distinguished four categories (procedural practice, conceptual understanding, problem-solving, and special requirements). Son and Senk (2010) extended the second aspect of cognitive expectation from Li (2000) into 5 subcategories to analyze the characteristics of problems. Their categories are conceptual knowledge, procedural knowledge, mathematical reasoning, representation, and problem solving.

Likewise, Wijaya et al. (2015) examined the opportunity-to-learn that textbooks provide for students by analyzing four aspects of problems: the types of context used in tasks (no context, camouflage context, and relevant and essential context), the purposes of context-based tasks (application and modeling), the types of information provided in tasks (matching, missing, and superfluous), and the types of cognitive demands of tasks (reproduction, connection, and reflection). Wijaya et al. (2015) investigated the opportunity-to-learn that three Indonesian textbooks provided for solving context-based mathematics problems and examined the relationship between this opportunity-to-learn and students' difficulties in solving those problems.

Tabachneck, Koedinger, and Nathan (1995) stated that the mathematical and contextual features of problems are two important aspects of analyzing mathematical problems. Their study presented problem analysis based on the identification of mathematical situational problem difficulty factors. The difficulty factors included unknown position, connectedness (with other knowledge), number of operators (one or more), number types of quantity (integer, real, and complex), kinds of operators (e.g., addition, subtraction, multiplication, and division), number-fact facilitation, and situational factors (problem representation).

Bao (2009) also analyzed five aspects of problems to develop a model of problem difficulty: a pentagon model of problem difficulty. The five aspects are: exploration, context, operation, reasoning, and knowledge amount. Each aspect has several sub-categories. The exploration aspect includes three levels: memorization, understanding, and exploring. The context aspect includes four categories: no context, personal life, common knowledge, and scientific situation. The operation aspect includes four levels: no operation, numerical operation, simple symbol operation, and complex symbol operation. The reasoning aspect includes no reasoning, sample reasoning, and complex reasoning. The amount of knowledge topics refers to the number of knowledge topics behind a problem.

For problems' CR, Nicely Jr. (1985) described 10 cognitive levels of tasks. Each level is introduced by descriptive verbs, such as observe and read (level 0); recall, recognize, repeat, and copy (level 1); and iterate (level 2). Additionally, Smith and Stein (1998) classified mathematical tasks into four cognitive levels: memorization, procedures without connections, procedures with connections, and doing mathematics. This classification of cognitive levels is popular and used by many researchers. This framework was originally used to analyze the tasks that teachers employed in teaching. It was also used to analyze the cognitive levels of problems in textbooks (e.g., Jones & Tarr, 2007; Ubuz, Erbaş, Çetinkaya, & Özgeldi, 2010).

Based on the above literature review, I found that a close relationship exists between the other five aspects of problems and CR. Fist, the representation of problems plays an important role in determining problems' CR. Problems in the textbooks are shown in different representations or combinations. Teachers often employ real-world situations (e.g., word problems) to teach new mathematical concepts and ideas. Real-world situations with illustrations, as compared to abstract explanations, may offer richer real-life situations so that students can understand concepts better. While some studies (e.g., Cai, 2004; Kaminski, Sloutsky, & Hecker, 2008) have found that abstract representations (symbol expressions) have advantages over concrete representations (word problems) in solving complex problems, there is a common belief that teachers and curriculum designers believe word problems are more difficult than computation problems (Nathan, Long, & Alibali, 2002). This inconsistency of the function of problem representations may be produced by the differences in students' experiences.

Some students may be familiar with a given problem context while others may not be. The ability to solve daily-life-based problems is seen as a core goal of mathematics education, but students have difficulties (a) understanding the statement of the problem, (b) distinguishing between relevant and irrelevant information, and (c) identifying the mathematical procedures required to solve a problem (Wijaya et al. 2015).

In this study, the representations include daily language expression, pictorial expression, and symbol mathematical expression. The cognitive level can be changed by converting the representations of problems into the representations with which students are familiar. An example of this is the way in which a problem or question is posed to a student; the statement of a problem can be written language, a picture, or a table. However, some students have trouble understanding written language. Therefore, they draw a picture to reduce the difficulty involved in understanding the problem and to finally understand the problem's meaning.

Second, problems' response is an important factor in problems' CR. For example, generally speaking, students prefer to be assigned closed-ended problems. Regarding open-ended problems, students may need to provide explanations for answers or detailed solutions, which may be troublesome because open-ended problems may have several reasonable answers. Accordingly, the modification of a problem' response can alter the problem's CR.

Third, the requirements of arithmetic operation, connections to other mathematical knowledge, and reasoning levels are positive relationships with problems' CR. Based on the statement of levels of demands on task cognitive level (Smith & Stein, 1998), I find evidence of close relationships between task cognitive levels and operation, reasoning, and connection. For example, in reference to *memorization level* (lower level), the description shows "have no connections to concepts or meaning that underlies the fact, rules, formulas, or definitions being learned or reproduced" (Smith & Stein, p. 348). This means connection is an important index for task cognitive levels. Additionally, the level of *procedure without connections* describes "require no explanations or explanations that focus solely on describing the procedure that was used" (Smith & Stein, p. 348) and the use of algorithms. This means the level of *procedures without connections* does not require reasoning or complex thinking, only a focus on the algorithms used. By contrast, the level of *doing mathematics* requires "complex and non-algorithmic thinking" (Smith & Stein, p. 348).

Additionally, in the framework for textbook analysis in Wijaya et al. (2015), the types of cognitive demand consist of three sub-categories: reproduction, connection, and reflection. These are closely related to the five aspects mentioned above. For example, the category of connection referred to: (a) integrating and connecting across content, situation or representation, (b) non-routine problem solving, (c) interpretation of problem situations and mathematical statements, and (d) engaging in simple mathematical reasoning. These four categories of connection mentioned connection, response type (e.g., interpretation), reasoning, and representation.

In summary, any changes to the five aspects of a problem can lead to a change in the problem's CR. For example, the problem's cognitive requirement can be reduced when the problem representations change from a word problem to a pictorial or abstract representation. Elementary students solve the pictorial fraction problems relatively easily when they have learned the definition of fractions. Accordingly, the problem about the symbolic expression of the sum of two fractions will become easy for students after they learn the procedural rule of the addition of fractions. On the other hand, students may find solving complicated word problems to be rather difficult when these problems are designed to practice the application of mathematical content because concrete representation has different roles in two teaching stages: learning and revisit (Ding & Li, 2014).

In this study, I defined the variation of problems' cognitive requirement as altering five aspects of problems: representation, response, operation, connection, and reasoning. The purpose of employing the variation of problems is to modify the problems' CR which is suited for students' cognitive levels. During the modification of CR, teachers can change one or more aspects. I use an example including fractions to illustrate how to change five aspects of problems' responses, representations, reasoning, operation, and connections, respectively, to modify problems' CR (see Table 3.1).

For example, the original problem is 1/8 + 3/8 = ? Teachers can employ the strategies of the variation of problems to modify this problem so that it is suited to students' cognitive levels. Teachers can choose from among many different strategies of combinations. Teachers can change one aspect or more aspects. To simplify the explanation of the application of the variation of problems, I changed only one aspect at a time. I assume that I must increase CRs' levels. Then I could transfer the abstract representation to concrete representation, like a1) and a2) (see Table 3.1). I list many modified cases in Table 3.1. Take *operation* as an example I can change the number of 3/8 into 3/5 to increase the requirements of operations and transfer the abstract representation to the concrete representation (see c1 in Table 3.1). Take another example. If I can change the two aspects of representation and response, the modified problem can be “Calculate the sum of 1/8 + 3/8, and use the pictures to explain your results.” In general, there are many cases that I did not list in Table 3.1.

Table 3.1 Examples of Changing Problems’ Cognitive Requirements by Altering the Five Aspects of Problems.

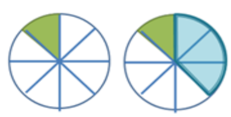
	Example
	1/8 + 3/8 = ?
Representat	 (1) (2) a1) What fraction of the circle area is the shaded green part and blue part altogether? a2) Ann ate 1/8 of a box of strawberries and her brother ate 3/8 of this box. What fraction of the box did they eat altogether?
Response	b1) Ann ate 1/8 of a box of strawberries and her brother ate 3/8 of the same box. What fraction of the box did they eat altogether? Could you use area models to explain your answer? b2) Ann ate 1/8 of a box of strawberries and her brother ate 3/8 of the same box. Ann and her brother ate ____ of the box altogether.
Operation	c1) Ann ate 1/8 of a box of strawberries and her brother ate 3/5 of the same box. What fraction of the box did they eat altogether? c2) Ann ate 1/8 of a box of strawberries, her brother ate 2/7 of the same box, and her sister ate 1/4 of the same box. What fraction of the box did they eat altogether?
Connection	d1) Ann ate 1/8 of a box of strawberries and her brother ate 3/8 of the reminder of the box. What fraction of the box did they eat altogether? d2) There are 40 strawberries in a box. Ann ate 1/8 of a box of strawberries and her brother ate 1/5 of the same box. How many strawberries did they eat altogether?
Reasoning	e1) Ann and her brother ate 3/8 of a box of strawberries. However, Ann ate only 1/8 of the same box. What fraction of the box did Ann’s brother eat? e2) There are 40 strawberries in a box. Ann ate 1/8 of a box of strawberries and her brother ate 8 strawberries. What fraction of the box did they eat altogether?

Table 3.2 The Framework for the Variation in Problems.

Factor	Subcategory	Instruction
Cognitive requirement	Recall/reproduce	This level involves the recall of information (fact, definition, term, or property) or the application of an algorithm or formula.
	Conceptual understanding	Skills and concepts involve more than one step, demonstrating conceptual understanding through models, comparing and classifying information, estimating, and interpreting data from a simple graph.
	Strategic thinking	This involves reasoning, planning, and using evidence to solve a problem or an algorithm.
Representation	Real world	The problem is related to daily experience and presented by daily language. Pictures are not necessary for students to solve problems.
	Pictures	Pictures show information that is necessary for students to solve the given problem.
	Written math language or symbol language	The problem is presented by written math language or symbol language.
Response	Numerical answer only	Students need only to give a number.
	Numerical expression required	Students must use a numerical expression to show the solution.
	Explanation or solution required	Students must explain the solution.
Reasoning	No reasoning	Reasoning is not necessary to solve the problem.
	Simple reasoning	One-step reasoning is necessary to solve the problem.
	Complex reasoning	Two or more steps of reasoning are necessary to solve the problem.
Operation	No operation	No arithmetic operation.
	Sample numerical operation	Operation steps are less than or equal to two steps; either addition/subtraction or multiplication/division.
	Complex numerical operation	Operation steps are more than two steps, or operation includes no fewer than two types of operations.
Connection	No connection	The problem has no connection to other concepts.
	One connection	The problem involves one mathematical concept or is connected to one mathematical topic.
	More than one connection	The problem involves more than one mathematical concept or is connected to more than one mathematical topic.

3.2.2. A Conceptual Framework for the Problems' Cognitive Requirements

Based on the above literature on the analysis of problems, I developed a framework for the variation of problems. The framework includes six aspects: problems' representations, response, connection, operation, reasoning, and CR. This framework system integrates the three aspects from Li (2000) and five aspects from Bao (2009) into our six aspects. Additionally, based on the framework for measuring problems' CR in Webb, Horton, and O'Neal (2002) and Son (2012), our problems' CR refers to the depth of problems' CR. I coded CR into three levels: recall/reproduce, conceptual understanding, and strategic thinking (see Table 3.2).

According to the framework for the variation of problems and the relationships between CR and the other five aspects, I built a conceptual framework for the variation of problems' CR to analyze the coherence of the variation of example and practice problems in terms of the five aspects' contribution to problems' CR (see Figure 3.1). The dependent variable is the problems' CR, which is to be explained by five variables corresponding to the five aspects: representation, response, connection, operation, and reasoning. Then I not only examined the relationship between problems' CR and the five aspects in the three series of mathematics textbook but also examined the difference among example and practice problems' CR models in the three series of textbooks from the perspective of textbook coherence. According to the framework of textbook coherence in the first article, micro-level coherence refers to the coherence of example and practice problems. In this study, I define the coherence of example and practice problems as the coherence of example and practice problems' CR in terms of five aspects. My goal is that the findings of this study will extend the theoretical framework used in previous studies (e.g., Cai, 2004; Ding & Li, 2014; Li, 2000; Son & Senk, 2010) and provide a new perspective from which to examine the coherence of example and practice problems.

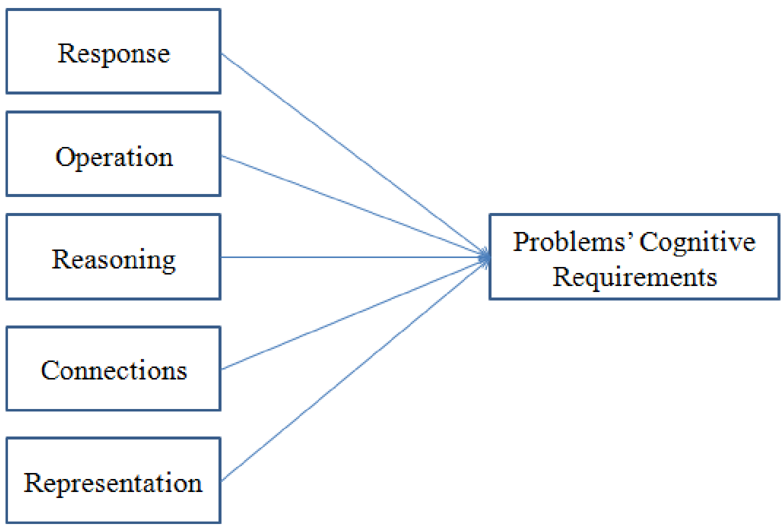


Figure 3.1 The model of the variation of problems' CR.

3.2.3. Research Questions

In this study, I examined the similarities and differences in the example and practice problems' CR in the three models for the variation of problems from the three series of textbooks. The following research questions guided this study:

1. How do the three series of mathematics textbooks specify problems' CR in the model for the variation of problems' cognitive requirements?

2. What are the similarities and differences in example problems and the practice problems' CR in the three models for the variation of problems' cognitive requirements?

3.3. Method

3.3.1. Textbook Selection

In this present study, I selected a series of school mathematics textbooks from three regions: California, Singapore, and Shanghai. Each region has its own educational system. California is one of the richest states in the U.S. and is the most populous state. California has its own educational system and curriculum standard. About 25% of K-12 students are English learners. I take the California educational system as representative of the U.S education system. Therefore, one popular series of textbook, *California Mathematics*--Macmillan/McGraw-Hill (Altieri et al., 2007) was selected (Koedel, Li, Polikoff, Hardaway, & Wrabel, 2017).

Singapore is a unique country with a small national territorial area. English is the first language and the language of instruction. Singaporean education is strongly influenced by Confucian culture but also has includes Western educational ideas. In addition, Singaporean students performed at the top level of international surveys (e.g., PISA and TIMSS). The US and UK have imported the Singaporean mathematics curriculum into their elementary stage to improve the quality of mathematics curricula [5]

[5] SingaporeMath.com, a company that has distributed the "Primary Mathematics" books in the United States since 1998, reports that it now has sales to more than 1,500 schools, about twice as many as in 2008. (The New York Times, 2010/10/01, Retrieved from https://www.nytimes.com/2010/10/01/education/01math.html).

(Borisovich, 2018[6]; Forsythe, 2017). Therefore, the series of *My Pals are Here* (2nd Edition) (Kheong, Ramakrishnan, & Wah, 2010) was selected because it is one of the most commonly used textbooks in Singapore (Yang, Reys, & Wu, 2010).

Shanghai is one of the biggest international cities in China and has its own educational system and mathematics curriculum standards. Chinese Mandarin is the primary language. There is only one series of mathematics textbooks in Shanghai, which was published by the Shanghai Education Press (SEP) (Shanghai Education Press, 2014a, 2014b). Additionally, Shanghai is the only city in mainland China that has participated in international assessments three times in the past 10 years (e.g., PISA). Shanghai elementary mathematics textbooks have been imported by the UK educational system. The Shanghai teaching style will become standard in England (Daily Mail, 12 July 2016). The hope is to stop British youth from falling behind their Asian counterparts. Thus, it is meaningful to analyze the features of these textbooks. The selected SEP is a revised version based on the new Shanghai Elementary and Secondary Mathematics Curriculum Standard (Shanghai Education Press, 2004).

To simplify the research range, I selected the topic of fractions because this topic is usually introduced across grade levels from the third to sixth grades. According to the Common Core State Standards (Common Core State Standard Initiative, 2010), fractions instruction begins in third grade and continues through middle school in the US. Meanwhile, fraction instruction begins in third grade and stops in sixth grade in

[6] 2018 November Retrieved from https://math.berkeley.edu/~giventh/diagnosis.pdf

Shanghai (Shanghai Department of Education, 2017). The Singapore syllabus mentioned that students learn fractions from second grade to sixth grade. Therefore, the topic of fractions is an appropriate medium through which to study problems' CR in different textbooks from different education systems. Furthermore, the content of fractions is a very important component of elementary school mathematics because fractions are students' first experience with a mathematical concept beyond what they have learned before. When learning fractions, students across the world tend to misconceive aspects, such as the concept of fractions, equivalent fractions, the addition of unlike fractions, and the division of fractions (e.g., Kamii & Clark, 1995; Li, 2008; Stafylidou & Vosniadou, 2004; Tirosh, 2000). Therefore, analysis of the problems from the chapters on fractions was judged to be an appropriate choice for the purposes of comparing textbooks.

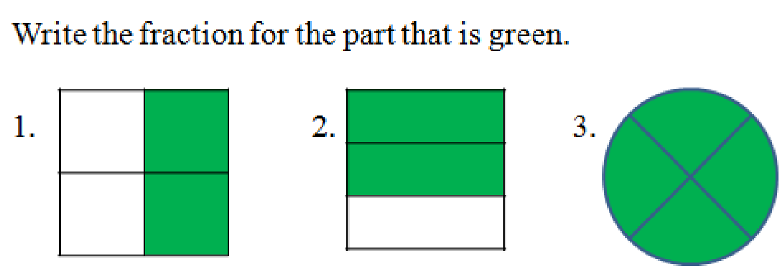


Figure 3.2 One example of the count of fraction problems.

3.3.2. A Coding Instance

First, I identified what a problem is and then coded all problems from the chapters about fractions in the three textbook series. Second, I invited a graduate student to randomly select and code 15% of the problems. Finally, I checked the agreement between my coding and the graduate student's coding and recoded the disagreed ones after discussing with each other.

A problem would be coded as one instance even if the problem included several sub-questions. For example, take the problem in Figure 3.2, "Write the fraction for the part that is green." There are three problems in Figure 3.2. In this study, I coded the three problems as one problem because they have one statement of problems.

Table 3.3 Selected Variables and Measures and Their Descriptive Statistics.

Variable	Representation	Coding
R	Region	Categorical scale
Type	Problem types	Categorical scale
Cog	Cognitive requirements	Continuous on a 3-point scale[7]
Res	Response types	Continuous on a 3-point scale
Ope	Problem operations	Continuous on a 3-point scale
Rea	Reasoning requirements	Continuous on a 3-point scale
Con	Connections with other concepts	Continuous on a 3-point scale
Rep	Problem representations	Continuous on a 3-point scale

Based on the conceptual framework of problems' CR (see Table 3.2), I coded all problems from all fraction chapters. All variables in this study are listed in Table 3.3. Into the order scale, I coded six aspects that can be treated as continuous variables except for regions and types of problems. All sub-categories of aspects were coded into ordinal data. For example, in reference to the cognitive requirements, the three sub-categories were coded into different levels: (1) recall/produce, (2) conceptual

[7] Continuous on a 3-point scale means that the variables are regarded as ordinal data.

understanding, and (3) strategic thinking. Therefore, the process of coding was to transfer all fraction problems to samples with six values of ordinal variables and two values of categorical variables (grade level and problem types).

