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Dirac Quantisation Conditions and Kaluza-Klein Reduction

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ABSTRACT

We present the form of the Dirac quantisation condition for the p -form charges carried by p -brane solutions of supergravity theories. This condition agrees precisely with the conditions obtained in lower dimensions, as is necessary for consistency with Kaluza-Klein dimensional reduction. These considerations also determine the charge lattice of BPS soliton states, which proves to be a universal modulus-independent lattice when the charges are defined to be the canonical charges corresponding to the quantum supergravity symmetry groups.

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1 Introduction

The global (*i.e.* rigid) supersymmetry algebra of $D = 11$ supergravity takes the form [1, 2]

$$\begin{aligned} \{Q, Q\} &= (C\Gamma^{\underline{M}})P_{\underline{M}} + \frac{1}{2}(C\Gamma^{\underline{M}_1\underline{M}_2})Z_{\underline{M}_1\underline{M}_2} \\ &+ \frac{1}{5!}(C\Gamma^{\underline{M}_1\underline{M}_2\dots\underline{M}_5})Y_{\underline{M}_1\underline{M}_2\dots\underline{M}_5}, \end{aligned} \quad (1.1)$$

where $\underline{M} = (0, M)$ are tangent-space $D = 11$ indices, C is the charge conjugation matrix, $P_{\underline{M}}$ is the $D = 11$ ADM momentum and $Z_{\underline{M}_1\underline{M}_2}$ and $Y_{\underline{M}_1\underline{M}_2\dots\underline{M}_5}$ are the analogues of the ‘central charges’ of the $D = 4$ supersymmetry algebra. These charges are not ‘central,’ in $D = 11$, evidently since they carry non-trivial Lorentz indices. Upon Kaluza-Klein dimensional reduction to $D = 4$, these indices become labels for the various Lorentz-scalar central charges of the descendant $N = 8$, $D = 4$ supergravity theory. The occurrence of such tensorial charges is one of the striking features of the $D = 11$ theory, and is fundamental for the subject of p -branes, which are the carriers of such charges.

In this paper, we shall study the implications of the Dirac quantisation condition for tensorial charges such as $Z_{\underline{M}_1\underline{M}_2}$ and $Y_{\underline{M}_1\underline{M}_2\dots\underline{M}_5}$, both in their original higher-dimensional incarnations and also in regard to the quantisation conditions on their dimensionally-reduced descendants. Along the way, we shall cast further light on the structure of the p -brane charge lattice, showing the way in which charge-unit scales are set.

The paper is organised as follows: in section 2, we shall discuss with some care the construction of the tensorial charges, focusing in particular on the topological class of curves in the p -brane transverse space that leads to ostensibly scalar charges nonetheless being labeled by p -forms. In section 3, we shall use this information to derive the Dirac quantisation condition for p -branes, following [3] and [4], but emphasising the existence of ‘Dirac-insensitive’ configurations. In section 4, we shall show how this charge-quantisation picture accords perfectly with the quantisation conditions obtained in lower dimensions by dimensional reduction. In section 5 we shall extend this picture to include quantisation of wave and NUT solutions by classical arguments which nonetheless fit in neatly with the Dirac condition. This is then extended in section 6 and section 7 to cover dyonic and self-dual p -branes and the relation between Dirac insensitive configurations and intersecting p -branes. This leads us in section 8 to a discussion of the web of relations between charge scales for all p -branes in various dimensions. This discussion will be similar to that presented in Refs [6] and [7], but we shall make the point that by considering the implications of T duality together with some special ‘scale-setting’ p -brane species, namely the self-dual 3-branes in $D = 10$ type IIB theory and the dual pairs of D0 branes and $(D - 4)$ branes, charge scales in $D \leq 10$

supergravity theories are in fact determined without recourse to M theory. (In fact the relations that we derive can be interpreted as supporting evidence for the M-theory conjecture.) The resulting charge lattices will then be discussed in section 9. In the Appendix, we discuss the structure of Dirac quantisation conditions for dyons in even dimensions, which have the familiar antisymmetric structure in $D = 4k$ dimensions, but become symmetric in $D = 4k + 2$ dimensions such as for the self-dual cases of strings in $D = 6$ and 3-branes in $D = 10$.

