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**Dualisation of Dualities II:  
Twisted self-duality of doubled fields and superdualities**

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ABSTRACT

We introduce a doubled formalism for the bosonic sector of the maximal supergravities, in which a Hodge dual potential is introduced for each bosonic field (except for the metric). The equations of motion can then be formulated as a twisted self-duality condition on the total field strength  $\mathcal{G}$ , which takes its values in a Lie superalgebra.

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# 1 Introduction

The study of the rigid (global) symmetry groups of the various supergravities has provided many insights into the understanding of the structure of the theories [1, 2]. In recent years the global symmetries have acquired a new significance, in the context of the conjectured non-perturbative U-duality symmetries of string theories and M-theory [3, 4].

It is useful to develop a universal framework within which the rigid symmetries can be studied, in which, for example, all the maximal supergravities can be discussed in an essentially dimension-independent way. Some steps in that direction were taken in [5], where a formalism for describing the bosonic sector of all the maximal supergravities was developed. Then, in [6], the global symmetries of the scalar sector were discussed in full generality, but the higher-rank fields were treated on more or less a “case by case” basis. In this paper we shall present a new description of the bosonic equations of motion in maximal supergravities, including all the higher-rank fields. It has been realised long ago that spacetime symmetries become internal symmetries upon dimensional reduction, in [6] the dualisation of dualities has been understood to exchange internal symmetries with gauge symmetries. It is a natural idea, in order to achieve a deeper understanding of duality symmetries that would be immune to dualisation, to treat more uniformly gauge and internal symmetries: this can be largely achieved.

It was clear for many years that the twisted self-duality structure of supergravities had to be more general than the few cases already known in the 1980’s. For instance the two-dimensional case with its affine symmetry enlarged by the Moebius subgroup of the circle diffeomorphisms was a confirmation of that hope [7, 8] in the Moebius sector. Furthermore the central extension of the affine symmetry originates in the gravity sector of the theory and suggests that gravity and matter could be unified by the magic of these theories even without invoking any supersymmetry. Our approach can be motivated also by considering the situation in even spacetime dimensions, where, as is well known, the rigid symmetries can usually be realised only in terms of local field transformations of solutions of the equations of motion, where they act on the field strengths themselves rather than on the potentials. This feature is especially starkly illustrated by the consideration of a field strength of degree  $n$  in  $D = 2n$  dimensions, where the field and its Hodge dual are members of an irreducible multiplet under the rigid symmetry. For example, in  $D = 8$  the 4-form field strength  $F_{(4)}$  and its Hodge dual form a doublet under the  $SL(2, \mathbb{R})$  factor in the  $SL(2, \mathbb{R}) \times SL(3, \mathbb{R})$  rigid U-duality group. Obviously, therefore, it is not possible to realise the  $SL(2, \mathbb{R})$  symmetry in terms of local transformations on the potential  $A_{(3)}$  for

$F_{(4)}$ , and so in particular the  $SL(2, \mathbb{R})$  cannot be realised at the level of the standard action. However, as we showed in [6], it *is* possible to introduce a formalism in which the symmetry is realised on potentials, by introducing a second 3-form potential  $\tilde{A}_{(3)}$ , with field strength  $\tilde{F}_4$ . The ensuing doubling of the degrees of freedom is counterbalanced by imposing, after varying the “doubled” Lagrangian, a constraint that the original and the doubled fields strengths are related by Hodge duality. (Actually, because of the presence of a dilaton  $\phi$  in the eight-dimensional theory, the constraint takes the form  $\tilde{F}_{(4)} = e^{-\phi} *F_{(4)}$  [6].) In fact although it can be useful to consider the Lagrangian for the doubled system it is in some sense a “gilding of the lilly,” since the constraint itself already implies all the equations for  $F_{(4)}$ . In other words, the Bianchi identities for  $F_{(4)}$  and  $\tilde{F}_{(4)}$ , together with the constraint, imply the equations of motion for  $\tilde{F}_{(4)}$  and  $F_{(4)}$  respectively.

In this paper we pursue the idea of introducing “doubled” fields for all fields including the dilatonic scalars. This can be also motivated by arguing that U-duality is not invariant under dualisation [9, 6] but is transmuted partly into gauge symmetries so F-duality [10] must include all the latter so as to deserve the name of Full duality.<sup>1</sup> Thus every bosonic field equation, with the exception of the Einstein equation, can be expressed as the statement that each field strength is equal to the dual of its double. We shall show that by introducing generators for each field and its double, we can write a combined single field  $\mathcal{G}$ , such that the equations of motion read simply

$$*\mathcal{G} = \mathcal{S} \mathcal{G} . \tag{1.1}$$

Here  $*$  denotes the Hodge dual, and  $\mathcal{S}$  is an involution or a map of square minus one (let us say a pseudo-involution) that exchanges the generators for fields and those for their partners under doubling. The field  $\mathcal{G}$  can itself be written in terms of the exponential  $\mathcal{V}$  of linear combinations of the generators mentioned above, with potentials (including the doubled potentials) as coefficients, as  $\mathcal{G} = d\mathcal{V} \mathcal{V}^{-1}$ . This is a generalisation of the parameterisation of scalar group manifolds in the Borel gauge, discussed in [6] and known as the Iwasawa decomposition in Mathematics. In this viewpoint, the Cartan-Maurer equation

$$d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0 \tag{1.2}$$

follows as an identity. One can also take another viewpoint and instead view (1.1) as the *definition* of the doubled field strengths such that the Cartan-Maurer equation (1.2) gives the equations of motion for the fields. In this alternative viewpoint the ability to write

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<sup>1</sup>We do, however, postpone the extension to the gravity sector. Also, we are considering only the bosonic theories here.

$\mathcal{G} = d\mathcal{V}\mathcal{V}^{-1}$  is viewed as a consequence of (1.2). Equation (1.2) is a zero curvature equation but where the field strengths play the role of (generalised) Yang-Mills potentials. Note that in (1.1) also, the potentials do not appear. Previous attempts to use differential algebras in Physics, see for instance [11], have imposed extra restrictions requiring freeness, disallowing Hodge duals or considering potentials and not field strengths as the basic objects. The equations we just presented in (1.2) can be interpreted as the defining equations of a minimal (defined here as quadratically nonlinear) differential algebra in the sense of Sullivan [12]. Actually (1.1) spoils the “freeness” by imposing relations beyond those of graded commutativity and most essential, the basic fields are the field strengths not the potentials. Lagrangian realisations of these theories require the choice of independent potentials, solutions of (1.1), and make use of half as many potentials as one starts with.

The paper is organised as follows. In section 2 we present a detailed discussion for  $D = 11$  supergravity, showing how the equation of motion for the 4-form field can be re-expressed in the doubled formalism. This illustrates many of the basic ideas that will recur in the later sections, including the fact that the generator associated with the original 3-form potential is an odd (fermionic) one, and thus the extended algebra of the doubled formalism is a superalgebra. In section 3 we extend the discussion to the important case of ten-dimensional type IIA supergravity, and then in section 4 we generalise to cover all the  $D$ -dimensional maximal supergravities that come from  $D = 11$ . We show that the underlying algebras in these cases are deformations of  $G \times G^*$ , where  $G$  itself is the semi-direct product of the Borel subalgebra of the superalgebra  $SL(11 - D|1)$  and a rank-3 tensor representation, and  $G^*$  is the co-adjoint representation of  $G$ . In section 5, we apply the doubled formalism to scalar coset manifolds and group manifolds, beginning with a detailed study of the symmetries for the  $O(2) \backslash SL(2, \mathbb{R})$  coset, and finishing with the general case, there one can verify the full  $G$  invariance of the doubled formalism. In section 6, we obtain the doubled formalism for type IIB supergravity in  $D = 10$ . Interestingly, this is the only example among the maximal supergravities where the generators are all exclusively bosonic.

## 2 $D = 11$ Supergravity

Our first example of a theory that can be expressed in terms of a doubled field equation is the bosonic sector of eleven-dimensional supergravity. The Lagrangian is given by [13]

$$\mathcal{L}_{11} = R * \mathbf{1} - \frac{1}{2} * F_{(4)} \wedge F_{(4)} - \frac{1}{6} F_{(4)} \wedge F_{(4)} \wedge A_{(3)} , \quad (2.1)$$

where  $F_{(4)} = dA_{(3)}$ , and the bracketed suffices denote the degrees of the differential forms. Varying with respect to  $A_{(3)}$ , we obtain the equation of motion

$$d*F_{(4)} + \frac{1}{2}F_{(4)} \wedge F_{(4)} = 0 . \quad (2.2)$$

Note that the action, and hence the equations of motion, are invariant under the abelian gauge transformation  $\delta A_{(3)} = d\lambda_{(2)}$ . Eqn. (2.2) can be written as  $d(*F_{(4)} + \frac{1}{2}A_{(3)} \wedge F_{(4)}) = 0$ , and so we can write the field equation in the first-order form

$$*F_{(4)} = \tilde{F}_{(7)} \equiv d\tilde{A}_{(6)} - \frac{1}{2}A_{(3)} \wedge F_{(4)} , \quad (2.3)$$

where we have introduced a dual potential  $\tilde{A}_{(6)}$ . Taking the exterior derivative of this equation gives rise to the second-order equation of motion (2.2). Note that it is not possible to eliminate the 3-form potential  $A_{(3)}$  and write the equation of motion purely in terms of the dual potential  $\tilde{A}_{(6)}$ ; nevertheless the equation of motion could still be rewritten as the closure of a form. (After we obtained these results a  $D = 11$  Lagrangian involving both a 3-form and a 6-form potential, together with further auxiliary fields, was proposed in [14].)

It is easily checked that the first-order equation (2.3) is invariant under the following infinitesimal gauge transformations:

$$\delta A_{(3)} = \Lambda_{(3)} , \quad \delta \tilde{A}_{(6)} = \tilde{\Lambda}_{(6)} - \frac{1}{2}\Lambda_{(3)} \wedge A_{(3)} \quad (2.4)$$

where  $\Lambda_{(3)}$  and  $\tilde{\Lambda}_{(6)}$  are 3-form and 6-form gauge parameters, satisfying  $d\Lambda_{(3)} = 0$  and  $d\tilde{\Lambda}_{(6)} = 0$ . (Note that we work with ‘‘gauge parameters’’ that are closed forms of degrees equal to the associated potentials, see [15]. This leads to a more uniform treatment when we discuss the global symmetries of 0-form potentials later.)

The commutators of infinitesimal gauge transformations are therefore given by

$$\begin{aligned} [\delta_{\Lambda_{(3)}}, \delta_{\Lambda'_{(3)}}] &= \delta_{\tilde{\Lambda}''_{(6)}} , & \tilde{\Lambda}''_{(6)} &= \Lambda_{(3)} \wedge \Lambda'_{(3)} , \\ [\delta_{\Lambda_{(3)}}, \delta_{\tilde{\Lambda}_{(6)}}] &= 0 , & [\delta_{\tilde{\Lambda}_{(6)}}, \delta_{\tilde{\Lambda}'_{(6)}}] &= 0 . \end{aligned} \quad (2.5)$$

Note that the introduction of the dual potential  $\tilde{A}_{(6)}$  has the consequence that the realisation of the originally abelian gauge symmetry of the potential  $A_{(3)}$  has now become non-abelian as acting on the dual potential  $\tilde{A}_{(6)}$ . However it is not a Yang-Mills type non-linearity but rather an odd one involving anticommutation rather than commutators, which can be traced back to the Chern-Simons term in  $D = 11$  supergravity. As we mentioned above, the field strengths are now quadratically coupled as if they were Yang-Mills connections. If we introduce generators  $V$  and  $\tilde{V}$  for the  $\Lambda_{(3)}$  and  $\tilde{\Lambda}_{(6)}$  transformations respectively, we see

that the commutation relations (2.5) translate into the (Lie) superalgebra

$$\{V, V\} = -\tilde{V} \ , \quad [V, \tilde{V}] = 0 \ , \quad [\tilde{V}, \tilde{V}] = 0 \ . \quad (2.6)$$

Note that the generators are even or odd according to whether the degrees of the associated field strengths are odd or even. In other words, the product  $A_{(3)} V$  and the product  $\tilde{A}_{(6)} \tilde{V}$  are both even elements, but  $V$  is an odd generator. Also, when the exterior derivative passes over a generator, the latter acquires a minus if it is odd. Thus  $d(V X) = -V dX$ , while  $d(\tilde{V} X) = \tilde{V} dX$ , for any  $X$ . We shall return to the general structure of the superalgebras corresponding to (2.6) and identify them in section 4, but let us right away discuss the simplest one given here.  $\tilde{V}$  is even and in the centre so it can be diagonalised. For each of its eigenvalue one has a Clifford algebra in one generator. It can be viewed as a deformation (quantisation) of the Grassmann superalgebra on one generator. This deformation is precisely the result of adding the Chern-Simons term into the Lagrangian. It will of course operate in any dimension of spacetime.

Moving a stage further, we can combine the doubled set of fields that describe the non-gravitational degrees of freedom of the extended  $D = 11$  supergravity equations:

$$\mathcal{V} = e^{A_{(3)} V} e^{\tilde{A}_{(6)} \tilde{V}} \ . \quad (2.7)$$

This parameterisation is suggested by (2.3). By an elementary calculation one checks that the field strength  $\mathcal{G} = d\mathcal{V} \mathcal{V}^{-1}$  following from (2.7) is given by

$$\begin{aligned} \mathcal{G} &= dA_{(3)} V + (d\tilde{A}_{(6)} - \frac{1}{2} A_{(3)} \wedge dA_{(3)}) \tilde{V} \ , \\ &= F_{(4)} V + \tilde{F}_{(7)} \tilde{V} \ . \end{aligned} \quad (2.8)$$

The gauge transformations (2.4) can be re-expressed in the simple form

$$\mathcal{V}' = \mathcal{V} e^{\Lambda_{(3)} V} e^{\tilde{\Lambda}_{(6)} \tilde{V}} \ . \quad (2.9)$$

It is straightforward to see that  $\mathcal{G}$  is invariant under these gauge transformations, since  $\Lambda_{(3)}$  and  $\tilde{\Lambda}_{(6)}$  are closed forms. The gauge transformations act on the right; they are analogous to the rigid  $G$  action on the right of the scalar coset space  $K \backslash G$ , and  $\mathcal{G}$  is the analogue of the  $K$ -tensor  $dgg^{-1}$ . Let us notice that in the scalar sector the local  $K$  action on the left will be fixed in the Borel gauge so it does not count as a gauge invariance in the new sense of differential algebras. The first-order field equation (2.3) can now be compactly written as a twisted self-duality condition:

$$*\mathcal{G} = \mathcal{S} \mathcal{G} \ , \quad (2.10)$$

where  $\mathcal{S}$  is a pseudo-involution that maps between the generators  $V$  and  $\tilde{V}$ :

$$\mathcal{S}V = \tilde{V} , \quad \mathcal{S}\tilde{V} = -V . \quad (2.11)$$

Note that here we have  $\mathcal{S}^2 V = -V$  and  $\mathcal{S}^2 \tilde{V} = -\tilde{V}$ , so  $\mathcal{S}^2 = -\text{id}$ . In general, the eigenvalue of  $\mathcal{S}^2$  on a given generator is the same as the eigenvalue of  $*^2$  on the associated field strength. In the more general examples in lower dimensions, we shall see that  $\mathcal{S}^2$  acts sometimes as an involution, and sometimes as a pseudo-involution. Let us insist however that  $\mathcal{S}$  does not preserve the commutation relations (2.6) but it is analogous to the scalar case situation where  $\mathcal{S}$  is a  $K$ -tensor [10].

There is another view of the above construction. Since the doubled field strength  $\mathcal{G}$  is written as  $\mathcal{G} = d\mathcal{V}\mathcal{V}^{-1}$ , it follows by taking an exterior derivative that we have the Cartan-Maurer equation  $d\mathcal{G} = -d\mathcal{V}d\mathcal{V}^{-1} = d\mathcal{V}\mathcal{V}^{-1}d\mathcal{V}\mathcal{V}^{-1}$ , and hence

$$d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0 . \quad (2.12)$$

Now, substituting (2.3) into (2.8), we can write the doubled field as

$$\mathcal{G} = F_{(4)} V + *F_{(4)} \tilde{V} . \quad (2.13)$$

It follows from this that

$$\begin{aligned} \mathcal{G} \wedge \mathcal{G} &= F_{(4)} V F_{(4)} V + F_{(4)} V *F_{(4)} \tilde{V} + *F_{(4)} \tilde{V} F_{(4)} V , \\ &= \frac{1}{2}\{F_{(4)} V, F_{(4)} V\} + \{F_{(4)} V, *F_{(4)} \tilde{V}\} , \\ &= \frac{1}{2}F_{(4)} \wedge F_{(4)} \{V, V\} - F_{(4)} \wedge *F_{(4)} [V, \tilde{V}] \\ &= -\frac{1}{2}F_{(4)} \wedge F_{(4)} \tilde{V} , \end{aligned} \quad (2.14)$$

and so the original second-order equation of motion (2.2) can be obtained simply by substituting (2.13) into the Cartan-Maurer equation (2.12). Note that in (2.14), we temporarily suspended the writing of the wedge-product symbols  $\wedge$ . In getting from the first line to the second, we used that for *any*  $X$ , we can write  $X X$  as  $\frac{1}{2}\{X, X\}$ . Passing to the third line, we used that  $V$  is odd (*i.e.* it behaves like an odd-degree differential form), while  $\tilde{V}$  is even. Thus in particular, we acquired a minus sign in turning  $V *F_{(4)}$  into  $-*F_{(4)} V$ . Finally, to reach the last line, we used that  $V$  and  $\tilde{V}$  satisfy the (anti)-commutation relations given in (2.6).

### 3 Type IIA Supergravity

The formalism developed above may be extended straightforwardly to the maximal supergravities obtained by dimensional reduction from eleven-dimensional supergravity. We shall

give the general  $D$ -dimensional results in the next section. Here, we consider the important special case of type IIA supergravity. The Lagrangian for the bosonic fields can be written as

$$\begin{aligned} \mathcal{L}_{10} = & R * \mathbb{1} - \frac{1}{2} * d\phi \wedge d\phi - \frac{1}{2} e^{-\frac{3}{2}\phi} * \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} - \frac{1}{2} e^\phi * F_{(3)} \wedge F_{(3)} \\ & - \frac{1}{2} e^{-\frac{1}{2}\phi} * F_{(4)} \wedge F_{(4)} + \frac{1}{2} dA_{(3)} \wedge dA_{(3)} \wedge A_{(2)} , \end{aligned} \quad (3.1)$$

where  $F_{(4)} = dA_{(3)} - dA_{(2)} \wedge \mathcal{A}_{(1)}$ ,  $F_{(3)} = dA_{(2)}$  and  $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$ . From this, it follows that the equations of motion for the antisymmetric tensor and scalar fields are:

$$\begin{aligned} d(e^{-\frac{1}{2}\phi} * F_{(4)}) &= -F_{(4)} \wedge F_{(3)} . \\ d(e^\phi * F_{(3)}) &= -\mathcal{F}_2 \wedge (e^{-\frac{1}{2}\phi} * F_{(4)}) - \frac{1}{2} F_{(4)} \wedge F_{(4)} , \\ d(e^{-\frac{3}{2}\phi} * \mathcal{F}_{(2)}) &= -F_{(3)} \wedge (e^{-\frac{1}{2}\phi} * F_{(4)}) , \\ *d\phi &= \frac{1}{4} F_{(4)} \wedge (e^{-\frac{1}{2}\phi} * F_{(4)}) + \frac{1}{2} F_{(3)} \wedge (e^\phi * F_{(3)}) + \frac{3}{4} \mathcal{F}_{(2)} \wedge (e^{-\frac{3}{2}\phi} * \mathcal{F}_{(2)}) . \end{aligned} \quad (3.2)$$

It is not hard to re-write these second-order field equations in first-order form, by extracting an overall exterior derivative from each equation. This means that all the equations of motion are (generalised) conservation laws [15, 16]. (This is best done by starting with the equation for  $F_{(4)}$ , and working down through the degrees of the fields, ending with the dilaton.) Thus, introducing a “doubled” set of potentials  $\{\psi, \tilde{\mathcal{A}}_{(7)}, \tilde{A}_{(6)}, \tilde{A}_{(5)}\}$  dual to  $\{\phi, \mathcal{A}_{(1)}, A_{(2)}, A_{(3)}\}$  respectively, we can write (at least locally) the following first-order equations:

$$\begin{aligned} e^{-\frac{1}{2}\phi} * F_{(4)} &\equiv \tilde{F}_{(6)} = d\tilde{A}_{(5)} - A_{(2)} \wedge dA_{(3)} , \\ e^\phi * F_{(3)} &\equiv \tilde{F}_{(7)} = d\tilde{A}_{(6)} - \frac{1}{2} A_{(3)} \wedge dA_{(3)} - \mathcal{A}_{(1)} \wedge (d\tilde{A}_{(5)} - A_{(2)} \wedge dA_{(3)}) , \\ e^{-\frac{3}{2}\phi} * \mathcal{F}_{(2)} &\equiv \tilde{\mathcal{F}}_{(8)} = d\tilde{\mathcal{A}}_{(7)} - A_{(2)} \wedge (d\tilde{A}_{(5)} - \frac{1}{2} A_{(2)} \wedge dA_{(3)}) , \\ *d\phi &\equiv \tilde{P} = d\psi + \frac{1}{2} A_{(2)} \wedge d\tilde{A}_{(6)} + \frac{1}{4} A_{(3)} \wedge (d\tilde{A}_{(5)} - A_{(2)} \wedge dA_{(3)}) \\ &\quad + \frac{3}{4} \mathcal{A}_{(1)} \wedge (d\tilde{\mathcal{A}}_{(7)} - A_{(2)} \wedge (d\tilde{A}_{(5)} - \frac{1}{2} A_{(2)} \wedge dA_{(3)})) . \end{aligned} \quad (3.3)$$

It is straightforward to check that by taking the exterior derivatives of these equations, and substituting the first-order equations back in where appropriate, we recover precisely the equations of motion (3.2).

In principle, we could now follow the same strategy that we described for  $D = 11$  supergravity in section 2, and derive the enlarged set of gauge transformations for all the potentials, including the doubled potentials for the dual field strengths. From the commutators of these gauge transformations we could then derive a superalgebra of generators



associated with the potentials, which would be the analogue of (2.5). In practice, this is a cumbersome procedure, and it is easier to derive the superalgebra by instead looking at the field equations. We first note that these can be written, using the tilded field strengths that are defined in a natural fashion in terms of the duals of the untilded ones in (3.3), as

$$\begin{aligned}
d\tilde{F}_{(6)} &= -F_{(4)} \wedge F_3 , \\
d\tilde{F}_{(7)} &= -\mathcal{F}_{(2)} \wedge \tilde{F}_{(6)} - \frac{1}{2}F_{(4)} \wedge F_{(4)} , \\
d\tilde{\mathcal{F}}_8 &= -F_{(3)} \wedge \tilde{F}_6 , \\
d\tilde{P} &= \frac{1}{4}F_{(4)} \wedge \tilde{F}_{(6)} + \frac{1}{2}F_{(3)} \wedge \tilde{F}_{(7)} + \frac{3}{4}\mathcal{F}_{(2)} \wedge \mathcal{F}_{(8)} .
\end{aligned} \tag{3.4}$$

The fact that the right-hand sides are all simply bilinear in field strengths suggests that it should be possible again to write the equations in the Cartan-Maurer form  $d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0$ , as in the case of eleven-dimensional supergravity treated in section 2. Thus let us introduce the doubled field strength  $\mathcal{G}$ :

$$\begin{aligned}
\mathcal{G} &= \frac{1}{2}d\phi H + e^{-\frac{3}{4}\phi} \mathcal{F}_{(2)} W_1 + e^{\frac{1}{2}\phi} F_{(3)} V^1 + e^{-\frac{1}{4}\phi} F_{(4)} V \\
&\quad + e^{\frac{1}{4}\phi} \tilde{F}_{(6)} \tilde{V} + e^{-\frac{1}{2}\phi} \tilde{F}_{(7)} \tilde{V}_1 + e^{\frac{3}{4}\phi} \tilde{\mathcal{F}}_{(8)} \tilde{W}^1 + \frac{1}{2}\tilde{P} \tilde{H} .
\end{aligned} \tag{3.5}$$

Note that  $\mathcal{G}$  is defined so as to be invariant under the gauge symmetries of the original Lagrangian, including the constant shift symmetry of the dilaton  $\phi$  together with the corresponding constant scalings of the other gauge potentials. (It is the requirement that  $\mathcal{G}$  be invariant under this shift symmetry that determines the exponential factors in the various terms in (3.5).) The generators  $H, V^1, \tilde{V}_1, \tilde{H}$ , being associated with field strengths whose potentials are of even degree, are themselves even. On the other hand the generators  $W_1, V, \tilde{V}$  and  $\tilde{W}^1$  are associated with potentials of odd degrees, and they will therefore be odd. (The notation here will be generalised in section 4, when we discuss the dimensional reduction to  $D$  dimensions. The “1” suffices and superscripts on generators indicate that they are associated with potentials arising in the first step of the reduction from  $D = 11$ .)

We may again introduce the (pseudo)-involution operator  $\mathcal{S}$ , which is defined to act on an untilded generator  $X$  to give the corresponding tilded generator  $\tilde{X}$  associated with the dual potential;  $\mathcal{S}X = \tilde{X}$ . Acting on  $\tilde{X}$ , we have  $\mathcal{S}\tilde{X} = \pm X$ ; the operator  $\mathcal{S}^2$  has eigenvalue  $+1$  or  $-1$  in accordance with the eigenvalue of  $*^2$  on the corresponding field strength. Thus the field  $\mathcal{G}$  defined in (3.5) automatically satisfies the twisted self-duality equation  $*\mathcal{G} = \mathcal{S}\mathcal{G}$ .