Thinking 1

$$\frac{1}{2}+\frac{1}{3}=?$$

Student A: $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$

Student B: $\frac{1}{2}$ is one half, $\frac{2}{5}$ is less than one half......Is this equation right?

Question: Please draw pictures and observe them or use other methods to think about it.

Figure 3.3 A coded problem (Translated from Shanghai Education Press, Copyright, 2010, G6A, p. 42).

Take an example from the Shanghai sixth-grade mathematics textbook (see Figure 3.3) to illustrate the process of coding. In Figure 3.3, this problem without a solution was in unit 4 from the fraction chapter in the sixth-grade Shanghai mathematics textbook. The representation of this problem is mathematical language, and the picture does not contain any information, so this problem's representation is coded into the third category, written math language or symbol language. The response type is labeled as "explanation or solution." Based on this categorization, the response aspect is then coded into the third category (see Table 3.4). The cognition requirement of this problem is strategic thinking because students must think about the reasons for the incorrect expression of $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$ and provide their evidence to explain their judgments. Also, students must use a simple numerical operation (therefore, it is coded into the second category) to check the correction of the answer of 2/5 by transferring the two fractions (1/2 and 1/3) into the other two fractions with the same denominator.

Table 3.4 The Coded Example in the Shanghai Textbook.

Aspect	Category	Code
Region	Shanghai	3
Grade	Grade 6	6
Type	Practice problem	2
Cognition	Strategic thinking	3
Response	Explanation or solution	3
Operation	Simple numerical operation	2
Reasoning	Complex reasoning	3
Connection	Connected to equivalent fractions, the definition of fractions, the comparison of fractions	3
Representation	Mathematical language	3

However, regarding the solution to this problem, students must use complex reasoning (thus, it is coded into the third category of the reasoning aspect) because they not only must understand the definition of fractions based on the part-whole relationship but also must know the sizes of 1/2 and 1/3. Then students must assess the reasoning behind the two students' answers in Figure 3.3 and give the correct explanation. Finally, this problem is connected to equivalent fractions, the definition of fractions, and the comparison of fractions, so the connection aspect was coded into three. I list the coded categories in Table 3.4. In addition, the two categorical variables (regions and problem types) were used as the grouping variables to answer the research questions. The regions include California, Singapore, and Shanghai. The problem types include example and practice problems.

3.3.3. Reliability

I coded all problems in the fraction chapters and then, two months later, recoded them. A few missed instances were added after the second coding. Then the reliability of the coding was checked through additional coding by a graduate student. This graduate student coded a random selection of 15% of the pages using the same coding framework. Cohen's kappa was computed to check the inter-rater reliability. (Usually, kappa, should be 0.7, Landis & Koch, 1977.) The average kappa of 0.91 indicated high agreement between my coding and the graduate student's coding. Regarding the coding with initial disagreement between the two coders, the difference was resolved through discussion.

3.3.4. Data Analysis

Multiple-group regression analysis and different Z-tests were used to address the research questions. First, the beta weight, *p*-values, and structure coefficients were used to examine the differences in factors' contributions to problems' CR in the different series of textbooks.

First, the structure coefficient is the bivariate Pearson *r* (Zero-order relationship) of a measured predictor with the latent dependent variable $\hat{Y}$ (*not* with the *Y* scores). It (r_s) is computed using a formula: $r_s = r_{X with \hat{Y}} = r_{X with Y} / R$ (Thompson, 2006, pp. 240-241), then $r_s^2 = r^2_{X with \hat{Y}} = r^2_{X with Y} / R^2$, where $r_{X with Y}$ is the bivariate correlation between a measured predictor (*X*) and the dependent variable (*Y*), and *R* is the square root of multiple R^2

effect size for the regression containing all independent variables. Therefore, "Squared structure coefficients can show information about the proportion of $\hat{Y}$ (i.e., the explained portion of *Y*) variance explained by the predictors" (Thompson, 2006, p. 240). Unlike regression coefficients (i.e., beta weights), which are affected by correlations among predictor variables, structure coefficients are not affected by the collinearity among predictors because it is the Zero-order relationship of *X* with $\hat{Y}$ (*not* with *Y*). Based on this formula of structure coefficients, I can infer that other independent variables contribute indirectly to structure coefficient values. The difference between a zero-order correlation and a structure coefficient is "that the structure coefficient is scaled to remove the difference of the multiple R^2" (Nathans, Oswald, & Nimon, 2012). As Courville and Thompson (2001) claimed, beta weights and structure coefficients ought to be interpreted in multiple regression. Further, I defined a factor as a significant one when the factor met one of two conditions: a) the factor' *p*-value was less than 0.05; b) when the percentage of a factor's contribution to the variation of the dependent variable in a model (the square of structure coefficient) was equal to or over 10% of the explained variation of the dependent variable even if its *p*-value was more than 0.05. Therefore, both beta weights and structure coefficients are used to evaluate the importance of a predictor rather than relying only on beta weights.

Second, Fishers' *Z*-test was used to examine how well the predictors predict problems' CR in the three series of textbooks, while Hotellings t/ Steiger's *Z*-test was used to examine the structures of the problems' CR models (see Garbin, 2014). Third, I selected the formulas for "$SE_{b\text{-}difference}$" (Paternost, Brame, Mazerolle, & Piquero, 1998) to examine the weights' differences in the different problems' CR models. That is, p-values, structure coefficients, and different *Z*-tests were used to address the two research questions.

Table 3.5 Example and Practice Problems' Distributions in Different Grades.

Region	Grade	Chapter	Example	Practice	Practice/Example	Total
Shanghai	3	1	11	20	1.881	31
	4	1	15	21	1.400	36
	6	1	58	76	1.310	134
	Subtotal	3	84	117	1.393	201
Singapore	2	1	18	47	2.611	65
	3	1	12	22	1.833	34
	4	1	21	45	2.143	66
	5	1	39	102	2.615	141
	6	1	7	37	5.286	44
	Subtotal	6	97	253	2.608	350
California	2	1	8	64	8.000	72
	3	1	20	142	7.100	162
	4	3	31	175	5.645	206
	5	1	77	295	3.831	372
	6	1	44	235	5.341	279
	Subtotal	7	180	911	5.061	1091
	Total	16	361	1281	3.548	1642

3.4. Results

Table 3.5 shows the information about the fraction problems' distributions in the different grades in the three textbook series. There are seven chapters on fractions in California textbooks and five chapters in Singapore textbooks from second grade to sixth grade, while there are three chapters in Shanghai textbooks in the third, fourth, and sixth grades, respectively. Furthermore, there is only one chapter on fractions in each grade level in California, Shanghai, and Singapore textbooks, except that there are three chapters on fractions for fourth graders in California textbooks (see Table 3.5). In addition, the number of problems in Shanghai textbooks is the lowest among the three series of textbooks, while the number of problems in California textbooks is the highest: five times as many as in Shanghai textbooks and over three times as many as in Singapore textbooks. The ratio of practice and example problems in Shanghai textbooks is smallest, while the ratios of practice and example problems in Singapore and California textbooks are almost 3 and 5, separately (see Table 3.5).

Means, standard deviations, and inter-correlations among all variables are shown in Table 3.6. Only the correlation coefficients between representation and the other five factors are negative, except for the categorical variables of region and type. The comparative results of problems' CR models and the differences in example-practice problems' CR models in the three series of mathematics textbooks are shown and explained as follows:

Table 3.6 Correlation Matrix.

Factor	R	Type	Res	CR	Ope	Rea	Con	Rep
R	1.000							
Type	.208**	1.000						
Res	.179**	.036	1.000					
CR	-.066**	-.017	.464**	1.000				
Ope	.108**	-.005	.453**	.285**	1.000			
Rea	-.250**	-.010	.393**	.537**	.298**	1.000		
Con	-.091**	.020	.508**	.552**	.587**	.555**	1.000	
Rep	.005	-.003	-.179**	-.072**	-.065**	-.093**	-.169**	1.000
Mean	2.54	1.78	2.16	2.06	2.00	1.48	1.98	2.14
SD	.702	.414	.789	.572	.746	.578	.724	1.198

Note. R = Regions; Type = Problems' types (example or practice); Res = Problems' responses; CR = Problems' cognitive requirements; Ope = Operation; Rea = Reasoning; Con = Connections; Rep = Problems' representations; SD = Standard deviation; *p < .05; **p < .01.

3.4.1. The Difference in Problems' CR Among the Three Series of Textbooks

3.4.1.1. Multiple-Group Regression Analysis.

This analysis was used to examine the differences among the five factors' contributions to problems' CR among California, Singapore, and Shanghai mathematics textbooks. The results are shown in Table 3.7. Based on the beta weights and *p*-values, the four factors of responses, reasoning, connection, and representation are statistically significant in contributing to problems' CR in California textbooks. Only operation is not statistically significant. In Singapore textbooks, responses, reasoning, and connections are statistically significant for problems' CR (see Table 3.7). Meanwhile, all factors are significant contributors to problems' CR in Shanghai textbooks.

Table 3.7 The Multiple Regression Weights, Construct Coefficients, and *p*-Values in the Three Models.

Factor	California R^2 (44.5%)			Singapore R^2 (28.5%)			Shanghai R^2 (52.2%)		
	β	r_s	p	β	r_s	p	β	r_s	p
Res	.286	.818	.000	.148	.619	.006	.179	.603	.008
Ope	-.035	.516	.244	-.097	.464	.097	-.314	.153	.000
Rea	.308	.823	.000	.162	.849	.017	.378	.796	.000
Con	.251	.809	.000	.362	.933	.000	.423	.761	.000
Rep	.054	-.096	.020	-.025	-.496	.622	-.158	-.253	.002

However, based on structure coefficients, the factor of operation is also an important predictor because its contribution to problems' CR occupies up to 26.6% (0.516^2) of the R^2 value in the California model. Also, the factors of operation and representation are significant for problems' CR in Singapore textbooks because their contributions to problems' CR are 21.5% and 24.6% of the R^2 value, respectively. In general, from the perspective of structure coefficients, all factors are significant in the three series of textbooks. The findings from the perspective of structure coefficients are in line with the theory of variation.

3.4.1.2. Differences Among the R^2 Values.

The *FZT* program was used to complete a Fisher's *Z* test to compare three pairs of R^2 values from three different textbooks (see Table 3.8). As Table 3.8 shows, the R^2 values of the California and Shanghai models are not significantly different. The R^2 value of the Singapore model is significantly smaller than the R^2 values of the California and Shanghai models. This finding means the five factors do not explain problems' CR in Singapore textbooks as efficiently as they do in California and Shanghai textbooks.

Table 3.8 The *Z*-values and *p*-values from Fisher's *Z*-tests for the Three Models.

Pair	R_{G1}	N_{G1}	R_{G2}	N_{G2}	*Z*-value	*p*-value
Ca&Si	.667	1091	.534	350	3.4	.000
Ca&Sh	.667	1091	.722	201	-1.378	.084
Si&Sh	.534	350	.722	201	-3.549	.000

Note. Ca = California textbooks; Si = Singapore textbooks; Sh = Shanghai textbooks.

3.4.1.3. Difference Among "Structures" of the Three Models.

This study selected two-group data from California, Singapore, and Shanghai to form three pairs. The detailed results are shown in Table 3.9. According to the *p*-values, I find that there are different structures between the California and Singapore models and between the California and Shanghai models. The structures of the Singapore and Shanghai models are not significantly different. This means that Shanghai and Singapore textbooks employ the five factors to mediate problems' CR in a similar way, while

California textbooks have a different way by which mediate the contributions from five factors to problems' CR.

Table 3.9 The *Z*-values and *p*-values from Steiger's *Z*-tests for the Three Models.

Pair	Direct R_{G1}	Crossed R_{G2}	Model correlation	*N*	*Z*	*P*
Ca&Si	.667	.636	.954	1091	4.484	.000
Ca&Sh	.667	.571	.856	1091	7.72	.000
Si&Sh	.534	.520	.973	350	1.324	.093

Note. Ca = California textbooks; Si = Singapore textbooks; Sh = Shanghai textbooks.

Table 3.10 The Values of $SE_{b\text{-diff}}$, *Z*-tests, and *p*-values of Brame/Colgg *Z*-tests for the Three Models.

Factor	Ca&Si			Ca&Sh			Si&Sh		
	$SE_{b\text{-diff}}$	*Z*	*p*	$SE_{b\text{-diff}}$	*Z*	*p*	$SE_{b\text{-diff}}$	*Z*	*p*
Res	.138	-2.799	.005	.163	-2.359	.018	.199	.000	1.000
Ope	.040	2.691	.007	.056	1.162	.245	.063	-.688	.491
Rea	.051	1.010	.312	.056	4.193	.000	.069	2.667	.008
Con	.068	2.507	.012	.068	-.514	.607	.088	-2.348	.019
Rep	.063	-1.176	.240	.070	-3.307	.001	.086	-1.835	.067

Note. Ca = California textbooks; Si = Singapore textbooks; Sh = Shanghai textbooks.

3.4.1.4. Differences Between Predictors' Contributions

The results in Table 3.10 illuminate that the contributions of response, operation, and connection to problems' CR between California and Singapore textbooks are significantly different; the contributions of response, reasoning, and representation to problems' CR between California and Shanghai textbooks are significantly different; and the contributions of reasoning and connection to problems' CR between Singapore and Shanghai textbooks are significantly different. This means the three textbook series have different emphases on the five factors. This finding further provides evidence of the differentiations in the three series of textbooks in terms of balancing the five factors to alter problems' CR.

3.4.2. Differences in Example and Practice Problems' CR in the Three Models

3.4.2.1. Multiple-Groups Regressions.

For example-practice comparison in each textbook series, based on beta weights and *p*-values, the two factors of response and reasoning are significant for both example and practice problems' CR in California textbooks. Meanwhile, connection is the only significant factor for both example and practice problems' CR in Singapore textbooks. In Shanghai textbooks, the three factors of operation, reasoning, and connection are also significant for both example and practice problems' CR. Therefore, the descending orders for the total number of significant factors for example and practice problems' CR is Shanghai, California, and Singapore textbooks.

Table 3.11 The Multiple Regression Weights, Construct Coefficients, and *p*-values for the Example and Practice Groups.

Factor		California R^2 (30.7%)			Singapore R^2 (33.3%)			Shanghai R^2 (52.1%)		
		β	r_s	p	β	r_s	p	β	r_s	p
Example	Res	.221	.642	.010	.071	.683	.566	.313	.746	.006
	Ope	-.019	.490	.818	.004	.545	.970	-.329	.408	.006
	Rea	.373	.876	.000	.288	.915	.022	.317	.774	.003
	Con	.164	.694	.055	.285	.923	.043	.426	.779	.001
	Rep	.156	-.121	.042	.001	-.377	.993	-.205	-.219	.012
			R^2 (46.6%)			R^2 (27.4%)			R^2 (53.0%)	
Practice	Res	.294	.827	.000	.159	.587	.010	.095	.492	.284
	Ope	-.049	.520	.130	-.134	.426	.054	-.321	-.049	.000
	Rea	.300	.813	.000	.109	.815	.185	.419	.791	.000
	Con	.274	.821	.000	.398	.930	.000	.411	.744	.000
	Rep	.042	-.090	.090	-.052	-.531	.403	-.114	-.245	.103

For the example comparison, based on the beta weights and *p*-values (see Table 3.11), response, reasoning, and representation are significant factors for the example problems' CR model in California textbooks; meanwhile, reasoning and connection are significant factors in the example problems' CR model in Singapore textbooks. All factors are significant in the example problems' CR models in Shanghai textbooks. According to the structure coefficients, the two factors of operation and connection are also important in the example problems' CR model in California textbooks because their structure coefficients are .490 and .694 (24% and 48.2% of R^2). From the perspective of structure coefficients, all factors are significant for the example problems' CR in the three series of textbooks.

For the practice comparison, according to the beta weights and *p*-values (see Table 3.11), the three factors of response, reasoning, and connection are significant factors in the practice problems' CR model in California textbooks. Also, only two factors of response and connection are significant for Singapore textbooks. Compared with the example problems' CR, response and representation are not significant in the practice problems' CR model in Shanghai textbooks. Even from the perspective of structure coefficient, representation is not significant in the practice problems' CR model in California or Shanghai textbooks.

3.4.2.2. The Differences Between the R^2 Values.

For the comparison of the R^2 of example and practice problems' CR in each textbook series, the findings (see Table 3.12) show only that the R^2 values from example and practice problems' CR models in California textbooks are significantly different, while the other comparison of R^2 values from the problems' CR models in the Shanghai and Singapore textbooks reveals no significant differences.

For the example comparisons, the results show that the R^2 values of the California and Singapore models are not significantly different, which means that the factors' contributions to the example problems' CR in California textbooks are not statistically different as compared to Singapore textbooks. Similarly, the factors' contributions to the example problems' CR in California and Singapore textbooks are significantly smaller than their contributions to the examples problems' CR in Shanghai textbooks (see Table 3.12).

Table 3.12 The *Z*-values and *p*-values of Fisher's Z-tests for the Example and Practice Groups.

Type	Pair	R_{G1}	N_{G1}	R_{G2}	N_{G2}	*Z*-value	*p*-value
California	Ex & Pr	.554	180	.683	911	-2.563	.005
Shanghai	Ex & Pr	.702	84	.728	117	-.366	.357
Singapore	Ex & Pr	.577	97	.523	253	.64	.261
Example	Ca &Si	.554	180	.577	97	-.265	.396
	Ca & Sh	.554	180	.721	84	-2.129	.017
	Si & Sh	.577	97	.721	84	-1.661	.048
Practice	Ca & Si	.683	911	.523	253	3.560	.000
	Ca & Sh	.683	911	.728	117	-.903	.183
	Si & Sh	.523	253	.728	117	-3.044	.001

Note. Ex = Example; Pr = Practice; Ca = California; Sh = Shanghai; Si = Singapore.

For the practice comparisons, the factors' contributions to the practice problems' CR in California textbooks are not statistically different compared to Shanghai textbooks, while the factors' contributions to the example problems' CR in California and Shanghai textbooks are significantly larger than their contributions to the example problems' CR in Singapore textbooks. In general, the five factors better explain the example and practice problems' CR in Shanghai textbooks and better explain the practice problems' CR in California textbooks.

3.4.2.3. Differences Between Model Structures.

The results in Table 3.13 indicate that the structures of example and practice problems' CR models in California textbooks are significantly different, while the structures of example and practice problems' CR models in Shanghai and Singapore textbooks are not significantly different.

Table 3.13 The *Z*-values and *p*-values of Steiger's *Z*-tests for the Example and Practice Groups.

Type	Pair	Direct R_{G1}	Crossed R_{G2}	Model correlation	*N*	*Z*	*p*
California	Ex & Pr	.648	.617	.792	911	1.939	.026
Shanghai	Ex & Pr	.710	.728	.976	117	-1.271	.102
Singapore	Ex & Pr	.501	.523	.958	253	-1.403	.08
	Ca &Si	.554	.517	.933	180	1.604	.054
Example	Ca & Sh	.554	.413	.745	180	3.076	.001
	Si & Sh	.577	.535	.927	97	1.294	.098
	Ca & Si	.683	.633	.927	911	5.319	.000
Practice	Ca & Sh	.683	.580	.849	911	7.514	.000
	Si & Sh	.523	.500	.955	253	1.417	.078

Furthermore, the results of a comparison of example problems' CR models of the three textbook series illuminate the structures of the example problems' CR models in California and Shanghai textbooks are significantly different, while the two other pairs are not significantly different. This no significant result suggests that California and Shanghai textbooks employ the five factors to mediate the example problems' CR in a different way (see Table 3.13).

