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The Hochschild cohomology problem for von Neumann algebras

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ABSTRACT In 1967, when Kadison and Ringrose began the development of continuous cohomology theory for operator algebras, they conjectured that the cohomology groups $H^n(\mathcal{M}, \mathcal{M})$, $n \geq 1$, for a von Neumann algebra $\mathcal{M}$, should all be zero. This conjecture, which has important structural implications for von Neumann algebras, has been solved affirmatively in the type *I*, II_∞, and *III* cases, leaving open only the type II_1 case. In this paper, we describe a positive solution when $\mathcal{M}$ is type II_1 and has a Cartan subalgebra and a separable predual.

1. Introduction

The study of the continuous Hochschild cohomology groups $H^n(\mathcal{M}, \mathcal{M})$, $n \geq 1$, of a von Neumann algebra $\mathcal{M}$ with coefficients in itself was begun in a series of papers (1–4) by Johnson, Kadison, and Ringrose. Their work was an outgrowth of the Kadison–Sakai Theorem on derivations (5, 6), which proved, in an equivalent formulation, that $H^1(\mathcal{M}, \mathcal{M}) = 0$ for all von Neumann algebras. It was natural to conjecture that the higher cohomology groups $H^n(\mathcal{M}, \mathcal{M})$ also vanish, and this was settled affirmatively for hyperfinite von Neumann algebras in ref. 4. These authors established many general results on cohomology, some of which are reviewed below. One particular consequence is that it suffices to consider separately the cases when $\mathcal{M}$ is type *I*, II_1, II_∞, or *III* in the Murray–von Neumann classification scheme; the general von Neumann algebra is a sum of these four types. Because type *I* von Neumann algebras are hyperfinite (but by no means exhaust this class), attention has been focused on the remaining three types. Considerable progress on the problem has been made recently by the introduction of the completely bounded cohomology groups $H^n_{cb}(\mathcal{M}, \mathcal{M})$. Christensen and Sinclair (7) used the structure theory of completely bounded multilinear maps to show that $H^n_{cb}(\mathcal{M}, \mathcal{M}) = 0$ for all von Neumann algebras (see chapter 4 of ref. 8). These authors and Effros (9) also proved that the continuous and completely bounded cohomology groups coincide when $\mathcal{M}$ is type II_∞, *III*, or II_1 and stable under tensoring with the hyperfinite type II_1 factor, showing that $H^n(\mathcal{M}, \mathcal{M}) = 0$ in these cases. Thus the conjecture remains open only for type II_1 von Neumann algebras.

There have been some partial results in this direction, mainly concerned with the lower order groups. The vanishing of $H^2(\mathcal{M}, \mathcal{M})$ was proved by Christensen and Sinclair for the type II_1 factors with property Γ (see chapter 6 of ref. 8), while the same conclusion was reached for a type II_1 von Neumann algebra with a Cartan subalgebra in ref. 10 (for $n = 2$) and ref. 11 (for $n = 3$ and a separable predual). The von Neumann algebras having Cartan subalgebras form a rich class (12), but this class does not contain the von Neumann algebra arising from the free group on two generators (13). The main result of this paper is that $H^n(\mathcal{M}, \mathcal{M}) = 0$, $n \geq 1$, for type II_1 von Neumann algebras with a Cartan subalgebra and separable predual, generalizing the results of refs. 10 and 11. Recent approaches to cohomology (8) have focused on proving that the relevant cocycles are completely bounded as multilinear maps, and this was successful in refs. 10 and 11 for the lower order groups. The principal idea of the present paper is to recognize that we need only establish complete boundedness in the last variable, a much weaker requirement.

In the second section of the paper, we review the basic definitions of cohomology theory, and we include a brief discussion of completely bounded maps. Various forms of averaging over amenable groups play a fundamental and continuing role in the theory, so we have taken the opportunity to recall the most important aspects in the third section. *Theorem 3.1* shows the equivalence of several cohomology groups, *Theorems 3.2* and *3.3* present useful inequalities based on the Pisier–Haagerup–Grothendieck inequality, and *Theorem 3.4* concerns the existence of a projection of completely bounded maps onto the subspace of right module maps which subsequently produces cobounding maps. The last section gives a sketch of our main result, and we indicate how the previously quoted theorems can be combined with some important work of Popa (14, 15) to establish that $H^n(\mathcal{M}, \mathcal{M}) = 0$ when $\mathcal{M}$ has a Cartan subalgebra and a separable predual. Complete details will appear elsewhere.