2 p -form charges

A charged p -brane embedded in a D -dimensional supergravity background naturally carries a conserved $(p + 1)$ -form current¹ J_{p+1} . For a p -brane carrying electric (magnetic) charge Q_e (Q_m) under some n -form field strength F_n , where $n = p + 2$ ($n = D - p - 2$), this current appears as a source term on the RHS of the field equation (Bianchi identity) for the field strength. The conservation condition for the current can be concisely formulated as $d * J_{p+1} = 0$. It implies that the charge

$$Q_\Sigma = \int_\Sigma * J_{p+1}, \quad (2.1)$$

where Σ is a $(D - p - 1)$ -dimensional spacelike surface, is conserved in time, provided that the current flowing through the boundary $\partial\Sigma$ vanishes. Note that unless $p = 0$, the integration surface Σ is a subsurface of the chosen spacelike hypersurface that serves as the integration domain for ordinary scalar charge integrals.² Thus, for $p \neq 0$, the integration surface Σ is not unique, i.e. there is no unique embedding of Σ into this spacelike hypersurface. The p -brane charge Q_Σ may thus in general be expected to depend on the choice of integration surface Σ . We shall see in the following, however, that this dependence is essentially topological.

Consider accordingly now the dependence of the p -brane charge on the choice of the integration surface Σ . In a coordinate system $x^M = \{t, x^M\}$ ($M = 1, 2, \dots, (D - 1)$) the integral (2.1) can be written as

$$Q_\Sigma = \frac{1}{p!} \int_\Sigma J^{0M_1 M_2 \dots M_p} d\Sigma_{M_1 M_2 \dots M_p}, \quad (2.2)$$

¹Such a current may be considered to exist even for non-singular p -brane solutions to the supergravity field equations. Even though the singularity structure of the solution does not then necessitate the introduction of source terms, it is nevertheless possible to couple a p -brane source consistently to a non-singular p -brane supergravity background. For further detail on source placement in supergravity solutions, cf. [8].

²In this paper, we shall mostly consider static solutions for which a natural ‘rest frame’ set of coordinates exists, thus defining a preferred spacelike hypersurface as the general arena for charge integrals such as (2.1).

where $d\Sigma_{M_1 M_2 \dots M_p}$ is the coordinate volume element on Σ and the p -brane current density $J^{0M_1 M_2 \dots M_p}$ is given by an integral over the p -brane worldvolume \mathcal{W}_{p+1}

$$J(x)^{0M_1 M_2 \dots M_p} = Q_{e/m} \int_{\mathcal{W}_{p+1}} \delta(t - T) \delta(x - X) dT \wedge dX^{M_1} \wedge dX^{M_2} \wedge \dots \wedge dX^{M_p}. \quad (2.3)$$

Here, $X^{\underline{M}} = (T, X^M)$ are the coordinates of the p -brane and $Q_{e/m}$ is the electric or magnetic source charge. Owing to the presence of the δ -functions in the integral (2.3), the non-zero contribution to Q_Σ comes from the intersection of Σ with the worldvolume \mathcal{W}_{p+1} of the p -brane [9]. The dimension of this intersection is zero and hence $\Sigma \cap \mathcal{W}_{p+1}$ consists in general of a finite number of discrete points. Each point in the intersection contributes $\pm Q_{e/m}$ to the integral, according to the orientation with which \mathcal{W}_{p+1} pierces Σ , *i.e.* from ‘above’ or from ‘below’. It is also possible for the integration surface Σ to be tangential to the p -brane worldvolume \mathcal{W}_{p+1} at the intersection point, or indeed for Σ and \mathcal{W}_{p+1} to overlap partially (in which case there are an infinite number of intersection points). However, in such cases an infinitesimal deformation of Σ near the intersection point would result in no intersection at all, or in pairs of intersections, with one from ‘above’ and one from ‘below’ in each pair. The net contribution to the corresponding charge Q_Σ is always zero in such cases. Hence, one may effectively ignore both ‘tangential’ intersection points and ‘overlapping’ integration surfaces. The two possible values of Q_Σ are thus $\{Q_{e/m}, 0\}$, according to whether the intersection consists of an odd or even number of points. These two values for Q_Σ naturally partition the set of all integration surfaces into two distinct subsets corresponding to the two values of the p -brane charge. We shall next show that these subsets can also be thought of as topological equivalence classes of integration surfaces.