For the comparison of practice problems' CR models, the findings show that the structures of practice problems' CR models in Shanghai and Singapore textbooks are not significantly different. This suggests that Shanghai textbooks use the five factors to modify practice problems' CR in the same way that Singapore textbooks do, while California textbooks use a different path to balance practice problems' CR.

3.4.2.4. Differences Between Factors' Contributions

For the comparison of beta weights in the example and practice problems' CR models from the three textbook series, the findings indicates that the beta weights of the five factors in example and practice problems' CR models for each textbook series do not have any significant differences.

Table 3.14 The Values of $SE_{b\text{-diff}}$, *Z*-tests, and *p*-values of Brame/Colgg Z-tests for the Example and Practice Groups.

T	Factor	California			Shanghai			Singapore		
		$SE_{b\text{-diff}}$	*Z*	*p*	$SE_{b\text{-diff}}$	*Z*	*p*	$SE_{b\text{-diff}}$	*Z*	*p*
Example & Practice	Res	.065	-.826	.409	.108	1.432	.152	.096	-.569	.570
	Ope	.057	.432	.666	.112	.237	.812	.106	1.016	.310
	Rea	.074	.433	.665	.128	-.728	.467	.139	1.249	.212
	Con	.062	-1.811	.070	.145	.002	.998	.127	-.538	.590
	Rep	.038	1.508	.132	.078	-.425	.671	.068	.439	.661
		California & Singapore			California & Shanghai			Singapore & Shanghai		
Example	Res	.106	1.018	.309	.102	-.706	.480	.119	-1.508	.132
	Ope	.105	-.146	.884	.103	2.339	.019	.128	2.011	.044
	Rea	.135	.582	.561	.118	.443	.658	.152	-.175	.861
	Con	.121	-.962	.336	.132	-2.354	.019	.162	-1.207	.227
	Rep	.069	1.093	.274	.063	3.282	.001	.077	1.700	.089
		California & Singapore			California & Shanghai			Singapore & Shanghai		
Practice	Res	.046	2.329	.020	.074	1.842	.065	.081	.356	.722
	Ope	.059	1.149	.251	.072	3.392	.001	.087	2.037	.042
	Rea	.081	2.722	.006	.089	-.818	.413	.112	-2.610	.009
	Con	.074	-.978	.328	.086	-2.298	.022	.105	-1.201	.230
	Rep	.036	1.292	.196	.060	1.917	.055	.069	.995	.320

For the example comparisons, Table 3.14 shows that none of the factor's contributions to the example problems' CR are significantly different in California and Singapore textbooks. However, the contributions of operation and connection to the example problems' CR in California textbooks are significantly smaller than those in Shanghai textbooks, while the contribution of representation is bigger than that in Shanghai textbooks. This finding about the comparison between example problems' CR in California and Shanghai textbooks suggests Shanghai textbooks emphasize the factor of the examples' operation and connection while California textbooks emphasize the examples' representation. When compared to Shanghai textbooks, Singapore textbooks place more emphasis on the examples' operation.

Similarly, in terms of the practice comparisons in three series of textbooks, California textbooks place more emphasis on the practice problems' response and reasoning than do Singapore textbooks (see Table 3.14). When comparing the practice problems in California textbooks to the practice problems in Shanghai textbooks, I find that California textbooks emphasize operation while Shanghai textbooks emphasize connection. Compared to Singapore textbooks which emphasize practice problems' operation, Shanghai textbooks emphasize reasoning.

3.5. Discussion

David Hilbert (1862-1943) noted that mathematical problems are the spirit of mathematics. Similarly, mathematical problems are the main body of mathematics school textbooks. Correspondingly, the variation of problems' CR should be one important aspect of textbook analysis (e.g., Li, 2000; Zhang et al., 2017). In this study, I have analyzed the variation of problems' CR in the three series of textbooks in terms of five factors. Next, I discuss the findings and implications.

3.5.1. Differences and Similarities

The differences among the important aspects between the three textbook series are not significant. The findings reveal that the five aspects are essential for problems' CR in the three series of textbooks, while the factor of representation is not an important contributor to practice problems' CR in California and Shanghai textbooks. Furthermore, representation has a negative relationship with problems' CR, while the other four factors have positive relationships. The negative relationship between representation and problems' CR aligns with the finding from Kaminski, Sloutsky, and Hecker (2008) that abstract representations have advantages over concrete representations in terms of solving complex problems. Nathan, Long, and Alibali (2002) noted that word problems are more difficult than computations. In other words, word problems highlight the translation between concrete and abstract representations, while computations highlight operation. For a problem with a concrete representation, students often need to translate a concrete representation into an abstract representation and use a rule to solve it. Finding a solution to a real-world problem requires students to use more representations. This aligns with the finding from Huang and Cai (2011), whose study of pedagogical representations found a positive relationship between problems' CR and the number of representations used for solving problems.

The emphasis placed on the five factors' contributions to problems' CR in the three textbook series varies widely. On the one hand, Shanghai textbooks put more emphasis on the balanced contributions from five factors than do the other two textbooks. As Sun (2013) found, Chinese mathematics textbooks provide opportunities to make connections and emphasize the underlying rationale behind the algorithm by using variation problems. On the other hand, California textbooks weaken the operation's contribution while Singapore textbooks weaken the two factors of operation and representation. The above finding provided evidence for the statement that U.S. standards-based mathematics textbooks emphasized the solving of real-world problems and the development of conceptual understanding (Senk & Thompson, 2003).

The percentages of the five factors' contributions to problems' CR between Singapore and Shanghai textbooks have no significant differences, while those of California and Singapore textbooks are significantly different, as are those in California and Shanghai textbooks. This means Singapore and Shanghai textbooks use a similar means to mediate problems' CR by balancing the five factors. The reason for this might be that they are both Asian countries and have similar cultures. Furthermore, the difference between California textbooks and Singapore/Shanghai textbooks might provide evidence for the criticism that traditional US textbooks are repetitive and undemanding, especially in terms of practice problems (e.g., Flanders, 1987; Zhu & Fan, 2006). The amount of practice in US mathematics textbooks is greater than those in Shanghai and Singapore textbooks (e.g., Alajmi, 2012; Schmidt, McKnight, & Raizen, 1997). This feature is related to the fact that the US has a more varied population of students, which may necessitate more practice to meet the needs of this diverse population.

The differences in the five factors' contributions to problems' CR in the three series of textbooks are significant. This finding further reveals that different textbooks place different emphases on the five factors. California textbooks emphasize the variation of operation and connection to modify problems' CR while Singapore textbooks highlight the variation of response. Also, the findings show that California textbooks emphasize the variation of reasoning while Shanghai textbooks underline the various responses and representations. A comparison of Singapore textbooks to Shanghai textbooks show that Singapore textbooks emphasize the variation of reasoning while Shanghai textbooks highlight the improvement of connection.

3.5.2. The Coherence Between Example Problems' CR and Practice Problems' CR

For the comparison of example and practice problems' CR in each of the three textbook series, the contributions of the five factors to the example and practice problems' CR in California textbooks are significantly different. Similarly, the structures of the variation of example and practice problems are significantly different. Based on these findings, I could infer that the coherence of the variation of example and practice problems' CR in California textbooks in terms of the five factors is weaker than those in Shanghai and Singapore textbooks. This finding supports the statement that mathematics textbooks' editorial teams must pay more attention to the coherence of mathematics textbooks (Schoenfeld, 2004; Wu, 2011).

To be specific, first, the differences in the factors' contributions in example problems' CR in the three series of textbooks are relatively little while those in the practice problems' CR are varied. The factor of representation is not an important contributor to practice problems' CR in California and Shanghai textbooks. As Son and Senk (2010) found, around 70% of problems in *Everyday Mathematics* and the 7th Korean mathematics curriculum are presented as pure mathematical language. Similarly, Sherman, Walkington, and Howell (2016) found that the symbol procedure view (i.e., symbolic equations are easier to solve and should be taught before verbal problems) still prevailed in US textbooks. On the other hand, Singapore textbooks often have the feature of translation among concrete-pictorial-abstract representation. However, this feature highlights the fact that the solutions to problems in Singapore textbooks are presented by means of using the translation of presentations from concrete to pictorial to abstract representation. Actually, most problems are presented in terms of pure mathematical language. Singapore textbooks often show example and practice problems in a manner of *concrete fading*. The transformation from concrete to pictorial to abstract representation is clearly shown in Singapore textbooks. The strategy of concrete fading has been advocated by some studies (e.g., Goldstone & Son, 2005; McNeil & Fyfe, 2012). In our study, I defined problem representation as the statements of problems. This definition could explain that representation is not an important factor in determining practice problems' CR in Shanghai textbooks. This finding about representation is also in line with the feature of Chinese mathematical textbooks that emphasizes the learning and application of abstract knowledge (e.g., Ding & Li, 2014; Huang & Cai, 2011).

Second, the differences in the five factors' contribution to example problems' CR in the three textbook series are relatively small. There are no significant differences between California and Singapore textbooks. The contributions of operation and representation to example problems' CR in California textbooks are significantly larger than those in Shanghai textbooks, while the contribution of example problems' connection is smaller than that in Shanghai textbooks. This result is in line with the finding from Ding (2016) that the nature of concrete representations in Chinese examples is contextual and that Chinese textbooks stress structural relations while US textbooks place a lesser emphasis on connections. Likewise, Sun (2011) argued that Chinese mathematics provided more opportunities to make connections by using the variation of problems. A comparison of Shanghai textbooks and Singapore textbooks reveals that the two textbook series use a similar strategy of mediating example problems' CR, which may be related to the similarities in the Singapore and Shanghai educational systems. Their education is strongly influenced by Confucian culture. Additionally, Singapore textbooks emphasize example problems' operation a little more than do Shanghai textbooks. This emphasis of operation supports the finding from Yang, Chang, and Sianturi (2017) that 75% of questions in Singapore textbooks are shown in purely mathematical forms.

Finally, the differences in the five factors' contribution to practice problems' CR in the three textbook series are significant. California textbooks emphasize practice problems' response and reasoning more than Singapore textbooks do. Additionally, when comparing practice problems in California textbooks and practice problems in Shanghai textbooks, I find that California textbooks emphasize operation while Shanghai textbooks emphasize connection. As Zhu and Fan (2006) found, US textbooks have a large number of practice problems and more variation of non-traditional problems than do Chinese textbooks. Non-traditional problems including problem-posing, puzzles, project, and journal-writing, require more response types. Furthermore, when compared to Singapore textbooks emphasizing practice problems' operation, Shanghai textbooks emphasize reasoning. As Seah and Bishop (2000) claimed, the editorial team of Singapore textbooks has the value that even practice problems appear in the form of drills.

3.6. Conclusion

This study aims to examine the variation of fraction problems' CR in the three series of mathematics textbook. To address this question, I developed a framework for the variation of problems with six aspects. Based on this framework, I built a conceptual model for problems' CR to analyze the coherence of the variation in California, Shanghai, and Singapore textbooks. Next, I employed multiple-group regression and various Z-tests to analyze the differences among the three problems' CR models in two aspects: example and practice problems. The results of this study provide detailed information about problems' CR model, including five other aspects (response, operation, reasoning, connection, and representation) by comparing the five aspects of fraction problems in the three textbook series.

The results showed that operation was not a significant factor contributing to problems' CR in California textbooks. Shanghai textbooks more effectively balanced the five aspects to mediate problems' CR. Second, the differences between example and practice problems' CR models in California textbooks were significant, while those in Shanghai and Singapore textbooks were not significant. Third, comparison analysis found that the emphases of the textbooks varied in three aspects. California textbooks emphasized operation and connection more, while Singapore textbooks emphasized response. On the other hand, Shanghai textbooks emphasized response and representation while California textbooks emphasized reasoning. Singapore textbooks emphasized reasoning while Shanghai textbooks emphasized connection.

In general, the present study provides a distinct perspective from which textbook designers, mathematics educators, and teachers can evaluate and use their textbooks by deeply analyzing problems' CR with respect to the five aspects. First, textbook designers can improve the quality of textbooks by examining problems' CR in terms of the five aspects. In designing textbooks, they can consider the five aspects and create problems that are better suited to students' cognitive development. Additionally, textbook writers might benefit from the results of comparative studies because they can learn from the features of others' textbooks and obtain more information about the variation of problems in mathematics textbooks from different education systems. As this study showed, the five aspects' contributions to problems' CR are significantly different in the three series of textbooks. Different textbooks have their own emphases on the five aspects. California textbooks emphasize problems' connections, response types, and reasoning. Shanghai textbooks emphasize the balance of problems' five aspects. Singapore textbooks underline problems' connection and response.

Moreover, this study may be helpful as a means by which mathematics educators can further analyze and explain the nature of mathematics textbooks (such as textbook coherence between example and practice problems) from the perspective of quantization.

On the one hand, based on the conceptual model of problems' CR, mathematics educators can provide pre-service and in-service teachers with professional development programs about problem-posing and problem-solving. Specifically, teachers can become informed by seeing the alteration of problems' five aspects and creating new problems based on a given problem. The five aspects can also be regarded as strategies of problem-posing. On the other hand, based on the findings of this study, mathematics educators can further explore the similarities and differences in the variation of problems among multiple mathematics textbooks and examine the coherence between example and practice problems in terms of the five aspects by selecting all problems in textbooks.

Finally, teachers can better use mathematics textbooks in their teaching after they understand the relationships between problems' CR and the five aspects. The gap between the intended and implemented curriculum has been examined in the past decade, and teachers can narrow this gap by creating appropriate problems suited for their students' cognitive levels, based on problems in textbooks, by altering the problems' five aspects. This study provides opportunities for teachers to understand problems' cognitive requirements and to examine the rationality of curriculum standards with respect to problem-solving by showing the differences between example and practice problems' CR in term of five aspects.

However, I caution against simple generalization of the findings due to the limitations of this study. One limitation is that this study examines only problems' CR in the chapters on fractions from the three series of mathematics textbooks rather than the whole series of textbooks. Future studies may examine additional chapters and versions of the textbooks. Another limitation is that the coding system is not perfect. Each factor includes only three sub-categories. More sub-categories could be added to the framework to vary problems based on the prior studies. The last limitation is that this study compares three different series of textbooks rather than the models between the example and practice problems in each textbook. Additional studies may explore a more complicated structure among these factors. Researchers can also explore the mediation effect of the factors of reasoning and connection in this conceptual framework for problems' CR because these factors have high correlation coefficients with problems' CR.

3.7. References

Altieri, M. B., Balka, D. S., Day, R. D., Gonsalves, P. D., Grace, E. C., Krulik, S. et al., (2007). *California mathematics: Concepts, skills, and problem solving* (grade 2-6). Columbus, OH: McGraw-Hill.

Bao, J. (2009). *A comparative study on the comprehensive difficulties of mathematics curriculum in junior middle school between China and the U.K.* Nanning, Guangxi: Guangxi Education Press.

Callahan, L., Nicely, Jr., R. F., Fiber, H. R., & Bobango, J. C. (1986). The cognitive content of elementary school mathematics textbooks. *Arithmetic Teacher*, *34*(2), 60-61.

Cai, J. (2004). Why do US and Chinese students think differently in mathematical problem solving? Impact of early algebra learning and teachers' beliefs. *The Journal of Mathematical Behavior*, *23*(2), 135-167.

Daily Mail (2016, July 12). *Half of primary schools set to teach maths Chinese-style: Children will be required to practice sums and exercises until they can prove they have mastered them* [News]. Retrieved from http://www.dailymail.co.uk/news/article-3685552/Half-primary-schools-set-teach-maths-Chinese-style-Children-required-practise-sums-exercises-prove-mastered-them.html

Ding, M. (2016). Opportunities to learn: Inverse relations in US and Chinese textbooks. *Mathematical Thinking and Learning, 18*(1), 45-68.

Ding, M., & Li, X. (2014). Transition from concrete to abstract representations: the distributive property in a Chinese textbook series. *Educational Studies in Mathematics*, *87*(1), 103-121.

Drake, C., Land, T. J., & Tyminski, A. M. (2014). Using educative curriculum materials to support the development of prospective teachers' knowledge. *Educational Researcher*, *43*(3), 154-162.

Fan, L., & Zhu, Y. (2007). Representation of problem-solving procedures: A comparative look at China, Singapore, and US mathematics textbooks. *Educational Studies in Mathematics*, *66*(1), 61-75.

Flanders, J. R. (1987). How much of the content in mathematics textbooks is new?. *Arithmetic teacher*, *35*(1), 18-23.

Forsythe, S. (2017). How do some teachers of primary mathematics view the impact of the Singapore textbook schemes? *Primary Mathematics*, *21*(1), 9-14.

Garner, R. (1992). Learning from school texts. *Educational Psychologist*, *27*(1), 53-63.

Goldstone, R. L., & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *The Journal of the Learning Sciences, 14*(1), 69-110.

Gu, F., Huang, R., & Gu, L. (2017). Theory and development of teaching through variation in mathematics in China. In R. Huang, & Y. Li (Eds.), *Teaching and learning mathematics through variation* (pp. 13-41). Rotterdam, Netherlands: Sense Publishers.

Huang, R., & Cai, J. (2011). Pedagogical representations to teach linear relations in Chinese and US classrooms: Parallel or hierarchical? *The Journal of Mathematical Behavior*, *30*(2), 149-165.

Huang, R., Mok, I. A. C., & Leung, F. K. S. (2006). Repetition or variation: Practicing in the mathematics classroom in China. In D. J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: The insider's perspective* (pp. 263–274). Rotterdam: Sense Publishers.

Jones, D. L., & Tarr, J. E. (2007). An examination of the levels of cognitive demand required by probability tasks in middle grades mathematics textbooks. *Statistics Education Research Journal*, *6*(2), 4-27.

Kamii, C., & Clark, F. B. (1995). Equivalent fractions: Their difficulty and educational implications. *The Journal of Mathematical Behavior, 14*(4), 365-378.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). The advantage of abstract examples in learning math. *Science, 320*, 454–455.

Kheong, F. H., Ramakrishnan, C., & Wah, L. P. (2010). *My pals are here! Maths* (2nd ed., grade 2-6). Singapore: Marshall Cavendish Education.

Koedel, C., Li, D., Polikoff, M. S., Hardaway, T., & Wrabel, S. L. (2017). Mathematics curriculum effects on student achievement in California. *AERA Open*, *3*(1), 2332858417690511.

Kolovou, A., Van den Heuvel-Panhuizen, M., & Bakker, A. (2009). Non-routine problem solving tasks in primary school mathematics textbooks---a needle in a haystack. *Mediterranean Journal for Research in Mathematics Education*, *8*(2), 31-68.

Landis, J. R., & Koch, G. G. (1977). The measurement of observer agreement for categorical data. *Biometrics, 33*(1), 159-174.

Li, Y. (2000). A comparison of problems that follow selected content presentations in American and Chinese mathematics textbooks. *Journal for Research in Mathematics Education, 31*(2), 234-241.

Li, Y. (2008). What do students need to learn about division of fractions?. *Mathematics Teaching in the Middle School*, *13*(9), 546-552.