We refer the reader to ref. 16 for an early survey of cohomology theory, and to a later account in ref. 8 that contains all the necessary background material for this paper, as well as a discussion of applications.

2. Preliminaries

The matrix algebras $M_k(\mathcal{A})$, $k \geq 1$, over a C^*-algebra $\mathcal{A} \subseteq B(H)$ carry natural norms, defined by viewing $M_k(\mathcal{A})$ as a subalgebra of $M_k(B(H))$ and identifying the latter algebra with $B(H \oplus \cdots \oplus H)$ (k-fold direct sum). Thus a bounded linear map $\phi: \mathcal{A} \to B(H)$ induces a family $\phi_k: M_k(\mathcal{A}) \to M_k(B(H))$, $k \geq 1$, of bounded maps on the matrix algebras by applying ϕ in each entry, and ϕ is said to be completely bounded if

$$\sup\{\|\phi_k\|: k \geq 1\} < \infty. \quad \mathbf{[2.1]}$$

This supremum then defines the completely bounded norm $\|\phi\|_{cb}$. The spaces $\mathrm{Row}_k(\mathcal{A})$ and $\mathrm{Col}_k(\mathcal{A})$ are, respectively, rows and columns of length k with entries from $\mathcal{A}$, and are obviously identified with subspaces of $M_k(\mathcal{A})$. Then $\phi: \mathcal{A} \to B(H)$ is said to be row bounded if the following quantity (which then defines the row bounded norm) is finite:

$$\|\phi\|_r = \sup\{\|\phi_k(E)\|: E \in \mathrm{Row}_k(\mathcal{A}), \quad \|E\| \leq 1, k \geq 1\}. \quad \mathbf{[2.2]}$$

There is a substantial literature on completely bounded maps (see ref. 8 and the references therein), but row bounded maps

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are much less studied. Nevertheless, they will play a crucial role subsequently. We note that the inequalities

$$\|\phi\| \leq \|\phi\|_{\mathrm{r}} \leq \|\phi\|_{\mathrm{cb}} \quad \textbf{[2.3]}$$

are immediate from the definitions.

Now let $\mathcal{A}^n$ denote the n-fold Cartesian product of copies of $\mathcal{A}$. An n-linear map ϕ: $\mathcal{A}^n \rightarrow B(H)$ may be lifted to an n-linear map ϕ_k: $M_k(\mathcal{A})^n \rightarrow M_k(B(H))$, $k \geq 1$. For clarity we take $n = 2$ because this case contains the essential ideas. For matrices $X = (x_{ij})$, $Y = (y_{ij}) \in M_k(\mathcal{A})$, the (i, j)-entry of $\phi_k(X, Y) \in M_k(B(H))$ is defined to be $\Sigma_{r=1}^{k} \phi(x_{ir}, y_{rj})$. Following the linear case, the completely bounded norm is also defined by Eq. **2.1**. Such maps have important applications in cohomology theory (8, 11).

We will also require the notion of multimodular maps below. If $\mathcal{R} \subseteq \mathcal{M} \subseteq B(H)$ is an inclusion of algebras, then $\mathcal{R}$-multimodularity of ϕ: $\mathcal{M}^n \rightarrow B(H)$ is defined by the equations

$$r\phi(x_1, \ldots, x_n) = \phi(rx_1, x_2, \ldots, x_n), \quad \textbf{[2.4]}$$

$$\phi(x_1, \ldots, x_n)r = \phi(x_1, \ldots, x_n r), \quad \textbf{[2.5]}$$

$$\phi(x_1, \ldots, x_i r, x_{i+1}, \ldots, x_n) = \phi(x_1, \ldots, x_i, rx_{i+1}, \ldots, x_n), \quad \textbf{[2.6]}$$

where $r \in \mathcal{R}$ and $x_i \in \mathcal{M}$ for $1 \leq i \leq n$. A simple, but important, consequence of the definitions is that ϕ_k is $M_k(\mathcal{R})$-multimodular, for all $k \geq 1$, when ϕ is $\mathcal{R}$-multimodular.