We now find that the equations of motion (3.4) can indeed be written simply as the “curvature-free” condition  $d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0$ , where the generators satisfy the following commutation and anti-commutation relations. Firstly the commutators with the Cartan

generator  $H$ , governed by the weights of the various fields appearing in (3.5), are

$$\begin{aligned} [H, W_1] &= -\frac{3}{2}W_1, & [H, V^1] &= V^1, & [H, V] &= -\frac{1}{2}V, \\ [H, \widetilde{W}_1] &= \frac{3}{2}\widetilde{W}_1, & [H, \widetilde{V}^1] &= -\widetilde{V}^1, & [H, \widetilde{V}] &= \frac{1}{2}\widetilde{V}. \end{aligned} \quad (3.6)$$

Next, the commutators and anti-commutators associated with the bilinear structures on the right-hand sides of the Bianchi identity for  $F_{(4)}$ , and those for  $\widetilde{F}_{(6)}$ ,  $\widetilde{F}_{(7)}$  and  $\widetilde{\mathcal{F}}_{(8)}$  in (3.4), are

$$\begin{aligned} [W_1, V^1] &= -V, & \{W_1, \widetilde{V}\} &= -\widetilde{V}_1, & [V^1, V] &= -\widetilde{V}, \\ [V^1, \widetilde{V}] &= -\widetilde{W}^1, & \{V, V\} &= -\widetilde{V}_1. \end{aligned} \quad (3.7)$$

Finally, those associated with the right-hand side in the equation for  $\widetilde{P}$  in (3.4) are

$$\{W_1, \widetilde{W}^1\} = \frac{3}{8}\widetilde{H}, \quad [V^1, \widetilde{V}_1] = \frac{2}{8}\widetilde{H}, \quad \{V, \widetilde{V}\} = \frac{1}{8}\widetilde{H}. \quad (3.8)$$

Note that here and in the sequel, all commutators and anti-commutators that are not listed do vanish. We shall discuss the structure of this superalgebra in the next section, where superduality algebras for  $D$ -dimensional maximal supergravities are obtained.

As in the eleven-dimensional example of the previous section, the Cartan-Maurer equation  $d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0$  for the doubled field  $\mathcal{G}$  can be solved by writing  $\mathcal{G} = d\mathcal{V}\mathcal{V}^{-1}$ , with  $\mathcal{V}$  most conveniently given by

$$\mathcal{V} = e^{\frac{1}{2}\phi H} e^{A_{(1)}W_1} e^{A_{(2)}V^1} e^{A_{(3)}V} e^{\widetilde{A}_{(5)}\widetilde{V}} e^{\widetilde{A}_{(6)}\widetilde{V}_1} e^{\widetilde{A}_{(7)}\widetilde{W}^1} e^{\frac{1}{2}\psi\widetilde{H}}. \quad (3.9)$$

A detailed calculation of  $d\mathcal{V}\mathcal{V}^{-1}$  gives precisely (3.5), where the tilded field strengths are now given by the right-hand sides of the first-order equations in (3.3). From this viewpoint, where  $\mathcal{G}$  is defined to be  $d\mathcal{V}\mathcal{V}^{-1}$ , the equation  $d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0$  is trivially satisfied, and the field equations, in the first-order form (3.3), arise from the twisted self-duality equation  $*\mathcal{G} = \mathcal{S}\mathcal{G}$ .

The type IIA supergravity has a classical global  $\mathbb{R}$  symmetry, which corresponds to continuous shifts of the dilaton and rescalings of the higher-degree potentials. It is straightforward to see that this symmetry is preserved in the doubled equation  $*\mathcal{G} = \mathcal{S}\mathcal{G}$ , since  $\mathcal{G}$ , given by (3.5), is invariant under this global symmetry provided that the dual fields rescale accordingly. In fact the doubled-equation formalism puts the local gauge symmetries of the higher-degree fields and the constant shift symmetry of the dilaton on an equal footing. The transformation rules for these symmetries can be expressed as

$$\mathcal{V}' = \mathcal{V} e^{\frac{1}{2}\Lambda_{(0)}H} e^{\Lambda_{(1)}W_1} e^{\Lambda_{(2)}V^1} e^{\Lambda_{(3)}V} e^{\widetilde{\Lambda}_{(5)}\widetilde{V}} e^{\widetilde{\Lambda}_{(6)}\widetilde{V}_1} e^{\widetilde{\Lambda}_{(7)}\widetilde{W}^1} e^{\frac{1}{2}\Lambda_{(8)}\widetilde{H}}, \quad (3.10)$$

where the gauge parameters  $\Lambda_{(i)}$  and  $\tilde{\Lambda}_{(i)}$  are all closed forms. The commutators of these transformations generate the algebra presented in (3.6), (3.7) and (3.8). The superalgebra of gauge symmetries of this bosonic theory seems to grow out of control, nevertheless it can be reorganised into a manageable form (that is by human beings). Again we may contract away the trilinear Chern-Simons term and its associated commutators, namely those producing tilded generators out of untilded ones. Then the untilded generators form a subalgebra  $G$ , and its dual space  $G^*$  transforms under it as its contragredient or dual representation. In other words we are actually considering a deformation (by the Chern-Simons term) of the Lie (super-)algebra  $G \times G^*$  where the semi-direct product is a standard construction for any linear representation of  $G$  to be treated as an abelian algebra. We may note in passing that the coadjoint representation is equivalent to the adjoint one for a semisimple algebra, or more generally for an algebra admitting a non degenerate invariant quadratic form in the adjoint representation; these are sometimes called contragredient (super-)algebras [17, 18]. Let us note also that the commutators (3.8) represent a central extension by  $\tilde{H}$ . It too can be contracted away; we shall return to this deformation later. If one were to contract away both  $\tilde{H}$  and  $\tilde{W}^1$  one would find a  $\mathbb{Z}/3\mathbb{Z}$  grading.

## 4 $D$ -dimensional Maximal Supergravity

In this section, we consider the general case of maximal supergravity in  $3 \leq D \leq 9$  dimensions, obtained by spacelike toroidal dimensional reduction from either  $D = 11$  supergravity or type IIB supergravity. We shall adopt the notation and conventions of [5, 6], where the dimensional reductions from  $D = 11$  are discussed. (But note that the sign of the Chern-Simons term in (2.1) is taken to be the opposite of the one chosen in those references.) Since the general case is quite complicated, we divide the analysis into four subsections and three appendices. First, we obtain the first-order equations of motion for the doubled systems of fields in each dimension. Then, we discuss the associated coset constructions. Next is the introduction of an unexpected twelfth fermionic dimension and finally the discussion of the deformation theory. Certain dimension-dependent details of the constructions as well as a more general discussion of the fermionic dimension are relegated to the appendices.

### 4.1 First-order equations for $D$ -dimensional supergravity

The  $D$ -dimensional Lagrangian, in the language of differential forms, is given by [5]

$$\mathcal{L} = R * \mathbb{1} - \frac{1}{2} * d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} e^{\vec{a} \cdot \vec{\phi}} * F_{(4)} \wedge F_{(4)} - \frac{1}{2} \sum_i e^{\vec{a}_i \cdot \vec{\phi}} * F_{(3)i} \wedge F_{(3)i}$$

$$\begin{aligned}
& -\frac{1}{2} \sum_{i < j} e^{\vec{a}_{ij} \cdot \vec{\phi}} * F_{(2)ij} \wedge F_{(2)ij} - \frac{1}{2} \sum_i e^{\vec{b}_i \cdot \vec{\phi}} * \mathcal{F}_{(2)}^i \wedge \mathcal{F}_{(2)}^i - \frac{1}{2} \sum_{i < j < k} e^{\vec{a}_{ijk} \cdot \vec{\phi}} * F_{(1)ijk} \wedge F_{(1)ijk} \\
& -\frac{1}{2} \sum_{i < j} e^{\vec{b}_{ij} \cdot \vec{\phi}} * \mathcal{F}_{(1)j}^i \wedge \mathcal{F}_{(1)j}^i + \mathcal{L}_{FFA} .
\end{aligned} \tag{4.1}$$

Let us recall that the vector notation represents vectors in the root space of  $GL(n, \mathbb{R})$ , with  $n = (11 - D)$ . The Chern-Simons terms  $\mathcal{L}_{FFA}$  are given for each dimension in [5, 6]. An important property of these terms is that their variation with respect to the various potentials  $A_{(3)}$ ,  $A_{(2)i}$ ,  $A_{(1)ij}$  and  $A_{(0)ijk}$  takes, up to a total derivative, the form

$$-\delta \mathcal{L}_{FFA} = dX \wedge \delta A_{(3)} + dX^i \wedge \delta A_{(2)i} + \frac{1}{2} dX^{ij} \wedge \delta A_{(1)ij} + \frac{1}{6} dX^{ijk} \delta A_{(0)ijk} , \tag{4.2}$$

where the quantities  $X$ ,  $X^i$ ,  $X^{ij}$  and  $X^{ijk}$  can be determined easily in each dimension. They are given in appendix A. Again the existence of the  $X$ 's reflects the abelian gauge invariance and source-freeness of the Chern-Simons integral and the ensuing possibility to rewrite the would be equations of motion of the Lagrangian  $\mathcal{L}_{FFA}$  as total derivatives [15, 16]. Here source-freeness means that a choice of action can be made such that any given potential can appear always differentiated.

We shall work with the hatted  $\hat{\mathcal{A}}_1^i = \gamma^i_j \mathcal{A}_1^j$  Kaluza-Klein potentials, introduced in (A.18) of ref. [6]. Thus the various field strengths are given by

$$\begin{aligned}
\mathcal{F}_{(2)}^i &= \tilde{\gamma}^i_j \hat{\mathcal{F}}_{(2)}^j , & \mathcal{F}_{(1)j}^i &= \gamma^k_j d\mathcal{A}_{(0)k}^i , \\
F_{(2)ij} &= \gamma^k_i \gamma^\ell_j \hat{F}_{(2)k\ell} , & F_{(3)i} &= \gamma^j_i \hat{F}_{(3)j} , & F_{(4)} &= \hat{F}_{(4)} ,
\end{aligned} \tag{4.3}$$

where these (Kaluza-Klein modified) field strengths read

$$\begin{aligned}
\hat{\mathcal{F}}_{(2)}^i &= d\hat{\mathcal{A}}_{(1)}^i , & \hat{F}_{(2)ij} &= dA_{(1)ij} - dA_{(0)ijk} \wedge \hat{\mathcal{A}}_{(1)}^k , \\
\hat{F}_{(3)i} &= dA_{(2)i} + dA_{(1)ij} \wedge \hat{\mathcal{A}}_{(1)}^j + \frac{1}{2} dA_{(0)ijk} \wedge \hat{\mathcal{A}}_{(1)}^j \wedge \hat{\mathcal{A}}_{(1)}^k , \\
\hat{F}_{(4)} &= dA_{(3)} - dA_{(2)i} \wedge \hat{\mathcal{A}}_{(1)}^i + \frac{1}{2} dA_{(1)ij} \wedge \hat{\mathcal{A}}_{(1)}^i \wedge \hat{\mathcal{A}}_{(1)}^j - \frac{1}{6} dA_{(0)ijk} \wedge \hat{\mathcal{A}}_{(1)}^i \wedge \hat{\mathcal{A}}_{(1)}^j \wedge \hat{\mathcal{A}}_{(1)}^k .
\end{aligned} \tag{4.4}$$

(Here  $\tilde{\gamma}^i_j = \delta^i_j + \mathcal{A}^i_{(0)j}$ , and  $\gamma^i_j$  is its inverse. See (A.19) and (A.29) in ref. [6].)

We are now in a position to start constructing the first-order equations, by writing down the second-order equations following from (4.1), and then stripping off a derivative in each case. If we handle the various field equations in the appropriate order, this turns out to be a fairly straightforward deductive process. The order to follow is first to look at the equation of motion coming from varying  $A_{(3)}$  in (4.1), then  $A_{(2)i}$ , then  $A_{(1)ij}$ , then  $A_{(0)ijk}$ , then  $\hat{\mathcal{A}}_1^i$ , then  $\mathcal{A}^i_{(0)j}$ , and finally  $\vec{\phi}$ . We shall look explicitly here at the first two of these, and then present only the results for the others. Thus varying (4.1) with respect to  $A_3$  we get

$$-\delta \mathcal{L} = e^{\vec{a} \cdot \vec{\phi}} * F_4 \wedge d\delta A_{(3)} + dX \wedge \delta A_{(3)} . \tag{4.5}$$

Integrating by parts, this gives<sup>2</sup>

$$-(-1)^D d(e^{\vec{a}\cdot\vec{\phi}} * F_4) + dX = 0 . \quad (4.6)$$

Thus we can immediately strip off the derivative, and write the first-order equation

$$e^{\vec{a}\cdot\vec{\phi}} * F_4 \equiv \tilde{F}_{(D-4)} = d\tilde{A}_{(D-5)} + (-1)^D X . \quad (4.7)$$

Varying (4.1) with respect to  $A_{(2)j}$ , and integrating by parts, we get the field equation

$$(-1)^D \sum_i d(e^{\vec{a}_i\cdot\vec{\phi}} * F_{(3)i} \gamma^j_i) + (-1)^D d(e^{\vec{a}\cdot\vec{\phi}} * F_4 \wedge \hat{\mathcal{A}}_1^j) + dX^j = 0 . \quad (4.8)$$

We can now strip off the derivative, and then use the previous result (4.7), to give the first-order equation

$$e^{\vec{a}_i\cdot\vec{\phi}} * F_{(3)i} = \tilde{\gamma}^i_j \tilde{F}_{(D-3)}^j , \quad (4.9)$$

where

$$\tilde{F}_{(D-3)}^j = d\tilde{A}_{(D-4)}^j - d\tilde{A}_{(D-5)} \wedge \hat{\mathcal{A}}_1^j - (-1)^D (X^j + X \wedge \hat{\mathcal{A}}_1^j) . \quad (4.10)$$

(We have also multiplied by a  $\tilde{\gamma}$  here, which has allowed us to obtain equations for each  $i$  value separately.)

Proceeding in a similar vein, we obtain the first-order equations

$$e^{\vec{a}_{ij}\cdot\vec{\phi}} * F_{(2)ij} = \tilde{\gamma}^i_k \tilde{\gamma}^j_\ell \tilde{F}_{(D-2)}^{k\ell} , \quad (4.11)$$

$$e^{\vec{a}_{ijk}\cdot\vec{\phi}} * F_{(1)ijk} = \tilde{\gamma}^i_\ell \tilde{\gamma}^j_m \tilde{\gamma}^k_n \tilde{F}_{(D-1)}^{\ell mn} , \quad (4.12)$$

where

$$\begin{aligned} \tilde{F}_{(D-2)}^{k\ell} &= d\tilde{A}_{(D-3)}^{k\ell} - d\tilde{A}_{(D-4)}^k \hat{\mathcal{A}}_{(1)}^\ell + d\tilde{A}_{(D-4)}^\ell \hat{\mathcal{A}}_{(1)}^k + d\tilde{A}_{(D-5)} \hat{\mathcal{A}}_{(1)}^k \hat{\mathcal{A}}_{(1)}^\ell \\ &\quad + (-1)^D (X^{k\ell} + X^k \hat{\mathcal{A}}_{(1)}^\ell - X^\ell \hat{\mathcal{A}}_{(1)}^k + X \hat{\mathcal{A}}_{(1)}^k \hat{\mathcal{A}}_{(1)}^\ell) , \end{aligned} \quad (4.13)$$

$$\begin{aligned} \tilde{F}_{(D-1)}^{\ell mn} &= d\tilde{A}_{(D-2)}^{\ell mn} - 3d\tilde{A}_{(D-3)}^{[\ell m} \hat{\mathcal{A}}_{(1)}^{n]} + 3d\tilde{A}_{(D-4)}^{[\ell} \hat{\mathcal{A}}_{(1)}^m \hat{\mathcal{A}}_{(1)}^{n]} - d\tilde{A}_{(D-5)} \hat{\mathcal{A}}_{(1)}^\ell \hat{\mathcal{A}}_{(1)}^m \hat{\mathcal{A}}_{(1)}^n \\ &\quad - (-1)^D (X^{\ell mn} + 3X^{[\ell m} \hat{\mathcal{A}}_{(1)}^{n]} + 3X^{[\ell} \hat{\mathcal{A}}_{(1)}^m \hat{\mathcal{A}}_{(1)}^{n]} + X \hat{\mathcal{A}}_{(1)}^\ell \hat{\mathcal{A}}_{(1)}^m \hat{\mathcal{A}}_{(1)}^n) . \end{aligned} \quad (4.14)$$

(As usual, we drop the  $\wedge$  symbols when the going gets tough.)

Now we turn to the equation of motion coming from varying  $\hat{\mathcal{A}}_{(1)}^k$  in (4.1). For this, we again make use of the first-order equations obtained previously in order to simplify the result. We then arrive at the equation

$$(-1)^D \sum_i d(e^{\vec{b}_i\cdot\vec{\phi}} * \mathcal{F}_{(2)}^i \tilde{\gamma}^i_k) = -\frac{1}{2} \tilde{F}_{(D-2)}^{ij} dA_{(0)ijk} + \tilde{F}_{(D-3)}^i \hat{F}_{(2)ik} - \tilde{F}_{(D-4)} \hat{F}_{(3)k} . \quad (4.15)$$

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<sup>2</sup>When the degree of the field strength becomes larger than or equal to  $D$  we take the dual to vanish in this paper.

This is still a bit messy-looking, since the terms here involve a lot of  $\hat{\mathcal{A}}_{(1)}^i$  Kaluza-Klein potentials. But, remarkably, if we substitute the definitions of the  $\tilde{F}$  and  $\hat{F}$  fields in terms of the potentials, we find that all the  $\hat{\mathcal{A}}_{(1)}^i$  potentials cancel out in (4.15), and we are left simply with:

$$\begin{aligned} \sum_i d(e^{\vec{b}_i \cdot \vec{\phi}} * \mathcal{F}_{(2)}^i \tilde{\gamma}^i_k) &= (-1)^D \left( -\frac{1}{2} d\tilde{A}_{(D-3)}^{ij} dA_{(0)ijk} + d\tilde{A}_{(D-4)}^i dA_{(1)ik} - d\tilde{A}_{(D-5)} dA_{(2)k} \right) \\ &\quad - \frac{1}{2} X^{ij} dA_{(0)ijk} - X^i dA_{(1)ik} - X dA_{(2)k} . \end{aligned} \quad (4.16)$$

We still have the task of stripping the derivative off this equation. This is clearly straightforward for the upper line on the right-hand side. For the lower line, it requires some more detailed analysis of the Chern-Simons terms in each dimension, which give rise to the quantities  $X$ ,  $X^i$  and  $X^{ij}$ , defined in (4.2). We find that the lower line is always a closed form, and so a derivative can indeed be stripped off. In other words, there exists a  $Y_k$  such that

$$X dA_{(2)k} + X^i dA_{(1)ik} + \frac{1}{2} X^{ij} dA_{(0)ijk} = dY_k , \quad (4.17)$$

where the quantities  $Y_k$  in each dimension  $D$  are given in appendix A. Thus we can now strip off the derivative in (4.16), to get the first-order equation

$$e^{\vec{b}_i \cdot \vec{\phi}} * \mathcal{F}_2^i = \gamma^j_i \tilde{\mathcal{F}}_{(D-2)j} , \quad (4.18)$$

where

$$\tilde{\mathcal{F}}_{(D-2)j} = d\tilde{\mathcal{A}}_{(D-3)j} - A_{(2)j} d\tilde{\mathcal{A}}_{(D-5)} + (-1)^D A_{(1)kj} d\tilde{\mathcal{A}}_{(D-4)}^k - \frac{1}{2} A_{(0)jkl} d\tilde{\mathcal{A}}_{(D-3)}^{kl} - Y_j . \quad (4.19)$$

Now we turn to the equation of motion coming from varying  $\mathcal{A}_{(0)k}^i$ . This will receive various contributions coming from the fact that the field strengths  $F_{(3)i}$ , etc., involve  $\gamma$ . After a little calculation, and substitution of the previous results for first-order equations, the field equation can be put in the form

$$(-1)^D \sum_j d(e^{\vec{b}_{ij} \cdot \vec{\phi}} * \mathcal{F}_{(1)j}^i \gamma^k_j) - \sum_{\ell < j} e^{\vec{b}_{\ell j} \cdot \vec{\phi}} * \mathcal{F}_{(1)j}^\ell \gamma^k_j \mathcal{F}_{(1)i}^\ell = \gamma^j_i \mathcal{B}^k_j , \quad (4.20)$$

where

$$\mathcal{B}^k_j = \tilde{F}_{(D-3)}^k \hat{F}_{(3)j} + \tilde{F}_{(D-2)}^{k\ell} \hat{F}_{(2)j\ell} + \frac{1}{2} \tilde{F}_{(D-1)}^{k\ell m} \hat{F}_{(1)j\ell m} - \tilde{\mathcal{F}}_{(D-2)j} d\hat{\mathcal{A}}_{(1)}^k . \quad (4.21)$$

When re-expressed in terms of the potentials, this is again an expression that undergoes “miraculous” simplifications, giving

$$\begin{aligned} \mathcal{B}^k_j &= d\tilde{\mathcal{A}}_{(D-5)} d(\hat{\mathcal{A}}_{(1)}^k A_{(2)j}) + d\tilde{\mathcal{A}}_{(D-4)}^k dA_{(2)j} - d\tilde{\mathcal{A}}_{(D-4)}^\ell d(\hat{\mathcal{A}}_{(1)}^k A_{(1)j\ell}) - \frac{1}{2} d\hat{\mathcal{A}}_{(1)}^k d\tilde{\mathcal{A}}_{(D-3)j} \\ &\quad + d\tilde{\mathcal{A}}_{(D-3)}^{k\ell} dA_{(1)j\ell} + \frac{1}{2} d\tilde{\mathcal{A}}_{(D-3)}^{\ell m} d(A_{(0)j\ell m} \hat{\mathcal{A}}_{(1)}^k) + \frac{1}{2} d\tilde{\mathcal{A}}_{(D-2)}^{k\ell m} dA_{(0)j\ell m} \\ &\quad + (-1)^D \left( X dA_{(2)j} + X^\ell dA_{(2)\ell j} + \frac{1}{2} X^{\ell m} dA_{(0)j\ell m} \right) \hat{\mathcal{A}}_{(1)}^k + Y_j d\hat{\mathcal{A}}_{(1)}^k \\ &\quad + (-1)^D \left( -X^k dA_{(2)j} + X^{k\ell} dA_{(1)j\ell} - \frac{1}{2} X^{k\ell m} dA_{(0)j\ell m} \right) . \end{aligned} \quad (4.22)$$

It is manifest that the first two lines on the right-hand side can be written as exact differentials. It is also clear that we can do this for the third line, after recognising that the three terms in the bracket are nothing but the exact form  $dY_j$  defined in (4.17). For the final line, we can write it as an exact form if we can find quantities  $Q^k_j$ , such that

$$X^k dA_{(2)j} + X^{k\ell} dA_{(1)\ell j} + \frac{1}{2} X^{k\ell m} dA_{(0)j\ell m} = -dQ^k_j . \quad (4.23)$$

We find that this can indeed be done, and the results are presented for each dimension in appendix A. In fact the structure of (4.23) is quite analogous to that of (4.17), the closure of their left-hand sides is equivalent upon integration by parts to that of expressions of the form  $\sum dX^B R_B^C A_C$  with  $B$  and  $C$  collective indices but we have seen that  $dX^B$  is the term in the equation of motion of  $A_B$  that comes from  $\mathcal{L}_{FFA}$ , and hence these mysterious equations reflect nothing but invariances of the pure Chern-Simons action under various diffeomorphisms of the compactified coordinates.

Thus we have that  $\mathcal{B}^k_j = dW^k_j$ , where

$$\begin{aligned} W^k_j &= Q^k_j + A_{(2)j} \hat{\mathcal{A}}_{(1)}^k d\tilde{A}_{(D-5)} - (-1)^D A_{(2)j} d\tilde{A}_{(D-4)}^k - (-1)^D A_{(1)j\ell} \hat{\mathcal{A}}_{(1)}^k d\tilde{A}_{(D-4)}^\ell \\ &\quad + A_{(1)j\ell} d\tilde{A}_{(D-3)}^{k\ell} + \frac{1}{2} A_{(0)j\ell m} \hat{\mathcal{A}}_{(1)}^k d\tilde{A}_{(D-3)}^{\ell m} - \frac{1}{2} (-1)^D A_{(0)j\ell m} d\tilde{A}_{(D-2)}^{k\ell m} \\ &\quad - \hat{\mathcal{A}}_{(1)}^k d\tilde{A}_{(D-3)j} + Y_j \hat{\mathcal{A}}_{(1)}^k . \end{aligned} \quad (4.24)$$

We have found the non-trivial result that it is possible to strip off a derivative in the second-order equations (4.20), by writing:

$$e^{\vec{b}_{ij} \cdot \vec{\phi}} * \mathcal{F}_{(1)j}^i = \gamma^\ell_i \tilde{\gamma}^j_k \tilde{\mathcal{F}}_{(D-1)\ell}^k , \quad (4.25)$$

where

$$\tilde{\mathcal{F}}_{(D-1)\ell}^k = d\tilde{\mathcal{A}}_{(D-2)\ell}^k + (-1)^D W^k_\ell . \quad (4.26)$$

This gives us the first-order equation for the  $\mathcal{F}_{(1)j}^i$  fields.