McNeil, N. M., & Fyfe, E. R. (2012). "Concreteness fading" promotes transfer of mathematical knowledge. *Learning and Instruction*, *22*(6), 440-448.

Nathan, M. J., Long, S. D., & Alibali, M. W. (2002). The symbol precedence view of mathematical development: A corpus analysis of the rhetorical structure of textbooks. *Discourse Processes*, *33*(1), 1-21.

Nicely, Jr., R. F. (1985). Higher-order thinking skills in mathematics textbooks. *Educational Leadership*, *42*(7), 26-30.

Park, K., & Leung, F. K. S. (2006). Mathematics lessons in Korea: Teaching with systematic variation. In D. J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: The insider's perspective* (pp. 247–261). Rotterdam: Sense Publishers.

Piaget, J. (1957). *Construction of reality in the child.* London: Routledge & Kegan Paul.

Schmidt, W. H., McKnight, C. C., & Raizen, S. A. (1997). *Splintered vision: An investigation of US mathematics and science education.* Boston, MA: Kluwer Academic Publishers.

Schmidt, W. H., & Houang, R. T. (2012). Curricular coherence and the common core state standards for mathematics. *Educational Researcher*, *41*(8), 294-308.

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of US mathematics and science content standards from an international perspective. *Journal of Curriculum Studies, 37*(5), 525-559.

Schoenfeld, A. H. (2004). The math wars. *Educational Policy*, *18*(1), 253-286.

Seah, W. T., & Bishop, A. J. (2000). *Values in mathematics textbooks: A view through two Australasian regions.* Paper presented at the 81st Annual Meeting of the American Educational Research Association, New Orleans, LA. (ERIC Document Reproduction Service No. ED 440870).

Senk, S. L., & Thompson, D. R. (2003). School mathematics curricula: recommendations and issues. In S. L. Senk & D. R. Thompson (Eds.), *Standards-based school mathematics curricula: what are they? What do students learn?* (pp. 3–27). Mahwah: Lawrence Erlbaum Associates.

Shanghai Education Press (2004). *Shanghai elementary and secondary mathematics curriculum standards* (Trial version). Shanghai, Shanghai Education Press.

Shanghai Education Press (2014a). *Mathematics textbook* (Grade 3-4). Shanghai, Shanghai Education Press.

Shanghai Education Press (2014b). *Mathematics textbook* (Grade 6). Shanghai, Shanghai Education Press.

Sherman, M. F., Walkington, C., & Howell, E. (2016). A comparison of symbol-precedence view in investigative and conventional textbooks used in algebra courses. *Journal for Research in Mathematics Education*, *47*(2), 134-146.

Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School, 3*(5), 344-350.

Son, J. W. (2012). A cross-national comparison of reform curricula in Korea and the US in terms of cognitive complexity: The case of fraction addition and subtraction. *ZDM, 44*(2), 161-174.

Son, J. W., & Senk, S. L. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics*, *74*(2), 117-142.

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*(5), 503-518.

Sun, X. (2011). "Variation problems" and their roles in the topic of fraction division in Chinese mathematics textbook examples. *Educational Studies in Mathematics, 76*(1), 65-85.

Tabachneck, H. J., Koedinger, K. R., & Nathan, M. J. (1995). A cognitive analysis of the task demands of early algebra. In *Proceedings of the seventeenth annual conference of the Cognitive Science Society* (Vol. 17, p. 397). Psychology Press.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5-25.

Tornroos, J. (2005). Mathematics textbooks, opportunity to learn and student achievement. *Studies in Educational Evaluation, 31*(4), 315-327.

Ubuz, B., Erbaş, A. K., Çetinkaya, B., & Özgeldi, M. (2010). Exploring the quality of the mathematical tasks in the new Turkish elementary school mathematics curriculum guidebook: the case of algebra. *ZDM*, *42*(5), 483-491.

Van den Heuvel-Panhuizen, M. (2005). The role of context in assessment problems in mathematics. *For the learning of mathematics, 25*(2), 2-9 and 23.

Webb, N. L., Horton, M., & O'Neal, S. (2002, April). *An analysis of the alignment between language arts standards and assessments in four states*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Wijaya, A., van den Heuvel-Panhuizen, M., & Doorman, M. (2015). Opportunity-to-learn context-based tasks provided by mathematics textbooks. *Educational Studies in Mathematics*, *89*(1), 41-65.

Wu, H. H. (2011). Phoenix rising: Bringing the Common Core State Mathematics Standards to life. *American Educator, 35*(3), 3-13.

Xiao, L. Z. (2000). A theory construction of variation model. *Mathematics in the Middle school*, *9*, 15–20.

Xin, Y. P. (2007). Word problem solving tasks in textbooks and their relation to student performance. *Journal of Educational Research, 100*(6), 347-359.

Yang, D. C., Reys, R. E., & Wu, L. L. (2010). Comparing the development of fractions in the fifth-and sixth-graders' textbooks of Singapore, Taiwan, and the USA. *School Science and Mathematics*, *110*(3), 118-127.

Yang, D. C., Chang, M. C., & Sianturi, I. A. (2017). The study of addition and subtraction for two digit numbers in grade one between Singapore and Taiwan. *American Journal of Education and Learning*, *2*(1), 75-82.

Zhang, J., Wang, R., Huang, R., & Kimmins, D. (2017). Strategies for using variation tasks in selected mathematics textbooks in China. In R. Huang, & Y. Li (Eds.), *Teaching and learning mathematics through variation* (pp. 213-239). Rotterdam, Netherlands: Sense Publishers.

Zhu, Y., & Fan, L. (2006). Focus on the representation of problem types in intended curriculum: A comparison of selected mathematics textbooks from Mainland China and the United States. *International Journal of Science and Mathematics Education*, *4*(4), 609-626.

4. EXAMINING US AND CHINESE MATHEMATICS TEACHERS' KNOWLEDGE FOR TEACHING WITH A FOCUS ON INSTRUCTIONAL COHERENCE: THE CASE OF EQUIVALENT FRACTIONS

4.1. Introduction

After the release of a greater number of international mathematics assessments (e.g., TIMSS and PISA), various concerns about students' mathematics achievements have grown throughout the world. Many countries have begun turning their attention toward mathematics education. Educators are working to improving students' achievement, particularly in areas like the US and the UK where students' mathematics achievement levels are below the international level.

Previous studies have explored the reasons for the differences in students' mathematics achievement. Few would disagree that the quality of mathematics teaching depends on teachers' content and pedagogical knowledge. NCTM (2000) stated, "Effective teaching requires knowing and understanding mathematics, students as learners, and pedagogical strategies" (p. 17). Also, Hill, Bowen, and Ball (2005) found a significant positive relationship between teachers' mathematical knowledge and their students' achievement. To be specific, the fact that "the mathematical knowledge of many teachers is dismayingly thin" (Ball, Hill, & Bass, 2008, p. 14) is one of the reasons for students' low achievement. To explore some possible reasons for US students' unsatisfactory mathematics achievement compared to Asian students, Ma (2010) examined US and Chinese mathematics teachers' understanding of fundamental mathematics. She concluded that to improve students' performance, mathematics educators must help teachers improve their understanding of mathematical knowledge. To better improve the level of teachers' content and pedagogical knowledge, it is essential that the educators of mathematics teacher education assess those teachers' mathematical knowledge for teaching.

n the past decade, many studies have reported differences in teachers' knowledge for mathematics teaching between the US and East Asia (e.g., An, Kulm, & Wu, 2004; Ma, 1999; Kim, Ham & Paine, 2011; Schmidt et al., 2007; Leung, 2006; Senk et al., 2012). An important reason for this lies in the difference between US and Chinese education systems. US students often have different teachers who teach their mathematics in various grades, while Chinese students often have the same teacher for several years. Specifically, US mathematics teachers are often assigned to teach one or two grade levels and they focus on teaching a limited number of content topics for several years. By contrast, Chinese mathematics teachers often teach the same mathematics to the same students throughout the grade levels. For example, each elementary mathematics teacher has an opportunity to teach different grades for several years. This means each teacher can teach all grade levels over the course of several years. Second, for US secondary students, even those in the same grade, different mathematics subjects could be taught by different mathematics teachers. For Chinese secondary students, all mathematics content is taught by one teacher. Therefore, US teachers teach only individual grade levels with no continuity, resulting in a system that provides inconsistent student learning.

In China, the feature of teachers teaching students for several years could contribute to two strengths. One is that Chinese mathematics teachers could know their students better than US mathematics teachers know their students. The other is that Chinese mathematics teachers could have coherent knowledge about what they teach. However, Chinese students can also encounter incoherent instructions when they have a new teacher or enter a new school, such as during the transition from elementary to middle school or from high school to college/university. Therefore, it is a challenge for middle school teachers teaching first-year middle school students to implement coherent instruction between the last-year of elementary school and the first year of middle school.

Few studies focus on examining the US and Chinese mathematics teachers' knowledge for teaching from the perspective of instructional coherence (Cai, Ding, & Wang, 2014; Chen & Li, 2010). There are at least two reasons for this topic of study in the existing literature. First, there is no clear consensus on the definition of instructional coherence due to the complex nature of mathematics knowledge for teaching. This conceptual issue becomes even more complicated when teachers are from different educational systems. Second, mathematics educators still know little about the examination of mathematics teachers' knowledge from the perspective of instructional coherence. Therefore, there is both theoretical and practical significance in studying teachers' mathematics knowledge in different countries through a comparative analysis from a perspective of instructional coherence.

To simplify the research range, this study selected the topic of equivalent fractions (EF) because this topic is usually introduced in the third or fourth grade and is repeatedly reviewed later in both fifth and sixth grades in the US and China. This means learning EF will be done across elementary and middle school. In the elementary stage, knowledge of EF can also be used to check students' understanding of fractions in terms of different pictorial models (e.g., area model, length model, set model, and number line). In the middle school stage, EF is the foundation of the addition and subtraction of unlike fractions. Students are then expected to master EF and to generate EF for learning the four operations of fractions. According to the Common Core State Standards (Common Core State Standard Initiative [CCSSI], 2010), fraction instruction begins in third grade and continue through low-secondary school in the US. Meanwhile, in China, fraction instruction begins in third grade and stops in sixth grade (Shanghai Department of Education, 2017). Therefore, the topic of EF is an appropriate medium through which research teachers' knowledge across grades from a perspective of coherence.

This study examines US and Chinese mathematics teachers' knowledge for teaching EF by using a cross-national comparative approach. First, this study develops a conceptual framework for instructional coherence in mathematics teaching by reviewing the pertinent literature. Then, this study analyzes and compares US and Chinese teachers' knowledge in the instruction of EF. To be specific, I will examine US and Chinese teachers' knowledge of EF in terms of two aspects: EF instruction in the elementary school and the differences between EF instruction in elementary school and in middle school. Therefore, the following two questions guide this study:

- What is the selected US and Chinese teachers' knowledge of teaching EF?
- What are the similarities and differences between the US and Chinese teachers' knowledge of teaching EF?

This study is significant because it can inform our understanding of the differences and similarities in US and Chinese mathematics teachers' knowledge for teaching EF from the perspective of instructional coherence as well as provide a new perspective on instructional coherence to improve teachers' mathematical knowledge for teaching.

4.2. Literature Review

4.2.1. Mathematics Knowledge for Teaching

Shulman's (1986) conceptualization of teacher knowledge led a new phase--one that continues to the present--of research into teacher knowledge. Shulman and his colleagues proposed different categories of teacher knowledge that are necessary for effective teaching. Although the specific boundaries and the names of the categories vary across publications, Shulman's is the most detailed. He proposed seven different categories of teacher knowledge: a) general pedagogical knowledge; b) knowledge of learners' characteristics; c) knowledge of educational context; d) knowledge of educational purposes and values; e) content knowledge; f) curriculum knowledge; and g) pedagogical content knowledge (PCK). PCK is a special domain of teacher knowledge. It refers to the distinctive bodies of knowledge for teaching--a kind of subject-matter-specific professional knowledge (Ball, Thame, & Phelps, 2008). Shulman (1987) further defined PCK as a special amalgam of content knowledge and pedagogy. One part of

PCK is the knowledge of teaching strategies and representation while the other is the knowledge of the students' understanding of the subject. After Shulman's definition of PCK, the term PCK was widely accepted across the world. However, PCK does not have an accepted definition or conceptualization. Many studies have expanded or re-conceptualized the definition of PCK (e.g., An, Kulm, & Wu, 2004; Ball et al., 2008; Cochran, DeRuiter, & King, 1993; Marks, 1990).

For example, Deborah L. Ball and her colleagues' work was accepted as the most prominent studies on re-conceptualizing PCK in the field of mathematics education. Building on Sulman's conceptualization of the teachers' knowledge base, Ball and her colleagues proposed the new name of *Mathematical Knowledge for Teaching* (MKT) based on an analysis of teaching on a large scale (Ball et al., 2008). The generation of MKT was characterized as work from the bottom to the top (from practice to theory). Ball and her colleagues tested their hypothesis about this "*professional*" knowledge of mathematics by creating special measures of teachers' professional mathematical knowledge. In addition, they analyzed the relationships between those measurements and students' mathematical achievement to provide more evidence of the different categories of knowledge; that is, students' high performance is one of the most important results of effective teaching.

Specifically, MKT emphasizes the use of knowledge in/for teaching rather than teachers themselves (Ball et al., 2008). The MKT model includes two major categories of knowledge: subject matter knowledge and PCK. Subject matter knowledge includes specialized content knowledge (SCK), common content knowledge (CCK), and horizon content knowledge. PCK includes knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC).

For several reasons, the MKT model was accepted as an important model for mathematics teachers' knowledge base. First, four components of knowledge (SCK, CCK, KCS, and KCT) in the MKT model were confirmed by the results of many empirical studies that measured teachers' MKT. Thus, the MKT model is relatively reliable. Second, in the MKT model, PCK includes three components (KCS, KCT, and KCC). Third, the MKT model was developed based on several MKT tests that focused on specialized content knowledge, such as number concepts, operations, and patterns (Hill, Ball, & Schilling, 2004). Finally, the development of the MKT model provides empirical evidence of the positive relationship between student achievement and teachers' MKT. During hypothesis testing for CCK and SCK, findings revealed and verified the positive relationship between MKT and students' performance (Hill et al., 2005). All of these facts help explain the validity of the MKT model.

Shulman's content knowledge and pedagogical content knowledge were questioned for a lack of clarity and their static view of knowledge. For example, Fennema and Franke (1992) claimed that knowledge is the production of the interaction between students and teachers in a given context. To be specific, the interaction among teachers' knowledge of content, teachers' knowledge of pedagogy, students' cognition, and teachers' beliefs produces a knowledge package. This knowledge package determines teachers' behavior and teaching in the classroom. Fennema and Franke (1992) proposed a model of mathematical knowledge for teaching that includes four components: knowledge of the content, knowledge of pedagogy, knowledge of students' cognition and teachers' beliefs.The model focuses on teacher knowledge as it occurs in the context of the classroom. Moreover, they stated that knowledge is dynamic and that teaching is a way to change teachers' existing knowledge system.

4.2.2. Horizon Content Knowledge

It is widely acknowledged that many teachers will do anything possible to support their instruction. In terms of mathematics, teachers must know curriculum mathematics, understand connections between mathematical ideas, and learn how to uncover the mathematical ideas behind mathematical content (Kilpatrick, Swafford, & Findell, 2001). Only then can they successfully solve teaching problems and help students deeply understand mathematical content. However, based on the MKT model, Ball and her colleagues only provisionally added Shulman's third category, curricular knowledge, to PCK. They noted uncertainty about its validity. Additionally, they reluctantly added the third component of knowledge, "horizon content knowledge (HCK)" into subject matter knowledge. They originally described horizon content knowledge as:

> An awareness of how mathematical topics are related over the span of mathematics included in the curriculum. First grade teachers, for example, may need to know how the mathematics they teach is related to the mathematics students will learn in third grade to be able to set the mathematical foundation for what will come later. It also includes the vision useful in seeing connections to much later mathematics ideas (Ball et al., 2008, p. 403).

However, this definition of HCK includes two levels of understanding. The first level of understanding is its relationship to content and curriculum, which is a curricular mathematical knowledge across grades. The second level of understanding refers to advanced mathematics (Wasserman & Stockton, 2014).

Ball and Bass (2009) further delineated the definition of HCK. They divided HCK into three sub-categories: topics which concern connections, both within the field of mathematics and with other disciplines; practice pertaining to how mathematics is constructed; and values specifying its usefulness when completing mathematical tasks.

On the other hand, Martinez et al. (2011) proposed that HCK was “not another sub-domain of MKT but rather mathematical knowledge that actually shapes the MKT from a continuous mathematical education point of view” (p. 2646). They also claimed that HCK can be classified into three types: a) intra-conceptual connections or junctions between different ideas associated with a particular mathematical concept, constituting the essence of mathematics; b) inter-conceptual connections or junctions to different mathematical concepts; and c) temporal connections or relationships between mathematical concepts at different stages of the curriculum, that is, between what has been studied and what will be studied.

Meanwhile, Ma (2010) proposed a concept of longitudinal coherence and explained,

> Teachers with a profound understanding of fundamental mathematics (PUFM) are not limited to the knowledge that should be taught in a certain grade; rather, they have achieved a fundamental understanding of the whole elementary mathematics curriculum. With PUFM, teachers are ready at any time to exploit an opportunity to

> review crucial concepts that students have studied previously. They also know what students are going to learn later, and take opportunities to lay the proper foundation for it. (p. 122)

The description of longitudinal coherence is similar to the statement of HCK from Ball et al. (2008). Teachers must realize that mathematical topics are related across the different chapters or the different grades in the curriculum.

Furthermore, Jakobsen, Thames, Ribeiro, and Delaney (2012) defined HCK and stated, “Horizon [content] knowledge relevant for teaching is typically about appreciating structure, both in the sense of gaining familiarity with important mathematical structures of the discipline and with understanding them and being able to use them as structures” (p. 9). This definition is related to the definition of curriculum coherence in Schmidt et al. (2002). A coherent curriculum should be one in which a set of content standards evolves from particulars to inherent deeper structures that serve as a means of connecting the particulars.

Moreover, Carreno, Ribeiro and Climent (2013) also defined HCK,

> As an awareness of how the current mathematical topic fits into the overall scheme of the students’ mathematical education, how the various topics relate to the others, and the way in which the learning of a particular topic may relate with others as one moves up the school. (p. 3)

Then they explained that teachers with HCK could have a large mathematical environment out of the subjects they teach and perceive different connections among different topics or in one topic across chapters/grades. Meanwhile, HCK can be regarded as advanced mathematical knowledge because teachers with HCK can know what (out of their curriculum) is mathematically important and worthwhile to pursue when they teach a topic (Carreno et al., 2013)

However, Zazkis and Mamolo (2011) claimed that the notion of HCK can be perceived as advanced mathematics knowledge. That is, “knowledge of the subject matter acquired during undergraduate studies at colleges or universities” (p. 1). Guberman and Gorev (2015) claimed that knowledge of the mathematical horizon can be considered a separate category based on their findings. They found that this knowledge has three characteristics: a) insight of subject matter, b) mathematical connections, and c) understanding of meta-mathematics.

In summary, the different definitions of HCK include one or two kinds of knowledge: the content across grades or advanced mathematics knowledge. However, scholars have different definitions of advanced mathematics knowledge. One is knowledge of the subject matter acquired during undergraduate studies at universities, while the other is a large mathematical environment out of the subjects they teach in the field of school mathematics.