We recall from refs. 1 and 8 the basic definitions of Hochschild cohomology theory. Let $\mathcal{M}$ be a von Neumann algebra (or C^*-algebra) and denote by $\mathcal{L}^n(\mathcal{M}, \mathcal{X})$ the space of n-linear bounded maps ϕ: $\mathcal{M}^n \rightarrow \mathcal{X}$ into a Banach $\mathcal{M}$-bimodule $\mathcal{X}$. The coboundary operator ∂: $\mathcal{L}^n(\mathcal{M}, \mathcal{X}) \rightarrow \mathcal{L}^{n+1}(\mathcal{M}, \mathcal{X})$ is defined by

$$\begin{aligned} \partial\phi(x_1, \ldots, x_{n+1}) = {} & x_1\phi(x_2, \ldots, x_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i \phi(x_1, \ldots, x_{i-1}, x_i x_{i+1}, \\ & \times x_{i+2}, \ldots, x_n) \\ & + (-1)^{n+1}\phi(x_1, \ldots, x_n)x_{n+1} \end{aligned} \quad \textbf{[2.7]}$$

for $x_1, \ldots, x_{n+1} \in \mathcal{M}$. Then ϕ is an n-cocycle if $\partial\phi = 0$, while ϕ is an n-coboundary if $\phi = \partial\psi$ for some $\psi \in \mathcal{L}^{n-1}(\mathcal{M}, \mathcal{X})$. A short algebraic calculation shows that $\partial\partial = 0$, and thus coboundaries are cocycles. The cohomology group $H^n(\mathcal{M}, \mathcal{X})$ is then defined to be the space of n-cocycles modulo the space of n-coboundaries (for $n \geq 2$). For $n = 1$, $H^1(\mathcal{M}, \mathcal{X})$ is defined to be the space of bounded derivations modulo the space of inner derivations. The definition gives rise to a related family of cohomology groups by imposing further restrictions on the bounded maps. We might require ultraweak continuity ($H_w^n(\mathcal{M}, \mathcal{X})$), $\mathcal{R}$-multimodularity ($H^n(\mathcal{M}, \mathcal{X}, :/\mathcal{R})$), complete boundedness ($H_{cb}^n(\mathcal{M}, \mathcal{X})$), or any combination of these. The interplay between these various cohomology theories gives an important tool for the determination of $H^n(\mathcal{M}, \mathcal{X})$ (see *Theorem 3.1*).

3. Averaging of Maps

One of the earliest and most fruitful techniques in cohomology theory is to replace a given cocycle with an equivalent one that has several desirable properties. This is often achieved by averaging over a suitable amenable group $\mathcal{G}$ of unitary operators in the von Neumann algebra $\mathcal{M}$, using an invariant mean β. If ϕ is a bounded n-cocycle, we may define a bounded $(n - 1)$-linear map α by

$$\alpha(x_1, \ldots, x_{n-1}) = \int_{\mathcal{G}} u^*\phi(u, x_1, \ldots, x_{n-1})d\beta(u), \quad \textbf{[3.1]}$$

where the action of β is denoted by integration. The invariance of β yields

$$(\phi - \partial\alpha)(x_1, \ldots, x_n) = 0 \quad \textbf{[3.2]}$$

whenever $x_1 \in \mathcal{G}$, and thus whenever $x_1 \in \mathcal{B}$, the C^*-algebra generated by $\mathcal{G}$. Further applications of unitary averaging lead to the conclusion that ϕ is equivalent to an n-cocycle ψ with the property that

$$\psi(x_1, \ldots, x_n) = 0 \quad \textbf{[3.3]}$$

when at least one of the variables is in $\mathcal{B}$. As a consequence, such a ψ is $\mathcal{B}$-multimodular. We use $n = 2$ to illustrate this point: the cocycle equation

$$b\psi(x, y) - \psi(bx, y) + \psi(b, xy) - \psi(b, x)y = 0, \quad \textbf{[3.4]}$$

$b \in \mathcal{B}$, $x, y \in \mathcal{M}$, reduces to

$$b\psi(x, y) = \psi(bx, y) \quad \textbf{[3.5]}$$

because the last two terms in **3.4** are 0.