This completes the derivation of first-order equations for all the field strengths (including non-dilatonic 1-form field strengths) in  $D$ -dimensional supergravity. It now remains to obtain the first-order equations for the dilatonic scalars  $\vec{\phi}$ . After varying (4.1) with respect to  $\vec{\phi}$ , and using the various first-order equations already obtained, we can write the second-order equations of motion as

$$\begin{aligned} -(-1)^D d*d\vec{\phi} &= \frac{1}{2} \vec{a} \tilde{F}_{(D-4)} F_{(4)} + \frac{1}{2} \sum_i \vec{a}_i \tilde{\gamma}^i_j \tilde{F}_{(D-3)}^j F_{(3)i} + \frac{1}{2} \sum_{i<j} \vec{a}_{ij} \tilde{\gamma}^i_k \tilde{\gamma}^j_\ell \tilde{F}_{(D-2)}^{k\ell} F_{(2)ij} \\ &\quad + \frac{1}{2} \sum_{i<j<k} \vec{a}_{ijk} \tilde{\gamma}^i_\ell \tilde{\gamma}^j_m \tilde{\gamma}^k_n \tilde{F}_{(D-1)}^{\ell mn} F_{(1)ijk} + \frac{1}{2} \sum_i \vec{b}_i \gamma^j_i \tilde{\mathcal{F}}_{(D-2)j} \mathcal{F}_2^i \\ &\quad + \frac{1}{2} \sum_{i<j} \vec{b}_{ij} \gamma^\ell_i \tilde{\gamma}^j_k \tilde{\mathcal{F}}_{(D-1)\ell}^k \mathcal{F}_{(1)j}^i . \end{aligned} \quad (4.27)$$

This is the trickiest equation to turn into first-order form.

We must consider the structure of the second-order equation (4.27), after replacing the field strengths by their expressions in terms of potentials. Firstly, we find that all terms of bilinear or higher order in  $\hat{\mathcal{A}}_{(1)}^i$  cancel out. Secondly, we note that the only occurrences of a differentiated  $\hat{\mathcal{A}}_{(1)}^i$  are from the  $\mathcal{F}_{(2)}^i$  field strength in the penultimate term in (4.27). This means that if we are to be able to strip off a derivative, the set of terms linear in an undifferentiated  $\hat{\mathcal{A}}_{(1)}^i$  must themselves assemble into the form  $Z \hat{\mathcal{A}}_{(1)}^i$ , where  $Z$  itself is the total derivative of the factor multiplying  $d\hat{\mathcal{A}}_{(1)}^i$ . It is easiest first to consider the case when we temporarily set the axions  $\mathcal{A}_{(0)j}^i$  to zero. It then becomes clear that in each dimension there must exist a vector of  $(D-1)$ -forms  $\vec{Q}$ , whose exterior derivatives satisfy

$$\vec{a} X dA_{(3)} - \sum_i \vec{a}_i X^i dA_{(2)i} + \frac{1}{2} \sum_{ij} \vec{a}_{ij} X^{ij} dA_{(1)ij} - \frac{1}{6} \sum_{ijk} \vec{a}_{ijk} X^{ijk} dA_{(0)ijk} = d\vec{Q}. \quad (4.28)$$

One can indeed find such quantities  $\vec{Q}$  in each dimension. (In doing this, and in proving the other previously-mentioned results for stripping off a derivative from (4.27), it is necessary to make extensive use of the various ‘‘sum rules’’ satisfied by the dilaton vectors. These are of the form  $\vec{b}_{ij} = \vec{b}_i - \vec{b}_j$ ,  $\vec{a}_i = \vec{a} - \vec{b}_i$ , *etc.* See [5, 6].) It may become less surprising if we make the same integration by parts as we did after (4.23); the corresponding sum with the  $\mathcal{A}_{(0)j}^i$  reinserted is now of the form  $\sum dX^B \vec{R}_B^B A_B$  and now vanishes by (Weyl) dimensional analysis again of the internal coordinates.

We now allow the axions  $\mathcal{A}_{(0)j}^i$  to be non-zero again. It is not hard to see that the expressions for  $\vec{Q}$  remain identical in structure, but now all indexed quantities are ‘‘dressed’’ with  $\gamma$  matrices (for downstairs indices) or  $\tilde{\gamma}$  matrices (for upstairs indices). The underlying reason for this ‘‘dressing’’ phenomenon is a silver rule of supergravity, namely that twisted self-duality holds for the ‘‘flattened’’ field strengths that transform under the subgroup  $K$ , whereas the natural potentials transform under the full group  $G$  (see, for instance, [10]). This will become obvious when we study the coset construction of the doubled field in the next subsection. It may be useful to recognise that the matrices  $\gamma$  and  $\tilde{\gamma}$  are actually moving frames intertwining between analogues of curved and flat indices, which unfortunately have not been distinguished here.

The expressions for  $\vec{Q}$  that we find in each dimension are given in appendix A. We can then strip off a derivative from (4.27), giving the first-order equation

$$\begin{aligned} *d\vec{\phi} &= -\frac{1}{2}(-1)^D \vec{a} A_{(3)} d\tilde{A}_{(D-5)} + \frac{1}{2} \sum_i \vec{a}_i \gamma^j{}_i \tilde{\gamma}^i{}_k A_{(2)j} d\tilde{A}_{(D-4)}^k \\ &\quad - \frac{1}{2}(-1)^D \sum_{i<j} \vec{a}_{ij} \gamma^\ell{}_i \gamma^m{}_j \tilde{\gamma}^i{}_p \tilde{\gamma}^j{}_q A_{(1)\ell m} d\tilde{A}_{(D-3)}^{pq} \end{aligned}$$



$$\begin{aligned}
& +\frac{1}{2} \sum_{i<j<k} \vec{a}_{ijk} \gamma^\ell_i \gamma^m_j \gamma^n_k \tilde{\gamma}^i_p \tilde{\gamma}^j_q \tilde{\gamma}^k_r A_{(0)\ell mn} d\tilde{A}_{(D-2)}^{pqr} \\
& -\frac{1}{2}(-1)^D \sum_i \vec{b}_i \tilde{\gamma}^i_j \gamma^k_i \hat{\mathcal{A}}_{(1)}^j \left( d\tilde{\mathcal{A}}_{(D-3)k} - A_{(2)k} d\tilde{\mathcal{A}}_{(D-5)} + (-1)^D A_{(1)\ell k} d\tilde{\mathcal{A}}_{(D-4)}^\ell \right. \\
& \qquad \qquad \qquad \left. -\frac{1}{2} A_{(0)k\ell m} d\tilde{\mathcal{A}}_{(D-3)}^{\ell m} - Y_k \right) \\
& +\frac{1}{2} \sum_{i<j} \vec{b}_{ij} \gamma^k_i \mathcal{A}_{(0)j}^i d\tilde{\mathcal{A}}_{(D-2)k}^j - \frac{1}{2} \vec{Q} .
\end{aligned} \tag{4.29}$$

Detailed calculation shows that the exterior derivative of this equation indeed gives (4.27).

## 4.2 “Coset” construction for $D$ -dimensional supergravity

Having obtained the first-order equations in the previous section, we can now write down the doubled field  $\mathcal{G}$ , and then look for a parameterisation for  $\mathcal{V}$ , such that  $\mathcal{G} = d\mathcal{V}\mathcal{V}^{-1}$ .

Thus as before, the doubled field is written as

$$\begin{aligned}
\mathcal{G} &= \frac{1}{2} d\vec{\phi} \cdot \vec{H} + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij} \cdot \vec{\phi}} \mathcal{F}_{(1)j}^i E_i^j + \sum_i e^{\frac{1}{2}\vec{b}_i \cdot \vec{\phi}} \mathcal{F}_{(2)}^i W_i + \sum_{i<j<k} e^{\frac{1}{2}\vec{a}_{ijk} \cdot \vec{\phi}} F_{(1)ijk} E^{ijk} \\
& + \sum_{i<j} e^{\frac{1}{2}\vec{a}_{ij} \cdot \vec{\phi}} F_{(2)ij} V^{ij} + \sum_i e^{\frac{1}{2}\vec{a}_i \cdot \vec{\phi}} F_{(3)i} V^i + e^{\frac{1}{2}\vec{a} \cdot \vec{\phi}} F_{(4)} V + e^{-\frac{1}{2}\vec{a} \cdot \vec{\phi}} \tilde{F}_{(D-4)} \tilde{V} \\
& + \sum_i e^{-\frac{1}{2}\vec{a}_i \cdot \vec{\phi}} \hat{F}_{(D-3)}^i \tilde{V}_i + \sum_{i<j} e^{-\frac{1}{2}\vec{a}_{ij} \cdot \vec{\phi}} \hat{F}_{(D-2)}^{ij} \tilde{V}_{ij} + \sum_{i<j<k} e^{-\frac{1}{2}\vec{a}_{ijk} \cdot \vec{\phi}} \hat{F}_{(D-1)}^{ijk} \tilde{E}_{ijk} \\
& + \sum_i e^{-\frac{1}{2}\vec{b}_i \cdot \vec{\phi}} \hat{\mathcal{F}}_{(D-2)i} \tilde{W}^i + \sum_{i<j} e^{-\frac{1}{2}\vec{b}_{ij} \cdot \vec{\phi}} \hat{\mathcal{F}}_{(D-1)i}^j \tilde{E}_j^i + \frac{1}{2} F_\psi \vec{H} ,
\end{aligned} \tag{4.30}$$

where the hatted dual field strengths are dressed (one could say flattened on all their internal indices) with  $\gamma$  and  $\tilde{\gamma}$  matrices in the systematic way:

$$\begin{aligned}
\hat{F}_{(D-3)}^i &= \tilde{\gamma}^i_j \tilde{F}_{(D-3)}^j , & \hat{F}_{(D-2)}^{ij} &= \tilde{\gamma}^i_k \tilde{\gamma}^j_\ell \tilde{F}_{(D-2)}^{k\ell} , & \hat{F}_{(D-1)}^{ijk} &= \tilde{\gamma}^i_\ell \tilde{\gamma}^j_m \tilde{\gamma}^k_n \tilde{F}_{(D-1)}^{\ell mn} , \\
\hat{\mathcal{F}}_{(D-2)i} &= \gamma^j_i \tilde{F}_{(D-2)j} , & \hat{\mathcal{F}}_{(D-1)i}^j &= \gamma^k_i \tilde{\gamma}^j_\ell \tilde{F}_{(D-1)k}^\ell ,
\end{aligned} \tag{4.31}$$

and  $F_{\vec{\psi}}$  is defined to be the right-hand side of the first-order dilaton equation (4.29).

We shall parameterise  $\mathcal{V}$  as

$$\begin{aligned}
\mathcal{V} &= e^{\frac{1}{2}\vec{\phi} \cdot \vec{H}} h e^{\hat{\mathcal{A}}_{(1)}^i} W_i e^{\frac{1}{6}A_{(0)ijk}} E^{ijk} e^{\frac{1}{2}A_{(1)ij}} V^{ij} e^{A_{(2)i}} V^i e^{A_{(3)}} V \times \\
& \times e^{\tilde{A}_{(D-5)}} \tilde{V} e^{\tilde{A}_{(D-4)}^i} \tilde{V}_i e^{\frac{1}{2}\tilde{A}_{(D-3)}^{ij}} \tilde{V}_{ij} e^{\frac{1}{6}A_{(D-2)}^{ijk}} \tilde{E}_{ijk} e^{\tilde{A}_{(D-3)i}} \tilde{W}^i e^{\tilde{A}_{(D-2)i}^j} \tilde{E}_j^i e^{\frac{1}{2}\vec{\psi} \cdot \vec{H}} ,
\end{aligned} \tag{4.32}$$

where  $h$  is defined as the product

$$h = \prod_{i<j} e^{A_{(0)j}^i} E_i^j , \tag{4.33}$$

with the terms arranged in anti-lexical order, namely

$$(i, j) = \cdots (3, 4), (2, 4), (1, 4), (2, 3), (1, 3), (1, 2) . \quad (4.34)$$

Note that only in the terms involving  $\mathcal{A}_{(0)j}^i$  is it necessary to separate the individual fields of an  $SL(11 - D, \mathbb{R})$  multiplet into separate exponential factors.

By comparing the terms in the doubled field (4.30) that are bilinear in fields with the bilinear terms coming from  $d\mathcal{V}\mathcal{V}^{-1}$ , we can read off all the commutation and anti-commutation relations for the generators. Note that a generator is *even* if it is associated in (4.32) with a potential of *even* degree, and it is *odd* if it is associated with a potential of *odd* degree. Two odd generators satisfy an anticommutation relation, while all other combinations satisfy commutation relations. We have seen in the previous subsection that the bosonic Lie derivatives become partly fermionic when we change the statistics of the internal coordinates by treating them separately from the remaining spacetime coordinates.

The (anti)-commutators divide into two sets. There are those that are independent of the dimension  $D$ ; we shall present these first. Then, there are additional (anti)-commutators that are specific to the dimension; these are all associated with terms in (4.30) coming from the Chern-Simons terms  $\mathcal{L}_{FFA}$ , and consequently they all involve the epsilon tensor. These are given dimension by dimension in appendix B. We find that the dimension-independent commutators are as follows. Firstly, we have the commutators of all generators with  $E_i^j$ , which characterise their  $SL(11 - D, \mathbb{R})$ -covariance properties:

$$\begin{aligned} [E_i^j, E_k^\ell] &= \delta_k^j E_i^\ell - \delta_i^\ell E_k^j , & [E_i^j, E^{k\ell m}] &= -3\delta_i^{[k} E^{\ell m]j} , \\ [E_i^j, V^k] &= -\delta_i^k V^j , & [E_i^j, V^{k\ell}] &= 2\delta_i^{[k} V^{\ell]j} , & [E_i^j, W_k] &= \delta_k^j W_i , \\ [E_i^j, \tilde{E}_{k\ell m}] &= 3\delta_{[k}^j \tilde{E}_{\ell m]i} , & [E_i^j, \tilde{W}^k] &= -\delta_i^k \tilde{W}^j , \\ [E_i^j, \tilde{V}_k] &= \delta_k^j \tilde{V}_i , & [E_i^j, \tilde{V}_{k\ell}] &= -2\delta_{[k}^j \tilde{V}_{\ell]i} , \\ [E_i^j, \tilde{E}^i_\ell] &= -\tilde{E}^j_\ell , \text{ no sum on } i, j \neq \ell , \\ [E_i^j, \tilde{E}^k_j] &= \tilde{E}^k_i , \text{ no sum on } j, i \neq k . \end{aligned} \quad (4.35)$$

Next, we have the commutators of all generators with  $\vec{H}$ , which are expressed in terms of the roots under the chosen Cartan subalgebra:

$$\begin{aligned} [\vec{H}, E_i^j] &= \vec{b}_{ij} E_i^j , & [\vec{H}, E^{ijk}] &= \vec{a}_{ijk} E^{ijk} , & [\vec{H}, V^{ij}] &= \vec{a}_{ij} V^{ij} , \\ [\vec{H}, V^i] &= \vec{a}_i V^i , & [\vec{H}, V] &= \vec{a} V , & [\vec{H}, W_i] &= \vec{b}_i W_i , \\ [\vec{H}, \tilde{E}^i_j] &= -\vec{b}_{ij} \tilde{E}^i_j , & [\vec{H}, \tilde{E}_{ijk}] &= -\vec{a}_{ijk} \tilde{E}_{ijk} , & [\vec{H}, \tilde{V}_{ij}] &= -\vec{a}_{ij} \tilde{V}_{ij} , \\ [\vec{H}, \tilde{V}_i] &= -\vec{a}_i \tilde{V}_i , & [\vec{H}, \tilde{V}] &= -\vec{a} \tilde{V} , & [\vec{H}, \tilde{W}^i] &= -\vec{b}_i \tilde{W}^i . \end{aligned} \quad (4.36)$$

The rest of the dimension-independent commutators are:

$$\begin{aligned}
[W_i, E^{jk\ell}] &= -3\delta_i^{[j} V^{k\ell]} , & \{W_i, V^{jk}\} &= -2\delta_i^{[j} V^{k]} , & [W_i, V^j] &= -\delta_i^j V , \\
[W_i, \tilde{V}] &= -\tilde{V}_i , & [W_i, \tilde{V}_j] &= \tilde{V}_{ij} , & [W_i, \tilde{V}_{jk}] &= -\tilde{E}_{ijk} , \\
[V^i, \tilde{V}] &= -\tilde{W}^i , & [V^{ij}, \tilde{V}_k] &= -2\delta_k^{[i} \tilde{W}^{j]} , & [E^{ijk}, \tilde{V}_{\ell m}] &= -6\delta_\ell^{[i} \delta_m^j \tilde{W}^{k]} , \\
[V^i, \tilde{V}_j] &= -\tilde{E}^i_j , & [V^{ij}, \tilde{V}_{k\ell}] &= 4\delta_{[k}^{[i} \tilde{E}^{j]}_{\ell]} , & & (4.37) \\
[E^{ijk}, \tilde{E}_{\ell mn}] &= -18\delta_{[\ell}^{[i} \delta_m^j \tilde{E}^{k]}_{n]} , & [W_i, \tilde{W}^j] &= -\tilde{E}^j_i , \\
[V, \tilde{V}] &= -\frac{1}{4}\vec{a} \cdot \vec{H} , & [V^i, \tilde{V}_i] &= \frac{1}{4}\vec{a}_i \cdot \vec{H} , & [V^{ij}, \tilde{V}_{ij}] &= -\frac{1}{4}\vec{a}_{ij} \cdot \vec{H} , \\
[E^{ijk}, \tilde{E}_{ijk}] &= \frac{1}{4}\vec{a}_{ijk} \cdot \vec{H} , & [W_i, \tilde{W}^i] &= -\frac{1}{4}\vec{b}_i \cdot \vec{H} , & [E_i^j, \tilde{E}^i_j] &= \frac{1}{4}\vec{b}_{ij} \cdot \vec{H} .
\end{aligned}$$

It is straightforward to verify, with the aid of a computer, that these commutation and anti-commutation relations indeed satisfy the Jacobi identities. We have also verified that the augmented set of commutation relations in each dimension  $D$ , where we include also the dimension-dependent ones given in appendix B, satisfy the Jacobi identities.<sup>3</sup>

### 4.3 A twelfth (fermionic) dimension

These algebras in  $D$  dimensions can be written in a considerably more elegant and transparent form. To do this, we first extend the range of the  $i, \dots$  indices to  $\alpha = (i, 0)$ , where 0 will, for convenience, be formally defined to be larger than any of the values taken by  $i$ . Thus we have an enlargement from  $n = (11 - D)$  bosonic dimensions to  $(n|1)$ , with the extra dimension turning out to be fermionic. We define the extended generators  $E_\alpha^\beta$ ,  $\tilde{E}^\alpha_\beta$ ,  $V^{\alpha\beta\gamma}$  and  $\tilde{V}_{\alpha\beta\gamma}$  as follows:

$$\begin{aligned}
E_i^j &= E_i^j , & E_i^0 &= W_i , \\
V^{ijk} &= E^{ijk} , & V^{ij0} &= V^{ij} , & V^{i00} &= V^i , & V^{000} &= V , \\
\tilde{E}^i_j &= \tilde{E}^i_j , & \tilde{E}^i_0 &= -\tilde{W}^i , \\
\tilde{V}_{ijk} &= \tilde{E}_{ijk} , & \tilde{V}^{ij0} &= -\tilde{V}_{ij} , & \tilde{V}_{i00} &= 2\tilde{V}_i , & \tilde{V}_{000} &= -6\tilde{V} .
\end{aligned} \tag{4.38}$$

The generators  $V^{\alpha\beta\gamma}$  and  $\tilde{V}_{\alpha\beta\gamma}$  are graded antisymmetric, *i.e.*  $V^{\alpha\beta\gamma} = V^{[\alpha\beta\gamma]}$  and  $\tilde{V}_{\alpha\beta\gamma} = \tilde{V}_{[\alpha\beta\gamma]}$ . In other words, the index 0 is symmetrised with itself but the other  $i$  indices are antisymmetrised.

In terms of these generators, the dimension-independent part (4.35), (4.36) and (4.37) of the algebras can be re-written as

$$[E_\alpha^\beta, E_\gamma^\delta] = \delta_\gamma^\beta E_\alpha^\delta - \delta_\alpha^\delta E_\gamma^\beta , \quad [\vec{H}, E_\alpha^\beta] = \vec{b}_{\alpha\beta} E_\alpha^\beta ,$$

<sup>3</sup>In the next subsection, after rewriting the (anti)-commutation relations in a more transparent form, we shall be able to present a more digestible proof of the Jacobi identities.

$$\begin{aligned}
[E_\alpha^\beta, V^{\gamma\delta\sigma}] &= -3 \delta_\alpha^{[\gamma} V^{\delta\sigma]\beta} , & [\vec{H}, V^{\alpha\beta\gamma}] &= \vec{a}_{\alpha\beta\gamma} V^{\alpha\beta\gamma} , \\
[E_\alpha^\beta, \tilde{E}^\gamma_\delta] &= -\delta_\alpha^\gamma \tilde{E}^\beta_\delta + \delta_\delta^\beta \tilde{E}^\gamma_\alpha + \frac{1}{4} \delta_\delta^\beta \delta_\alpha^\gamma \vec{b}_{\alpha\beta} \cdot \vec{H} , & [\vec{H}, \tilde{E}^\alpha_\beta] &= -\vec{b}_{\alpha\beta} \tilde{E}^\alpha_\beta , \\
[E_\alpha^\beta, \tilde{V}_{\gamma\delta\sigma}] &= 3 \delta_{[\gamma}^\beta \tilde{V}_{\delta\sigma]\alpha} , & [\vec{H}, \tilde{V}_{\alpha\beta\gamma}] &= -\vec{a}_{\alpha\beta\gamma} \tilde{V}_{\alpha\beta\gamma} , \\
[V^{\alpha\beta\gamma}, \tilde{V}_{\delta\sigma\lambda}] &= -18 \delta_{[\delta}^{[\alpha} \delta_\sigma^\beta \tilde{E}^{\gamma]\lambda]} + \frac{3}{2} \delta_{\delta\sigma\lambda}^{\alpha\beta\gamma} \vec{a}_{\alpha\beta\gamma} \cdot \vec{H} , & & (4.39)
\end{aligned}$$

while the dimension-dependent (anti)-commutators, given in appendix B, can all be expressed in the simple form

$$[V^{\alpha_1\alpha_2\alpha_3}, V^{\beta_1\beta_2\beta_3}] = \frac{1}{6} (-1)^{11-D} \epsilon^{\beta_1\beta_2\beta_3\alpha_1\alpha_2\alpha_3\gamma_1\gamma_2\gamma_3} \tilde{V}_{\gamma_1\gamma_2\gamma_3} . \quad (4.40)$$

Here, the 9-index  $\epsilon$  tensor is graded antisymmetric with  $\epsilon^{i_1\dots i_n 0\dots 0}$  simply equal to  $\epsilon^{i_1\dots i_n}$  when the number of 0's is  $D-2$  but vanishes otherwise. In (4.39) we have defined

$$\vec{b}_{i0} = \vec{b}_i , \quad \vec{a}_{ij0} = \vec{a}_{ij} , \quad \vec{a}_{i00} = \vec{a}_i , \quad \vec{a}_{000} = \vec{a} . \quad (4.41)$$

As we shall now explain, the algebra (4.39) contains the superalgebra  $SL_+(n|1)$  (the Borel subalgebra of  $SL(n|1)$ ), generated by  $\vec{H}$  and  $\tilde{E}_\alpha^\beta$  with  $\alpha < \beta$  (subject, as before, to the formal rule that the 0 value is regarded as being greater than any other value  $i$ ). It is clear that the Cartan generators  $\vec{H}$  should be expressible in terms of the diagonal generators  $E_\alpha^\beta$  with  $\alpha = \beta$ , and likewise we would expect that  $\vec{H}$  should be expressible in terms of the diagonal generators amongst the  $\tilde{E}^\alpha_\beta$ . The expressions for the Cartan generators of  $SL(n|1)$  in terms of the  $E_\alpha^\alpha$  are presented in appendix C, equation (C.9). In our case, we may therefore write

$$\vec{H} = \sum_{\alpha=0}^n (\vec{b}_\alpha + \vec{c}) E_\alpha^\alpha , \quad (4.42)$$

where  $\vec{c} = \frac{9}{n-1} \vec{s}$ , and by definition  $\vec{b}_0 \equiv 0$ . It is useful also to “invert” this expression, and give the diagonal generators  $E_\alpha^\beta$  in terms of  $\vec{H}$ . Of course there will not be a unique result, since there are only  $n$  Cartan generators  $\vec{H}$  while there are  $(n+1)$  diagonal generators. In fact the sum  $\sum_\alpha E_\alpha^\alpha$  commutes with all generators in  $SL(n|1)$ . Rather than simply setting it to the identity (or zero), however, it is useful instead to introduce one additional generator  $\mathcal{D}$ , a “dilatation,” which commutes with all the generators  $E_\alpha^\beta$  for all  $\alpha$  and  $\beta$ . In terms of this, it turns out that the solution for the diagonal generators is

$$\begin{aligned}
E_0^0 &= \frac{1}{2}(D-2) \vec{s} \cdot \vec{H} - \mathcal{D} , \\
E_i^i &= \frac{1}{2}(\vec{b}_i + \vec{s}) \cdot \vec{H} + \mathcal{D} .
\end{aligned} \quad (4.43)$$

Another useful expression is obtained by considering the simple-root Cartan generators  $K_\alpha$ , defined by

$$K_0 \equiv E_n^n + E_0^0 , \quad K_i \equiv E_i^i - E_{i+1}^{i+1} . \quad (4.44)$$

In terms of these, we find that

$$\vec{H} = \vec{c} K_0 + \sum_i \vec{\beta}_i K_i, \quad \text{where} \quad \vec{\beta}_i = \sum_{j=1}^i (\vec{b}_j + \vec{c}). \quad (4.45)$$

For the diagonal  $\tilde{E}^\alpha_\alpha$  generators, we find that they are related to the generators  $\vec{H}$  by

$$\tilde{E}^\alpha_\alpha = \frac{1}{4}(\vec{b}_\alpha + \vec{c}) \cdot \vec{H}. \quad (4.46)$$

With the introduction of the additional generator  $\mathcal{D}$ , the original  $SL_+(n|1)$  algebra generated by  $\vec{H}$  and  $E_\alpha^\beta$  with  $\alpha < \beta$  is in fact enlarged to  $GL_+(n|1)$ . By doing this, it turns out that the 3-tensors  $V^{\alpha\beta\gamma}$  and  $\tilde{V}_{\alpha\beta\gamma}$ , which were previously seen to be irreducible representations of the Borel subalgebra  $SL_+(n|1)$  of  $SL(n|1)$ , can be viewed also as irreducible (tensor density) representations of the full algebra  $GL(n|1)$ . To do this, we assign dilatation weights to  $V^{\alpha\beta\gamma}$  and  $\tilde{V}_{\alpha\beta\gamma}$  as follows:

$$[\mathcal{D}, V^{\alpha\beta\gamma}] = \frac{1}{6}(D-3) V^{\alpha\beta\gamma}, \quad [\mathcal{D}, \tilde{V}_{\alpha\beta\gamma}] = -\frac{1}{6}(D-3) \tilde{V}_{\alpha\beta\gamma}. \quad (4.47)$$

We then find that the algebra (4.39) is a subalgebra of the simpler superalgebra:

$$\begin{aligned} [E_\alpha^\beta, E_\gamma^\delta] &= \delta_\gamma^\beta E_\alpha^\delta - \delta_\alpha^\delta E_\gamma^\beta, \\ [E_\alpha^\beta, V^{\gamma\delta\sigma}] &= -3 \delta_\alpha^{[\gamma} V^{\delta\sigma]\beta} + \frac{1}{6}(D-1) \delta_\alpha^\beta V^{\gamma\delta\sigma}, \\ [E_\alpha^\beta, \tilde{E}^\gamma_\delta] &= -\delta_\alpha^\gamma \tilde{E}^\beta_\delta + \delta_\delta^\beta \tilde{E}^\gamma_\alpha, \\ [E_\alpha^\beta, \tilde{V}_{\gamma\delta\sigma}] &= 3 \delta_{[\gamma}^\beta \tilde{V}_{\delta\sigma]\alpha} - \frac{1}{6}(D-1) \delta_\alpha^\beta \tilde{V}_{\gamma\delta\sigma}, \\ [V^{\alpha\beta\gamma}, \tilde{V}_{\delta\sigma\lambda}] &= -18 \delta_{[\delta}^\alpha \delta_\sigma^\beta \tilde{E}^\gamma]_\lambda, \end{aligned} \quad (4.48)$$

where now the index ranges on  $E_\alpha^\beta$  and  $\tilde{E}^\alpha_\beta$  are simply restricted by the Borel subalgebra conditions  $\alpha \leq \beta$  (again with the understanding that 0 is larger than any  $i$ ).