4.2.3. Coherence of Mathematical Knowledge

Mathematics is shown as a coherent, consistent, structured, hierarchical, and organized knowledge system of concepts, laws, and propositions (Suppe, 1977; Koponen & Pehkonen, 2010). The Common Core State Standards Initiative (CCSSI) stated that “mathematics is not a list of disconnected topics, tricks, or mnemonics; it is a coherent body of knowledge made up of interconnected concepts” (CCSSI, 2017).

Coherence of knowledge can ensure that the system of knowledge becomes a whole rather than a set of isolated concepts, theorems, propositions or models (Koponen & Pehkonen, 2010). Coherent knowledge is relatively readable and understandable for learners. The coherentists claimed that knowledge is a globally connected structured system. Most importantly, when new knowledge is added to the system, the large parts of the structure are involved, and the structure itself is also affected by the newly added knowledge (Kosso, 2009).

CCSSI (2017) also defined the coherence in mathematics curriculum as "linking topics and thinking across grades." CCSSI further explained,

> The standards are designed around coherent progressions from grade to grade. Learning is carefully connected across grades so that students can build new understanding onto foundations built in previous years... Each standard is not a new event, but an extension of previous learning. Coherence is also built into the standards in how they reinforce a major topic in a grade by utilizing supporting, complementary topics. For example, instead of presenting the topic of data displays as an end in itself, the topic is used to support grade-level word problems in which students apply mathematical skills to solve problems (CCSSI, 2017).

The CCSSI-described coherence focuses on the structure of mathematics and the natural pathways through that structure. The natural pathways should follow the logical sequence and cognitive development in designing the sequence of mathematics content. Meanwhile, Koponen and Pehkonen (2010) claimed that the coherence of knowledge is restricted in explanatory and deductive coherence. They further explained, "The explanatory coherence is quite naturally connected to the methodology of the experiments. The deductive coherence, on the other hand, is closely related to the deductive use of models and model-type symbolic relations" (p. 262).

4.2.4. The Conceptual Framework for Instructional Coherence

Researchers commonly acknowledge that a teacher's subject matter knowledge and general pedagogical knowledge are important ingredients for high-quality teaching. To implement effective instruction, teachers must have subject matter knowledge and general pedagogical knowledge for teaching a topic. They also must attend to instructional coherence to help students build the connections among the different mathematical topics within a grade or across grades. The precondition of the implementation of instructional coherence is that teachers must have coherence of content knowledge. Therefore, based on Koponen and Pehkonen's (2010) pedagogical coherence and deductive coherence, this study proposed a conceptual framework for instructional coherence including two aspects: coherence of mathematical content and coherence of mathematical pedagogy.

Coherence of mathematical content is defined as an awareness of how mathematical topics are conveyed over the span of mathematics. It also includes the vision of seeing connections to later mathematics content. On the other hand, coherence of mathematical pedagogy means an awareness of the way in which mathematical topics are related throughout the cognitive development and teaching strategies across grades. For example, first-grade teachers may need to know how the mathematics they teach is related to the students' cognitive levels and which kinds of teaching strategies are useful.

4.2.5. Literature About Learning Equivalent Fractions

Learning fractions is a challenging topic for students. The understanding of fractions includes multifaceted constructs of fractions (Behr, Harel, Post, & Lesh, 1992; Kieren, 1993). Five sub-constructs have been identified: part-whole relationship, measurement, quotient[8] (fractions as division), operator, and ratio. To be specific, first, for the construct of a part-whole relationship, the fraction 2/3 can be conceived of as two parts of a whole including three equal parts. Three main models are used to teach fractions when using a part-whole relationship. They are: area model, length model, set model, and line number model. Second, a fraction can be considered a quotient. The fraction 2/3 means two divided by three. Teachers often use a concrete problem to illustrate this construct. For example, *Mrs. Ana has two pizzas, and three children will share the two pizzas equally. How many pizzas will each get? Third, a fraction is regarded as an operator. There are 12 apples in a box. How many apples do you have if you take two-thirds of a box? Fourth, a fraction is a ratio. There are two girls and three boys on a team. What is the ratio of girls to boys?* Finally, fractions are considered as measurements; for example, if 1/3 is located on a number line, then where is the location of 2/3 on this number line (Pantziara & Philippou, 2012)?

Fractions are regarded not only as numbers but also the relation between parts and a whole or between two objects. These factors could be confusing for students. Meanwhile, the understanding of EF is based on the various conceptualizations of

[8] In this study, I used the term *fractions as division* to instead of *quotient*.

fractions. The understanding of EF can be regarded as the extension of the understanding of fractions. In other words, the use of EF to describe different pictorial models (e.g., area model, length model, set model, and number line) can check students' understanding of fractions.

Chan, Leu, and Chen (2007) claimed that the understanding of fractions included three primary sub-concepts: equal sharing, units, and EF. Therefore, the learning difficulties involving fractions extend to the learning of EF. Van de Walle (2004) identified three main indicators to examine students' understanding of EF: rename a fraction into its simplest form, generate sets of EF, and determine fraction equivalence. Furthermore, Chan et al. (2017) summarized previous studies on students' conceptual deficiencies in fractions: a fraction is conceptualized as a quantity, the understanding of equal parts, identify a unit or whole, simplify fractions and represent EF, and representations model distraction.

Ni (2001) argued that students' conception of EF depends on the sub-construct of fractions involved in the learning process. For example, the part-whole construct about area and length embodiments is easier because students can perceive the equivalence of two fractions in their areas and length embodiments. Meanwhile, when using part-whole to explain EF, students have trouble in understanding the set models. As English and Halford (1995) claimed, the equivalence of fractions represented in set embodiments was not suitable for students to learn EF because set embodiments are discrete.

From the perspective of cognition, Piaget (1983/1987) claimed that the understanding of EF involves two types of operative thinking: multiplication thinking and the conservation of parts and the whole. Kamii and Clark (1995) said that EF also involves hierarchical and simultaneous thinking. Specifically, when understanding the rule of EF, students must understand the multiplicative relationship in terms of two aspects: numerators and denominators from different fractions and the numerator and denominator from a single fraction (Moss, 2005). Ding, Li, Capraro and Klum (2012) explained students' misconception of EF. That is, some students' conceptions of EF are at the participatory stage because they cannot understand why fractions are equivalent in the different concrete contexts where they learn EF.

4.2.6. Background Information on the Educational System in the US and in China

This study aims to examine US and Chinese teachers' knowledge of EF. Therefore, it is necessary to describe the differences between the US and Chinese educational systems.

First, China has a center-educational system with three different stage levels for first through twelfth grades. Generally, the elementary stage includes first through sixth grades; the low-secondary school stage includes seventh through ninth grades, and the high-secondary school stage includes tenth through twelfth grades. However, in Shanghai, the elementary stage includes first through fifth grades, the low-secondary school stage includes sixth through ninth grades, and the high-secondary school stage includes tenth through twelfth grades.

In general, when students go to high school (or other stages), they stay in the same class with their classmates for three years from tenth grade to twelfth grade. Students' mobility is very low. A mathematics teacher is assigned to teach them for three years. Most importantly, this teacher must teach all mathematics subjects in the high school stage. In China, mathematics is one subject, and teachers must teach all sub-mathematics subjects, such as algebra I, algebra II, geometry, analytic geometry, calculus, pre-calculus, statistics, and probability. In specific situations, the mathematics teacher may be replaced by colleagues. This can happen if the teacher is sick, or if a student's parents complain, and student achievement is poor. Therefore, the Chinese educational system has many more opportunities for teachers and students to get to know each other in three or more years. In addition, all students in a province (state) use a textbook published by a local education press. All teachers use the same textbook and a teacher version of the textbook. Teachers must complete the teaching of all content required by curriculum standards.

On the other hand, the US has a decentralized educational system. First, there are different classifications of grade level stages. For example, in Texas, the elementary stage includes kindergarten to fifth grade; low-secondary school includes sixth through eighth grades; high-secondary school includes ninth through twelfth grades (TEKS, 2012). Additionally, each state has its own curriculum standard. There is no unified textbook for all students and teachers. That is, US teachers have more freedom in selecting their teaching materials according to the curriculum standard. Furthermore, teachers can make decisions about teaching content according to their students' real situations. For example, some teachers do not use textbooks in teaching. In that case, students do not need to use a textbook in class and instead receive materials from teachers. Additionally, most districts have guidelines/scopes and sequences, as teachers do not have full autonomy in teaching.

4.3. Method

4.3.1. Participants

This study includes 10 Chinese mathematics teachers and 10 US mathematics teachers. The Chinese teachers were selected from Shanghai. Half of them taught at the low-secondary level, while the other half taught at the elementary level. The 10 Chinese teachers were recommended by my former colleagues. The 10 US teachers were recruited from Texas through an email. Four of them taught at the low-secondary level and others taught at the elementary level. In this study, I targeted experienced teachers; therefore, all mathematics teachers must have over three years of teaching experience and have experience teaching EF.

Table 4.1 shows the teachers' background information. Both samples had similar proportions of male and female teachers. All the teachers had over three years of teaching experience; 70% of the selected US teachers and 80% of the selected Chinese teachers had over 10 years of teaching experience. This is intentional in our sampling because this study wanted to include teachers who specialized in the elementary and low-secondary levels separately, as our study topic is EF. In Shanghai, only fourth graders and sixth graders learn EF. However, in the US, those who learn EF are generally from third to seventh grades. Therefore, in the participants, 40% of US teachers are from low-secondary school, while 60% of US teachers are from the elementary stage.

Table 4.1 Background Information About US and Chinese Mathematics Teachers.

C	#	G	Experience (Year)	Fraction teaching	Teaching grade	Taught grades	Teaching Stage
Chinese	1	M	16	16	5	2-5 G	Elementary
	2	F	10	10	3	1-5G	Elementary
	3	F	13	6	4	1-5 G	Elementary
	4	F	12	6	4	1-5 G	Elementary
	5	M	15	10	5	2-6 G	Elementary
	6	F	19	4	9	6-9 G	Low-secondary
	7	F	13	4	6	6-9 G	Low-secondary
	8	F	6	2	7	6-9 G	Low-secondary
	9	F	11	3	6	6-9 G	Low-secondary
	10	F	7	1	9	6-9 G	Low-secondary
US	11	F	20	16	7	6-7-12 G	Low-secondary
	12	F	15	4	7	6-7 G	Low-secondary
	13	F	13	11	6	3-4, 6 G	Low-secondary
	14	F	4	4	7	7 G	Low-secondary
	15	F	5	3	5	5 G	Elementary
	16	F	7	7	5	5 G	Elementary
	17	F	15	7	4	K-4 G	Elementary
	18	F	11	11	4	3-4 G	Elementary
	19	F	15	10	N/A	Coach-K-6	Elementary
	20	M	12	7	5	3-5 G	Elementary

Note. C = Country, G = Gender, Fraction teaching = Years of fraction teaching experience.

The significant difference between the two groups was that 80% of US teachers only had teaching experience at only two grade levels, while, on the other hand, each elementary Chinese teacher had taught from second through fifth grades (almost all grade levels). All Chinese low-secondary school teachers had taught all low-secondary grade levels.

4.3.2. Interview Tasks

4.3.2.1. Instruction of Interview Tasks.

According to the conceptual framework for instructional coherence, I designed four interview tasks to examine teachers’ mathematical knowledge for teaching in terms of two aspects: instructional coherence with a grade and instructional coherence across grades. For instructional coherence with a grade, the first three interview tasks were developed based on relevant literature (see Table 4.2). The three interview tasks match the three main aspects of EF instruction: prior knowledge of EF, teaching EF, and teaching expectations.

Table 4.2 The Instruction of Interview Tasks.

#	Interview task	Purpose	Instructional coherence
1	1. According to your teaching experience, what kind of knowledge do your students need to know before they learn equivalent fractions?	Examine teachers' understanding of the prior knowledge of EF in the elementary stage.	Instructional coherence within grades
	2. In your teaching, how do you interpret the meaning of 2/3 = 4/6 to your students? Can you interpret 2/3 = 4/6 in other ways?	Examine teachers' instructional process about EF.	
	3. What are your teaching expectations on the topic of equivalent fractions in your teaching?	Examine teachers' teaching objectives.	
2	4. What are the similarities and differences in teaching equivalent fractions between the different grade levels? Please provide your explanation as to why.	Examine the coherence of teachers' knowledge of EF teaching across grades	Instructional coherence across grades

In the first interview task, teachers were asked, "According to your teaching experience, what kind of knowledge do your students need to know before they learn equivalent fractions?" Then teachers were asked to respond to the second interview task: "In your teaching, how do you interpret the meaning of 2/3 = 4/6 to your students? Can you interpret 2/3 = 4/6 in other ways?" The third interview task is, "What are your teaching expectations on the topic of equivalent fractions in your teaching?" The three interview tasks are to answer the first research question: "What are the similarities and differences between the selected US and Chinese teachers' knowledge of EF teaching in the elementary stage?" Through the responses to the three interview tasks, I can evaluate teachers' general knowledge for teaching EF.

For instructional coherence across grades, I created one interview task: "Could you share your opinions about the similarities and differences in teaching EF in different grades?" This interview task is to answer the second research question, "Examine teachers' mathematical knowledge for teaching EF across grades." The last interview task is to answer the second question: "What are the similarities and differences between the selected US and Chinese teachers' knowledge of the differences of EF teaching in elementary and middle school?"

4.3.2.2. The Translation of Interview Tasks

Two people who are literate in both Chinese and English double-checked the translations of the interview questions. The first person translated the original questions from English to Chinese. The second person then translated them back into English. I compared their equivalence and consistency. The Chinese version was reviewed by two Chinese mathematics teachers. The English version was reviewed by two US professors in mathematics education in my department. The final version of the interview questions was edited and slightly revised to reflect the feedback from these reviewers.

4.3.3. Data Collection and Analysis

The interview tasks underwent multiple phases of revisions and were piloted by two volunteers who were interviewed to check for possible misunderstandings. Once the revisions were made, the final version of the tasks was emailed to potential in-service teachers.

4.3.3.1. Recruitment

I asked my Chinese former colleagues to recruit mathematics teachers in China. Meanwhile, to recruit US mathematics teachers, I emailed recruitment letters to US mathematics teachers through the websites of College Station and Bryan, Katy Independent School District in Texas. An email including recruitment information (an interview outline and an informed content document) was sent to my former Chinese colleagues and some professors. The volunteer's contact information (i.e., email address, social media, QQ or WeChat) was emailed to us. Once volunteers were selected, interviews were scheduled at mutually convenient times and took place via audio interview. I also used the snowball method to recruit mathematics teachers. I asked interested volunteers to help me recruit additional volunteers until the total number of teachers satisfied the study requirements.

Each interview was conducted either face-to-face or via online video. Before the interview, the interviewers answered any questions that the mathematics teachers might have had and also explained how the teachers would benefit from spending extra time answering the interview questions. I also asked for the teachers' consent to use the interview content for this study. Interviewers conducted the semi-structured interviews by interviewing one participant at a time. Participants received the e-version interview outline before the interview so that they would have enough time to think about the questions. Interviewers asked the volunteers these questions and probed them to elaborate on their answers. When they got stuck, I asked follow-up questions or modified the questions to lead them to answer. All interviews were recorded by voice recording equipment to help the researcher understand and explore these mathematics teachers' understanding of EF.

4.3.3.2. Data Collection

The data comes from voluntary online videotaped interviews. The interviews have been transcribed for analysis. I transcribed all the audio files into one "restored view" file with the help of transcription software. Specifically, I started by analyzing the interview data from each question to compare American and Chinese mathematics teachers' answers and explanations for the same questions. This involved listening to the audio with the transcript and stepping through the mathematics teachers' answers line-by-line to interpret their reasoning and explanations. Additionally, I wrote analytic memos to describe segments of the audio and the transcript, as well as looked for differences that suggested variations in the understanding of curriculum coherence. Finally, I examined similarities and differences that may have indicated whether teachers' realizations about EF were multi-faceted and coherent.

4.3.3.3. Data Coding

I analyzed the teachers' responses to each question using a constant comparison method from Gay and Airasian (2000). I began by generating a list of key terms based on each teacher's responses. For example, when coding the response from Chinese teacher number 1 (CH1) to Q1 regarding prior knowledge before students learn EF, I obtained codes of participants' responses, such as "What do the numerator and denominator mean?", "Have an understanding of the relationship from part to whole." "What is a fraction?", and "Multiplication and division." When coding another teacher's response, I added new codes if the existing codes were not included (see Table 4.3).

Table 4.3 A Coding Example.

Response	Procedure
They have to really understand, there are the part-whole relationship, what the numerator represents, what the denominator represents, because that's the help. You know the idea especially when they first started again, look at models, and they're going to see how they can visually see that as the same amount shaded. And then you will talk about how the parts are different. When you have more parts, and there would be more of them shaded, for that to stay the same. And they have to have multiplication skills, but that's not, we have difficulty for them. But others really understand what is a numerator and denominator, is critical.	Original responses to the first interview question
Part-whole relationship; what the numerator represents; what the denominator represents, multiplication skills; look at models; visually see the same amount shaded; how the parts are different.	Highlight words
Take the highlighted word out and add them to a table and label with either Ch 1 or US1 (see Table 4.4).	Coding
Third grade, they may have looked at two pictures, your models, and recognized that they were equivalent but they are not asked to generate equivalent fractions until fourth grade (When US3 talked about her experience with fractions). We started very elementary. It's a bit. In fourth grade we really start with those models, the paper folding (when US3 explained why 2/3 is equal to 4/6).	Cross-checking: Teaching expectations

During the process of coding each individual question, I triangulated the teachers' responses across questions. For example, when US3 answered the question about the teaching expectations regarding EF, she mentioned that students must know fractions by visually seeing the models. Also, when US3 answered the question about teaching expectations regarding EF, she mentioned, "Third grade, they may have looked at two pictures, your models, and recognized that they were equivalent, but they are not asked to generate equivalent fractions until fourth grade." She said that third graders should know how to label a fraction based on a given picture. That is, third graders must know what a numerator and a denominator are. Similarly, when US3 answered the last interview task regarding the differences in teaching EF among different grades, she stated, "We start with the model, and we want them to see it, that's really more to see, the stage of what we're doing, exercise, foundation" (see Table 4.3). Consistencies and inconsistencies within teachers' responses were recorded. Almost all the teachers' responses were consistent.

After developing the codes, I combined similar codes and sorted the combined codes into several categories, such as the understanding of EF, the application of EF, and concrete, pictorial, and abstract representations. After analyzing the responses of the teachers in each country, I compared US and Chinese teachers' responses and identified the commonalities and differences. In addition, I analyzed the differences in elementary and low-secondary teachers' responses to four interview tasks separately.

To ensure reliability, two months after my first coding, I reviewed the complete data set again using similar procedures. The codes achieved an agreement of 87%. The disagreements were resolved through discussion with other researchers.

4.4. Results

I present the findings on teachers' responses to the interview tasks in terms of two research questions: similarities and differences in US and Chinese[9] teachers' understanding of EF within and across grades.

4.4.1. Similarities and Differences in Teachers' Knowledge of EF Teaching

4.4.1.1. Prior Knowledge of EF

The question is intended to examine teachers' knowledge about the instruction of EF. Generally, teachers must have prior knowledge of a topic that students learn so that they can design an appropriate lesson plan for this topic. Then students can easily learn the topic by connecting it to their prior knowledge. When students cannot connect prior knowledge and a new topic, they might become confused about the topic and not deeply understand the topic's content. In the following, I summarize the findings related to the differences between US and Chinese teachers' understandings of prior knowledge of EF in terms of the elementary and low-secondary stages.