To this end, consider deforming the integration surface Σ infinitesimally to a nearby surface Σ' . Mathematically, this is achieved by considering the flow Φ_V induced by a spacelike vector field V normal to Σ . The flow Φ_V maps a point $\mathcal{P} \in \Sigma$ to a nearby point \mathcal{P}' , which is obtained by going an infinitesimal parameter distance along the integral curve (through p) of the vector field V .³ Mapping each point $\mathcal{P} \in \Sigma$ to a nearby point \mathcal{P}' , we get the deformed surface Σ' . The infinitesimal change in the charge Q_Σ is then given by the integral (over Σ) of the Lie derivative along V of $*J_{p+1}$. Because the p -brane current is conserved, $\mathcal{L}_V * J_{p+1} = d \iota_V * J_{p+1}$, where ι_V denotes the interior product with the vector field V . Using Stokes’ theorem, we may write the change in Q_Σ as an integral over the

³The integral curve of V through a point \mathcal{P} with coordinates $x^M(\mathcal{P})$ is given by the solution to the differential equation $dx^M/d\sigma = V^M(x(\sigma))$, with initial condition $x^M(0) = x^M(\mathcal{P})$.

boundary of Σ :

$$\delta Q_\Sigma = \int_{\partial\Sigma} \iota_V * J_{p+1} = \frac{1}{p!} \int_{\partial\Sigma} J^{0M_1 M_2 \dots M_p} V^N d\Sigma_{M_1 M_2 \dots M_p N}. \quad (2.4)$$

Because the p -brane current density $J^{0M_1 M_2 \dots M_p}$ is non-zero only on the p -brane worldvolume \mathcal{W}_{p+1} , δQ_Σ thus vanishes unless $\partial\Sigma$ intersects the p -brane. Since the charge integrals are always carried out within a fixed spacelike hypersurface in spacetime (*i.e.* at fixed ‘time’), the question of whether $\partial\Sigma$ intersects the p -brane at a fixed time is more precisely the question whether $\partial\Sigma$ intersects the intersection of \mathcal{W}_{p+1} with the chosen spacelike hypersurface. The vanishing of δQ_Σ implies that two integration surfaces Σ_1 and Σ_2 give rise to the same p -brane charge if their *boundaries* can be continuously deformed into one another *without intersecting the p -brane* in the course of the deformation. The set of integration surfaces is therefore naturally partitioned into equivalence classes of surfaces whose boundaries can be continuously deformed into one another without crossing the p -brane. All surfaces belonging to a given equivalence class give rise to the same value of the p -form charge. If the spacetime manifold is simply connected, the converse of this statement also holds. Any two integration surfaces Σ_1 and Σ_2 such that $Q_{\Sigma_1} = Q_{\Sigma_2}$ must belong to the same equivalence class, *i.e.* their boundaries can be continuously deformed into one another without intersecting the p -brane worldvolume.⁴ We therefore conclude that the set of equivalence classes of integration surfaces for a given p -brane solution consists of only two points, which are naturally associated with the two values $\{Q_{e/m}, 0\}$ of the p -brane charge.