Now let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space H, and let $\mathcal{M}$ be its ultraweak closure. There is a central projection $z \in \mathcal{A}^{**}$ such that $\mathcal{A}^{**}z$ and $\mathcal{M}$ are isomorphic and, employing this isomorphism, $\mathcal{M}$ becomes a dual normal $\mathcal{A}^{**}$-bimodule. By using second dual techniques, it is then possible to replace a cocycle ϕ: $\mathcal{A}^n \rightarrow \mathcal{M}$ by an equivalent cocycle ψ: $\mathcal{A}^n \rightarrow \mathcal{M}$ that is separately ultraweakly-weak* continuous in each variable. Moreover, such a cocycle extends to a cocycle $\bar{\psi}$: $\mathcal{M}^n \rightarrow \mathcal{M}$ that is separately normal in each variable. In the particular case when $\mathcal{A} = \mathcal{M}$, we conclude that each cocycle is equivalent to one that is separately normal in each variable.

Now consider a hyperfinite subalgebra $\mathcal{R}$ of a von Neumann algebra $\mathcal{M}$. Because $\mathcal{R}$ is the ultraweak closure of an increasing family of finite dimensional subalgebras, it is possible to find an amenable group $\mathcal{G}$ of unitary operators in $\mathcal{R}$ that generates a C^*-algebra $\mathcal{B}$ whose ultraweak closure is $\mathcal{R}$. The averaging and second dual techniques can be applied in tandem to replace a cocycle ϕ: $\mathcal{M}^n \rightarrow \mathcal{M}$ by an equivalent cocycle ψ that is both $\mathcal{B}$-multimodular and separately normal in each variable. Of course, ψ is then $\mathcal{R}$-multimodular by ultraweak continuity.

All the results discussed above are due to refs. 2–4 and may also be found in chapter 3 of ref. 8. They are summarized by the following theorem, which is undoubtedly the most important for cohomological calculations.

THEOREM 3.1. *(See ref. 8.) Let $\mathcal{M} \subseteq B(H)$ be a von Neumann algebra with a hyperfinite von Neumann subalgebra $\mathcal{R}$ and an ultraweakly dense C^*-subalgebra $\mathcal{A}$. Then the cohomology groups*

$$H^n(\mathcal{A}, \mathcal{M}),\ H_w^n(\mathcal{A}, \mathcal{M}),\ H^n(\mathcal{M}, \mathcal{M}),\ H_w^n(\mathcal{M}, \mathcal{M}),$$

$$\text{and } H_w^n(\mathcal{M}, \mathcal{M} :/\mathcal{R})$$

are pairwise isomorphic, for each $n \geq 1$.

As will be seen subsequently, this theorem gives several options for the determination of $H^n(\mathcal{M}, \mathcal{M})$.

A second application of averaging over an amenable unitary group leads to a very useful inequality for bilinear maps. Grothendieck's inequality for abelian C^*-algebras was extended to C^*-algebras with the approximation property by Pisier (17), and then to all C^*-algebras by Haagerup (18). The latter formulation, appropriate for von Neumann algebras, is as follows. Given a bounded bilinear form θ: $\mathcal{M} \times \mathcal{M} \to \mathbf{C}$, separately normal in each variable, there exist four states $f_1, f_2, g_1, g_2 \in \mathcal{M}_*$, the predual of $\mathcal{M}$, such that, for $x, y \in \mathcal{M}$,

$$|\theta(x, y)| \le \tfrac{1}{2}\|\theta\|(f_1(x^*x) + f_2(xx^*) + g_1(y^*y) + g_2(yy^*)). \quad \textbf{[3.6]}$$