4.4.1.1.1. US and Chinese Elementary Teachers

Four Chinese elementary teachers mentioned that students' prior knowledge included the meaning of fractions (CH 2, 3, 4, and 5). The four teachers used few words

---

[9] US and Chinese teachers in this study refers to the selected US and Chinese teachers in the following sections.

to summarize the prior knowledge of EF and did not provide an additional explanation, such as what the numerator and denominator represent and what the parts and whole are (see Table 4.4). Only one Chinese teacher (CH1) said, "Students need to label fractions on a number line, identify fractions based on pictures, and use fractions to represent quantities in real life."

Table 4.4 Elementary Math Teachers' Responses to Prior Knowledge of EF.

Prior knowledge	Chinese	US
A fraction is a part of a whole	5	5,9,10
What is a fraction		6,7,8,10
The meaning/concept of fractions	2,3,4,5	
Know fractions in terms of pictures or manipulatives	1,4	6,7,8,10
More parts than it takes to make a whole the smaller the pieces		5,
What equivalence mean		6
Draw, label, explain fractions	1	7
The whole has to be the same		7,8
What is the whole		9
Less than one or greater than one		8

All US elementary teachers not only mentioned what a fraction is (US6, 7, 8, and 10) but also mentioned more detailed information about the concept of fractions such as "a fraction is a part of a whole" (US5, 9, and 10) and "students need to know fractions in terms of area models, manipulatives, and number lines" (US6, 7, 8, and 10). To be specific, US8 said, "They would need a basic understanding of 'greater than,' 'less than.' They need to understand, what a fraction is, what it represents, and it would be important that they understand how to locate fractions on a number line in third grade."

In addition, US elementary teachers mentioned other knowledge that Chinese elementary teachers did not mention. For example, US6 mentioned that students must know equivalence. US5 claimed that students must know the idea of "taking more parts to make a whole and proportional the smaller the piece is."

Finally, a US teacher stated that the content of EF is involved in the range of the topic of comparison. The Chinese teachers thought that the learning of EF is one means of reviewing and applying fractions in terms of different models: the area model, length model, set model, and number lines (CH1 and 4). US8 claimed that students had to know the ideas of less than or greater than one and the idea of the whole has to be the same: "They need to know 'greater than,' 'less than,' 'a whole number,' they would need to understand, because if I start talking about comparing fractions, because in order for them to be equivalent. I need to be able to compare them."

4.4.1.1.2. US and Chinese Low-Secondary Teachers

All Chinese low-secondary teachers stated that they did not know elementary mathematics (see Table 4.5). Four of them mentioned that students must know the meaning of fractions (CH6, 7, 9, and 10). For example, CH9 said, "I really do not know elementary mathematics. I thought students need to know the quotient property of invariant[10] and the definition of fractions." Actually, Chinese low-secondary teachers tend to use more abstract symbols to teach mathematical content. This behavior might be due to Chinese teachers' teaching styles and low-secondary school mathematics content. Students will learn algebra and geometry; then they must use more mathematical reasoning instead of multiple representations. All Chinese low-secondary teachers

---

[10] The quotient property of invariant means $a \div b = (a \times k) \div (b \div k) = (a \div n) \div (b \div n), b \neq 0, k \neq 0, n \neq 0$

mentioned that students must know the quotient property of invariant and fractions as divisions before they learn the rule of EF in sixth grade. Additionally, one teacher said students must know fractions as numbers (CH7). Finally, two Chinese low-secondary teachers said that students must know how to label fractions on a number line (CH8 and 10).

Table 4.5 Low-Secondary Math Teachers' Responses to Prior Knowledge on EF.

Prior knowledge	Chinese	US
What do numerator and denominator mean		1,2,3
Parts and whole	6	1, 2,3,4
The definition/concept of fractions	9,10	2,4
The meaning of fractions	6,7	
Fractions as quantities	7	
The property of equality	8	
Multiplication skills		3,4
Less than one or greater than one, mixed numbers		2
Know fractions in terms of pictures or manipulatives	8,10	1,2,3,4
I never taught elementary math; I did not know exactly	6,7,8,9,10	1,2,4

However, all US low-secondary school teachers said that students must know the meaning of fractions and know fractions in terms of different representations (US1, 2, 3, and 4). For instance, US3 said,

> They have to really understand the part-whole relationship, what the numerator and denominator represent because that's the idea especially when they first start to look at models. And they're going to see how they can visually see that as the same amount shaded, and then you will talk about how the parts are different. When you have more parts and there would be more of them shaded for that to stay the same.

Furthermore, two US low-secondary teachers (US3 and 4) claimed that basic multiplication skills were necessary for students to learn EF. For example, US4 said, “They need to know the concept of part to whole, what fractions are. They need to know multiplication and division and have some experience with fractions in terms of either pictures or manipulative, at the end of early grade levels.” One US low-secondary teacher implied that students must know “less than one or greater than one” before they learn EF (US2).

Three low-secondary teachers (US1, 2, and 3) emphasized that students must know what the numerator and denominator represent. Low-secondary teachers may have thought that concrete representations were necessary for students to learn mathematics. For example, a seventh-grade teacher (US2) said, “I tend to go with a geometric model more than a number line model. I know that they are supposed to get in both ways: circles, rectangles, as opposed to the number line. Place it somewhere on the number line.” US2 also explained the reasons for using representations: “I think if they have the connection between what’s going on in an area model, then making the connection to EF, is easier than if they’re just doing abstract like with manipulating the symbols.”

In general, all Chinese elementary teachers stated that the meaning/concept of fractions is the prior knowledge of EF, but they did not further explain what the concept of fractions is. Meanwhile, US elementary teachers explained the prior knowledge in more detail. US elementary teachers further emphasized the application of representations to understand EF.

All Chinese low-secondary teachers had not taught elementary mathematics, while one in four US low-secondary teachers had teaching experience in elementary mathematics. Also, US low-secondary teachers emphasized the application of representations and explained prior knowledge in more detail, while Chinese low-secondary teachers explained prior knowledge in abstract expressions, such as the concept of fractions, the meaning of fractions, and the definition of fractions.

4.4.1.2. Responses to Teaching Expectations of EF

Different grade levels have various requirements for students in terms of EF. Both US and Chinese teachers discussed teaching expectations at the understanding and application levels. Based on teachers' responses, I classified them into two sub-categories: understanding and application levels. Understanding level refers to comprehending EF in terms of the rule, format, and conception. Application level refers to applying the rule and understanding of fractions to generate EF.

4.4.1.2.1. US and Chinese Elementary Teachers

All Chinese elementary mathematics teachers expected students to be able to not only identity EF by pictures but also use the rule of EF to find EF (see Table 4.6). For example, CH5 said,

> We always use pictures to explain why two fractions are equal. It is impossible for students to understand EF based on mathematics logical reasoning. Maybe for some high-level students, you can introduce the rule of EF. It means that you introduce the topic from sixth grade in advance. That is, the value of the fraction is the same when the numerator and denominator of the fraction are multiplied/divided by a number. This is only one requirement for high-level students. Most of the students can understand EF by pictures. I only expect each student to be able to master and identify which fractions are equal based on pictures. That's all.

Specifically, all Chinese elementary mathematics teachers design an in-class activity through which students can find the pattern of EF. Although this approach to EF is often used in the instruction about the rule of EF in sixth grade (low-secondary school), it is also used in the elementary stage in China. For example, CH2 said, “I ask students to find the pattern when using models to find EF. Students can find that the numerator and denominator of a fraction are multiplied and divided by a number; the new fraction is equal to the original one.” However, Chinese elementary teachers expected their students only to be able to use the rule (as a skill) to generate EF rather than to understand the rule of EF.

Table 4.6 Elementary Math Teachers' Teaching Expectations.

	Expectation	Chinese	US
Understanding	The rule of finding EF	1,2,3,4,5	6,8,9,10
	The definition of fractions	1,2	
	Different-format fractions can be equal	2,4,5	
Application	Compare unlike fractions	1,2	7
	Find and explain fractions by drawing/reading pictures	1,2,3,4,5	5,6,7,9,10
	Cancel, reduction, simplifying		5,8,9,10
	Solve real problems		7,8
	Generate EF		6,8

Five US elementary mathematics teachers (US5, 6, 7, 9, and 10) expected students to be able to understand EF by pictures. Four of six US elementary teachers also emphasized the application of the rule of EF in the elementary stage (US6, 8, 9, and 10). However, the expectations in different grade levels are different and overlap in third grade to seventh grade.

Based on the Curriculum Standard (Texas Essential Knowledge and Skills), third graders must identify EF based on models and pictures. Then, fourth graders must generate EF by using the rule of EF and simplifying fractions. A fourth-grade teacher (US7) said,

> In third grade they're going to be doing it with models. They're given the models, maybe a fraction wall or are diagrams of pictures, so they are really just matching the EF with a visual. But in fourth grade, they may or may not have a visual, so they have to have a deeper understanding of it...; they need to be able to do that without the model and could be able to prove it mathematically basically.

However, a fifth-grade teacher (US6) also taught the topic of EF in a series of lessons. First, she used manipulatives and modeling. Then she taught the rule of EF. Finally, she expected students to be able to solve word problems. Another fifth-grade teacher (US5) claimed that students must be able to add and subtract fractions of unlike denominators and to simplify fractions. US5 said,

> The fifth graders have to be able to add and subtract fractions with unlike denominators, so after they go through the process of first finding an equivalent fraction with the common denominator; then they have to be able to solve it. And then they have to be able to simplify it, so they have to find another equivalent fraction that's more simplified, so it's a very long process of getting them to understand both.

4.4.1.2.2. US and Chinese Low-Secondary Teachers

All Chinese low-secondary school mathematics teachers emphasized the application of EF. They thought the generation of EF is a necessary step for students to calculate fractions, such as canceling, reducing, simplifying, and the four operations of fractions (see Table 4.7). As CH6 said, "After learning the basic property of fractions [the rule of equivalent fractions], students should master canceling (reduction of fractions) and reduction of fractions into a common denominator. The purpose of reduction is to add and subtract fractions."

For US low-secondary mathematics, seventh graders must learn a topic on ratio and proportion that is close to EF. They must apply the rule of EF to solve problems, including ratio and proportion. Based on the US low-secondary teachers' responses, they all expected their students to be able to apply the rule to find or generate EF. For example, a seventh-grade mathematics teacher (US4) said, "We are reviewing EF but we don't teach it. We just have a review at the beginning of the year. We use them throughout the year. ... Briefly, we spend a little time on the modeling."

Table 4.7 Low-Secondary Math Teachers' Teaching Expectations.

	Expectation	Chinese	US
Understanding	The rule of finding EF		1,3,4
	Two directions of EF	6	
Application	Compare unlike fractions	10	
	Explain fractions by drawing/reading pictures		4
	Cancel, reduction, and simplifying	6,7,8,9,10	1,3
	Four operations of fractions	6,9,10	
	Solve real problems	6	2
	Generate EF	7,9	1,2,3

Additionally, US1 and US3 expect their students to be able to simplify fractions. For example, a seventh-grade teacher (US1) stated, "I'm expecting them to have learned it in the fourth, fifth, or sixth grade when they come to me. We're doing things with EF. I expect them to be able to simplify, recognize, and generate EF with no pictures."

Another sixth-grade teacher (US2) stated that she hoped students could use the rule of EF to solve real problems.

In general, both US and Chinese elementary teachers expected students to not only understand and identify EF by drawing/reading pictures but also to generate EF by using the rule of EF. These expectations align with the requirements of the curriculum standards (see Table 4.7). Four of the six US elementary teachers also expect their students to be able to simplify fractions, like canceling and reducing fractions, while none of the Chinese elementary teachers mentioned this aspect. Additionally, Chinese elementary teachers emphasized that students must understand EF in terms of different formats and representations of fractions, while none of the US elementary teachers stated that aspect.

The student level expected by Chinese low-secondary teachers is a little more advanced than that of US low-secondary teachers. Chinese low-secondary teachers expect students to be able to apply the rule of EF to calculate the four operations of fractions by simplifying fractions and using reduction of fractions, while US low-secondary teachers expect students to be able to generate EF to a far greater extent. Both US and Chinese low-secondary teachers' expectations align with the requirements of the curriculum standards (see Table 4.8).

Finally, there are clear teaching expectations in the different stages in China, while US teachers' teaching expectations are more intricate. Third-grade through seventh-grade teachers taught the topic of EF, and their expectations are slightly overlapping, perhaps because their students have different knowledge levels. As a sixth-grade teacher (US2) said, “That’s why I spend the time with the kids who come to me and can’t do it; we go back and draw pictures, and we model it.”

Table 4.8 The Requirements of Curriculum Standards on EF.

Curriculum standard	G3	G4	G6
Represent EF with denominators of 2, 3, 4, 6, and 8 using a variety of objects and pictorial models, including number lines	US		
Explain that two fractions are equivalent if and only if they are both represented by the same point on the number line or represent the same portion of a same-size whole for an area model	US		
Determine if two given fractions are equivalent using a variety of methods		US	
Compare two fractions with different numerators and different denominators and represent the comparison using the symbols >, =, or <		US	
Generate equivalent forms of fractions, decimals, and percents using real-world problems, including problems that involve money (Proportionality)			US
Use EF, decimals, and percents to show equal parts of the same whole (Proportionality)			US
Preliminary know EF by observing pictures		CH	
Further understand EF		CH	
Understand the rule of EF by understanding the relationship between fractions and division and the principle of division [$a \div b = (a \times k) \div (b \times k) = (a \div n) \div (b \div n)$, ($b \neq 0, k \neq 0, n \neq 0$)]			CH
Express a fraction in its simplest form by using the rule of EF			CH
Understand the concept of a fraction in its simplest form			CH
Use the rule of EF to solve sample word problems			CH

Note. G = Grade; US = United States; CH = Chinese.

4.4.1.3. Teachers’ Explanations About EF

The explanations of EF vary based on the five different interpretations of fractions. In third grade, teachers can use the part-whole relationship to model fractions and EF (area, length, and set model). Teachers can use the measurement model (number line) to explain EF in fourth grade. Teachers can also use fractions as division and fraction as ratios to explain EF at the high grade levels. I show the findings in terms of the elementary and low-secondary stages as follows.

4.4.1.3.1. US and Chinese Elementary Teachers

All Chinese elementary teachers employed different models (area, linear, and discrete) to explain EF. Three of them also used the interpretation of fractions as quotients to explain them. However, two Chinese elementary teachers mentioned the part-whole relationship (see Table 4.9).

Table 4.9 Elementary Math Teachers' Explanations of EF.

Explanation		Chinese	US
Concrete	Manipulatives		6,7,9,10
	Real problems		
Pictorial	Draw pictures		
	Area model	1,2,3,4,5	5,6,7,8,9,10
	Linear model	1,2.3,4,5	8,10
	Discrete model	1,3,4,5	
Abstract	The rule		5,7,8,9
Fraction Schemes	Part-whole		
	Quotient	2,3,4	
	Proportion /ratio		

All elementary Chinese teachers claimed that the three main types of models (area, linear, discrete) could be used to explain EF. Meanwhile, three of them (CH2, 3, and 4) mentioned that they could also use decimals to explain that the values of the two fractions are equivalent for high-grade-level students because they know sixth graders learn the rule of EF and fractions as division. For example, CH2 mentioned that she could use fraction circles and fraction walls to explain the equivalent. She also said, "We can use decimals to explain EF for high-grade-level students. Our elementary students did not learn fractions as division. After they learn fractions as division, we can use the values of the fractions to explain that."

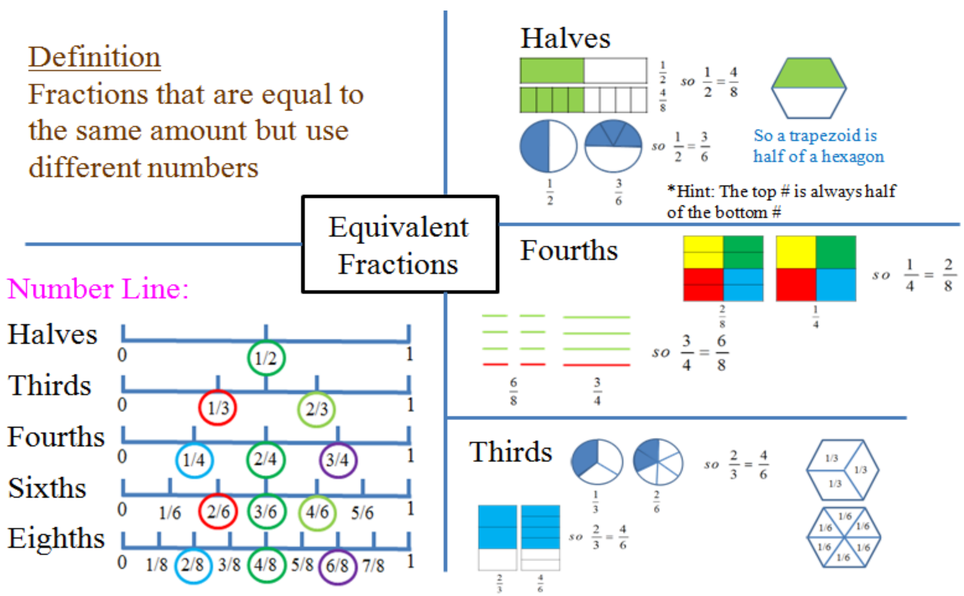


Figure 4.1 Explanation for "2/3 =4/6" (from US10).

All of the elementary US teachers in the interview process emphasized using concrete models (manipulatives and area models) to explain that two fractions are equal. For example, US7 stated, "I would say it is more concrete; we do a lot of hands-on and we do work with a fractions bar and a circle. We start out by looking at what is a fraction and what's not a fraction?" Also, four elementary US teachers (US6, 7, 9, and 10) mentioned manipulatives are an important tool for them to teach EF. For instance, US10 said, “I’m just an advocate for manipulatives, in visuals and hands on learning. I mean, I do that with everything.” US10 used Figure 4.1 to show his preference.

Additionally, four elementary US teachers (US5, 7, 8, and 9) used the rule of EF to explain that the two fractions are equivalent. For example, US5 stated, “We tried to keep it very concrete at this point, but some students do really grasp it very quickly. So, I can tell them if you multiply 2/3 by two halves, because two halves are one, and get 4/6.”

Table 4.10 Low-Secondary Math Teachers’ Explanations of EF.

Explanation		Chinese	US
Concrete	Manipulatives		2,3
	Real problems	7,8,10	
Pictorial	Draw pictures	6,9,10	
	Area model	7,8	1,2,3,4
	Linear model		
	Discrete model	8	
Abstract	The rule	6,7,8,9,10	1,4
Fraction Schemes	Part-whole		
	Quotient	6,7,8,9,10	1,4
	Proportion/ratio		1

4.4.1.3.2. US and Chinese Low-Secondary Teachers

All Chinese low-secondary teachers stated that they can use the invariant property of quotients and fractions as division to explain EF. Meanwhile, these teachers also mentioned that they used models to help students understand “2/3=4/6.” However, Chinese teachers use general words such as “pictures” and “shapes,” rather than the names of “area models” (see Table 4.10). For example, CH9 said, “We also can use drawing pictures, a rectangle or a circle to represent 2/3 and 4/6. There is an introduction to pictures in our textbook. The introduction uses pictures to explain EF."

Most importantly, three Chinese low-secondary teachers (CH6, 8 and 9) claimed that their students have no difficulty understanding the rule of EF. CH9 stated, "Our sixth graders naturally understand the transformation from the quotient property of in-variety to the rule of EF, because it is consistent with their recognition ... Students can easily find the results of 2÷3 and 4÷6 are the same."