The  $D$ -dimensional algebra (4.39) has the form

$$\begin{aligned} [T^A, T^B] &= f^{AB}{}_C T^C + g^{ABC} \tilde{T}_C, \\ [T^A, \tilde{T}_B] &= f^{CA}{}_B \tilde{T}_C, \\ [\tilde{T}_A, \tilde{T}_B] &= 0, \end{aligned} \quad (4.49)$$

where  $T^A$  represents the untilded generators  $\{E_\alpha^\beta, \vec{H}, V^{\alpha\beta\gamma}\}$ , while  $\tilde{T}_A$  represents the tilded (dual) generators  $\{\tilde{E}^\alpha_\beta, \vec{H}, \tilde{V}_{\alpha\beta\gamma}\}$ . This dual structure is the central point of our discussion and we are naturally led to consider a splitting (a quantum mechanics or symplectic geometer would say a polarisation) of the doubled set of potentials into two halves; on the one hand the fundamental fields to be used in a Lagrangian, and on the other their dual

potentials. This splitting is not unique, as our previous paper has testified [6], but the most obvious choice is the one implied by selecting precisely the untilded generators as the fundamental ones. Let us call  $\mathcal{P}$  the vector space of superalgebra generators of the fundamental fields. The dual potentials live in the dual space  $\mathcal{P}^*$ . Let us remark that the second line of eq. (4.49) becomes just the transformation law of the super-coadjoint representation [19] (in other words super-contragredient to the adjoint) after we set the  $g^{ABC}$  equal to zero.

It is worth remarking that the terms appearing on the right-hand side of  $[T^A, \tilde{T}_B]$  can be deduced very easily from the right-hand side of  $[T^A, T^B]$ , by using the following ‘‘Jade Rule.’’ This rule states that if we have untilded generators  $X, Y$  and  $Z$  where  $[X, Y] = Z$ , then it follows that we will necessarily also have  $[X, \tilde{Z}] = (-1)^{XY+1} \tilde{Y}$ . One can easily see from (4.49) that the jade rule is equivalent to the statement that  $f^{AB}{}_C$  is graded (anti)-symmetric in its upper indices. In our algebras, the origins of the jade rule can be traced back to the supergravity theories from which we first derived the (anti)-commutation relations: If  $A_X, A_Y$  and  $A_Z$  are the three potentials associated with the generators  $X, Y$  and  $Z$ , then a ‘‘Chern-Simons type’’ modification in a field strength, of the form  $F_Z = dA_Z + A_Y \wedge F_X$ , leads both to a ‘‘Bianchi identity contribution’’  $dF_Z \sim F_Y \wedge F_X$  and a ‘‘field equation contribution’’  $d\tilde{F}_Y \sim \tilde{F}_Z \wedge F_X$ , where  $\tilde{F}_Y$  and  $\tilde{F}_Z$  are the duals of the field strengths  $F_Y$  and  $F_Z$ .

It is easy to check the jade rule in examples. For instance, we see from (4.37) that  $[W_i, V^j] = -\delta_i^j V$ . By the jade rule, there should therefore also be an (anti)-commutator  $[W_i, \tilde{V}] = -\tilde{V}_i$ , and indeed we see it too in (4.37). (On has to be careful to make the sign reversals indicated in the bottom line of (4.38) before applying the jade rule, in order to get precise agreement.)

Before discussing further specific details of our algebra, it is of interest to consider the general class of algebras defined by (4.49). We now turn to this in the next subsection.

#### 4.4 Deformation of ‘‘cotangent algebras’’

It is easily seen that the Jacobi identities for (4.49) impose the following requirements on the structure constants:

$$\begin{aligned} f^{[AB}{}_D f^{D|C]}{}_E &= 0, \\ f^{[AB}{}_D g^{D|C]}{}^E + f^{E[C}{}_D g^{AB]}{}^D &= 0. \end{aligned} \tag{4.50}$$

The first is just the usual requirement for the  $T^A$  generators to form a superalgebra after contracting away the tilded generators.<sup>4</sup> There is no unique solution for the additional structure constants  $g^{ABC}$ . One solution would be to take  $g^{ABC}$  to be proportional to  $f^{ABC}$ , where the lower index of  $f^{AB}{}_C$  is raised using the Cartan-Killing metric  $h^{AB} \equiv f^{AC}{}_D f^{BD}{}_C$ ;  $f^{ABC} = h^{CD} f^{AB}{}_D$ . However, this is not the solution that arises in our case, as we shall see in detail below. In our algebras the  $g^{ABC}$  are the structure constants associated with the dimension-dependent commutation relations given by (4.40). These terms have their origin in the  $\mathcal{L}_{FFA}$  terms in the Lagrangian.

In order to understand these algebras, we found it useful first to contract them by setting  $g^{ABC} = 0$ . In this way the vector space  $\mathcal{P}$  acquires a Lie algebra structure  $G$ . This can be achieved by rescaling the  $\tilde{T}_A$  generators to zero, giving rise to

$$[T^A, T^B] = f^{AB}{}_C T^C, \quad [T^A, \tilde{T}_B] = f^{CA}{}_B \tilde{T}_C, \quad [\tilde{T}_A, \tilde{T}_B] = 0. \quad (4.51)$$

We shall keep the same notation for the (rescaled) generators  $\tilde{T}_A$  as there will be no ambiguity. Then the full (contracted) algebra becomes  $G \ltimes G^*$ , where  $G$  is generated by  $T^A$ , and in the semi-direct product  $G^*$ , generated by  $\tilde{T}_A$ , denotes the co-adjoint representation of  $G$ . We will see in the next section that it has the same general structure as the algebra (5.35) in the doubled formalism describing the sigma model Lagrangians.

Here  $G$  is generated by  $\vec{H}$ ,  $E_\alpha^\beta$ , and  $V^{\alpha\beta\gamma}$ . In particular  $\vec{H}$  and  $E_i^j$  form the Borel subalgebra of  $GL(n, \mathbb{R})$ , denoted by  $GL_+(n, \mathbb{R})$ , where  $n = 11 - D$ . The  $E_i^0 = W_i$  generators are odd, and are associated with internal diffeomorphisms. Thus the generators  $\vec{H}$ ,  $E_\alpha^\beta$ , which are all associated with Kaluza-Klein fields, generate the Borel subalgebra  $SL_+(n|1)$  of the superalgebra  $SL(n|1)$  (see appendix C for details and a more general and purely bosonic discussion). The generators  $V^{\alpha\beta\gamma} = \{V, V^i, V^{ij}, E^{ijk}\}$ , associated with the fields coming from the 3-form potential in  $D = 11$ , form a linear graded antisymmetric

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<sup>4</sup>Note that the  $\tilde{T}_A$  generators carry the same indices as the  $T^A$  generators. In odd dimensions,  $\tilde{T}_A$  and  $T^A$  have opposite statistics, and so it becomes important in general to distinguish the statistics of the indices from that of the generators. Any graded representation leads to a BF exchanged representation upon shifting the gradation by one, and the corresponding semi-direct products are different. The symmetry property of the structure constants changes accordingly. Our convention will be that an index  $A$  *always* has the statistics corresponding to  $T^A$ , regardless of whether in a particular commutator it is actually associated with  $T^A$  or  $\tilde{T}_A$ . In other words, the statistics factor  $(-1)^{T^A}$  is associated with the index  $A$ , where we use the standard convention that an odd generator  $X$  has  $(-1)^X = -1$ , while if it is even it has  $(-1)^X = +1$ . An example is the first Jacobi identity in (4.50), which requires that the symmetry of the indices appearing on  $f^{CA}{}_B$  in the second line in (4.49) is the same as in the first line in (4.50), where none of the indices on  $f^{AB}{}_C$  is linked to a tilded generator.

3-tensor representation of  $SL(n|1)$ . Thus the algebra  $G$  for  $D$ -dimensional supergravity can be denoted by

$$G = SL_+(n|1) \ltimes (\wedge v)^3 . \quad (4.52)$$

with  $v$  the appropriate fundamental representation. Note that  $G$  is associated with the complete set of “gauge” symmetries of the  $D$ -dimensional supergravity coming from the dimensional reduction from  $D = 11$  (comprising both local gauge symmetries for vector and tensor potentials, and global symmetries for scalars and axionic potentials). The contraction occurs by considering only the action on the fundamental potentials obtained without any dualisation.

Having understood the contracted algebra where the  $g^{ABC}$  coefficients are set to zero, we may now study the deformation of this algebra where the  $g^{ABC}$  are non-vanishing. It is useful first to note that the analysis of the algebra (4.49) can be further refined. We may divide the generators into two subsets, labelled generically by  $T^a$  to denote the subset  $\{\vec{H}, E_\alpha^\beta\}$  which are the  $SL_+(n|1)$  generators, and by  $T^{\bar{a}}$  to denote the 3-tensor representation generators  $V^{\alpha\beta\gamma}$ . The dual generators  $\tilde{T}_A$  are correspondingly split as  $\tilde{T}_a$  representing  $\vec{H}$  and  $\tilde{E}_\alpha^\beta$ , and  $\tilde{T}_{\bar{a}}$  representing  $\tilde{V}_{\alpha\beta\gamma}$ . In the algebras that we encounter in the dimensionally-reduced supergravities, certain of the sets of structure constants in the refined form of (4.49) vanish, and we have

$$\begin{aligned} [T^a, T^b] &= f^{ab}{}_c T^c , & [T^a, T^{\bar{b}}] &= f^{a\bar{b}}{}_{\bar{c}} T^{\bar{c}} , & [T^{\bar{a}}, T^{\bar{b}}] &= g^{\bar{a}\bar{b}\bar{c}} \tilde{T}_{\bar{c}} , \\ [T^a, \tilde{T}_b] &= f^{ca}{}_b \tilde{T}_c , & [T^a, \tilde{T}_{\bar{b}}] &= f^{\bar{c}a}{}_{\bar{b}} \tilde{T}_{\bar{c}} , & [T^{\bar{a}}, \tilde{T}_{\bar{b}}] &= f^{c\bar{a}}{}_{\bar{b}} \tilde{T}_c . \end{aligned} \quad (4.53)$$

In particular, we see that the deformation  $g^{ABC}$  arises in the sector  $g^{\bar{a}\bar{b}\bar{c}}$ , describing (anti)-commutators involving only the tensor-representation generators  $V^{\alpha\beta\gamma}$ .

Before discussing the specific algebras of the doubled formalism in more detail, it is again worthwhile to give the Jacobi identities for general algebras of the form (4.53). Aside from the obvious Jacobi requirements for the structure constants  $f^{ab}{}_c$  and  $f^{a\bar{b}}{}_{\bar{c}}$  themselves, we find that the conditions on the deformations  $g^{\bar{a}\bar{b}\bar{c}}$  can be written as

$$\begin{aligned} f^{e[\bar{a}}{}_d g^{\bar{b}\bar{c}]\bar{d}} &= 0 , \\ (-1)^{\bar{c}\bar{d}} f^{a\bar{b}}{}_{\bar{d}} [g^{\bar{c}\bar{d}\bar{e}} + (-1)^{\bar{e}(\bar{c}+\bar{d})+\bar{c}\bar{d}} g^{\bar{c}\bar{e}\bar{d}}] - (-1)^{a\bar{b}} f^{a\bar{c}}{}_{\bar{d}} [g^{\bar{b}\bar{d}\bar{e}} + (-1)^{\bar{e}(\bar{b}+\bar{d})+\bar{b}\bar{d}} g^{\bar{b}\bar{e}\bar{d}}] &= 0 . \end{aligned} \quad (4.54)$$

The first condition is, modulo the second condition, the statement that the deformation constants  $g^{\bar{a}\bar{b}\bar{c}}$  form an invariant tensor of  $G$ . A simple way of satisfying the second condition is by requiring that each of the two terms vanishes separately, which is achieved by the single requirement

$$g^{\bar{a}\bar{b}\bar{c}} = (-1)^{\bar{a}\bar{b}+\bar{b}\bar{c}+\bar{a}\bar{c}+1} g^{\bar{a}\bar{c}\bar{b}} . \quad (4.55)$$



The symmetry in the first two indices is of course dictated by (4.53), *i.e.*  $g^{\bar{a}\bar{b}\bar{c}} = (-1)^{\bar{a}\bar{b}+1}g^{\bar{b}\bar{a}\bar{c}}$ . It should be emphasised again that the statistics of the indices are determined by the untilded generators  $T^{\bar{a}}$ , and not the tilded generators  $\tilde{T}_{\bar{a}}$ . Using this graded antisymmetry it is now an easy exercise to rewrite (4.55) as graded antisymmetry of the  $g^{\bar{a}\bar{b}\bar{c}}$  in  $\bar{a}$  and  $\bar{c}$ . Returning now to our concrete situation we can check explicitly all the Jacobi identities.

The structure constants  $f^{\bar{a}\bar{b}}_{\bar{c}}$  and  $f^{\bar{a}\bar{b}}_{\bar{c}}$  appear in the commutation relations given in (4.39), and, as we discussed above, they are the structure constants of the superalgebra  $G \times G^*$ , where  $G$  is given in (4.52). The deformation coefficients  $g^{\bar{a}\bar{b}\bar{c}}$  can be read off from the dimension-dependent commutation relations given in (4.40); it can now be easily verified that they satisfy the requirement (4.55), and satisfy the Jacobi conditions (4.54).

Having obtained all the commutators by comparing the bilinear terms in the doubled field (4.30) with the bilinear terms from  $d\mathcal{V}\mathcal{V}^{-1}$ , it is now a matter of detailed computation to verify that the complete calculation of  $d\mathcal{V}\mathcal{V}^{-1}$  gives rise to the complete expression (4.30) for the doubled field. In particular, this depends crucially on the ordering of the various factors in the expression (4.32) for  $\mathcal{V}$ . This ordering is dictated by the strategy that we followed when stripping off the derivatives from the second-order equations to obtain first-order equations. Namely, whenever a derivative was to be extracted from an expression such as  $dB \wedge dB'$ , where  $B$  and  $B'$  represent two potentials in the theory, we always extracted the derivative so that the potential *further to the left* in (4.32) lost its derivative. Thus if  $B$  appears further to the left than  $B'$ , we extract the derivative by writing  $dB \wedge dB'$  as  $d(B \wedge dB')$ : One can see by considering the detailed calculation of  $d\mathcal{V}\mathcal{V}^{-1}$  that when the  $d$  lands on a particular factor in (4.32), the factors sitting further to the right cancel out, while the factors sitting further to the left provide a “dressing” of undifferentiated potentials, in a pattern governed by the details of the commutation relations.

Most of the “dressings” alluded to above come from the action of the coset factor  $h$  for the axions  $\mathcal{A}_{(0)j}^i$ , given in (4.33). This has the effect of dressing all upstairs indices with factors of  $\tilde{\gamma}^i_j$ , and all downstairs indices with factors of  $\gamma^i_j$ . This follows from the fact that the generators  $T_{i_1 \dots i_p}^{j_1 \dots j_q}$  associated with a field  $B^{i_1 \dots i_p}_{j_1 \dots j_q}$  satisfy the  $SL(11-D, \mathbb{R})$  commutation relations

$$\begin{aligned} [E_k^\ell, T_{i_1 \dots i_p}^{j_1 \dots j_q}] &= \delta_{i_1}^\ell T_{ki_2 \dots i_p}^{j_1 \dots j_q} + \delta_{i_2}^\ell T_{i_1 ki_3 \dots i_p}^{j_1 \dots j_q} + \dots \\ &\quad - \delta_k^{j_1} T_{i_1 \dots i_p}^{\ell j_2 \dots j_q} - \delta_k^{j_2} T_{i_1 \dots i_p}^{j_1 \ell j_3 \dots j_q} - \dots \end{aligned} \quad (4.56)$$

Then  $h$  acts on  $T_{i_1 \dots i_p}^{j_1 \dots j_q}$  valued fields as follows:

$$h (B^{i_1 \dots i_p}_{j_1 \dots j_q} T_{i_1 \dots i_p}^{j_1 \dots j_q}) h^{-1} = \gamma^{k_1}_{j_1} \dots \gamma^{k_q}_{j_q} \tilde{\gamma}^{i_1}_{\ell_1} \dots \tilde{\gamma}^{i_p}_{\ell_p} B^{\ell_1 \dots \ell_q}_{k_1 \dots k_p} T_{i_1 \dots i_p}^{j_1 \dots j_q} . \quad (4.57)$$

Consequently, all fields associated with terms in  $\mathcal{V}$  that get sandwiched between  $h$  and  $h^{-1}$  will acquire a “dressing” of  $\gamma$  and  $\tilde{\gamma}$  factors on their indices.

Another dressing that frequently occurs involves the Kaluza-Klein vectors  $\hat{\mathcal{A}}_{(1)}^i$ . For example, the sequence of commutators in the top line in (4.37) is responsible for generating the various higher-order terms in the hatted field strengths (4.3).

These observations, together with the fact that the right-hand sides of all commutation relations arise with “unit strength,” and that all higher-order terms in the doubled field (4.30) occur with the “expected” combinatoric factors, enable us to see that the all-orders computation of  $d\mathcal{V}\mathcal{V}^{-1}$  should indeed give (4.30).

## 5 Doubled Formalism for Scalar Cosets

In the previous sections, we studied the doubled-formalism for eleven-dimensional supergravity and lower dimensional maximal supergravities coming from dimensional reductions of eleven-dimensional supergravity. We showed that the complete set of gauge symmetries of the doubled formalism (including the constant shifts of dilatonic and axionic scalars) form a closed algebra that is a deformation of  $G \times G^*$ , where  $G$  is the superalgebra given in (4.52). The full set of gauge symmetries leaves the complete set of generalised field strengths  $\mathcal{G}$  invariant.

The full symmetry of the doubled equations of motion is larger than the symmetry described above, since there can also be transformations that change  $\mathcal{G}$  whilst nevertheless leaving the equation  $*\mathcal{G} = \mathcal{S}\mathcal{G}$  invariant. In the case of the doubled system of equations for maximal supergravity in  $D = 11 - n$  dimensions, the global part of the  $\mathcal{G}$ -preserving symmetry is in fact  $E_n^+$ , the Borel subalgebra of the familiar maximally-noncompact  $E_n$  symmetry. We expect that the doubled formalism, however, is in fact invariant under the full global  $E_n$  algebra, with the anti-Borel generators describing transformations under which  $\mathcal{G}$  varies but  $*\mathcal{G}$  is still equal to  $\mathcal{S}\mathcal{G}$ .

Owing to the fact that the global symmetries are realised non-linearly on the scalar manifold, the generalisation of the usual scalar coset to the “doubled” formalism is more complicated than the analogous generalisation of the discussion for higher-degree forms. In this section, therefore, we shall present a discussion of the global symmetries of the doubled system in the comparatively-simple example of an  $SL(2, \mathbb{R})$ -symmetric scalar theory. We shall then generalise the results to any symmetric space (in particular principal) sigma model. This construction encompasses the classical work of [22] and [23].

## 5.1 Doubled equations for $SL(2, R)$ coset

Let us consider the  $SL(2, \mathbb{R})$ -invariant scalar Lagrangian

$$\mathcal{L}_{10} = -\frac{1}{2}e (\partial\phi)^2 - \frac{1}{2}e e^{2\phi} (\partial\chi)^2 \quad (5.1)$$

in  $D$  spacetime dimensions. This can be written as  $\mathcal{L} = \frac{1}{4}e \text{tr}(\partial\mathcal{M}^{-1}\partial\mathcal{M})$ , where  $\mathcal{M} = \mathcal{V}_0^T \mathcal{V}_0$  and

$$\mathcal{V}_0 = \begin{pmatrix} e^{\frac{1}{2}\phi} & \chi e^{\frac{1}{2}\phi} \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix}. \quad (5.2)$$

In the language of differential forms, we have

$$\mathcal{L}_{10} = -\frac{1}{2}*d\phi \wedge d\phi - \frac{1}{2}e^{2\phi} *d\chi \wedge d\chi. \quad (5.3)$$

The resulting equations of motion are:

$$\begin{aligned} d*d\phi &= e^{2\phi} d\chi \wedge *d\chi, \\ d(e^{2\phi} *d\chi) &= 0. \end{aligned} \quad (5.4)$$

This enables us to write down two first order equations

$$\begin{aligned} *d\phi &= d\psi + \chi d\tilde{\chi}, \\ e^\phi *d\chi &= e^{-\phi} d\tilde{\chi}, \end{aligned} \quad (5.5)$$

where we have introduced dual  $(D-2)$ -forms  $\psi$  and  $\tilde{\chi}$  for the dilaton  $\phi$  and the axion  $\chi$  respectively. Taking the exterior derivatives of these first order equations, we obtain the second-order equations of motion (5.4).