If θ has the additional property of being inner $\mathcal{R}$-modular, in the sense that

$$\theta(xr, y) = \theta(x, ry) \quad \textbf{[3.7]}$$

for $x, y \in \mathcal{M}$, $r \in \mathcal{R}$, where $\mathcal{R}$ is a hyperfinite von Neumann subalgebra whose relative commutant $\mathcal{R}' \cap \mathcal{M}$ is the center $\mathcal{Z}$ of $\mathcal{M}$, then we may fix an amenable group $\mathcal{G}$ of unitary operators in $\mathcal{R}$ and average in **3.6**. The resulting inequality, when $\mathcal{M}$ is type II_1, is

$$\begin{aligned} |\theta(x, y)| &= \int_{\mathcal{G}} |\theta(xu^*, uy)| d\beta(u) \\ &\le \tfrac{1}{2}\|\theta\|\left(f_1\left(\int_{\mathcal{G}} ux^*xu^* d\beta(u)\right) + f_2(xx^*) + g_1(y^*y)\right. \\ &\quad \left. + g_2\left(\int_{\mathcal{G}} uyy^*u^* d\beta(u)\right)\right), \end{aligned} \quad \textbf{[3.8]}$$

leading to the existence of two states $F, G \in \mathcal{M}_*$ such that

$$|\theta(x, y)| \le \|\theta\|(F(xx^*) + G(y^*y)). \quad \textbf{[3.9]}$$

The x^*x and yy^* terms have disappeared from **3.6** because, for $m \in \mathcal{M}$,

$$\int_{\mathcal{G}} um^*mu^* d\beta(u) = \mathbf{E}(m^*m) = \mathbf{E}(mm^*), \quad \textbf{[3.10]}$$

where $\mathbf{E}$ is the tracial conditional expectation of $\mathcal{M}$ onto $\mathcal{Z}$. The following result from ref. 11, to which we refer for details, is a straightforward deduction from **3.9**.

THEOREM 3.2. *Let $\mathcal{M}$ be a type II_1 von Neumann algebra with a hyperfinite subalgebra $\mathcal{R}$ satisfying $\mathcal{R}' \cap \mathcal{M} = \mathcal{Z}$. If ψ: $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is inner $\mathcal{R}$-modular and separately normal in each variable, then*

$$\left\|\sum_{i=1}^n \psi(x_i, y_i)\right\| \le 2\|\psi\| \left\|\sum_{i=1}^n x_i x_i^*\right\|^{1/2} \left\|\sum_{i=1}^n y_i^* y_i\right\|^{1/2}. \quad \textbf{[3.11]}$$

for $x_i, y_i \in \mathcal{M}$.

This theorem applies to a normal right $\mathcal{R}$-module map ϕ: $\mathcal{M} \to \mathcal{M}$ by considering the inner $\mathcal{R}$-modular bilinear map

$$\psi(x, y) = \phi(x)\phi(y^*)^*, \quad x, y \in \mathcal{M}. \quad \textbf{[3.12]}$$

The next result follows from **3.11** by taking y_i to be x_i^*. The crucial point is the equivalence of the operator norm and the row bounded norm, which is immediate from **3.13**.

THEOREM 3.3. *Let $\mathcal{R} \subseteq \mathcal{M}$ satisfy the hypotheses of Theorem 3.2, and let ϕ: $\mathcal{M} \to \mathcal{M}$ be a bounded normal right $\mathcal{R}$-module map. Then*

(i) for $x_i \in \mathcal{M}$, $1 \le i \le n$,

$$\left\|\sum_{i=1}^n \phi(x_i)\phi(x_i)^*\right\| \le 2\|\phi\|^2 \left\|\sum_{i=1}^n x_i x_i^*\right\|, \quad \textbf{[3.13]}$$

(ii) ϕ is row bounded, and

$$\|\phi\| \le \|\phi\|_{\mathrm{r}} \le \sqrt{2}\,\|\phi\|. \quad \textbf{[3.14]}$$

A nonhyperfinite type II_1 von Neumann algebra will not be generated by an amenable group of unitaries, but nevertheless there is a notion of averaging that works in this situation, but only on completely bounded maps. The idea is to replace an average $\int_{\mathcal{G}} \phi(xu)u^* d\beta(u)$ by an ultraweak limit of maps of the form

$$x \mapsto \sum_{i=1}^{\infty} \phi(xm_i)m_i^* \quad \textbf{[3.15]}$$

where $m_i \in \mathcal{M}$ and $\sum_{i=1}^{\infty} m_i m_i^* = 1$, to obtain right $\mathcal{M}$-module maps from bounded maps. The next result, taken from ref. 19 but tailored to our needs, relies on the minimal invariant set technique pioneered by Kadison (5) and Sakai (6).