All US low-secondary teachers mentioned only that they used area models to explain 2/3=4/6. US1 and US4 mentioned using the rule of EF to explain that the two fractions are equivalent and claimed to use concrete models to show the equivalence. For example, US4 said, "I would show multiplication, multiply the numerator and denominator of 2/3, by two to get 4/6. I can also interpret it with a model. For example, a bar, a rectangle, 2/3, and the equivalent 4/6 bar. And interpret it."

In addition, only US3 suggested that different approaches can be introduced and used based on students' grade levels. She explained, "Depending on the grade level like in fourth grade, when this is more of an introductory skill, we will actually take a piece of paper and fold it and like a nail shade a part of it." Moreover, US1 mentioned the relationship between EF and proportion: "We solve proportion as a huge thing in seventh grade. We set up the definition of proportion, which is the ratio. They are equal. So, you can see if they're equal, by seeing, divide out to the same number."

In general, Chinese elementary teachers preferred to use pictorial representations to explain EF, while US elementary teachers tended to use three ways to explain EF:

area models, manipulatives of EF, and the rule of EF. Furthermore, Chinese elementary teachers realized that fractions as division are an advanced way of explaining EF for higher grade, while no US elementary teacher mentioned the use of fractions as division.

Chinese low-secondary teachers preferred to use the rule of EF and fractions as division when compared to the use of pictorial models to explain EF. However, US low-secondary teachers tend to use area models to explain EF rather than the rule of EF and fractions as division. All US teachers emphasize the employment of concrete representation in teaching, which might be related to curriculum standards. As TEKS stated, “Represent equivalent fractions with denominators of 2, 3, 4, 6, and 8 using a variety of objects and pictorial models, including number lines;” “determine if two given fractions are equivalent using a variety of methods.”

4.4.2. Differences in US and Chinese Teachers’ Knowledge for Teaching Across Grades

Based on the sequence of content about EF in different grade levels across countries, teachers should prepare differently when teaching EF. Therefore, I first summarized US and Chinese teachers’ responses to the fourth interview question. I then compared the differences in elementary and low-secondary teachers’ understandings of EF between the United States and China in terms of the four interview questions.

4.4.2.1. Differences in Teaching EF Across Grades

4.4.2.1.1. US and Chinese Elementary Teachers

Three Chinese elementary teachers mentioned the use of representations with different purposes (CH1, 3 and 5), and four Chinese elementary teachers mentioned the difference in the emphasis on interpretation of fractions (see Table 4.11). For example, CH3 stated, “Fourth graders learn EF by understanding the part-whole relationship, while sixth graders learn it by understanding the numerical values of EF: fractions as quotient. We transform fractions into decimals and help students understand that the fractions are equal.”

Table 4.11 Elementary Math Teachers’ Perceptions of the Differences in Teaching Styles Across Grades.

	Categories		Chinese	US
Teaching expectations	The degree of understanding of EF			
	Low	Understand and identify EF	1	
		Understand the rule of EF		7
		Use pictures to show EF	1,3,5	
	High	Application of EF	5	
		Addition and subtraction of fractions		10
		Prove the rule of EF		7
	The amount of the denominators			8
	Low	The denominators are limited		
	High	The denominators are unlimited		
Emphasis	Different emphasis on the interpretation of fractions		1,2,3,4	
	Low	Part-whole relationship		10
	High	The values of fractions		
Representation	The abstract degree of presentations			5,6,7,8, 9,10
	Low	More concrete		
	High	More and more abstract		

Note. Low means lower grade levels. High means higher grade levels. The numbers represent teachers. The table summarizes the differences in teaching styles between lower grade levels and higher grade levels.

Meanwhile, one Chinese elementary teacher (CH5) discussed differences based on the perspective of the relationships between fractions and EF. He said, “The learning of EF in fourth grade is to better understand the meaning of fractions, rather than generate EF. In sixth grade, teachers more emphasize the interpretation of fractions: the perspective of the numerical value [fractions as numbers].”

Similarly, five US elementary teachers (US5, 6, 7, 8, and 9) stated that the employment of representations was different. Lower-grade teachers should use more concrete representations while higher-grade teachers can employ more abstract representations. For example, US8 stated, “In third grade, there are many pictures for them. In fourth grade, there are pictures available, but it’s very abstract. They have to be able to understand.” Furthermore, she mentioned another difference: “In third grade, the denominators are limited. In fourth grade, the denominators are unlimited because there was so much pictorial in third grade. But in fourth grade, it isn’t.”

However, US7 mentioned the differences between the understanding and application of EF. She said that students must understand EF and improve this understanding of multiplying fractions by one to generate EF in fourth grade. Next, students must be able to generate EF and also must be able to prove the correctness of EF.

In general, Chinese elementary teachers emphasized the difference in interpretations of fractions, while US elementary teachers emphasized the differences between the uses of representations in different grade levels. Chinese elementary teachers cited the differences in teaching styles in different grades from the perspective of teaching expectations and the multiple interpretations of fractions, while US elementary teachers highlighted the differences in explanations in terms of representations.

4.4.2.1.2. US and Chinese Low-Secondary Teachers

Five Chinese low-secondary teachers said that the instructional differences in different grades were related to learning objectives, but they all mentioned the differences in the use of representations. They expected their students to be able to calculate fractions and solve real-word problems including fractions (see Table 4.12). For example, CH6 said, “I guess that elementary students use pictures to solve problems. For low-secondary students, they become more rational. They first learn to calculate the fractions, and then solve real problems including fractions.”

Table 4.12 Low-secondary Math Teachers’ Perceptions of the Differences in Teaching Styles across Grades.

	Categories		Chinese	US
Teaching expectations	The degree of understanding of EF			
	Low	Understand and identify EF	9	
		Use pictures to show EF	6,7,8	3
	High	Application of EF	6,7,8,9	
Representation	The abstract degree of presentations			1,2,3,4
	Low	More concrete		
	High	Increasingly abstract		

Note. Low means lower grade levels. High means higher grade levels. The numbers represent teachers. The table summarizes the differences in teaching styles between lower grade levels and higher grade levels. CH10 claimed there is no difference in elementary and sixth grade. She thought sixth grade is the extended stage of the elementary stage.

Meanwhile, CH6 also mentioned the difference in representations in teaching EF. She said, "More concrete representations are used in the elementary stage. Elementary students can touch fractions because fractions consisted of colorful shapes. In low-secondary school, students are required to use reasoning. Teachers might not draw pictures, circles, and segments."

Only CH10 believed that no significant instructional difference existed between fourth and sixth grades, but she noted a significant difference between sixth grade and seventh through ninth grades. She explained, "Sixth grade is more like the elementary stage; teachers need to teach mathematical knowledge in a more detailed approach to help students understand the content. In seventh-ninth grade, students need to practice more on mathematical thinking and reasoning."

All US low-secondary teachers mentioned differences in using concrete and abstract representations in teaching (see Table 4.12). For example, a sixth-grade teacher (US3) said, "In fourth grade you rely much more heavily on pictorial models. So, the kids are looking and visually seeing the fractions. In sixth grade, I will bring those [pictorial model] out ... whatever the algorithm is, I prefer them to use."

Similarly, only one US low-secondary teacher (US3) claimed that no significant difference existed in instructions in different grades. She said, "It's all very similar. It just grows every year. I would say it just becomes a little more rigorous, a little more complicated, and, a little was even more relying on the lower grade levels. Every year, they understood the foundation of stuff to force it. It should be a very easy connection.

Fourth to fifth [grade], there should be a very straight connection between the grade levels."

4.4.2.2. Differences in Responses Between Elementary and Low-Secondary Teachers

4.4.2.2.1. US Elementary and Low-Secondary Teachers

(1) *Prior knowledge of EF.* All US elementary and low-secondary teachers stated that the understanding of the concept of EF was based on the understanding of the concept of fractions. An understanding of EF can help students better understand the concept of fractions. US elementary teachers only used more detailed related words (e.g., *What is a whole? The whole has to be the same;* and *What does equivalence mean?*) to express their prior knowledge than did US low-secondary teachers. Also, US low-secondary teachers said that they did not have experience with elementary mathematics.

(2) *Teaching expectations of EF.* US elementary teachers' expectations include obtaining an understanding of EF by drawing pictures, simplifying fractions, and applying EF, while low-secondary teachers mainly expect their students to be able to generate EF and simplify fractions by applying the rule of EF.

(3) *Explanations of EF.* All US teachers like to use area models, manipulatives, and the rule of EF to explain EF. Some US elementary teachers tend to use several models, while the US low-secondary teachers mentioned only area models. Additionally, some low-secondary teachers use fractions as division or proportion to explain EF while no elementary teachers mentioned this kind of explanation.

(4) *Differences in teaching EF across grades.* Little significant difference in teaching styles exists between US elementary and low-secondary teachers. They all emphasized that the one difference in teaching is the use of presentations. With a rise in grade levels, teachers use more abstract representations in teaching. They did not mention the difference in the interpretations of fractions.

4.4.2.2.2. Chinese Elementary and Low-Secondary Teachers.

(1) *Prior knowledge of EF.* There is no difference between elementary and low-secondary Chinese teachers' prior knowledge of EF. They all thought the concept of fractions was essential for students to learn EF. Also, all Chinese low-secondary teachers claimed that they had never taught elementary mathematics like their US counterparts, so they did not know exactly what kind of knowledge students acquire about EF in elementary school. All Chinese elementary teachers know that sixth graders in low-secondary school would learn the rule of EF. Therefore, a gap exists in knowledge of content and students and horizon content knowledge between elementary and low-secondary Chinese teachers.

(2) *Teaching expectations of EF*. There are clear teaching expectations at the different grade stages in China, which are aligned with the Chinese Curriculum Standard (see Table 4.7). Chinese elementary teachers expected students to be able to understand EF by drawing or reading pictures, while Chinese low-secondary teachers expected students to be able to skillfully apply the rule of EF to simplify fractions and calculate the addition and subtraction of fractions. Also, Chinese elementary teachers expected that most students could apply the rule of EF to identify or generate EF. However, this

expectation is not necessary because the Chinese curriculum standard for third and fourth grades do not state this requirement (see Table 4.7). This finding shows that Chinese teachers required their students to learn more and practice more on EF compared to US elementary teachers. In other words, Chinese elementary teachers' expectations are more advanced than the Chinese curriculum standard.

(3) *Explanations of EF.* Chinese elementary and low-secondary teachers used several models to explain EF. Although some Chinese elementary teachers mentioned that they could use the interpretation of fractions as division, they tend to use models to explain EF. Chinese low-secondary teachers emphasized the use of the rule and fractions as division to explain EF, although they could use pictures and word problems to explain EF. The explanations from Chinese teachers are consistent with their teaching expectations of EF.

(4) *Differences in teaching styles across grades.* Chinese elementary teachers claimed that there are two significant differences between low and high grade levels. The interpretation of the part-whole relationship is taught in the elementary stage while the interpretation of the values of fractions is taught in sixth grade. Furthermore, the teaching emphasized the use of pictures to show and explain EF at the elementary stage, while emphasizing the application of EF at the low-secondary stage. Chinese low-secondary teachers stated that the differences focused on the understanding and application of EF in different grade stages.

4.5. Discussion

Mathematical knowledge for teaching a topic within a grade requires not only content knowledge but also pedagogical knowledge. However, for a topic covered across grades, teachers also must have coherent knowledge. In this study, I investigated US and Chinese teachers' knowledge for teaching EF from the perspective of instructional coherence. This study does not judge teachers but aims to demonstrate the importance of instructional coherence, examine gaps in knowledge across grades, and explore the approach to bridge the gap. These findings are expected to create insights into mathematical knowledge for teaching EF (Ball et al., 2008) and ways in which this study can improve teachers' mathematics knowledge for teaching this topic.

US and Chinese teachers all had a sound mathematical understanding of EF and recognized the variety and importance of representations and instructional coherence across grades. However, they lacked knowledge of the coherent instruction of EF, especially the coherence of the different representations and interpretations of EF within grades and across grades. The Chinese teachers emphasized the correct conceptual understanding of EF by relying on the development of procedures based on prior abstract mathematical knowledge. US teachers emphasized the use of a variety of representations to develop the concept of EF but often lacked the connection between manipulatives and abstract thinking. This finding aligns with the finding of An et al. (2004) and draws attention to the knowledge of instructional coherence when developing teachers' mathematical knowledge for teaching. The following discussion focuses on two ideas: the coherence of mathematical pedagogy and the coherence of mathematical content.

The discussion could help teachers implement the coherent mathematics instruction and promote a coherent development of students' understanding of mathematical content.

4.5.1. Coherence of Mathematical Pedagogy About Representations

Both US and Chinese teachers emphasized the importance of the concept of fractions in learning EF; they use multiple representations (various models) to explain the meaning of EF. This aligns with the statement from the National Council of Teachers of Mathematics (NCTM) (2000). NCTM (2000) emphasized the use of representations in teaching: "(a) create and use representations to organize, record, and communicate mathematical ideas; (b) select, apply, and translate among mathematical representations to solve problems; (c) use representations to model and interpret physical, social, and mathematical phenomena" (p. 76).

Representations are reliable tools with which teachers can explain abstract knowledge. Teachers often utilize a specific context to introduce a concept when students might be challenged. To that end, teachers must build the relationship between abstract knowledge and the specific context. However, concrete specific contexts have limitations. As Fyfe, McNeil, Son, and Goldstone (2014) stated, it is likely that students' knowledge would remain too tied to the concrete context and would not transfer to dissimilar situations. Concrete contexts limit students' advancement unless connections are rebuilt between concrete contexts and abstract knowledge to allow for free movement between them.

Furthermore, representations not only are teaching tools for US teachers but also foster students' ability to meet academic goals. US teachers emphasized the understanding of EF by using different models based on the interpretation of the part-whole relationship. Although US teachers placed more emphasis on the application of representations, they often did not realize the importance of the concrete representation fading in teaching and the use of students' prior abstract mathematical knowledge to solve new problems. These findings are consistent with those of other studies about US teachers' teaching strategies (An et al., 2004; Cai, 2005; Ma, 2010). Several US teachers mentioned that they enjoy using manipulatives to help students understand mathematical concepts. Most importantly, without a solid knowledge of the representations, teachers cannot produce a conceptually correct representation even if they have a rich knowledge of students' lives (Ma, 2010). Teachers should comprehend the differences between the use of the models (concrete, pictorial, abstract) to help students understand the concept of EF and the use of models to solve problems. Representations are the analogs that teachers use to explain the meaning of fractions to students (Ni, 2001). To better help students understand the meaning of fractions, teachers must explain the common properties between representations and fractions. US teachers hope to use models to help students understand the concept of fractions. In teaching, the models become an important component of learning goals for students. If teachers do not build a connection between the models and the abstract rule of EF, students might acquire a mistaken definition of fractions or EF and might only know how to solve problems with fractions only by using models. In fact, models and concrete contexts are only tools and representations that teachers can use to help students understand the abstract definition of fractions.

To Chinese teachers, representations are teaching tools, not teaching goals. The purpose of the use of manipulatives is to help students understand the concept of EF rather than inform students of a method to generate EF. Conversely, US teachers might unconsciously emphasize the notion that the use of drawings and manipulatives should be regarded as a method of finding an EF. Chinese teachers regarded the manipulatives as a tool for teaching. The purpose of instruction is to help students understand mathematical content. Chinese teachers expected their students to use the rule of EF to generate EF. Furthermore, Chinese teachers realized that the role of the introduction of equivalent fraction in the lower grade level is to help students better understand the concept of fractions as a part-whole relation. They also knew that students will learn the generation of EF in the higher grade level by using the rule of EF.

When explaining EF teaching, Chinese teachers tended to connect the topic more to related conceptual topics (e.g., the meaning of fractions, simplifying fractions, and the addition-subtraction of unlike fractions) and be less to students' lives. Also, Chinese teachers explained the rule of EF by reasoning, which reinforces student learning through the use of reasoning. Chinese teachers provided more related knowledge to support students' learning of the rule of EF. This finding echoes the statement from Cai (2005): "Teachers may start with concrete representations or physical manipulatives to encourage students to use their own strategies for solving problems and making sense of mathematics. But the students' further conceptual development requires that teachers help students to develop more generalized solution representations and strategies" (pp. 153-155).

There are two ways of finding the rule of EF based on teachers' explanations. One is the use of representations, while the other is the use of the prior knowledge of fractions as division and the invariance of quotients to derive the rule of EF. Chinese teachers introduced these two methods to instruction about the rule of EF and highlighted the reasoning of mathematical content. They expected students to understand the algorithms and apply them at a proficient level. In addition, Chinese teachers required that their students calculate addition and subtraction of unlike fractions not only correctly but also quickly by using the rule of EF. US teachers placed more emphasis on the requirement that students use the models to generate EF, which aligns with the findings from An et al. (2004). In other words, the use of representations becomes a learning goal in US classroom teaching. Previous studies showed the ineffectiveness of using concrete strategies to develop students' mathematical thinking and substantive content knowledge (Cai & Hwang, 2002; Rosli, Goldsby, & Capraro, 2015).

Neither US nor Chinese teachers realized the differences and similarities between area/region/length/set models and number lines. Area/region/length/set models are often used to explain the interpretation of the part-whole relationship during the process of teaching the concept of fractions. Then students could develop a deep impression of fractions as a part-whole relationship. The number line model is also used to explain the second aspect of fractions based on measurement interpretation because fractions as numbers can be labeled at the number line. Therefore, the functions of the models are different in teaching fractions. Although the number line model is recommended by

CCSSM in teaching fractions, the real function of the number line model is not described in CCSSM.

The finding of US and Chinese teachers' different beliefs about the use of representations in teaching can contribute to our understanding of the differences between US and Chinese students' mathematical thinking. Most importantly, the finding suggests the importance of teachers' coherence of pedagogical knowledge on representations in teaching.

4.5.2. Coherence of Mathematical Content

It is necessary for teachers to understand the five different interpretations of fractions in the instruction of EF across grades. The five interpretations of fractions include: part-whole, measurement, division, operator, and ratio. However, neither US teachers nor their Chinese counterparts recognized the importance of coherence and transformation among different interpretations.

Most Chinese teachers used at least one appropriate representation and also used mathematical reasoning to explain EF. They knew the part-whole and quotient interpretation and realized the two different interpretations had been introduced in different grades. US teachers did not explain EF from the perspective of reasoning and preferred to use models. However, US and Chinese teachers did not recognize the differences between the part-whole relationship and measurement interpretation of fractions. To be specific, the part-whole relationship is a way to explain the original definition of fractions in that they represent parts of a whole. Measurement interpretation is a way to explain fractions as numbers rather than fractions as a part-whole relationship.

The significant difference between these two interpretations of fractions corresponds to the double roles of fractions: fractions as relationships between the parts and a whole and fractions as numbers. The concrete models including area models, region models, length models, and set models, are often used to explain the part-whole relationship; however, number line models are often mistakenly placed into this category. In fact, number lines are used to explain the measurement interpretation of fractions. For example, a number line is similar to a ruler. Students must understand the concept of unit fractions; then they can understand general fractions.

However, the use of number lines might be at a higher grade level because students might not have strong knowledge about number lines. As Ni (2001) found, sixth graders performed best on area and line segment items, less efficiently on set items, and the poorest on number line items. Similarly, Clark and Kamii (1996) found that sixth graders performed better on equivalent fraction items by representing the part-whole relationship but performed poorly on those representing EF on number lines. Ni (2001) explained the reasons for this phenomenon: Sixth graders tended to treat the whole number line as a unit rather than a segment from 0 to 1.