It is well known that the Lagrangian (5.1) is invariant under the  $SL(2, \mathbb{R})$  global symmetry  $\tau \longrightarrow \tau' = (a\tau + b)/(c\tau + d)$ , where  $\tau = \chi + ie^{-\phi}$ . We shall now check that this symmetry can be extended to act on the dual forms and to preserve the first order equations (5.5). It is straightforward to see that (5.5) is invariant under the Borel subgroup of the  $SL(2, \mathbb{R})$ , corresponding to the matrices

$$\begin{pmatrix} 1/a & b \\ 0 & a \end{pmatrix} \quad (5.6)$$

acting from the right on the matrix  $\mathcal{V}_0$  given in (5.2). In fact the Borel subgroup is generated by the constant shift symmetries of the dilaton  $\phi$  and the axion  $\chi$

$$\begin{aligned} \phi \longrightarrow \phi' &= \phi - 2 \log a, & \chi \longrightarrow \chi' &= a^2 \chi + ab, \\ \tilde{\chi} \longrightarrow \tilde{\chi}' &= a^{-2} \tilde{\chi}, & \psi \longrightarrow \psi' &= \psi - b a^{-1} \tilde{\chi}. \end{aligned} \quad (5.7)$$

The higher-degree potentials  $\tilde{\chi}$  and  $\psi$  have, in addition, the (diagonal) local gauge symmetries

$$\tilde{\chi} \longrightarrow \tilde{\chi}' = \tilde{\chi} + \Lambda_{\tilde{\chi}} , \quad \psi \longrightarrow \psi' = \psi + \Lambda_{\psi} , \quad (5.8)$$

where  $d\Lambda_{\tilde{\chi}} = 0 = d\Lambda_{\psi}$ . In fact the Borel symmetries and the local gauge symmetries of  $\tilde{\chi}$  and  $\psi$  form a closed algebra. As in the appendix of [6], we can introduce generators  $H$  and  $E_{\pm}$  for  $SL(2, \mathbb{R})$ . Then the Cartan generator  $H$  and the positive-root generator  $E_+$  are associated with the dilaton and axion respectively. We also introduce the new generators  $\tilde{H}$  and  $\tilde{E}_+$ , associated with  $\psi$  and  $\tilde{\chi}$  respectively. The commutators of the transformations (5.7) and (5.8) implies that these generators satisfy the algebra

$$[H, E_+] = 2E_+ , \quad [H, \tilde{E}_+] = -2\tilde{E}_+ , \quad [E_+, \tilde{E}_+] = \frac{1}{2}\tilde{H} . \quad (5.9)$$

Note that the first commutator defines the Borel subalgebra of the  $SL(2, \mathbb{R})$  symmetry, and that  $\tilde{H}$  commutes with everything. Then we find that if we define

$$\mathcal{V} = e^{\frac{1}{2}\phi H} e^{\chi E_+} e^{\tilde{\chi} \tilde{E}_+} e^{\frac{1}{2}\psi \tilde{H}} , \quad (5.10)$$

where  $\tilde{\chi}$  is the  $(D-2)$ -form dual to the axion  $\chi$ , and  $\psi$  is the  $(D-2)$ -form dual to the dilaton  $\phi$ , then  $\mathcal{G} = d\mathcal{V}\mathcal{V}^{-1}$  is given by

$$\mathcal{G} = \frac{1}{2}d\phi H + e^{\phi} d\chi E_+ + e^{-\phi} d\tilde{\chi} \tilde{E}_+ + \frac{1}{2}(d\psi + \chi d\tilde{\chi}) \tilde{H} . \quad (5.11)$$

Thus the doubled equation  $*\mathcal{G} = \Omega\mathcal{G}$  gives precisely the two first-order equations (5.5). The Borel and gauge transformation rules (5.7) and (5.8) can be re-expressed as

$$\mathcal{V}' = \mathcal{V} e^{\log a H} e^{b E_+} e^{\Lambda_{\tilde{\chi}} \tilde{E}_+} e^{\frac{1}{2}\Lambda_{\psi} \tilde{H}} , \quad (5.12)$$

which leaves  $\mathcal{G}$  invariant. In particular each quantity coupled to each generator in (5.11) is independently invariant under the transformations in (5.12).

Thus we see that as in the case of eleven-dimensional supergravity, the first order equations (5.5) of the  $SL(2, \mathbb{R})$  coset can be re-expressed as the doubled equation  $*\mathcal{G} = \mathcal{S}\mathcal{G}$ , and that  $\mathcal{G}$  itself is invariant under the Borel subgroup of the global  $SL(2, \mathbb{R})$  scalar symmetry, and under the gauge symmetries of the  $(D-2)$ -form dual fields. Here, the involution  $\mathcal{S}$  acts on the generators by  $\mathcal{S}H = \tilde{H}$ ,  $\mathcal{S}E_+ = \tilde{E}_+$ . Obviously, therefore, the Borel transformations are also an invariance of the doubled equation of motion  $*\mathcal{G} = \mathcal{S}\mathcal{G}$ .

This equation is in fact invariant under a larger global symmetry group, namely the entire  $SL(2, \mathbb{R})$  global symmetry. To show this, we note that any  $SL(2, \mathbb{R})$  matrix can be decomposed as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = B_L \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B_R \quad (5.13)$$

where  $B_L$  and  $B_R$  are Borel matrices, given by

$$B_L = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \quad B_R = \begin{pmatrix} -c & -d \\ 0 & -1/c \end{pmatrix}. \quad (5.14)$$

Since  $\mathcal{G}$  itself is invariant under the Borel transformations, it remains only to verify that the equations are invariant under transformations generated by the inversion group element

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (5.15)$$

corresponding to  $\tau' = -1/\tau$ . Defining  $P = d\psi$  and  $Q = d\tilde{\chi}$ , it is straightforward to verify that the equations (5.5) are invariant under this transformation, provided that  $P$  and  $Q$  transform as

$$P' = -P, \quad Q' = -(\chi^2 + e^{-2\phi})Q - 2\chi P. \quad (5.16)$$

(Note that the Bianchi identities  $dP' = 0$  and  $dQ' = 0$  are indeed satisfied, modulo the equations (5.5).) Thus we have verified that the doubled equation  $*\mathcal{G} = \mathcal{S}\mathcal{G}$  is also invariant under the entire global  $SL(2, \mathbb{R})$ , although  $\mathcal{G}$  itself transforms non-trivially under the inversion generated by (5.15). The entire  $SL(2, \mathbb{R})$  symmetry is realised on the scalars  $\chi$  and  $\phi$ , but, locally, only on the derivatives  $P$  and  $Q$  of the potentials  $\psi$  and  $\tilde{\chi}$ . Note, incidentally, that the natural ‘‘field strengths’’ for the potentials  $\psi$  and  $\tilde{\chi}$  are

$$\bar{P} = P + \chi Q = d\psi + \chi d\tilde{\chi}, \quad \bar{Q} = e^{-\phi} d\tilde{\chi}. \quad (5.17)$$

In particular, it is these quantities that appear in the coefficients of the generators  $\tilde{H}$  and  $\tilde{E}_+$  in (5.11), and hence they are invariant under the Borel subgroup of transformations (5.7). Under the inversion  $\tau' = -1/\tau$ , these field strengths transform as a  $U(1)$  doublet

$$\begin{pmatrix} \bar{P}' \\ \bar{Q}' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \bar{P} \\ \bar{Q} \end{pmatrix} \quad (5.18)$$

with field-dependent parameter  $\sin \theta = 2\chi e^\phi / (1 + \chi^2 e^{2\phi})$ .

As in the case of  $D = 11$  supergravity that we discussed in section 2, we may present an alternative derivation of the algebra (5.9) which does not require the introduction of the potentials. Let us first introduce a field strength

$$\mathcal{G}_0 = d\mathcal{V}_0 \mathcal{V}_0^{-1} = \frac{1}{2} d\phi H + e^\phi d\chi E_+, \quad (5.19)$$

where  $\mathcal{V}_0$  is given by (5.2), and then define  $\mathcal{G} = \mathcal{G}_0 + \mathcal{S}*\mathcal{G}_0$ . Thus we have

$$\mathcal{G} = \frac{1}{2} d\phi H + e^\phi d\chi E_+ + e^\phi *d\chi \tilde{E}_+ + \frac{1}{2} *d\phi \tilde{H}. \quad (5.20)$$

The doubled equation  $*\mathcal{G} = \mathcal{S}\mathcal{G}$  is now trivially satisfied, and instead the second-order equations of motion follow from the Cartan-Maurer equation

$$d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0 , \quad (5.21)$$

provided that we take the generators to have the non-vanishing commutation relations given in (5.9).

## 5.2 Noether currents of the global symmetry

As we have mentioned earlier, the dual fields in the doubled formalism are introduced to equate the duals of the (generalised) Noether currents of the gauge symmetries. Thus the transformation of the dual fields of the doubled formalism under the full global symmetry can be derived from the transformation rules of the Noether currents. To see this explicitly, note that the first-order equations (5.5) can also be expressed as

$$\begin{aligned} d\psi &= *(d\phi - e^{2\phi}\chi d\chi) \equiv *J_0 , \\ d\tilde{\chi} &= e^{2\phi} * d\chi \equiv *J_+ , \end{aligned} \quad (5.22)$$

where  $J_0$  and  $J_+$  are precisely the conserved Noether currents, associated with the constant shift symmetries of the scalars, *i.e.* the Borel symmetries. Specifically,  $J_0$  is the Noether current associated with the Cartan generator, and  $J_+$  is the Noether current associated with the positive-root generator of  $SL(2, \mathbb{R})$ . In the case of  $SL(2, \mathbb{R})$ , there is a third Noether current  $J_-$ , associated with the negative-root generator of  $SL(2, \mathbb{R})$ , given by

$$\begin{aligned} J_- &= d\chi + 2\chi d\phi - e^{2\phi}\chi^2 d\chi \\ &= 2\chi J_0 + (\chi^2 + e^{-2\phi}) J_+ . \end{aligned} \quad (5.23)$$

The fact that  $J_-$  is a linear combination of the two Borel currents  $J_0$  and  $J_+$  is not surprising, since we have only two independent fields,  $\phi$  and  $\chi$ . This dependence is a general feature of sigma models resulting from the local gauge invariance without propagating gauge fields that can be restored there [1], and results from the vanishing of the would-be Noether currents of the gauge symmetry [16]. The complete set of Noether currents ( $J_0, J_+, J_-$ ) form the adjoint representation of  $SL(2, \mathbb{R})$ .<sup>5</sup> They transform as

$$X \longrightarrow X' = \Lambda X \Lambda^{-1} , \quad (5.24)$$

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<sup>5</sup>Here we are concerned with the Noether current  $\mathcal{J}$  with  $d*\mathcal{J} = 0$  giving rise to equations of motion. The dual currents associated with Bianchi identity  $d^2\phi = 0 = d^2\chi$  are given by  $d\chi$  and  $d\phi$ . Unlike the Noether currents, which form the adjoint representation, acting on the dual currents with  $SL(2, \mathbb{R})$  generates an infinite number of currents, forming an infinite-dimensional representation of  $SL(2, \mathbb{R})$  [20].

where

$$X = \begin{pmatrix} J_0 & -J_- \\ -J_+ & -J_0 \end{pmatrix} \quad (5.25)$$

and  $\Lambda$  is a constant  $SL(2, \mathbb{R})$  matrix. Since  $J_-$  is a linear combination of  $J_0$  and  $J_+$  with field-dependent coefficients, it follows that the linear  $SL(2, \mathbb{R})$  transformation of the full set of three Noether currents can be re-expressed as a non-linear transformation of the Borel currents  $J_0$  and  $J_+$ , namely

$$\begin{pmatrix} J_0 \\ J_+ \end{pmatrix} \longrightarrow \begin{pmatrix} J_0 \\ J_+ \end{pmatrix}' = \Lambda_{\phi, \chi} \begin{pmatrix} J_0 \\ J_+ \end{pmatrix}, \quad (5.26)$$

where  $\Lambda_{\phi, \chi}$  is some specific field-dependent  $2 \times 2$  matrix. Thus, the first-order equations (5.22) are invariant under  $SL(2, \mathbb{R})$ , provided that  $d\psi$  and  $d\tilde{\chi}$  transform in the same way as  $J_0$  and  $J_+$  under  $\Lambda_{\phi, \chi}$ . Thus we may assign this transformation rule to  $d\psi$  and  $d\tilde{\chi}$ . However, we must check that this is consistent with the Bianchi identities for  $d\psi$  and  $d\tilde{\chi}$ . This is in fact clearly the case, since, as we have already seen, the transformed  $J'_0$ ,  $J'_+$  and  $J'_-$  currents can be expressed as linear combinations of the three original Noether currents ( $J_0, J_+, J_-$ ), with constant coefficients. Thus it is manifest, since the Noether currents are conserved, that the transformed  $J'_0$  and  $J'_+$  currents are also conserved, even if we choose to express them in terms of the field-dependent combinations of the original  $J_0$  and  $J_+$  currents. Since the calculations for checking the Bianchi identities for the transformed  $d\psi$  and  $d\tilde{\chi}$  fields will be identical, the conclusion will also be identical, namely that the  $SL(2, \mathbb{R})$  transformations preserve the Bianchi identities.

### 5.3 Generalisation to general cosets and principal $\sigma$ -models

The above discussion can be easily generalised to an arbitrary coset  $K \backslash G$  with maximally non-compact (ie split) group  $G$  and  $K$  its maximal compact subgroup. The coset can be parameterised by the Borel subgroup elements, as in the  $SL(2, \mathbb{R})$  example. The dilatons  $\phi_i$  couple to the Cartan generators  $H_i$ , and the axions  $\chi_m$  couple to the positive-root generators  $E_+^m$ , so that the coset representative can be written as

$$\mathcal{V} = \exp\left(\frac{1}{2}\phi_i H_i\right) \exp(\chi_m E_+^m). \quad (5.27)$$

The Lagrangian is of the form (D.5), and is invariant under the full global symmetry  $G$ . This can be seen from the Iwasawa decomposition, which asserts that any group element  $\mathbf{g}$  in  $G$  can be written in the form  $\mathbf{k} \times \mathbf{g}_B$ , where  $\mathbf{k}$  is an element of the maximal compact subgroup and  $\mathbf{g}_B$  is an element of the Borel subgroup. Thus for any group element  $\mathbf{g}$ , there

is a (field-dependent) compensating transformation  $\mathbf{k}$  such that

$$\mathcal{V} \longrightarrow \mathcal{V}' = \mathbf{k} \mathcal{V} \mathbf{g} \quad (5.28)$$

is back in the Borel gauge. The scalar Lagrangian (D.5) can also be written as

$$\mathcal{L} = \frac{1}{4} \text{tr} (\partial \mathcal{M}^{-1} \partial \mathcal{M}) , \quad (5.29)$$

using [6, 10] the Cartan involution and  $K$ -invariant metric  $\eta$  with  $\mathcal{M} = \mathcal{V}^\# \eta \mathcal{V}$ , and so it is evident that it is invariant under the transformation (5.28) for any element  $\mathbf{g}$  in  $G$ .

We can calculate the Noether currents for the global symmetries  $G$ , by the standard procedure of replacing the global parameters by spacetime-dependent ones, and collecting the terms in the variation of the Lagrangian where derivatives fall on the parameters. Infinitesimally, we have  $\delta \mathcal{M} = \epsilon^\# \mathcal{M} + \mathcal{M} \epsilon$ , where  $\mathbf{g} = \mathbf{1} + \epsilon$ , and hence we find  $\delta \mathcal{L} = -\text{tr} (\partial \epsilon \mathcal{M}^{-1} \partial \mathcal{M})$ , implying that the Noether currents  $\mathcal{J}$  are given by

$$\mathcal{J} = \mathcal{M}^{-1} d\mathcal{M} . \quad (5.30)$$

Under the global  $G$  transformations, they therefore transform linearly:

$$\mathcal{J} \longrightarrow \mathcal{J}' = \mathbf{g}^{-1} \mathcal{J} \mathbf{g} . \quad (5.31)$$

The Noether currents are not all linearly independent, since the number of currents exceeds the number of scalar fields. In fact they satisfy the relations

$$\text{tr} (\mathcal{J} \mathcal{V}^{-1} h_i \mathcal{V}) = 0 , \quad (5.32)$$

where  $h_i$  denotes the generators of the maximal compact subgroup of the global symmetry group. This can be seen by substituting (5.30) into (5.32), and writing  $\mathcal{M}$  as  $\mathcal{V}^\# \mathcal{V}$ , giving

$$\text{tr} \left( (d\mathcal{V} \mathcal{V}^{-1} + \eta (d\mathcal{V} \mathcal{V}^{-1})^\# \eta) h_i \right) = 0 . \quad (5.33)$$

In this form, the relation is manifestly true since for any generator  $\mathcal{T}$  of  $G$ , it is the case that  $\mathcal{T} + \eta \mathcal{T}^\# \eta$  is non-compact, and hence orthogonal to  $h_i$ . (Note that (5.33) can be shown to follow from the statement that the Noether currents for transformations associated with the denominator gauge group vanish [16].) Thus the total number of relations in (5.32) on the  $\dim(G)$  Noether currents is equal to the dimension of the maximal compact subgroup of  $G$ . The total number of linearly-independent Noether currents is therefore equal to the dimension of the scalar coset manifold.

In the Borel parameterisation (5.27) of the scalar coset, the transformations in the Borel subgroup of  $G$  are generated by constant shifts of the dilatons and axions, implying that



the equations of motion can be expressed as  $d * J_0^i = 0 = d * J_+^m$ , where  $J_0^i$  and  $J_+^m$  are the Noether currents associated with these shift symmetries. The explicit forms for these currents for the  $E_{n(n)}$  global symmetry groups of the maximal supergravities are given in section 4. Thus the first-order equations can be expressed as

$$d\psi^i = *J_0^i, \quad d\tilde{\chi}^m = *J_+^m, \quad (5.34)$$

where  $\psi^i$  and  $\tilde{\chi}^m$  are the associated dual potentials. The Noether currents  $J_-^m$  of the transformations generated by the negative-root generators can be expressed as linear combinations of  $J_0^i$  and  $J_+^m$ , with scalar-dependent coefficients. Thus the linear transformation of the complete set of the Noether currents becomes a non-linear transformation when acting purely on the Borel currents. It follows that the first-order equation is invariant under the full group  $G$ , provided that  $d\psi^i$  and  $d\tilde{\chi}^m$  transform covariantly, with the same non-linear transformation rules as the Borel currents. Under these transformations, the Bianchi identities for the transformed dual fields are guaranteed, for the same reason that we discussed in the  $SL(2, \mathbb{R})$  example.

The full dualisation of the general coset model can be easily understood. In the Borel parameterisation of the coset, the scalars appear in the Lagrangian (D.5) through  $\mathcal{G}_0$ , which satisfies the Bianchi identity  $d\mathcal{G}_0 - \mathcal{G}_0 \wedge \mathcal{G}_0 = 0$ . Thus we can introduce Lagrange multipliers for this Bianchi identity. The resulting fully-dualised theory has no global symmetry. The disappearance of the global Borel subgroup can be easily understood, since the doubled field  $\mathcal{G}$  is invariant under the Borel subgroup, and hence so is its Lagrange multiplier. The rest of the transformations involve undifferentiated scalars as coefficients, and hence cannot be expressed in terms of the dual fields, the Noether currents have to be exchanged with topological charges and rigid invariance with gauge invariance.

The above analysis has so far concentrated on symmetric spaces for maximally non-compact groups. The situation is analogous for a general non-compact symmetric space. In such a case, the coset  $\mathcal{V}$  can be parameterised by using the Iwasawa decomposition and the so-called solvable subalgebra [24]. In other words, it can be parameterised by using a subset of the positive-root generators and the Cartan generators. Again the Noether currents for the symmetries associated with these generators can be used to construct first-order equations, and they are invariant under the full global symmetry group. This can be seen by means of the same argument as that discussed in the case of maximally non-compact groups. An example of a coset with a non-compact group that is not maximally non-compact is provided by the toroidal dimensional reduction of the heterotic string, which, in  $D$  dimensions has a  $O(26 - D, 10 - D)/((O(26 - D) \times O(10 - D)))$  scalar manifold.

Another interesting example is that of non-linear sigma models with (compact or non-compact) group-manifold target spaces. An important difference in this case is that some subgroup of global symmetry is linearly realised on the 1-form field strengths, although it is still non-linearly realised on the scalars, *i.e.* the 0-form potentials. Thus it is manifest that the first-order equations should have this global symmetry. In fact, owing to the linearity of the realisation of the global symmetry on the 1-forms, it follows that after fully dualising the sigma model fields to higher-degree forms, the Lagrangian will maintain this global symmetry. The specific example of such a dualisation in  $D = 4$  can be found in [22].

The algebras of the doubled formalism for the scalar Lagrangians all have the form

$$[T^a, T^b] = f^{ab}{}_c T^c, \quad [T^a, \tilde{T}_b] = -f^{ac}{}_b \tilde{T}_c, \quad [\tilde{T}_a, \tilde{T}_b] = 0, \quad (5.35)$$

where the generators  $T^a$  are associated with the scalars, while the generators  $\tilde{T}_a$  are associated with the duals of the scalars. Similar Lie algebras seem to be interesting from the point of view of integrable systems [25]. The algebra can be denoted again by  $G \ltimes G^*$ , where  $G$  is generated by  $T^\alpha$  and  $G^*$  is the co-adjoint of  $G$ .

In the case of a principal sigma model (*i.e.* one with a group manifold  $G$  as its target space),  $T^\alpha$  generates the group  $G$  of the sigma model, whilst in the case of a coset  $G/H$  in the Borel gauge,  $T^\alpha$  generates the Borel subalgebra  $G_+$  of  $G$ . When  $G = SL(2, \mathbb{R})$ , it is easy to verify that  $G_+ \ltimes G_+^*$  is a subalgebra of  $G \ltimes G^*$ . However, this statement is not true for generic groups  $G$ . The gauge invariant treatment of symmetric space sigma models including the principal models is sketched in Appendix D.

## 6 Type IIB Supergravity

There is no covariant Lagrangian for type IIB supergravity, since it includes a self-dual 5-form field strength. However one can write down covariant equations of motion [26]. In order to make manifest their global  $SL(2, \mathbb{R})$  symmetry, it is useful first to assemble the dilaton  $\phi$  and axion  $\chi$  into a  $2 \times 2$  matrix:

$$\mathcal{M} = \begin{pmatrix} e^\phi & \chi e^\phi \\ \chi e^\phi & e^{-\phi} + \chi^2 e^\phi \end{pmatrix} \quad (6.1)$$

Also, define the  $SL(2, \mathbb{R})$ -invariant matrix

$$\Xi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (6.2)$$

and the two-component column vector of 2-form potentials

$$A_{(2)} = \begin{pmatrix} A_{(2)}^1 \\ A_{(2)}^2 \end{pmatrix}. \quad (6.3)$$

Here  $A_{(2)}^1$  is the R-R potential, and  $A_{(2)}^2$  is the NS-NS potential. The bosonic matter equations of motion can then be written as [27]

$$\begin{aligned} d*H_{(5)} &= -\frac{1}{2}\epsilon_{ij} F_{(3)}^i \wedge F_{(3)}^j, \\ d(\mathcal{M}*H_{(3)}) &= H_{(5)} \wedge \Xi H_{(3)}, \\ d(e^{2\phi}*d\chi) &= -e^\phi F_{(3)}^2 \wedge *F_{(3)}^1, \\ d*d\phi &= e^{2\phi} d\chi \wedge *d\chi + \frac{1}{2}e^\phi F_{(3)}^1 \wedge *F_{(3)}^1 - \frac{1}{2}e^{-\phi} F_{(3)}^2 \wedge *F_{(3)}^2, \end{aligned} \quad (6.4)$$

where  $F_{(3)}^1 = dA_{(2)}^1 - \chi dA_{(2)}^2$ ,  $F_{(3)}^2 = dA_{(2)}^2$ ,  $H_{(3)} = dA_{(2)}$ , and  $H_{(5)} = dB_{(4)} - \frac{1}{2}\epsilon_{ij} A_{(2)}^i \wedge dA_{(2)}^j$ .

Introducing a ‘‘doubled’’ set of potentials  $\{\tilde{A}_{(6)}, \psi, \tilde{\chi}\}$  for the  $SL(2, \mathbb{R})$  doublet  $A_{(2)}$ , and for  $\phi$  and  $\chi$ , respectively, we find that one may write the equations of motion (6.4) in first-order form, as

$$\begin{aligned} *H_{(5)} &= dB_{(4)} - \frac{1}{2}\epsilon_{ij} A_{(2)}^i \wedge dA_{(2)}^j, \\ \mathcal{M}*dA_{(2)} &= d\tilde{A}_{(6)} - \frac{1}{2}\Xi A_{(2)} \wedge (dB_{(4)} - \frac{1}{6}\epsilon_{ij} A_{(2)}^i \wedge dA_{(2)}^j), \\ i e^\phi *d\tau &= P + \tau Q, \end{aligned} \quad (6.5)$$

where  $\tau = \chi + i e^{-\phi}$ , and the quantities  $P$  and  $Q$  are defined by

$$\begin{aligned} P &= d\psi + \frac{1}{2}A_{(2)}^1 d\tilde{A}_{(6)}^1 - \frac{1}{2}A_{(2)}^2 d\tilde{A}_{(6)}^2 - \frac{1}{4}A_{(2)}^1 A_{(2)}^2 dB_{(4)} - \frac{1}{24}A_{(2)}^2 A_{(2)}^2 A_{(2)}^1 dA_{(2)}^1, \\ Q &= d\tilde{\chi} + A_{(2)}^2 d\tilde{A}_{(6)}^1 - \frac{1}{4}A_{(2)}^2 A_{(2)}^2 dB_{(4)} - \frac{1}{36}A_{(2)}^2 A_{(2)}^2 A_{(2)}^2 dA_{(2)}^1. \end{aligned} \quad (6.6)$$

Note that we do not need to introduce a ‘‘double’’ potential for  $B_{(4)}$ ; the ‘‘doubling’’ in this case is automatically achieved by the fact that  $H_{(5)}$ , until the imposition of the self-duality constraint, already has twice the physical degrees of freedom. As usual, taking the exterior derivatives of the equations in (6.5) gives the second-order field equations (6.4).