THEOREM 3.4. *There exists a contractive projection ρ from $\mathcal{L}_{cb}(\mathcal{M}, \mathcal{M})$ onto the subspace $\mathcal{L}_{cb}(\mathcal{M}, \mathcal{M})_{\mathcal{M}}$ of right $\mathcal{M}$-module maps with the following properties*:

(i) There exists a net of maps ρ_α: $\mathcal{L}_{cb}(\mathcal{M}, \mathcal{M}) \to \mathcal{L}_{cb}(\mathcal{M}, \mathcal{M})$, each of the form **3.15**, *such that*

$$(\rho\phi)(x) = \lim_{\alpha}(\rho_\alpha\phi)(x) \quad \textbf{[3.16]}$$

ultraweakly for $\phi \in \mathcal{L}_{cb}(\mathcal{M}, \mathcal{M})$ and $x \in \mathcal{M}$.
(ii) For all $\phi \in \mathcal{L}_{cb}(\mathcal{M}, \mathcal{M})$,

$$\|\rho\phi\| \le \|\phi\|_{\mathrm{r}} \le \|\phi\|_{\mathrm{cb}}. \quad \textbf{[3.17]}$$

(iii) If $a \in \mathcal{M}$ is fixed and $\phi_a \in \mathcal{L}_{cb}(\mathcal{M}, \mathcal{M})$ is defined, for each $\phi \in \mathcal{L}_{cb}(\mathcal{M}, \mathcal{M})$, by

$$\phi_a(x) = \phi(ax), \quad x \in \mathcal{M}, \quad \textbf{[3.18]}$$

then

$$(\rho\phi_a)(x) = (\rho\phi)(ax), \quad x \in \mathcal{M}. \quad \textbf{[3.19]}$$

4. The Main Result

Throughout this section, $\mathcal{M}$ is a type II_1 von Neumann algebra with a separable predual, center $\mathcal{Z}$, and a faithful trace tr. We assume that $\mathcal{M}$ is represented on $L^2(\mathcal{M}, tr)$, in which case there is a conjugate linear isometry J: $[x] \mapsto [x^*]$ so that $J\mathcal{M}J = \mathcal{M}'$, the commutant of $\mathcal{M}$. We also assume that $\mathcal{M}$ has a Cartan subalgebra $\mathcal{A}$. This is a maximal abelian self-adjoint subalgebra of $\mathcal{M}$ whose unitary normalizer

$$\mathcal{U} \equiv \{u \in \mathcal{M}: u\mathcal{A}u^* = \mathcal{A}, \quad u \text{ unitary}\} \quad \textbf{[4.1]}$$

generates $\mathcal{M}$ as a von Neumann algebra (12).

THEOREM 4.1. *With the above assumptions on $\mathcal{M}$,*

$$H^n(\mathcal{M}, \mathcal{M}) = 0, \quad n \ge 1. \quad \textbf{[4.2]}$$

Sketch of Proof: From *Theorem 3.1*, it suffices to show that $H_w^n(C^*(\mathcal{U}), \mathcal{M}) = 0, n \ge 1$. Because the case $n = 1$ is a special case of the Kadison–Sakai derivation theorem (5, 6), we make the further restriction of $n \geq 2$. By refs. 14 and 20 there is a hyperfinite subalgebra $\mathcal{R}$ such that $\mathcal{A} \subseteq \mathcal{R} \subseteq \mathcal{M}$ and $\mathcal{R}' \cap \mathcal{M} = \mathcal{Z}$. The averaging techniques of the previous section allow us to consider a cocycle ϕ: $\mathcal{M}^n \to \mathcal{M}$ which is $\mathcal{R}$-multimodular and separately normal in each variable, and it is required to show that the restriction of ϕ to $C^*(\mathcal{U})^n$ is a coboundary.