From the perspective of coherent content, students can learn EF based on concrete models that they learned before. However, in the second stage, students must know fractions as division. They should then understand and use the rule of EF. The pictorial model can also be used to explain the reasoning of the rule of EF at this stage;

however, the emphasis of instruction is to help students apply the rule to solve problems, generate a common denominator for the addition and subtraction of unlike fractions, or to simplify a fraction in the lowest term. In the second stage, students must understand the relationship between fractions as division and the rule of EF. Therefore, teachers must realize the relationship between the fractions as division (a fraction is two numbers but placed in a special format as seen in Behr et al., 1984) and fractions as the relationship of part-whole. Then they can help students better understand fractions as numbers.

4.6. Conclusion

In this study, I found that US and Chinese teachers did not pay sufficient attention to the instructional coherence of EF. Some US teachers mentioned that they taught only one grade level. This might be related to the features of the United States education system: teachers have a fixed classroom and students do not have a fixed classroom. Similarly, Chinese teachers encounter the same challenges to coherence because of the gap between the elementary stage and low-secondary school. Low-secondary teachers do not know their students' prior knowledge before they start teaching mathematics at the beginning of sixth grade. In addition, some Chinese teachers mentioned that exploring the meaning of EF was a good opportunity to review and deepen the understanding of the concept of fractions. Therefore, to explain EF, teachers can use different models related to the definition of fractions. Teachers in the United States also mainly used the part-whole relationship to explain EF. Few US teachers discussed the connections between the different models and interpretations of fractions across grades, while most Chinese teachers mentioned the connections and different interpretations of fractions, but only superficially.

To implement a coherent instruction of EF, teachers must be versed in the fact that fractions have different interpretations that can be used to explain EF. This knowledge can then provide more opportunities for students to understand EF. Prior studies showed that the part-whole interpretation of fractions may limit one's interpretation of EF to "the same shaded parts of the same whole" (Ding et al., 2013, p. 67). Also, the interpretation of part-whole ignores discussions about the numerical relationships but regards EF as a procedural operation: A numerator represents the parts and a denominator represents the total number of parts in the whole (Simon & Tzur, 2004; Tzur, 2007).

Significant differences exist among the interpretations of fractions across grades. Some US and Chinese teachers believed that the content of EF should be included in the unit of the comparison of fractions: more than, less than, and equal. As Ma (2010) said, "a well-developed conceptual understanding of a topic also includes [an] understanding of another aspect of [the] structure of the subject--attitudes toward mathematics" (p. 24). The learning of fractions cuts across several grades in the elementary and low-secondary stages. The coherent interpretations of fractions could be an implicit structure through which students can learn fractions across grades. This aligns with one of three aspects of curriculum coherence in Schmidt et al. (2002): deep structure, "a coherent arrangement of topics might be achieved: through logical sequences, through evaluation from particulars to deep structures, and though using deep structures to make connections" (Cuoco & McCallum, 2018, p. 247).

Generally, the coherence of mathematical content is more than a deep conceptual understanding of elementary mathematics; it is the awareness of the conceptual structure and the coherence of mathematical content inherent in mathematics. When teachers have coherent mathematical knowledge about elementary mathematics, they can see the whole picture, and deeply understand the structure of elementary mathematics.

The coherence of mathematical pedagogy is awareness of the development of students' cognition and use of different representations based on the features of mathematical knowledge. Teachers with the coherence of mathematical pedagogy can reveal and represent connections between mathematical concepts and representations to students. They can show different aspects of a concept and various approaches to a solution, as well as their advantages and disadvantages. They can provide explanations to students in a manner that suits students' cognitive characteristics.

Teachers must explain mathematical content based on students' prior knowledge and capture students' current understanding of content. Then teachers must use, design, create, and construct pedagogical strategies. Examples include the construction of a situation, the use of multiple presentations, and the organization of group discussions to help students build a connection between current and prior knowledge. Teachers pursue strong mathematical knowledge for teaching by emphasizing both the coherence of mathematical content and the coherence of mathematical pedagogy.

Both within grades and across grade levels, teachers should know the underlying logical sequence of fraction topics instead of viewing mathematical knowledge as pieces. In addition, they should have the connections between the knowledge of across grades. This emphasis echoes the meaning of coherence in curriculum research (Schmidt et al., 2005).

4.7. Implications, Limitations, and Future Directions

In this study, the findings contribute to the understanding of teachers' knowledge of EF and recognize the importance of instructional coherence in mathematics teaching. However, our findings are based on the interviews with a relatively small group of experienced teachers, and thus should not be over-generalized. Also, I did not collect data on teachers' classroom teaching. Thus, the findings were based on an analysis of the teachers' responses, not their actual instructions.

The findings of the differentiation between elementary and low-secondary teachers' knowledge highlight the importance of instructional coherence for teachers and students. The importance of analyzing, across grades, the knowledge of teachers of different grades in regard to teaching from a dynamic and coherent perspective was also confirmed. In addition, the findings of this study have implications for classroom instruction and teacher professional development.

First, teachers can learn from the findings and deepen their understanding of the concept of EF within and across grade levels. Most importantly, they can realize the reasons for students' misconceptions of EF. Teachers can then build awareness of instructional coherence in their teaching. Furthermore, teachers might understand their knowledge of content and curriculum (Ball et al., 2008). With a deeper understanding of the coherence of mathematical content and mathematical pedagogy, teachers might become more mathematically knowledgeable about the design and thereby present more coherent instructions in their classrooms.

Second, curriculum designers can rethink the shape of the curriculum based on CCSSM and focus on showing explicit internal consistency for users to easily realize the coherent content and coherent pedagogy. For example, for Singapore math, there is an obvious characteristic of the representations of pedagogy: the use of the concrete→pictorial→abstract approach. Similarly, a coherent curriculum can help novice teachers and unqualified teachers improve instructional coherence (Chen & Li, 2010).

Finally, this study suggests that teacher educators should pay closer attention to instructional coherence. After all, mathematical content is coherent and must be taught using a coherent instructional approach. Teacher educators and teacher professional development affect the improvement of teachers' knowledge of instructional coherence. Teachers who have coherent content knowledge and pedagogical knowledge are able to build a connected network of mathematical knowledge for teaching.

The studies of curriculum from the perspective of coherence (Schmidt et al., 2005; NCTM, 2006) have confirmed the importance of coherence and provided a direction to improve teachers' instructional coherence. For future studies, it may be helpful to focus on cultivating teachers' beliefs in the significance of instructional coherence within grades and across grades. Then, not only could we examine the consistency between teachers' knowledge of fractions and their classroom teaching, but also analyze teachers' knowledge for teaching fractions to identify the instructional coherence between fractions and EF across grades. Meanwhile, future studies can investigate teachers' views of the relationship between instructional coherence and curriculum coherence to promote a better understanding of the MKT model in Ball et al (2008). Most importantly, future studies are needed to explore the approaches to help teachers improve instructional coherence and further fill the gaps produced by differences in different teachers' instruction across grades as well as differences in a teacher's instructions on different topics within grades.

4.8. References

Alajmi, A. H. (2012). How do elementary textbooks address fractions? A review of mathematics textbooks in the USA, Japan, and Kuwait. *Educational Studies in Mathematics*, *79*(2), 239-261.

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school, mathematics teachers in China and the US. *Journal of Mathematics Teacher Education*, *7*(2), 145-172.

Ball, D. L., Hill, H. C, & Bass, H. (2005, Fall). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, *29*, 14-22.

Ball, L. D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-233). New York, NY: Macmillan.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education*, *15*(5), 323-341.

Cai, J. (2005) U.S. and Chinese teachers' constructing, knowing, and evaluating representations to teach mathematics. *Mathematical Thinking and Learning*, *7*(2), 135-169.

Cai, J., & Hwang, S. (2002). Generalized and generative thinking in US and Chinese students' mathematical problem solving and problem posing. *The Journal of Mathematical Behavior*, *21*(4), 401-421.

Carreño, E., Ribeiro, C. M., & Climent, N. (2013). Specialized and horizon content knowledge–Discussing prospective teachers knowledge on polygons. Paper presented in the *Eighth Congress of European Mathematics Education* (pp. 2966-2975). Antalya, Turkey.

Chan, W. H., Leu, Y. C., & Chen, C. M. (2007). Exploring group-wise conceptual deficiencies of fractions for fifth and sixth graders in Taiwan. *The Journal of Experimental Education*, *76*(1), 26-57.

Cochran, K. F., DeRuiter, J. A., & King, R. A. (1993). Pedagogical content knowing: An integrative model for teacher preparation. *Journal of Teacher Education, 44*(4), 263-272.

Common Core State Standards Initiative (2010). *Common Core State Standards for Mathematics*. Washington: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Common Core State Standards Initiative (2017, December). *Key shifts in mathematics*. Retrieved from http://www.corestandards.org/other-resources/key-shifts-in-mathematics/

Cuoco, A., & McCallum, W. (2018). Curricular coherence in mathematics. In Y. Li, W. J. Lewis, & J. J., Madden (Eds.), *Mathematics matters in education* (pp. 245-256). Cham, Switzerland: Springer.

Ding, M. (2007). *Knowing mathematics for teaching: A case study of teacher responses to students' errors and difficulties in teaching equivalent fractions*. Unpublished doctoral dissertation, Texas A&M University.

Ding, M., Li, X., Capraro, M. M., & Kulm, G. (2012). A case study of teacher responses to a doubling error and difficulty in learning equivalent fractions. *Investigations in Mathematics Learning*, *4*(2), 42-73.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes*. Mahwah, NJ: Erlbaum.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147–164). New York, NY: Macmillan.

Gay, L. R., & Airasian, P. (2000). *Educational research* (6th ed.). Columbus, OH: Merrill.

Guberman, R., & Gorev, D. (2015). Knowledge concerning the mathematical horizon: A close view. *Mathematics Education Research Journal, 27*(2), 165-182.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*(2), 371-406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal*, *105*(1), 11-30.

Jakobsen, A., Thames, M. H., Ribeiro, C. M., & Delaney, S. (2012, July). Using practice to define and distinguish horizon content knowledge. In *12th International Congress in Mathematics Education (12th ICME)* (pp. 4635-4644).

Kamii, C., & Clark, R. B. (1995). Equivalent fractions: Their difficulty and educational implications. *Journal of Mathematics Behavior, 14,* 365-378.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49 - 84). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). The strands of mathematical proficiency. In *Adding it up: Helping children learn mathematics* (pp. 115–155). Washington, DC: National Academy Press.

Kim, R. Y., Ham, S. H., & Paine, L. W. (2011). Knowledge expectations in mathematics teacher preparation programs in South Korea and the United States: Towards international dialogue. *Journal of Teacher Education*, *62*(1), 48-61.

Koponen, I. T., & Pehkonen, M. (2010). Coherent knowledge structures of physics represented as concept networks in teacher education. *Science & Education*, *19*(3), 259-282.

Kosso, P. (2009). The large-scale structure of scientific method. *Science & Education*, *18*(1), 33-42.

Leung, K. S. F. (2006). Mathematics education in East Asia and the West: Does culture matter? In *Mathematics education in different cultural traditions-A comparative study of East Asia and the West* (pp. 21-46). Boston, MA: Springer.

Ma, L. (2010). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. New York, NY: Routledge.

Marks, R. (1990). Pedagogical content knowledge: From a mathematical case to a modified conception. *Journal of Teacher Education*, *41*(3), 3-11.

Moss, J. 2005. "Pipes, tubes, and beakers: New approaches to teaching the rational-number system". In M. S. Donovan & J. D. Bransford (Eds.), *How students learn: Mathematics in the classroom* (pp. 121–162). Washington, DC: National Academic Press.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through Grade 8 mathematics: A quest for coherence*. Reston, VA: NCTM.

Ni, Y. (2001). Semantic domains of rational numbers and the acquisition of fraction equivalence. *Contemporary Educational Psychology*, *26*(3), 400-417.

Pantziara, M., & Philippou, G. (2012). Levels of students' "conception" of fractions. *Educational Studies in Mathematics, 79*(1), 61-83.

Piaget, J. (1987). Possibility and necessity, Vol. 1: The role of possibility in cognitive development (H. Feider, Trans.), *Minneapolis, MN: University of Minnesota Press*.

Rosli, R., Goldsby, D., & Capraro, M. M. (2015) Using manipulatives in solving and posing mathematical problems. *Creative Education, 6*, 1718-1725.

Schmidt, W. H., Tatto, M. T., Bankov, K., Blömeke, S., Cedillo, T., Cogan, L., ... & Santillan, M. (2007). The preparation gap: Teacher education for middle school mathematics in six countries. *MT21 Report. East Lansing: Michigan State University*.

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of US mathematics and science content standards from an international perspective. *Journal of Curriculum Studies*, *37*(5), 525-559.

Senk, S. L., Tatto, M. T., Reckase, M., Rowley, G., Peck, R., & Bankov, K. (2012). Knowledge of future primary teachers for teaching mathematics: an international comparative study. *ZDM*, *44*(3), 307-324.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1-23.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*(2), 91-104.

Suppe, F. (1977). I. Introduction. *Teaching Philosophy*, *2*(2), 99-108.

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics*, *66*(3), 273-291.

Van de Walle, J. A. (2004). *Elementary and middle school mathematics: Teaching developmentally*. Boston, MA: Pearson.

Wasserman, N., & Stockton, J. (2013). Horizon content knowledge in the work of teaching: A focus on planning. *For the Learning of Mathematics*, *33*(3), 20-22.

Wijaya, A., van den Heuvel-Panhuizen, M., & Doorman, M. (2015). Opportunity-to-learn context-based tasks provided by mathematics textbooks. *Educational Studies in Mathematics*, *89*(1), 41-65.

Zazkis, R., & Mamolo, A. (2011). Reconceptualizing knowledge at the mathematical horizon. *For the Learning of Mathematics*, *31*(2), 8-13.

5. CONCLUSIONS

This dissertation utilized the perspective of coherence to analyze textbooks from three different regions and analyzed teachers' mathematical knowledge for teaching equivalent fractions. In this dissertation, I hope to provide a new perspective on coherence for researchers, textbook editors, and teachers to better implement their work, separately.

5.1. Additional Thoughts on Coherence

To illustrate the importance of curriculum and instructional coherence, I would like to use the example of a taxi driver and a map from Ma (2010). Let us assume that mathematical concepts are the buildings in towns. Several towns could be a chapter in a textbook, and several chapters in a textbook could create an entire metropolis. Students could be people who just moved into a town. Mathematical content in one stage (like elementary) could be the map of the metropolis, including many towns. Teachers could be the drawers of the map of the metropolis or the navigators, like GPS. There are many buildings, and their residents need clear instructions about the available paths between any two buildings or about other paths to their destinations. A wider selection of paths means more convenience. One building in one town is similar to a concept within a grade, while two buildings in two different towns refers to concepts across grades. In general, people prefer to have broad selections to advance from one point to another.

A GPS allows one to easily arrive at a destination. Similarly, a teacher who can provide more available paths for students to understand a new concept based on their prior knowledge must have a clear map outlining the principles taught in their grade, or even across grades, like Google Maps. Students can easily understand the new concept when more paths are available by which they can become well-versed in a new concept. Furthermore, teachers must understand the personalities and different backgrounds of their students to provide appropriate support for them to grow at their own paces.

In my research, I completed three related studies focusing on textbooks and teachers' knowledge by employing the perspective of coherence. I built a conceptual framework for textbook coherence including three aspects of coherence: macro-logical, meso-lesson-structure, and micro-example-practice. The purpose was to show the coherent characteristics of equivalent fractions in different textbooks and to provide an opportunity for textbook designers and teachers to better understand the approaches of three different educational systems.

Continuing on the study, I explored the coherence of the presentation of problems in a micro-level of example-practice problems. Based on the variation of problems, I built a conceptual framework for problems' cognitive requirements. Altering one or more aspects of problems (responses, reasoning, operation, connection, and representation) can lead to the modification of problems' cognitive requirements. I examined the differences in example and practice problems' cognitive requirements among the three series of mathematics textbooks to show their characteristics in terms of the variation of problems. The purpose was to provide a new perspective from which textbook designers can improve the quality of textbooks through an examination of problems' cognitive requirements in terms of five aspects. Designers can consider the five aspects and create problems that are better suited for students' cognitive development. Teachers can also employ, select, and create appropriate problems suited to their students' cognitive levels by altering the problems' five aspects.

Finally, in the third study, I employed the perspective of instructional coherence to examine teachers' knowledge for teaching equivalent fractions. According to the model of mathematical knowledge for teaching (Ball et al., 2008) and curriculum coherence (Schmidt et al., 2005), I built a conceptual model for instructional coherence based on two aspects: coherence of mathematical content and coherence of mathematical pedagogy. Following this framework, I designed four interview tasks to examine teachers' knowledge for teaching equivalent fractions. I then analyzed teachers' responses in terms of two aspects: instructional coherence in a specific grade and instructional coherence across grades. The findings provide evidence about instructional coherence in that selected US and Chinese teachers did not pay sufficient attention to the instructional coherence of equivalent fractions. For example, few US teachers discussed the connections between different models and interpretations of fractions across grades, while most selected Chinese teachers mentioned connections and different interpretations of fractions, but only superficially. To implement a coherent instruction of equivalent fractions, teachers must be well-versed in the fact that fractions have different interpretations that can be used to explain equivalent fractions. By using different interpretations of fractions, teachers can provide more opportunities for students to understand equivalent fractions from different perspectives, like fractions as numbers and fractions as quotients.

5.2. Future Research Plan

For my future research, I will continue exploring the relationship between curriculum coherence and instructional coherence based on teachers' interview data. Following my dissertation topic, I will investigate teachers' understanding of fractions based on the perspective of curriculum coherence. I also hope to further verify the relationship between teachers' perceptions of curriculum coherence and teachers' instructional coherence. Clearly, teachers' coherence of knowledge is a critical component of mathematical knowledge for teaching and professional development. This topic is still valuable to explore in the future.

Because the history of mathematics already provides detailed information about the development of mathematics, my long-term research goals include integrating the history of mathematics into teaching. To that end, I will develop a course for teachers that will improve their knowledge of content and curriculum. As Ma (2010) claimed, teaching should have longitudinal coherence, notably, "with profound understanding of foundational mathematics, teachers are ready at any time to exploit an opportunity to review crucial concepts that students have studies previously. They also know about what students are going to learn later, and take opportunities to lay the proper foundation for it" (p. 122).

Furthermore, based on my study about integrating the history of mathematics into teaching over the past several years, I have determined that this approach has the potential to provide an effective tool for educators to implement STEM education. During the past thousands of years, mathematics and science have been inextricably woven together. The history of mathematics exposes interrelations between mathematics and other subjects, providing us with solutions to the challenges we face in our daily lives. For example, I designed an activity for pre-service teachers to measure the width and length of a table using a pen. The purpose of this activity was to explore the reasons why students must learn fractions. During this process, pre-service teachers engaged in the problem and better understood the definition and application of fractions, simultaneously realizing the beauty of mathematical thinking/methods.

In summary, this dissertation is merely the starting point of my academic career. My ultimate goal is to be an internationally recognized scholar of mathematics education. I will endeavor to improve teachers' efficiency by improving their understanding of mathematical content.

5.3. References

Ball, L. D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Ma, L. (2010). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. New York, NY: Routledge.

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of US mathematics and science content standards from an international perspective. *Journal of Curriculum Studies*, *37*(5), 525-559.