The first-order equations (6.5) can, if desired, be written also in the form

$$\begin{aligned} *H_{(5)} &= H_{(5)} = dB_{(4)} - \frac{1}{2}\epsilon_{ij} A_{(2)}^i \wedge dA_{(2)}^j, \\ e^\phi *F_{(3)}^1 &\equiv \tilde{F}_{(7)}^1 = d\tilde{A}_{(6)}^1 - \frac{1}{2}A_{(2)}^2 (dB_{(4)} - \frac{1}{6}\epsilon_{ij} A_{(2)}^i dA_{(2)}^j), \\ e^{-\phi} *F_{(3)}^2 &\equiv \tilde{F}_{(7)}^2 = d\tilde{A}_{(6)}^2 + \frac{1}{2}A_{(2)}^1 (dB_{(4)} - \frac{1}{6}\epsilon_{ij} A_{(2)}^i dA_{(2)}^j) \\ &\quad - \chi (d\tilde{A}_{(6)}^1 - \frac{1}{2}A_{(2)}^2 (dB_{(4)} - \frac{1}{6}\epsilon_{ij} A_{(2)}^i dA_{(2)}^j)), \\ e^{2\phi} *d\chi &\equiv Q, \\ *d\phi &\equiv \tilde{P} = P + \chi Q, \end{aligned} \quad (6.7)$$

where  $P$  and  $Q$  are defined in (6.6).

The second-order equations of motion can be written in the bilinear form

$$\begin{aligned}
dH_{(5)} &= -F_{(3)}^1 \wedge F_{(3)}^2, \\
d\tilde{F}_{(7)}^1 &= H_{(5)} \wedge F_{(3)}^2, \\
d\tilde{F}_{(7)}^2 &= -H_{(5)} \wedge F_{(3)}^1 - d\chi \wedge \tilde{F}_7^1, \\
dQ &= -F_{(3)}^2 \wedge \tilde{F}_{(7)}^1, \\
d\tilde{P} &= d\chi \wedge Q + \frac{1}{2}F_{(3)}^1 \wedge \tilde{F}_{(7)}^1 - \frac{1}{2}F_{(3)}^2 \wedge \tilde{F}_{(7)}^2.
\end{aligned} \tag{6.8}$$

We may then define a doubled field strength  $\mathcal{G}$ , given by

$$\begin{aligned}
\mathcal{G} &= \frac{1}{2}d\phi H + e^\phi d\chi E_+ + e^{\frac{1}{2}\phi} F_{(3)}^1 V_+ + e^{-\frac{1}{2}\phi} F_{(3)}^2 V_- + H_{(5)} U \\
&\quad + e^{-\frac{1}{2}\phi} \tilde{F}_{(7)}^1 \tilde{V}_+ + e^{\frac{1}{2}\phi} \tilde{F}_{(7)}^2 \tilde{V}_- + e^{-\phi} Q \tilde{E}_+ + \frac{1}{2}\tilde{P} \tilde{H}.
\end{aligned} \tag{6.9}$$

As usual, the various untilded generators are mapped into their associated tilded ‘‘dual’’ generators by an (anti)-involution  $\mathcal{S}$ , and so  $\mathcal{G}$  automatically satisfies the equation  $*\mathcal{G} = \mathcal{S}\mathcal{G}$ . The equations of motion (6.8) can then be expressed in the Cartan-Maurer form  $d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0$ , provided that the generators have the non-vanishing commutation relations

$$\begin{aligned}
[H, E_+] &= 2E_+, & [H, V_+] &= V_+, & [H, V_-] &= -V_-, \\
[H, \tilde{E}_+] &= -2\tilde{E}_+, & [H, \tilde{V}_+] &= -\tilde{V}_+, & [H, \tilde{V}_-] &= \tilde{V}_-, \\
[E_+, V_-] &= V_+, & [E_+, \tilde{V}_+] &= -\tilde{V}_-, & [V_+, V_-] &= -U, \\
[V_+, U] &= \frac{1}{2}\tilde{V}_-, & [V_-, U] &= -\frac{1}{2}\tilde{V}_+, & [V_-, \tilde{V}_+] &= \tilde{E}_+, \\
[E_+, \tilde{E}_+] &= \frac{1}{2}\tilde{H}, & [V_+, \tilde{V}_+] &= \frac{1}{4}\tilde{H}, & [V_-, \tilde{V}_-] &= -\frac{1}{4}\tilde{H}.
\end{aligned} \tag{6.10}$$

The commutators in the first two lines are determined by the weights of the various fields, which are evident from (6.9). The next two lines give the commutators associated with the terms on the right-hand sides of various Bianchi identities given above. The last line gives the commutators associated with the equation for  $d\tilde{P}$ . Note that in the type IIB theory all the generators are even, since all the potentials are of even degree.

We can solve the Cartan-Maurer equation by writing  $\mathcal{G} = d\mathcal{V}\mathcal{V}^{-1}$ , where

$$\mathcal{V} = e^{\frac{1}{2}\phi H} e^{\chi E_+} e^{(A_{(2)}^1 V_+ + A_{(2)}^2 V_-)} e^{B_{(4)} U} e^{(\tilde{A}_{(6)}^1 \tilde{V}_+ + \tilde{A}_{(6)}^2 \tilde{V}_-)} e^{\tilde{\chi} \tilde{E}_+} e^{\frac{1}{2}\psi \tilde{H}}. \tag{6.11}$$

It is straightforward to check that all terms in (6.9) are correctly produced by (6.11) and (6.10).

In addition to the local gauge invariances of the higher-rank fields, the type IIB theory also has a global  $SL(2, \mathbb{R})$  symmetry, under the transformations

$$\begin{aligned} \tau &\longrightarrow \frac{a\tau + b}{c\tau + d}, & B_{(4)} &\longrightarrow B_{(4)}, \\ A_{(2)} &\longrightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} A_{(2)}, & \tilde{A}_{(6)} &\longrightarrow \begin{pmatrix} d & c \\ b & a \end{pmatrix} \tilde{A}_{(6)}. \end{aligned} \quad (6.12)$$

The  $SL(2, \mathbb{R})$  transformations of the field strengths  $\tilde{P}$  and  $Q$  are exactly the same as the ones given in the previous section.

As they stand, the commutation relations (6.10) for the type IIB theory do not quite fit the pattern of the general algebras we discussed in sections (4.3) and (4.4). Specifically, it is easily seen that they do not satisfy the jade rule that we discussed at the end of section (4.3). For example, given the commutator  $[V_+, V_-] = -U$  in (6.10), the jade rule would lead us to expect a non-vanishing commutator  $[V_+, \tilde{U}] = \tilde{V}_-$ , while from (6.10) we see that this does not occur. Similarly, from  $[V_+, U] = \frac{1}{2}\tilde{V}_-$ , we would expect from the jade rule that  $[V_+, V_-] \sim -\frac{1}{2}\tilde{U}$ , whereas in fact we have  $[V_+, V_-] = -U$ . In fact, of course, we do not even have a generator  $\tilde{U}$  in our theory. The reason for this is that in our expression (6.9) for the total field strength  $\mathcal{G}$ , we already made use of the fact that in the type IIB theory  $H_{(5)}$  is self dual, and so we did not have to introduce a “doubled” field for  $H_{(5)}$ . Related to this self-duality constraint is the fact that there exists no Lagrangian for the type IIB theory. In [28], it was shown that one could derive the type IIB equations of motion from a Lagrangian, if one initially relaxes the self-duality condition and allows the 5-form field strength to be unconstrained in the Lagrangian. After varying to obtain the equations of motion one can then impose self-duality as a consistent solution of the enlarged equations, thereby recovering the equations of motion for type IIB supergravity.

We can perform a similar enlargement of the system of fields in our description, and initially treat  $H_{(5)}$  as an unrestricted field with no self-duality constraint. We then have to introduce a doubled field, say  $\tilde{H}_{(5)}$ , and its associated generator  $\tilde{U}$ , in the same way as we do for all other field strengths. The resulting changes in the equations of motion will imply, for example, that the second line of (6.4) will become

$$d(\mathcal{M}*H_{(3)}) = \frac{1}{2}(H_{(5)} + *\tilde{H}_{(5)}) \wedge \Xi H_{(3)}. \quad (6.13)$$

Following through the consequences, we eventually find that the three commutation relations in (6.10) that involve  $U$  will be modified, so that they will be replaced by

$$\begin{aligned} [V_+, V_-] &= -\frac{1}{2}(U + \tilde{U}), & [V_{\pm}, U - \tilde{U}] &= 0, \\ [V_+, U + \tilde{U}] &= \frac{1}{2}\tilde{V}_-, & [V_-, U + \tilde{U}] &= -\frac{1}{2}\tilde{V}_+. \end{aligned} \quad (6.14)$$

With these replacements of commutation relations, the algebra (6.10) does now have the form (4.49). The original algebra (6.10) can be viewed as a subalgebra, obtained by taking  $U - \tilde{U}$ , which commutes with everything, to be vanishing. This choosing of a subalgebra is the analogue, at the level of the algebra, of the imposition of the  $H_{(5)} = *H_{(5)}$  constraint at the level of the field theory.

It is interesting to note that whereas the type IIB theory has an  $SL(2, \mathbb{R})$  global symmetry, the type IIA theory has an  $SL(1|1)$  global symmetry. This observation emphasises the similarity between the symmetries of type IIA and type IIB, which both suggest a twelve-dimensional origin (intriguingly, possibly fermionic).

## 7 Conclusions

In this paper, we have studied the bosonic sectors of the various  $D$ -dimensional maximal supergravities, in a unified formalism in which every field, with the exception of the metric itself, is augmented by the introduction of a “doubled” field, related to the original one by Hodge dualisation. This is done not only for the various antisymmetric tensor gauge fields but also for the dilatonic scalars, for which dual  $(D - 2)$ -form potentials are also introduced. The equations of motion for the various fields are then all expressible in the form of the simple twisted self-duality equation  $*\mathcal{G} = \mathcal{S}\mathcal{G}$ , where  $\mathcal{G}$  is the total field strength, written as a sum  $\mathcal{G} = \sum_i \mathcal{G}_i X^i$  of each individual field strength  $\mathcal{G}_i$  (including those for the doubled potentials) times an associated generator  $X^i$ . The equations of motion can equivalently be expressed as the zero-curvature condition  $d\mathcal{G} - \mathcal{G} \wedge \mathcal{G} = 0$ , provided that appropriate (anti)-commutation relations are imposed on the generators  $X^i$ . This condition can be interpreted as the Cartan-Maurer equation allowing  $\mathcal{G}$  to be written as  $\mathcal{G} = d\mathcal{V}\mathcal{V}^{-1}$ , where  $\mathcal{V}$  is expressed in the general form  $\mathcal{V} = \exp(\mathcal{A}_i X^i)$ . (The precise details of the parameterisation of  $\mathcal{V}$  depend upon the choice of field variables.)

In the simplest example, of  $D = 11$  supergravity, the two generators  $V$  and  $\tilde{V}$  associated with the field strengths  $F_{(4)}$  and  $\tilde{F}_{(7)}$  satisfy a Clifford algebra, with  $V$  being an odd element, and  $\tilde{V}$  a central element. More generally, the maximal supergravity in  $D = 11 - n$  has an algebra which is a deformation of  $G \times G^*$ , where  $G$  itself is the superalgebra  $SL_+(n|1) \times (\wedge V)^3$ , as described in section 4.4. An exception is the ten-dimensional type IIB theory, for which all the generators are even, and hence the algebra is purely bosonic.

The symmetries discussed above leave the total field strength  $\mathcal{G} = d\mathcal{V}\mathcal{V}^{-1}$  invariant, since they act on  $\mathcal{V}$  as  $\mathcal{V} \rightarrow \mathcal{V} \exp(\Lambda_i X^i)$ , where  $\Lambda_i$  are the gauge parameters, satisfying  $d\Lambda_i = 0$ . The 0-form parameters are associated with constant shifts of the various dilatonic

and axionic scalar fields, and hence describe global (*i.e.* rigid) symmetries, while the higher-degree parameters are associated with local gauge transformations of the higher-degree potentials. This somewhat unusual circumstance of having an algebra with both global and local transformations arises because the total field strength  $\mathcal{G}$  is the sum of various antisymmetric tensors of differing degrees including, in particular, degree 1 and degrees greater than 1.

The full symmetry of the doubled equations of motion is larger than the symmetry described above, since there can also be transformations that change  $\mathcal{G}$  whilst nevertheless leaving the equation  $*\mathcal{G} = \mathcal{S}\mathcal{G}$  invariant. In the case of the doubled system of equations for maximal supergravity in  $D = 11 - n$  dimensions, the global part of the  $\mathcal{G}$ -preserving symmetry turns out to be  $E_n^+$ , the Borel subalgebra of the familiar maximally-noncompact  $E_n$  symmetry.<sup>6</sup> We believe that the doubled formalism, however, is also invariant under the full global  $E_n$  algebra, with the anti-Borel generators describing transformations under which  $\mathcal{G}$  varies but  $*\mathcal{G}$  is still equal to  $\mathcal{S}\mathcal{G}$ . Indeed we have shown that the doubled formalism for the type IIB in  $D = 10$  is invariant under the full  $SL(2, \mathbb{R})$  global symmetry. We also showed in general that, at least in the scalar sectors, the introduction of the doubled formalism preserves the original full  $E_n$  global symmetry groups.

The global symmetries of the doubled formalism for the scalar sectors can be studied in detail in more general situations, for any principal sigma model. We discussed this in section 5. In both cases the doubled formalism retains the full global symmetries of the original formulation, with the scalars themselves transforming in an unaltered manner. For a group manifold  $G$ , the dual potentials are invariant under the right action  $G_R$  of the group  $G$ , but transform covariantly under the left action  $G_L$ . For a covariant formulation of a coset target space  $K \backslash G$  the situation is similar, with the dual potentials being invariant under the global right action of the symmetry  $G$ , and transforming covariantly under the local left action of the subgroup  $K$ . On the other hand, in a gauge-fixed formulation in terms of the Borel generators of  $G$ , the dual potentials are invariant only under the Borel subgroup of  $G$ , while the anti-Borel generators describe symmetries that act covariantly only on the doubled field strengths, but not on the doubled potentials. Thus the doubled gauge-fixed scalar coset theory retains a local action of the entire global symmetry  $G$  of the original formulation only at the level of the doubled field strengths, and not the doubled potentials. In appendix D, we studied the symmetries that remain when certain of the “redundant”

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<sup>6</sup>There is no necessity of dualising higher-degree field strengths here, since in the doubled formalism all possible dualised field strengths are automatically present. The  $E_n^+$  symmetry is realised as the global shift transformations of the full set of 0-form potentials of the doubled formalism.

fields of the doubled formalism are eliminated.

Other aspects of the symmetry algebras of the supergravity theories can also be abstracted and studied in their own right. In appendix C, we considered the Kaluza-Klein reduction of  $\tilde{D}$ -dimensional pure gravity to  $D$  dimensions on a torus of  $n = \tilde{D} - D$  dimensions. It is well known that the resulting theory has a global  $GL(n, \mathbb{R})$  symmetry, with the Kaluza-Klein vectors transforming linearly under the  $SL(n, \mathbb{R})$  subgroup. We showed that the gauge symmetry of the Kaluza-Klein vectors, together with the Borel subalgebra  $GL_+(n, \mathbb{R})$  of  $GL(n, \mathbb{R})$ , form the superalgebra  $SL_+(n|1)$ . In the special case of  $D = 3$  the Kaluza-Klein vectors can be dualised to give additional axionic scalars, and the full scalar Lagrangian then has a global  $SL(n + 1, \mathbb{R})$  symmetry.

## A Dimension-dependent terms in first-order equations

In section 4, we gave a general derivation of the first-order equations of motion for  $D$ -dimensional maximal supergravity. The contributions coming from the Chern-Simons terms in the Lagrangians are dimension-dependent, and here we present their detailed forms in each dimension. In each dimension, the variation of the appropriate Chern-Simons term with respect to the various potentials  $A_{(3)}$ ,  $A_{(2)i}$ ,  $A_{(1)ij}$  and  $A_{(0)ijk}$  takes the form

$$-\delta\mathcal{L}_{FFA} = dX \wedge \delta A_{(3)} + dX^i \wedge \delta A_{(2)i} + \frac{1}{2}dX^{ij} \wedge \delta A_{(1)ij} + \frac{1}{6}dX^{ijk} \delta A_{(0)ijk} , \quad (\text{A.1})$$

where the quantities  $X$ ,  $X^i$ ,  $X^{ij}$  and  $X^{ijk}$  can be determined easily in each dimension. First, we list the results for these quantities dimension by dimension:

$$D = 11 : \quad X = \frac{1}{2}A_{(3)} dA_{(3)}$$

$$D = 10 : \quad X = -A_{(2)1} dA_{(3)} , \quad X^1 = \frac{1}{2}A_{(3)} dA_{(3)}$$

$$\begin{aligned} D = 9 : \quad X &= \frac{1}{2}\epsilon^{ij} (A_{(1)ij} dA_{(3)} - A_{(2)i} dA_{(2)j}) \\ X^i &= -\epsilon^{ij} A_{(2)j} dA_{(3)} \\ X^{ij} &= \frac{1}{2}\epsilon^{ij} A_{(3)} dA_{(3)} \end{aligned}$$

$$\begin{aligned} D = 8 : \quad X &= \epsilon^{ijk} \left( -\frac{1}{6}A_{(0)ijk} dA_{(3)} - \frac{1}{2}A_{(1)ij} dA_{(2)k} \right) \\ X^i &= \frac{1}{2}\epsilon^{ijk} (A_{(1)jk} dA_{(3)} - A_{(2)j} dA_{(2)k}) \\ X^{ij} &= -\epsilon^{ijk} A_{(2)k} dA_{(3)} \end{aligned}$$



$$X^{ijk} = \frac{1}{2}\epsilon^{ijk} A_{(3)} dA_{(3)}$$

$$\begin{aligned} D = 7 : \quad X &= \epsilon^{ijk\ell} \left( \frac{1}{8} A_{(1)ij} dA_{(1)k\ell} - \frac{1}{6} A_{(0)ijk} dA_{(2)\ell} \right) \\ X^i &= -\epsilon^{ijk\ell} \left( \frac{1}{2} A_{(1)jk} dA_{(2)\ell} + \frac{1}{6} A_{(0)jkl} dA_{(3)} \right) \\ X^{ij} &= \frac{1}{2}\epsilon^{ijk\ell} \left( A_{(1)k\ell} dA_{(3)} - A_{(2)k} dA_{(2)\ell} \right) \\ X^{ijk} &= -\epsilon^{ijk\ell} A_{(2)\ell} dA_{(3)} \end{aligned}$$

$$\begin{aligned} D = 6 : \quad X &= -\frac{1}{12}\epsilon^{ijk\ell m} A_{(0)ijk} dA_{(1)\ell m} \\ X^i &= \epsilon^{ijk\ell m} \left( \frac{1}{8} A_{(1)jk} dA_{(1)\ell m} - \frac{1}{6} A_{(0)jkl} dA_{(2)m} \right) \\ X^{ij} &= -\epsilon^{ijk\ell m} \left( \frac{1}{2} A_{(1)k\ell} dA_{(2)m} + \frac{1}{6} A_{(0)k\ell m} dA_{(3)} \right) \\ X^{ijk} &= \frac{1}{2}\epsilon^{ijk\ell m} \left( A_{(1)\ell m} dA_{(3)} - A_{(2)\ell} dA_{(2)m} \right) \end{aligned}$$

$$\begin{aligned} D = 5 : \quad X &= -\frac{1}{72}\epsilon^{ijk\ell mn} A_{(0)ijk} dA_{(0)\ell mn} \\ X^i &= -\frac{1}{12}\epsilon^{ijk\ell mn} A_{(0)jk\ell} dA_{(1)mn} \\ X^{ij} &= \epsilon^{ijk\ell mn} \left( \frac{1}{8} A_{(1)k\ell} dA_{(1)mn} - \frac{1}{6} A_{(0)k\ell m} dA_{(2)n} \right) \\ X^{ijk} &= -\epsilon^{ijk\ell mn} \left( \frac{1}{2} A_{(1)\ell m} dA_{(2)n} + \frac{1}{6} A_{(0)\ell mn} dA_{(3)} \right) \end{aligned}$$

$$\begin{aligned} D = 4 : \quad X^i &= -\frac{1}{72}\epsilon^{ijk\ell mnp} A_{(0)jk\ell} dA_{(0)mnp} \\ X^{ij} &= -\frac{1}{12}\epsilon^{ijk\ell mnp} A_{(0)k\ell m} dA_{(1)np} \\ X^{ijk} &= \epsilon^{ijk\ell mnp} \left( \frac{1}{8} A_{(1)\ell m} dA_{(1)np} - \frac{1}{6} A_{(0)\ell mn} dA_{(2)p} \right) \end{aligned}$$

$$\begin{aligned} D = 3 : \quad X^{ij} &= -\frac{1}{72}\epsilon^{ijk\ell mnpq} A_{(0)k\ell m} dA_{(0)npq} \\ X^{ijk} &= -\frac{1}{12}\epsilon^{ijk\ell mnpq} A_{(0)\ell mn} dA_{(1)pq} \end{aligned}$$

$$D = 2 : \quad X^{ijk} = -\frac{1}{72}\epsilon^{ijk\ell mnpqr} A_{(0)\ell mn} dA_{(0)pqr}$$

The task now is to show that the lower line on the right-hand side of (4.16) can be written as the exterior derivative of something, in each dimension. In other words, we want to show that we can write

$$X dA_{(2)k} + X^i dA_{(1)ik} + \frac{1}{2} X^{ij} dA_{(0)ijk} = dY_k , \quad (\text{A.2})$$

and to find  $Y_k$  explicitly for each dimension  $D$ . The results are:

$$D = 10 : \quad Y_1 = -\frac{1}{2}A_{(2)1} A_{(2)1} dA_{(3)}$$

$$D = 9 : \quad Y_k = \epsilon^{ij} \left( \frac{1}{6}A_{(2)k} A_{(2)i} dA_{(2)j} - A_{(1)ik} A_{(2)j} dA_{(3)} \right) ,$$

$$D = 8 : \quad Y_k = \epsilon^{ij\ell} \left( -\frac{1}{2}A_{(0)j\ell k} A_{(2)i} dA_{(3)} - \frac{1}{4}A_{(1)j\ell} A_{(1)ik} dA_{(3)} - \frac{1}{2}A_{(1)ik} A_{(2)j} dA_{(2)\ell} \right)$$

$$D = 7 : \quad Y_k = \frac{1}{4}\epsilon^{ij\ell m} \left( A_{(0)ijk} A_{(2)\ell} dA_{(2)m} - A_{(0)ijk} A_{(1)\ell m} dA_{(3)} - A_{(1)ik} A_{(1)j\ell} dA_{(2)m} \right)$$

$$D = 6 : \quad Y_k = \epsilon^{ij\ell mn} \left( -\frac{1}{4}A_{(0)ijk} A_{(1)\ell m} dA_{(2)n} + \frac{1}{24}A_{(1)ik} A_{(1)j\ell} dA_{(1)mn} \right. \\ \left. - \frac{1}{12}A_{(0)ijk} A_{(0)\ell mn} dA_{(3)} \right)$$

$$D = 5 : \quad Y_k = \epsilon^{ij\ell mnp} \left( \frac{1}{24}A_{(0)ijk} A_{(0)\ell mn} dA_{(2)p} - \frac{1}{16}A_{(0)ijk} A_{(1)\ell m} dA_{(1)np} \right)$$

$$D = 4 : \quad Y_k = -\frac{1}{48}\epsilon^{ij\ell mnpq} A_{(0)ijk} A_{(0)\ell mn} dA_{(1)pq}$$

$$D = 3 : \quad Y_k = \frac{1}{432}\epsilon^{ij\ell mnpqr} A_{(0)ijk} A_{(0)\ell mn} dA_{(0)pqr}$$

$$D = 2 : \quad Y_k = 0$$

Now we turn to the quantities  $Q^k_j$ , which were introduced in the derivation of the first-order equations for the axions  $\mathcal{A}_{(0)j}^i$ . They were defined in (4.23), as the forms whose exterior derivatives would give

$$dQ^k_j = -X^k dA_{(2)j} + X^{k\ell} dA_{(1)j\ell} - \frac{1}{2}X^{k\ell m} dA_{(0)j\ell m} . \quad (\text{A.3})$$

We find that indeed such forms exist, and are given by

$$D = 9 : \quad Q^2_1 = -\frac{1}{2}A_{(2)1} A_{(2)1} dA_{(3)}$$

$$D = 8 : \quad Q^k_j = \epsilon^{k\ell m} \left( -A_{(1)j\ell} A_{(2)m} dA_{(3)} - \frac{1}{6}A_{(2)j} A_{(2)\ell} dA_{(2)m} \right)$$

$$D = 7 : \quad Q^k_j = \epsilon^{k\ell mn} \left( \frac{1}{2}A_{(0)j\ell m} A_{(2)n} dA_{(3)} - \frac{1}{4}A_{(1)mn} A_{(1)j\ell} dA_{(3)} + \frac{1}{2}A_{(1)nj} A_{(2)\ell} dA_{(2)m} \right)$$

$$D = 6 : \quad Q^k_j = \frac{1}{4} \epsilon^{klmnp} (A_{(0)j\ell m} A_{(1)np} dA_{(3)} - A_{(0)j\ell m} A_{(2)n} dA_{(2)p} - A_{(1)j\ell} A_{(1)np} dA_{(2)m})$$

$$D = 5 : \quad Q^k_j = \epsilon^{klmnpq} \left( \frac{1}{4} A_{(0)j\ell m} A_{(1)np} dA_{(2)q} \right. \\ \left. + \frac{1}{24} A_{(0)j\ell m} A_{(0)npq} dA_{(3)} + \frac{1}{24} A_{(1)j\ell} A_{(1)mn} dA_{(1)pq} \right)$$

$$D = 4 : \quad Q^k_j = \epsilon^{klmnpqr} \left( \frac{1}{16} A_{(0)j\ell m} A_{(1)np} dA_{(1)qr} - \frac{1}{24} A_{(0)j\ell m} A_{(0)npq} dA_{(2)r} \right)$$

$$D = 3 : \quad Q^k_j = \frac{1}{48} \epsilon^{klmnpqrs} A_{(0)j\ell m} A_{(0)npq} dA_{(1)rs}$$

$$D = 2 : \quad Q^k_j = -\frac{1}{432} \epsilon^{klmnpqrst} A_{(0)j\ell m} A_{(0)npq} dA_{(0)rst}$$

Note that they are defined only for  $k > j$ , and thus arise only in  $D \leq 9$ . Proving the above identities involves the use of Schoutens' "over-antisymmetrisation" identities on the lower indices. These can be used here, even though the number of lower indices is the same as  $11 - D$ , because all the lower indices are necessarily different from  $k$ .