The first step is to prove that, for fixed $u_1, \ldots, u_{n-1} \in \mathcal{U}$, the map

$$\mu(x) = \phi(u_1, \ldots, u_{n-1}, x), \quad x \in \mathcal{M}, \qquad \mathbf{[4.3]}$$

is completely bounded in the x-variable. The Cartan subalgebra and $\mathcal{R}$-multimodularity hypotheses are used to establish this. Given $X \in M_k(\mathcal{M})$, $R_0 \in \mathrm{Row}_k(\mathcal{A})$, $C \in \mathrm{Col}_k(\mathcal{A})$, all of norm 1, we may find rows $R_1, \ldots, R_{n-1} \in \mathrm{Row}_k(\mathcal{A})$, again of norm 1, so that

$$R_j(u_{j+1} \otimes I_k) = u_{j+1}R_{j+1}, \quad 0 \leq j \leq n-2, \qquad \mathbf{[4.4]}$$

because $u_1, \ldots, u_{n-1} \in \mathcal{U}$. Then elements of $\mathcal{A}$ can be passed through the variables of ϕ, as discovered by Rădulescu (21), giving

$$R_0\mu_k(X)C = \phi(u_1, \ldots, u_{n-1}, R_{n-1}XC). \qquad \mathbf{[4.5]}$$

Thus

$$\sup\{\|R_0\mu_k(X)C\|: \|R_0\| = \|C\| = 1\} \leq \|\phi\|. \qquad \mathbf{[4.6]}$$

By ref. 15, $\mathcal{A}$ and $J\mathcal{A}J$ generate a maximal abelian subalgebra of $B(L^2(\mathcal{M}, tr))$, and this is sufficient to conclude that the supremum in **4.6** is $\|\mu_k(X)\|$ (see theorem 2.1 of ref. 22). Thus $\|\mu\|_{\mathrm{cb}} \leq \|\phi\|$. It is then clear that, for any fixed $y_1, \ldots, y_{n-1} \in \mathrm{Alg}(\mathcal{U})$,

$$x \mapsto \phi(y_1, \ldots, y_{n-1}, x), \quad x \in \mathcal{M}, \qquad \mathbf{[4.7]}$$

is completely bounded in the x-variable, and is a normal right $\mathcal{R}$-module map. For $y_1, \ldots, y_n \in \mathrm{Alg}\ \mathcal{U}$, $x \in \mathcal{M}$, each term in the cocycle equation

$$\begin{aligned} &y_1\phi(y_2, \ldots, y_n, x) + \sum_{i=1}^{n-1} (-1)^i\phi \\ &\times (y_1, \ldots, y_{i-1}, y_iy_{i+1}, y_{i+2}, \ldots, y_n, x) \\ &+ (-1)^n\phi(y_1, \ldots, y_{n-1}, y_nx) \\ &+ (-1)^{n+1}\phi(y_1, \ldots, y_n)x = 0 \end{aligned} \qquad \mathbf{[4.8]}$$

is a completely bounded map in x, so the projection ρ of *Theorem 3.4* may be applied. Because $\mathcal{L}_{cb}(\mathcal{M}, \mathcal{M})_{\mathcal{M}}$ consists of maps of the form $x \mapsto m_0x$ for a fixed $m_0 \in \mathcal{M}$, we obtain α: $\mathrm{Alg}(\mathcal{U})^{n-1} \to \mathcal{M}$ so that **4.8** becomes

$$\begin{aligned} &y_1\alpha(y_2, \ldots, y_n)x + \sum_{i=1}^{n-1} (-1)^i\alpha \\ &\times (y_1, \ldots, y_{i-1}, y_iy_{i+1}, y_{i+2}, \ldots, y_n)x \\ &+ (-1)^n\alpha(y_1, \ldots, y_{n-1})y_nx \\ &+ (-1)^{n+1}\phi(y_1, \ldots, y_n)x = 0, \end{aligned} \qquad \mathbf{[4.9]}$$

and the estimate $\|\alpha\| \leq \sqrt{2}\|\phi\|$ follows from *Theorem 3.3* (*ii*) and *Theorem 3.4* (*ii*). Eq. **4.9**, with $x = 1$, gives $\phi = \partial((-1)^n\alpha)$ on $\mathrm{Alg}(\mathcal{U})^n$, and the same conclusion holds on $C^*(\mathcal{U})^n$ by continuity. This establishes that the restriction of ϕ to $C^*(\mathcal{U})$ is a coboundary, as required.

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