Finally, we turn to the dimension-dependent quantities  $\vec{Q}$ , which must satisfy equation (4.28). As discussed at the end of section 4.1, the expressions for  $\vec{Q}$  are written in terms of "dressed" fields, where all downstairs indices are dressed with  $\gamma$ , and all upstairs indices are dressed with  $\tilde{\gamma}$ . It is useful first to make the following definitions of dressed quantities:

$$\hat{A}_{(2)i} = \gamma^j_i A_{(2)j}, \quad \hat{A}_{(1)ij} = \gamma^k_i \gamma^\ell_j A_{(1)k\ell}, \quad \hat{A}_{(0)ijk} = \gamma^\ell_i \gamma^m_j \gamma^n_k A_{(0)\ell mn}. \quad (\text{A.4})$$

In terms of these, the required results for the  $\vec{Q}$  are found to be:

$$D = 10 : \quad \vec{Q} = \frac{1}{2} \vec{a} A_{(2)1} A_{(3)} dA_{(3)},$$

$$D = 9 : \quad \vec{Q} = -\frac{1}{2} \vec{a} \epsilon^{ij} \hat{A}_{(1)ij} A_{(3)} dA_{(3)} + \frac{1}{2} \sum_i \vec{a}_i \hat{A}_{(2)i} \hat{A}_{(2)j} dA_{(3)} \epsilon^{ij},$$

$$D = 8 : \quad \vec{Q} = -\frac{1}{6} \vec{a} \hat{A}_{(0)ijk} A_{(3)} dA_{(3)} \epsilon^{ijk} - \frac{1}{2} \sum_{ij} \vec{a}_{ij} \hat{A}_{(1)ij} \hat{A}_{(2)k} dA_{(3)} \epsilon^{ijk} \\ - \frac{1}{3} \sum_i \vec{a}_i \hat{A}_{(2)i} \hat{A}_{(2)j} d\hat{A}_{(2)k} \epsilon^{ijk},$$

$$D = 7 : \quad \vec{Q} = \sum_{ij} \vec{a}_{ij} \left( \frac{1}{8} \hat{A}_{(1)ij} \hat{A}_{(1)k\ell} dA_{(3)} - \frac{1}{4} \hat{A}_{(1)ij} \hat{A}_{(2)k} \gamma^m_\ell dA_{(2)m} \right) \epsilon^{ijk\ell}$$

$$\begin{aligned}
& +\frac{1}{6} \sum_{ijk} \vec{a}_{ijk} \hat{A}_{(0)ijk} \hat{A}_{(2)\ell} dA_{(3)} \epsilon^{ijkl} , \\
D = 6 : \quad \vec{Q} &= -\frac{1}{8} \sum_{ij} \vec{a}_{ij} \hat{A}_{(1)ij} \hat{A}_{(1)kl} \gamma^n{}_m dA_{(2)n} \epsilon^{ijklm} \\
& +\frac{1}{12} \sum_{ijk} \vec{a}_{ijk} \left( \hat{A}_{(0)ijk} \hat{A}_{(1)\ell m} dA_{(3)} - \hat{A}_{(0)ijk} \hat{A}_{(2)\ell} \gamma^n{}_m dA_{(2)n} \right) \epsilon^{ijklm} , \\
D = 5 : \quad \vec{Q} &= \frac{1}{48} \sum_{ij} \vec{a}_{ij} \hat{A}_{(1)ij} \hat{A}_{(1)kl} \gamma^p{}_m \gamma^q{}_n dA_{(1)pq} \epsilon^{ijklmn} \\
& + \sum_{ijk} \vec{a}_{ijk} \left( \frac{1}{72} \hat{A}_{(0)ijk} \hat{A}_{(0)\ell mn} dA_{(3)} + \frac{1}{12} \hat{A}_{(0)ijk} \hat{A}_{(1)\ell m} \gamma^p{}_n dA_{(2)p} \right) \epsilon^{ijklmn} , \\
D = 4 : \quad \vec{Q} &= \sum_{ijk} \vec{a}_{ijk} \left( \frac{1}{24} \hat{A}_{(0)ijk} \hat{A}_{(1)\ell m} \gamma^q{}_n \gamma^r{}_p dA_{(1)qr} \right. \\
& \quad \left. - \frac{1}{72} \hat{A}_{(0)ijk} \hat{A}_{(0)\ell mn} \gamma^q{}_p dA_{(2)q} \right) \epsilon^{ijklmnp} , \\
D = 3 : \quad \vec{Q} &= \frac{1}{144} \sum_{ijk} \vec{a}_{ijk} \hat{A}_{(0)ijk} \hat{A}_{(0)\ell mn} \gamma^r{}_p \gamma^s{}_q dA_{(1)rs} \epsilon^{ijklmnpq} . \tag{A.5}
\end{aligned}$$

## B Dimension-dependent commutators

In section 4.2 we derived the commutation and anticommutation relations for the various generators appearing in the construction of the doubled field  $\mathcal{G}$  in each dimension. Those associated with the contributions from the Chern-Simons terms in the Lagrangian are dimension dependent, and here we present the detailed results for each dimension  $D$ .

The commutation relations can be read off from the bilinear terms in the doubled field  $\mathcal{G}$  that involve the contributions from the Chern-Simons terms  $\mathcal{L}_{FFA}$ , and which therefore all involve the epsilon tensor. We find that they are as follows:

$$\begin{aligned}
D = 11 : \quad \{V, V\} &= -\tilde{V} , \\
D = 10 : \quad \{V, V\} &= -\tilde{V}_1 , \quad [V, V^1] = \tilde{V} , \\
D = 9 : \quad \{V^{ij}, V\} &= -\epsilon^{ij} \tilde{V} , \quad [V^i, V^j] = \epsilon^{ij} \tilde{V} , \quad [V^i, V] = \epsilon^{ij} \tilde{V}_j , \quad \{V, V\} = -\tilde{V}_{12} , \\
D = 8 : \quad [E^{ijk}, V] &= -\epsilon^{ijk} \tilde{V} , \quad [V^{ij}, V^k] = \epsilon^{ijk} \tilde{V} , \quad \{V^{ij}, V\} = -\epsilon^{ijk} \tilde{V}_k , \\
& [V^i, V^j] = \epsilon^{ijk} \tilde{V}_k , \quad [V^i, V] = -\frac{1}{2} \epsilon^{ijk} \tilde{V}_{jk} , \quad \{V, V\} = -\tilde{E}_{123} ,
\end{aligned}$$

$$\begin{aligned}
D = 7 : \quad & [E^{ijk}, V] = \epsilon^{ijkl} \tilde{V}_\ell, \quad [E^{ijk}, V^\ell] = \epsilon^{ijkl} \tilde{V}, \\
& \{V^{ij}, V^{kl}\} = -\epsilon^{ijkl} \tilde{V}, \quad [V^{ij}, V^k] = -\epsilon^{ijkl} \tilde{V}_\ell, \quad \{V^{ij}, V\} = -\frac{1}{2}\epsilon^{ijkl} \tilde{V}_{kl}, \\
& [V^i, V^j] = \frac{1}{2}\epsilon^{ijkl} \tilde{V}_{kl}, \quad [V^i, V] = \frac{1}{6}\epsilon^{ijkl} \tilde{E}_{jkl}, \\
D = 6 : \quad & [E^{ijk}, V^{\ell m}] = -\epsilon^{ijklm} \tilde{V}, \quad [E^{ijk}, V^\ell] = \epsilon^{ijklm} \tilde{V}_m, \quad [E^{ijk}, V] = -\frac{1}{2}\epsilon^{ijklm} \tilde{V}_{\ell m}, \\
& \{V^{ij}, V^{kl}\} = -\epsilon^{ijklm} \tilde{V}_m, \quad [V^{ij}, V^k] = \frac{1}{2}\epsilon^{ijklm} \tilde{V}_{\ell m}, \\
& \{V^{ij}, V\} = -\frac{1}{6}\epsilon^{ijklm} \tilde{E}_{klm}, \quad [V^i, V^j] = \frac{1}{6}\epsilon^{ijklm} \tilde{E}_{klm}, \\
D = 5 : \quad & [E^{ijk}, E^{\ell mn}] = \epsilon^{ijklmn} \tilde{V}, \quad [E^{ijk}, V^{\ell m}] = \epsilon^{ijklmn} \tilde{V}_n, \\
& [E^{ijk}, V^\ell] = \frac{1}{2}\epsilon^{ijklmn} \tilde{V}_{mn}, \quad [E_{ijk}, V] = \frac{1}{6}\epsilon^{ijklmn} \tilde{E}^{\ell mn}, \\
& \{V^{ij}, V^{kl}\} = -\frac{1}{2}\epsilon^{ijklmn} \tilde{V}_{mn}, \quad [V^{ij}, V^k] = -\frac{1}{6}\epsilon^{ijklmn} \tilde{E}_{\ell mn}, \\
D = 4 : \quad & [E^{ijk}, E^{\ell mn}] = \epsilon^{ijklmnp} \tilde{V}_p, \quad [E^{ijk}, V^{\ell m}] = -\frac{1}{2}\epsilon^{ijklmnp} \tilde{V}_{np}, \\
& [E^{ijk}, V^\ell] = \frac{1}{6}\epsilon^{ijklmnp} \tilde{E}_{mnp}, \quad \{V^{ij}, V^{kl}\} = -\frac{1}{6}\epsilon^{ijklmnp} \tilde{E}_{mnp}, \\
D = 3 : \quad & [E^{ijk}, E^{\ell mn}] = \frac{1}{2}\epsilon^{ijklmnpq} \tilde{V}^{pq}, \quad [E^{ijk}, V^{\ell m}] = \frac{1}{6}\epsilon^{ijklmnpq} \tilde{E}_{npq}. \tag{B.1}
\end{aligned}$$

We may observe that these commutation relations can all be summarised in the single expression

$$[T^{\bar{a}}, T^{\bar{b}}] = -(-1)^{[\bar{b}]} \epsilon^{\bar{c}\bar{a}\bar{b}} \tilde{T}_{\bar{c}}, \tag{B.2}$$

where, as in the notation of section 4.4, we use generic indices  $\bar{a}, \bar{b}, \dots$  to represent antisymmetrised sets of  $i, j, \dots$  indices. The symbol  $[\bar{a}]$  denotes the number of such  $i, j, \dots$  indices. Appropriate  $1/[\bar{a}]!$  combinatoric factors are understood in summations over repeated generic indices. Also, we have  $T = V, T^i = V^i, T^{ij} = V^{ij}, T^{ijk} = E^{ijk}$ , with a similar set of definitions for  $\tilde{T}_{\bar{a}}$ . It is useful also to define generators  $\tilde{U}^{\bar{a}}$ , by

$$\tilde{U}^{\bar{a}} = \epsilon^{\bar{b}\bar{a}} \tilde{T}_{\bar{b}}. \tag{B.3}$$

(In explicit notation, this means  $\tilde{U}^{i_1 \dots i_p} = 1/q! \epsilon^{j_1 \dots j_q i_1 \dots i_p} \tilde{T}_{j_1 \dots j_q}$ , where  $p = [\bar{a}]$ ,  $q = [\bar{b}]$ , and  $p + q = 11 - D$ .) In terms of  $\tilde{U}^{\bar{a}}$ , the commutators (B.1) can all be written in the form

$$[T^{\bar{a}}, T^{\bar{b}}] = -(-1)^{[\bar{b}]} \tilde{U}^{\bar{a}\bar{b}}. \tag{B.4}$$

The above algebras can be understood directly through dimensional reduction from the  $D = 11$  algebra  $\{V, V\} = -\tilde{V}$ . To see this, we define

$$T^{\bar{a}} = d^{\bar{a}}z V, \quad \tilde{U}^{\bar{a}} = d^{\bar{a}}z \tilde{V}, \quad (\text{B.5})$$

where  $d^{\bar{a}}z$  denotes  $1, dz^i, dz^i \wedge dz^j, dz^i \wedge dz^j \wedge dz^k$  corresponding to  $[\bar{a}] = 0, 1, 2, 3$ . Thus we will have

$$\begin{aligned} [T^{\bar{a}}, T^{\bar{b}}] &= [d^{\bar{a}}z V, d^{\bar{b}}z V] = (-1)^{[\bar{b}]} d^{\bar{a}}z d^{\bar{b}}z \{V, V\} \\ &= -(-1)^{[\bar{b}]} d^{\bar{a}}z d^{\bar{b}}z \tilde{V} = -(-1)^{[\bar{b}]} \tilde{U}^{\bar{a}\bar{b}}. \end{aligned} \quad (\text{B.6})$$

The peculiar sign in front of (B.2) follows from the fermionic character of  $V$ .

## C Kaluza-Klein reduction and $SL_+(n|1)$

Let us consider the dimensional reduction of the pure gravity Lagrangian  $\hat{\mathcal{L}} = \hat{e} \hat{R}$  in  $\tilde{D}$  dimensions on an  $n$ -torus to  $D = \tilde{D} - n$  dimensions. This will give the  $D$ -dimensional Lagrangian [5]

$$\mathcal{L} = e R - \frac{1}{2}e (\partial\vec{\phi})^2 - \frac{1}{4}e \sum_i e^{\vec{b}_i \cdot \vec{\phi}} (\mathcal{F}_{(2)}^i)^2 - \frac{1}{2}e \sum_{i < j} e^{\vec{b}_{ij} \cdot \vec{\phi}} (\mathcal{F}_{(1)j}^i)^2, \quad (\text{C.1})$$

where the dilaton vectors are given by

$$\vec{b}_i = -\vec{f}_i, \quad b_{ij} = -\vec{f}_i + \vec{f}_j. \quad (\text{C.2})$$

Here,  $\vec{f}_i$  is given by

$$\vec{f}_i = \left( \underbrace{0, 0, \dots, 0}_{i-1}, (\tilde{D} - 1 - i)s_i, s_{i+1}, s_{i+2}, \dots, s_n \right), \quad (\text{C.3})$$

where

$$s_i = \sqrt{\frac{2}{(\tilde{D}-1-i)(\tilde{D}-2-i)}}. \quad (\text{C.4})$$

It is also convenient to define

$$\vec{s} = (s_1, s_2, \dots, s_n). \quad (\text{C.5})$$

We note that  $\vec{f}_i$  satisfies the sum rule

$$\sum_i \vec{f}_i = (\tilde{D} - 2) \vec{s}. \quad (\text{C.6})$$

It is also easily established that  $\vec{f}_i$  and  $\vec{s}$  satisfy the relations

$$\vec{s} \cdot \vec{s} = \frac{2n}{(\tilde{D}-2)(D-2)}, \quad \vec{s} \cdot \vec{f}_i = \frac{2}{D-2}, \quad \vec{f}_i \cdot \vec{f}_j = 2\delta_{ij} + \frac{2}{D-2}. \quad (\text{C.7})$$

From these, the following lemma can also be derived:

$$\sum_i (\vec{f}_i \cdot \vec{x})^2 = 2\vec{x} \cdot \vec{x} + (\tilde{D} - 2) (\vec{s} \cdot \vec{x})^2 , \quad (\text{C.8})$$

where  $\vec{x}$  is an arbitrary vector.

It has been shown previously that the dilaton vectors  $\vec{b}_{ij}$  form the positive roots of the  $SL(n, \mathbb{R})$  global symmetry algebra of gravity compactified on the  $n$ -torus [21, 6]. In this appendix, we show that the extended system  $\vec{b}_{\alpha\beta}$ , with  $\alpha = (i, 0)$ , comprising  $\vec{b}_{ij}$  and  $\vec{b}_{i0} \equiv \vec{b}_i$ , form the positive roots of the superalgebra  $SL(n|1)$ . As in section 4.3, we extend the definition of the generators  $E_i^j$  (with  $i < j$ ) for the positive roots  $\vec{b}_{ij}$  to  $E_\alpha^\beta$  (with  $\alpha < \beta$ ), where  $E_i^0 = W_i$ , and  $W_i$  are the odd generators associated with the weights  $\vec{b}_i$ . (As before, we find it convenient to make the formal definition that 0 is larger than any of the values  $i$ .) We may take a representation where  $E_\alpha^\beta$  is the  $(n+1) \times (n+1)$  matrix which is zero everywhere except for a “1” at the  $\alpha$ 'th row and  $\beta$ 'th column. In this representation, it follows from the commutation relations  $[\vec{H}, E_\alpha^\beta] = \vec{b}_{\alpha\beta} E_\alpha^\beta$  that the Cartan generators  $\vec{H}$  are the  $(n+1) \times (n+1)$  matrices

$$\vec{H} = \text{diag}(\vec{b}_1 + \vec{c}, \vec{b}_2 + \vec{c}, \dots, \vec{b}_n + \vec{c}, \vec{c}) , \quad (\text{C.9})$$

where  $\vec{c}$  is an as-yet arbitrary vector. We can now show that  $\vec{H}$  and  $E_\alpha^\beta$  ( $\alpha < \beta$ ) form the Borel subalgebra  $SL_+(n|1)$  of the superalgebra  $SL(n|1)$ , provided that we choose

$$\vec{c} = \frac{\tilde{D}-2}{n-1} \vec{s} , \quad (\text{C.10})$$

so as to make the supertrace of  $\vec{H}$  vanish. (The  $SL(n|1)$  supertrace of the matrix  $X_{\alpha\beta}$  is given by  $\text{str}(X) = \sum_{i=1}^n X_{ii} - X_{00}$ .) It only remains to show that the vectors  $\vec{b}_{\alpha\beta}$  with  $\alpha < \beta$  indeed form the positive roots of  $SL_+(n|1)$ .

To do this, we must first construct the Cartan-Killing metric. We can get it up to a Casimir factor as

$$K_{ab} = \frac{1}{2} \text{str}(H_a H_b) , \quad (\text{C.11})$$

where  $H_a$  denotes the  $a$ 'th component of  $\vec{H}$ . We can then show, by making use of (C.8), that for any  $\vec{x}$

$$x_a x_b K_{ab} = \vec{x} \cdot \vec{x} - \frac{(D-1)(\tilde{D}-2)}{2(n-1)} (\vec{s} \cdot \vec{x})^2 , \quad (\text{C.12})$$

and hence  $K_{ab} = \delta_{ab} - \frac{1}{2}(D-1)(\tilde{D}-2) s_a s_b / (n-1)$ . The inverse metric is then easily seen to be

$$K^{ab} = \delta_{ab} - \frac{1}{2}(D-1)(D-2) s_a s_b . \quad (\text{C.13})$$

Thus the  $SL(n|1)$  inner product between weight vectors  $\vec{x}$  and  $\vec{y}$  is given by

$$K(\vec{x}, \vec{y}) \equiv K^{ab} x_a y_b = \vec{x} \cdot \vec{y} - \frac{1}{2(D-1)(D-2)} (\vec{s} \cdot \vec{x})(\vec{s} \cdot \vec{y}) . \quad (\text{C.14})$$

(Note that if we consider  $n = 1$ , we find that the inverse metric for  $SL(n|1)$  vanishes, this is the special case where  $SL(n|1)$  is not simple.) We can now deduce from (C.7) that

$$\begin{aligned} K(\vec{b}_{ij}, \vec{b}_{k\ell}) &= 2\delta_{ik} + 2\delta_{j\ell} - 2\delta_{i\ell} - 2\delta_{jk} , \\ K(\vec{b}_{ij}, \vec{b}_{k0}) &= 2\delta_{ik} - 2\delta_{jk} , \\ K(\vec{b}_{i0}, \vec{b}_{j0}) &= 2\delta_{ij} - 2 . \end{aligned} \quad (\text{C.15})$$

These are precisely the inner products of the positive-root vectors of  $SL(n|1)$  up to a normalisation factor that should follow from the Casimir number alluded to above. In particular, we may augment the set of simple roots  $\vec{b}_{i,i+1}$  of  $SL(n, \mathbb{R})$  by including the null vector  $\vec{b}_{n0}$ . These generate the Dynkin diagram for  $SL(n|1)$ , namely

$$\begin{array}{ccccccccc} \vec{b}_{12} & & \vec{b}_{23} & & \vec{b}_{34} & & & & \vec{b}_{n-1,n} & & \vec{b}_{n0} \\ \circ & - & \circ & - & \circ & - & \cdots & - & \circ & - & \otimes \end{array}$$

Table 1: The dilaton vectors  $\vec{b}_{i,i+1}$  and  $\vec{b}_{n0}$  generate the  $SL(n|1)$  Dynkin diagram

A new situation arises in the special case of a reduction to  $D = 3$  dimensions. If we leave the Kaluza-Klein 2-form field strengths  $\mathcal{F}_{(2)}^i$  undualised, then the theory will have the  $SL_+(n|1)$  symmetry described above. However, if we instead dualise the 2-form field strengths, we will gain  $n$  additional axionic scalars, while at the same time losing all the vector potentials. In this situation, the obvious  $GL(n, \mathbb{R})$  symmetry from the  $n$ -torus compactification can again be enlarged, but this time to the bosonic group  $SL(n+1, \mathbb{R})$  rather than the supergroup  $SL_+(n|1)$ . In other words the superalgebra  $SL_+(1|1)$  is the other side of the Ehlers coin, namely the still mysterious *a priori*  $SL(2, \mathbb{R})$  of gravity reduced to 3 dimensions; see [10] for a recent review. This can be foreseen by noting that the Lagrangian for the undualised theory will be

$$\mathcal{L} = e R - \frac{1}{2}e (\partial\vec{\phi})^2 - \frac{1}{4}e \sum_i e^{\vec{b}_i \cdot \vec{\phi}} (\mathcal{F}_{(2)}^i)^2 - \frac{1}{2}e \sum_{i < j} e^{\vec{b}_{ij} \cdot \vec{\phi}} (\mathcal{F}_{(1)j}^i)^2 . \quad (\text{C.16})$$

After dualising  $\mathcal{F}_{(2)}^i$ , the dilaton prefactors for the the dual fields  $G_{(1)i}$  will be  $e^{-\vec{b}_i \cdot \vec{\phi}}$ . It is now easily seen that the dilaton vectors  $\{-\vec{b}_i, \vec{b}_{ij}\}$  form the positive roots of  $SL(n+1, \mathbb{R})$ ,



and that the simple roots can be taken to be  $-\vec{b}_1$ , together with  $\vec{b}_{i,i+1}$  for  $1 \leq i \leq n-1$ . Thus we have the Dynkin diagram

$$\begin{array}{cccccccc} -\vec{b}_1 & & \vec{b}_{12} & & \vec{b}_{23} & & & & \vec{b}_{n-2,n-1} & & \vec{b}_{n-1,n} \\ \circ & - & \circ & - & \circ & - & \cdots & - & \circ & - & \circ \end{array}$$

Table 2: In  $D = 3$ ,  $-\vec{b}_1$  and  $\vec{b}_{i,i+1}$  and generate the  $SL(n+1, \mathbb{R})$  Dynkin diagram

A detailed calculation confirms that indeed we have an  $SL(n+1, \mathbb{R})$  symmetry. First, we note that the Bianchi identity for the field strengths  $\mathcal{F}_{(2)}^i$  can be written as  $d(\gamma^i_j \mathcal{F}_{(2)}^j) = 0$  [6]. The fields  $\mathcal{F}_{(2)}^i$  can therefore be dualised by introducing Lagrange multipliers  $\chi_i$ , and adding the term  $\chi_i d(\gamma^i_j \mathcal{F}_{(2)}^j)$  to the Lagrangian (C.16). We then treat  $\mathcal{F}_{(2)}^i$  as auxiliary fields, and solve for them giving  $*\mathcal{F}_{(2)}^i e^{\vec{b}_i \cdot \vec{\phi}} = G_{(1)i} \equiv \gamma^j_i d\chi_j$ , and hence the Lagrangian becomes

$$\mathcal{L} = e R - \frac{1}{2} e (\partial \vec{\phi})^2 - \frac{1}{2} e \sum_i e^{-\vec{b}_i \cdot \vec{\phi}} (G_{(1)i})^2 - \frac{1}{2} e \sum_{i < j} e^{\vec{b}_{ij} \cdot \vec{\phi}} (\mathcal{F}_{(1)j}^i)^2. \quad (\text{C.17})$$

It is now evident that we can extend the range of the  $i$  index to  $a = (0, i)$  (with  $0 < i$  here), and define axions  $\bar{\mathcal{A}}_{(0)b}^a$  for all  $a < b$ :

$$\bar{\mathcal{A}}_{(0)i}^0 = \chi_i, \quad \bar{\mathcal{A}}_{(0)j}^i = \mathcal{A}_{(0)j}^i. \quad (\text{C.18})$$

(The bar over the potential indicates the extended set.) Defining also  $\bar{\gamma}^a_b$  as in [6], but for the extended set of axionic potentials  $\bar{\mathcal{A}}_{(0)b}^a$ , by  $\bar{\gamma}^0_i = -\chi_i$  and  $\bar{\gamma}^i_j = \gamma^i_j$ , we see that (C.17) assumes the form

$$\mathcal{L} = e R - \frac{1}{2} e (\partial \vec{\phi})^2 - \frac{1}{2} e \sum_{a < b} e^{\vec{b}_{ab} \cdot \vec{\phi}} (\bar{\mathcal{F}}_{(1)b}^a)^2, \quad (\text{C.19})$$

where  $\bar{\mathcal{F}}_{(0)b}^a = \bar{\gamma}^c_b d\bar{\mathcal{A}}_{(0)c}^a$ . This is exactly of the form of the familiar scalar Lagrangian resulting from the dimensional reduction of pure gravity on a spacelike  $n$ -torus [6], except that now the index range is extended to include the value 0. Thus the usual proof of the existence of the  $SL(n, \mathbb{R})$  symmetry now establishes that we have an  $SL(n+1, \mathbb{R})$  symmetry in this three-dimensional case.

More generally, if any  $N$ -form potential is present in the original  $\tilde{D}$ -dimensional theory, in addition to gravity, then we expect that there would still be an  $SL(n+1, \mathbb{R})$  symmetry in  $D = 3$ , with the  $N$ th-degree potential yielding (after dualisation in  $D = 3$ ) an  $\binom{n+1}{N}$  dimensional irreducible representation of  $SL(n+1, \mathbb{R})$ . For example, if we consider  $D = 11$

supergravity reduced to  $D = 3$ , we will have an  $SL(9, \mathbb{R})$  global symmetry, with the Kaluza-Klein descendants of the  $D = 11$  3-form potential giving an irreducible 84-dimensional representation of  $SL(9, \mathbb{R})$ . Of course in this case the symmetry actually enlarges further to  $E_8$ , but this latter enlargement depends crucially on the presence (with the correct coefficient) of the  $FFA$  term in  $D = 11$ .

Another example is the interesting case of the dimensional reduction of the  $D = 4$  Einstein-Maxwell system to  $D = 3$ . It has been known that after dualising the vector potentials in  $D = 3$ , the resulting purely scalar theory then has an  $SU(2, 1)$  global symmetry (see the first reference in [2], but this group was actually known before; see [29]), which contains  $SL(2, \mathbb{R})$  as a subalgebra. Furthermore, if one considers  $N$  Maxwell fields in  $D = 4$  rather than just one, then after dualising the vectors to scalars in  $D = 3$  one obtains a three-dimensional purely scalar Lagrangian with an  $SU(N + 1, 1)$  symmetry [30]. Here, we present a very simple proof of this result, by showing that the target space of the scalar sigma model is the coset  $U(N + 1) \backslash SU(N + 1, 1)$ , which is a non-compact form of  $CP^{N+1}$ .

We start from the standard Lagrangian for  $N$  Maxwell fields  $\hat{F}_{(2)}^i$  coupled to gravity in  $D = 4$ :

$$\mathcal{L}_4 = \hat{R} * \mathbf{1} - \frac{1}{2} * \hat{F}_{(2)}^i \wedge \hat{F}_{(2)}^i . \quad (\text{C.20})$$

We now make a standard Kaluza-Klein reduction to  $D = 3$ , for which the metric ansatz will be  $d\hat{s}_4^2 = e^\varphi ds_3^2 + e^{-\varphi} (dz + \mathcal{A}_{(1)})^2$ . Thus the  $D = 3$  Lagrangian will be

$$\mathcal{L}_3 = R * \mathbf{1} - \frac{1}{2} * d\varphi \wedge d\varphi - \frac{1}{2} e^{-\varphi} * F_{(2)}^i \wedge F_{(2)}^i - \frac{1}{2} e^\varphi * F_{(1)}^i \wedge F_{(1)}^i - \frac{1}{2} e^{-2\varphi} * \mathcal{F}_{(2)} \wedge \mathcal{F}_{(2)} , \quad (\text{C.21})$$

where  $F_{(2)}^i = dA_{(1)}^i - dA_{(0)}^i \wedge \mathcal{A}_{(1)}$ ,  $F_{(1)}^i = dA_{(0)}^i$ , and  $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$ . (Here we have reduced the gauge fields using the standard prescription  $\hat{A}_1^i = A_{(1)}^i + A_{(0)}^i dz$ .)

Now we dualise all the 1-form potentials, to give  $(N + 1)$  further axions. Thus we add Lagrange multiplier terms  $-\chi d\mathcal{F}_{(2)} - \psi_i d(F_{(2)}^i - A_{(0)}^i \mathcal{F}_{(2)})$  to (C.21), to enforce the Bianchi identities  $d\mathcal{F}_{(2)} = 0$  and  $dF_{(2)}^i = F_{(1)}^i \wedge \mathcal{F}_{(2)}$ , and treat  $\mathcal{F}_{(2)}$  and  $F_{(2)}^i$  as auxiliary fields which we now eliminate. We find that  $e^{-2\varphi} * \mathcal{F}_{(2)} = d\chi - A_{(0)}^i d\psi_i$ , and  $e^{-\varphi} * F_{(2)}^i = d\psi_i$ . Substituting back into the Lagrangian, we obtain the fully-dualised result

$$e^{-1} \mathcal{L}_3 = R - \frac{1}{2} (\partial\varphi)^2 - \frac{1}{2} e^\varphi (\partial\psi_i)^2 - \frac{1}{2} e^\varphi (\partial A_{(0)}^i)^2 - \frac{1}{2} e^{2\varphi} (\partial\chi - A_{(0)}^i \partial\psi_i)^2 . \quad (\text{C.22})$$

To study the structure of the scalar manifold, we may simply consider the metric on the  $(2N + 2)$ -dimensional target space which, from (C.22), we read off to be

$$ds^2 = d\varphi^2 + e^\varphi (d\psi_i)^2 + e^\varphi (dA_{(0)}^i)^2 + e^{2\varphi} (d\chi - A_{(0)}^i d\psi_i)^2 . \quad (\text{C.23})$$

Now define the obvious orthonormal basis,

$$e^0 = d\varphi, \quad e^i = e^{\frac{1}{2}\varphi} d\psi_i, \quad e^{i'} = e^{\frac{1}{2}\varphi} dA_{(0)}^{i'}, \quad e^{0'} = e^\varphi (d\chi - A_{(0)}^i d\psi_i), \quad (\text{C.24})$$

which can be seen to satisfy

$$de^0 = 0, \quad de^i = \frac{1}{2}e^0 \wedge e^i, \quad de^{i'} = \frac{1}{2}e^0 \wedge e^{i'}, \quad de^{0'} = e^0 \wedge e^{0'} + e^i \wedge e^{i'}. \quad (\text{C.25})$$

We then easily see that the spin connection  $\omega_{ab}$  is given by

$$\begin{aligned} \omega_{0i} &= -\frac{1}{2}e^i, & \omega_{0i'} &= -\frac{1}{2}e^{i'}, & \omega_{00'} &= -e^{0'}, \\ \omega_{0'i'} &= -\frac{1}{2}e^{i'}, & \omega_{0'i} &= \frac{1}{2}e^{i'}, & \omega_{ij'} &= -\frac{1}{2}\delta_{ij} e^{0'}. \end{aligned} \quad (\text{C.26})$$

It is evident from (C.25) that the 2-form

$$J = e^0 \wedge e^{0'} + e^i \wedge e^{i'} \quad (\text{C.27})$$

is closed,  $dJ = 0$ , and that it is a complex structure, satisfying  $J^a{}_b J^b{}_c = -\delta_c^a$ , where  $a = (0, 0', i, i')$ . In fact,  $J$  is clearly therefore a Kähler form. After further elementary algebra, we find that the curvature 2-forms  $\Theta_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega_{cb}$  can be written as

$$\Theta_{ab} = -\frac{1}{4}e^a \wedge e^b - \frac{1}{4}J_{ac} J_{bd} e^c \wedge e^d - \frac{1}{2}J_{ab} J. \quad (\text{C.28})$$

This means that the components of the Riemann tensor are given by

$$R_{abcd} = -\frac{1}{4}(\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} + J_{ac} J_{bd} - J_{ad} J_{bc} + 2J_{ab} J_{cd}). \quad (\text{C.29})$$

This can be recognised as the curvature tensor for a space of constant (negative) holomorphic sectional curvature [31]. Had it been of positive curvature, it would have been  $CP^{N+1}$ , which is the coset space  $U(N+1) \backslash SU(N+2)$ . Since here the curvature is negative, we can recognise it as the non-compact form of the coset,  $U(N+1) \backslash SU(N+1, 1)$ . (This is related to  $CP^{N+1}$  in the same way as the hyperbolic space  $H^n$  is related to the  $n$ -sphere.<sup>7</sup>) Thus in particular, we see that the fully dualised scalar Lagrangian (C.22) in  $D = 3$ , coming from the Einstein Lagrangian coupled to  $N$  Maxwell fields in  $D = 4$ , has an  $SU(N+1, 1)$  global symmetry group. The symmetry contains once more the group  $SL(2, \mathbb{R})$  as a proper subgroup.

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<sup>7</sup>It is interesting that just as  $H^n$  admits a much simpler metric than  $S^n$ , namely the horospherical metric  $ds^2 = d\rho^2 + e^{2\rho} dx^i dx^i$ , so the non-compact form of the  $CP^{N+1}$  metric can be written in an analogous very simple ‘‘horospherical’’ form (C.23).

## D Fully-dualised $SL(2, \mathbb{R})$ coset and generalisations

In section 4, we established that the first-order equations (5.5) have an  $SL(2, \mathbb{R})$  global symmetry for the scalars, and two abelian gauge symmetries for the two dual fields. It is interesting to study the symmetries of the various second-order equations that can be obtained by integrating out auxiliary fields. For example,  $\psi$  and  $\tilde{\chi}$  appear in the first-order equations only through their field strengths  $P$  and  $Q$ , and the Bianchi identities  $dP = 0 = dQ$  correspond to the two second-order equations of motion for the scalar fields  $\phi$  and  $\chi$ . These second-order equations have only the  $SL(2, \mathbb{R})$  global symmetry. The disappearance of the local gauge symmetries is understandable since the scalars are invariant under these symmetries, even in the full doubled equations.

If, on the other hand, we integrate out the  $\chi$  and  $\psi$  fields, then the remaining fields  $\phi$  and  $\tilde{\chi}$  have the  $\mathbb{R}$  global symmetry corresponding to constant shifts of the dilaton, and the local gauge symmetry associated with  $\tilde{\chi}$ . This case is studied in [6]. Finally, let us study the case where both of the scalars  $\phi$  and  $\chi$  are integrated out. In order to do this, we must first make a field redefinition of the dual fields:

$$\bar{\psi} = \psi + \chi\tilde{\chi}, \quad \tilde{\tilde{\chi}} = e^{-\phi}\tilde{\chi}. \quad (\text{D.1})$$

Under this redefinition, the doubled equations (5.5) become

$$*g = d\bar{\psi} - \tilde{\tilde{\chi}} \wedge f, \quad *f = d\tilde{\tilde{\chi}} + g \wedge \tilde{\tilde{\chi}}, \quad (\text{D.2})$$

where  $f = e^\phi d\chi$  and  $g = d\phi$ . Thus we obtain two independent linear equations for the 1-form field strengths  $f$  and  $g$  for the scalars. From these equations, we can solve for  $f$  and  $g$  purely in terms of the redefined dual fields  $\bar{\psi}$  and  $\tilde{\tilde{\chi}}$ . The Bianchi identities

$$dg = 0, \quad df - g \wedge f = 0 \quad (\text{D.3})$$

then become equations of motion for the dual fields.

Note that the relation between the new dual fields  $(\bar{\psi}, \tilde{\tilde{\chi}})$  and the old fields  $(\psi, \tilde{\chi})$  can be also expressed as

$$\mathcal{V} = e^{\frac{1}{2}\phi H} e^{\chi E_+} e^{\tilde{\chi} \tilde{E}_+} e^{\frac{1}{2}\psi \tilde{H}} = e^{\tilde{\tilde{\chi}} \tilde{E}_+} e^{\frac{1}{2}\bar{\psi} \tilde{H}} e^{\frac{1}{2}\phi H} e^{\chi E_+}. \quad (\text{D.4})$$

The new dual fields are invariant under the Borel subgroup of the global  $SL(2, \mathbb{R})$  symmetry group. This has the consequence that the completely dualised theory of the  $SL(2, \mathbb{R})$  scalar manifold has no global symmetry, but it does have the abelian local gauge symmetries of the dual potentials, which are non-diagonally realised.

The doubled equation  $*\mathcal{G} = \mathcal{S}\mathcal{G}$ , or equivalently the first-order equations (5.5), enable us fully to dualise the coset, and write the equations of motion in terms of the sole dualised pair of potentials  $\bar{\psi}$  and  $\tilde{\chi}$ . The dualised theory no longer has any global symmetry, but it does retain the local gauge symmetry, which becomes non-linear. Now we show that this full dualisation of the scalar coset can also be achieved at the level of Lagrangian, and that in particular,  $\bar{\psi}$  and  $\tilde{\chi}$  are the precise Lagrangian multipliers. To see this, we note that the Lagrangian (5.1) can be written

$$\mathcal{L} = -\frac{1}{4}(\mathcal{G}_0 + \mathcal{G}_0^T)^2, \quad (\text{D.5})$$

where

$$\begin{aligned} \mathcal{G}_0 &= d\mathcal{V}_0\mathcal{V}_0^{-1} = \frac{1}{2}d\phi H + e^\phi d\chi E_+ \\ &= \frac{1}{2}g H + f E_+ = \begin{pmatrix} \frac{1}{2}g & f \\ 0 & -\frac{1}{2}g \end{pmatrix}, \end{aligned} \quad (\text{D.6})$$

and  $\mathcal{V}_0 = \exp(\frac{1}{2}\phi H + \chi E_+)$ . (Note that here we have  $E_+^T = E_-$ ,  $\text{tr}H^2 = 2$ ,  $\text{tr}E_+^2 = \text{tr}E_-^2 = 0$  and  $\text{tr}E_+E_- = 1$ . Note also that the doubled equation can be equivalently expressed as  $*(\mathcal{G} + \mathcal{G}^T) = \mathcal{S}(\mathcal{G} + \mathcal{G}^T)$ .) Thus we see that the dilaton  $\phi$  and axion  $\chi$  appear in the Lagrangian only through  $\mathcal{G}$ , *i.e.* through the quantities  $f$  and  $g$ . The Bianchi identities (D.3) for these two fields can be expressed as  $F \equiv d\mathcal{G}_0 - \mathcal{G}_0 \wedge \mathcal{G}_0 = 0$ . Treating  $f$  and  $g$  as a new set of basic fields, we can introduce a Lagrange multiplier  $\Sigma$

$$\Sigma = \bar{\psi} H + \tilde{\chi} E_- = \begin{pmatrix} \bar{\psi} & 0 \\ \tilde{\chi} & -\bar{\psi} \end{pmatrix}. \quad (\text{D.7})$$

The first-order Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}\text{tr}\{(\mathcal{G}_0 + \mathcal{G}_0^T)^2 + *F \wedge \Sigma\}. \quad (\text{D.8})$$

Varying the Lagrangian with respect to  $f$  and  $g$  gives rise to the equations of motion (D.2), which, as we have seen, enable us to solve for  $f$  and  $g$  in terms  $\bar{\psi}$  and  $\tilde{\chi}$ . Substituting the results into the above first-order Lagrangian, we obtain the fully-dualised Lagrangian for the  $SL(2, \mathbb{R})$  coset (in the Borel gauge).

The fully-dualised Lagrangian has no global symmetry, but it does have non-diagonally realised commuting local gauge symmetries. It follows from (5.8) and (D.1) that the gauge symmetries are

$$\delta\bar{\psi} = \Lambda_\psi + \chi\Lambda_{\tilde{\chi}}, \quad \delta\tilde{\chi} = e^{-\phi}\Lambda_{\tilde{\chi}}. \quad (\text{D.9})$$

However,  $\Lambda_\psi$  and  $\Lambda_{\tilde{\chi}}$  are bad choices of gauge parameters for the dualised theory, in that the transformations cannot be expressed purely in terms of the dual fields. However, if we

define  $\Lambda_\psi = d\lambda_1$  and  $\Lambda_{\tilde{\chi}} = d\lambda_2$ , the gauge transformations can now be expressed purely in terms of the dual potentials, namely

$$\delta\bar{\psi} = d\bar{\lambda}_1 - \bar{\lambda}_2 \wedge f, \quad \delta\bar{\chi} = d\bar{\lambda}_2 + \bar{\lambda}_2 \wedge g, \quad (\text{D.10})$$

where  $\bar{\lambda}_1 = \lambda_1 + \chi\lambda_2$  and  $\bar{\lambda}_2 = e^{-\phi}\lambda_2$ . This gauge invariance of the dualised field  $\Sigma$  is a consequence of the non-abelian Bianchi identity for  $F$ , namely

$$D_0 F \equiv dF - \mathcal{G}_0 \wedge F = 0. \quad (\text{D.11})$$

Taking into account the fact that  $\Sigma$  belongs to the anti-Borel Lie algebra, we find that  $\delta * \Sigma = (D\bar{\lambda})^\nabla$ , where  $\nabla$  denotes the projection onto the anti-Borel Lie algebra along the positive root generators, and  $\bar{\lambda}$  is parameter in the anti-Borel algebra. This gauge transformation rule is precisely the same as the one given in (D.10). Note that the relation between the fields  $\bar{\psi}$ ,  $\bar{\chi}$  and the original fields  $\psi$  and  $\tilde{\chi}$ , given by (D.1), can be expressed as a double projection into the anti-Borel Lie algebra. Denoting  $\Sigma' = \psi H + \tilde{\chi} E_-$ , we then have  $\Sigma' = (\mathcal{V}_0^{-1} \Sigma \mathcal{V}_0)^\nabla$ , which is in fact equivalent to the statement in (D.4). This procedure for dualisation does not work if we choose any other parameterisation for the coset  $SL(2, \mathbb{R})/O(2)$  which is not expressible as the exponentials of a Lie algebra. For example, the coset parameterisation in the so-called symmetric gauge, where  $\mathcal{V}_0$  is chosen to be a symmetric matrix, is not dualisable.

We could, however, start with the completely covariant formulation which admits a global  $SL(2, \mathbb{R}) \times \text{local } O(2)$  invariance. The Lagrangian is now given by

$$\mathcal{L} = \text{tr}\{(D_\mu \mathcal{V} \mathcal{V}^{-1})^2 + *F \wedge \Sigma\}, \quad (\text{D.12})$$

where  $D_\mu \mathcal{V} = (\partial_\mu - h_\mu) \mathcal{V}$ , the representative  $\mathcal{V}$  is  $SL(2, \mathbb{R})$ -valued, and  $h_\mu$  is the composite ‘gauge field’ for the local  $O(2)$  symmetry. Here,  $F$  is defined as previously in terms of  $\mathcal{G}$ , which is still given by  $\mathcal{G} = d\mathcal{V}\mathcal{V}^{-1}$ , but is now in the full Lie algebra of  $SL(2, \mathbb{R})$ .  $\Sigma$  is an arbitrary element of the Lie algebra of  $SL(2, \mathbb{R})$ , parameterised in terms of three fields. The Lagrangian (D.12) is invariant under the transformations

$$\begin{aligned} \text{local } O(2) \times SL(2, \mathbb{R}) : \quad & \mathcal{V} \longrightarrow O(x) \mathcal{V} U^{-1}, \\ & F \longrightarrow O(x) F O^{-1}(x), \\ & \Sigma \longrightarrow O(x) \Sigma O^{-1}(x), \\ \text{gauge} : \quad & \delta(*\Sigma) = D_{\mathcal{G}} \lambda = d\lambda - \mathcal{G} \wedge \lambda, \end{aligned} \quad (\text{D.13})$$

where  $O(x)$  belongs to the local  $O(2)$  and  $U$  belongs to the global  $SL(2, \mathbb{R})$ . Let us write  $\mathcal{G} = \mathcal{G}^\perp + \mathcal{G}^\parallel$ , where  $\mathcal{G}^\parallel$  belongs to Lie algebra of  $O(2)$ , and  $\mathcal{G}^\perp$  is in the orthogonal

complement. We can solve the equations for  $\mathcal{G}^\perp$ ,  $\mathcal{G}^\parallel$  and  $h$  ( $h = \mathcal{G}^\parallel$ ), and obtain a highly non-linear Lagrangian for  $\Sigma$ . Although the Lagrangian still has the local  $O(2)$  invariance (as well as the gauge invariance), it is not possible to write it in terms of only the two fields  $\bar{\phi}$  and  $\tilde{\chi}$ . However, the field equations still describe just two degrees of freedom.

Let us now compare the dualisations of the Borel-gauge coset, the covariant coset, and the principal sigma model. In all three cases the Lagrangian can be expressed in terms of  $\mathcal{G}$ , where  $\mathcal{G} = d\mathcal{V}\mathcal{V}^{-1}$ . In the case of the Borel-gauged coset,  $\mathcal{V}$  is parameterised by scalars associated with the Cartan and the positive-roots generators. The principal sigma models are parameterised either by the left-acting group or the group of right-shifts. For the covariant coset  $\mathcal{V}$  is parameterised by the scalars associated with the full root system. The Lagrangians are invariant under the following transformations

$$\begin{aligned}
\text{Principal Sigma Model :} & \quad \mathcal{V} \longrightarrow \mathcal{V}' = \mathcal{O}\mathcal{V}\Lambda^{-1} \\
\text{Covariant Coset :} & \quad \mathcal{V} \longrightarrow \mathcal{V}' = \mathcal{O}(x)\mathcal{V}\Lambda^{-1} \\
\text{Borel - gauge coset :} & \quad \mathcal{V} \longrightarrow \mathcal{V}' = \mathcal{O}(\text{scalar})\mathcal{V}\Lambda^{-1} .
\end{aligned} \tag{D.14}$$

In other words, the principal sigma model Lagrangian is invariant under the global symmetry  $G_L \times G_R$ , with  $\mathcal{O}$  and  $\Lambda$  being independent constant matrices belonging to  $G_L$  and  $G_R$  respectively. The covariant coset, on the other hand, is invariant under  $H(\text{local}) \times G(\text{global})$  transformations. Finally, the Borel-gauge coset is invariant only under the global symmetry group  $G$ , since the local group  $H$  has been gauged. Note that in this case the transformation  $\mathcal{O}(\text{scalar})$ , which is a scalar-dependent transformation, is not associated with an independent symmetry. Rather, it is a compensating transformation that is needed for implementing the global  $G$  symmetry, and is used for restoring  $\mathcal{V}$  to the Borel gauge after having performed the  $G$  transformation. In all the three cases  $\mathcal{G}$ , and hence  $F \equiv d\mathcal{G} - \mathcal{G} \wedge \mathcal{G}$ , transforms as

$$\mathcal{G} \longrightarrow \mathcal{G}' = \mathcal{O}\mathcal{G}\mathcal{O}^{-1} , \quad F \longrightarrow F' = \mathcal{O}F\mathcal{O}^{-1} . \tag{D.15}$$

In particular, note that both  $\mathcal{G}$  and  $F$  are invariant under the global symmetry group  $G$  acting from the right. Since all the fields appear in the Lagrangian through  $\mathcal{G}$ , which satisfies the Bianchi identity  $F = 0$ , it follows that for all three cases the first order Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_0(\mathcal{G}) + *F \wedge \Sigma , \tag{D.16}$$

where  $\Sigma$  are the dual  $(D-2)$ -form potentials. In the case of Borel-gauge coset, the number of dual potentials is  $\text{Dim}(G/H)$ . In the other two cases, the number is instead equal to the

dimension of the group  $G$ . Thus the Lagrangian (D.16) is invariant under (D.15), provided that  $\Sigma$  transforms as

$$\Sigma \longrightarrow \Sigma' = \mathcal{O}\Sigma\mathcal{O}^{-1} . \quad (\text{D.17})$$

If we solve for  $\mathcal{G}$  in terms of  $\Sigma$ , and substitute back into (D.16), we obtain a second order Lagrangian that is expressed purely in terms of  $\Sigma$ . In the cases of the principal sigma model and the covariant coset, the dualised Lagrangian is expressed in terms of the  $Dim(G)$  dual potentials, and the theories are additionally invariant under global  $G_L$  or  $H$ (local) symmetry groups respectively. In the case of the Borel-gauge coset,  $\mathcal{O}$  is a compensating transformation that depends on the original scalars, and it cannot be expressed locally in terms of the dual potentials  $\Sigma$ . In this case there is no remaining global symmetry that can be expressed in terms of local field transformations.

Note that in the coset case there are two dual Lagrangians, one obtained by dualising the covariant coset Lagrangian, and the other obtained by dualising the Borel-gauge Lagrangian. Although the two systems describe the same on-shell degrees of freedom, their off-shell degrees of freedom differ. To see this, we observe that the dualised covariant-coset Lagrangian contains a number  $Dim(G)$  of  $(D-2)$ -form potentials, each of which has  $(D-1)$  off-shell degrees of freedom. On the other hand the dualised Borel-gauge Lagrangian contains only  $Dim(G/H)$   $(D-2)$ -form potentials. The local  $H$  gauge symmetry of the former Lagrangian, which can be used to remove  $Dim(H)$  degrees of freedom, is not enough to remove the excess of  $Dim(H)$   $(D-2)$ -form potentials, which would need to be done in order to map it to the dualised Borel-gauge Lagrangian, since each potential has  $(D-1)$  off-shell degrees of freedom. This phenomenon should be contrasted with the situation for the original undualised coset Lagrangians. In that case, the local group  $H$  is precisely enough to fix the gauge and hence to map the covariant-coset Lagrangian to the Borel-gauge coset Lagrangian, since each scalar has just one degree of freedom, both off-shell as well as on-shell.

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