What remains to be done now is to find a representative member of the equivalence class of integration surfaces which give rise to a non-zero p -brane charge. One might think it natural for the p -brane charge to be labeled by this representative integration surface. However, it is clear from the above that the spatial p -brane section of the p -brane worldvolume determines the set of topologically equivalent integration surfaces that give rise to a non-zero charge. Hence, we expect that the form structure of the p -brane charge is ultimately characterised by the configuration of this spatial section itself. To make these ideas more precise, let us concentrate on p -branes of infinite spatial extent with \mathbb{R}^p topology. In fact, it will turn out that these, together with p -branes wrapped around compact dimensions, are the only ones that can carry non-zero p -form charge [9]. Now, in order for a p -brane of infinite spatial extent to have a finite energy density, it must be asymptotically flat. ‘Asymptotically

⁴Note the equality of the charges associated with the integration surfaces Σ_i implies that the numbers of intersections of the Σ_i themselves with the p -brane differ (at most) by an even number, where this even difference of contributions to the charges cancels out in positive/negative pairs. In such a case, however, the integration surface boundaries $\partial\Sigma_i$ may still be deformed into one another without intersecting the p -brane.

flat’ is here taken to consist of the following two conditions: (i) the deviation of the metric from the flat Minkowski metric vanishes ‘sufficiently rapidly’ as one approaches transverse spatial infinity *in spacetime* (for more details, see [10]), and (ii) the transverse vibrations of the p -brane worldvolume vanish ‘sufficiently rapidly’ as one approaches transverse spatial infinity *on the worldvolume*. The second condition in particular implies that the tangent vectors to the p -brane worldvolume at spatial infinity on the worldvolume must point along asymptotically flat directions in spacetime. These directions then define the asymptotic orientation of the p -brane. For sake of definiteness, we assume that the spatial section of the p -brane worldvolume is asymptotically oriented along the $\{x^1, x^2, \dots, x^p\}$ directions, where $x^M = \{t, x^i, y^m\}$ ($i = 1, 2, \dots, p$ and $m = (p + 1), \dots, (D - 1)$) is an asymptotically flat coordinate system. The remaining asymptotically flat spatial directions $\{y^m\}$ define the asymptotic transverse space, which we denote by $\Sigma^{12\dots p}$.

We now note that the asymptotic transverse space $\Sigma^{12\dots p}$ of a given p -brane solution is a $(D - p - 1)$ -dimensional spacelike surface which necessarily intersects a spatial section of the p -brane worldvolume in an *odd* number of points. It is therefore a representative integration surface for the equivalence class of integration surfaces that give rise to a non-zero p -brane charge. As this asymptotic transverse space is entirely determined by the asymptotic orientation of the spatial section of the p -brane worldvolume, it is natural to *label* the p -brane charge by the asymptotic orientation of this spatial section. Another way of seeing why this labeling is natural is to realise that the boundaries of all integration surfaces that give rise to a non-zero p -brane charge must have the same topology, because they can be continuously deformed into one another. The boundary of the asymptotic transverse space is topologically equivalent to the $(D - p - 2)$ -dimensional sphere S^{D-p-2} that totally surrounds the spatial section of the p -brane worldvolume. So, the equivalence class of integration surfaces that give rise to a non-zero p -brane charge can be characterised by the fact that the boundary of each surface in the equivalence class is topologically equivalent to a S^{D-p-2} that totally surrounds the p -brane. This topological condition on the boundaries of the integration surfaces is manifestly dependent on the asymptotic orientation of the p -brane worldvolume’s spatial section.

It now is clear why only infinite p -branes or p -branes that are wrapped around compact dimensions can carry non-zero p -form charges [9].⁵ Only such p -branes can be ‘captured’

⁵While it is true that only infinite or wrapped p -branes can carry the p -form charges appearing in the supersymmetry algebra (2.7), conserved charges defined by (2.1) with surfaces Σ that do not necessarily extend out to transverse infinity also have important rôles to play in the theory. For example, the conservation of charges of this sort, with $\partial\Sigma$ closely looping around a p -brane segment, may be used to derive

by a bounding S^{D-p-2} surface at infinity, recalling that continuous deformations of such a bounding surface can be made provided they do not intersect the p -brane. Thus, for a p -brane of finite extent, any candidate integration surface could have its boundary deformed and completely moved away from the p -brane at infinity, in which case the resulting charge is clearly zero. Another way of saying this is to note that integration volumes for p -form charge integrals will be intersected an even number of times by p -branes of finite extent, with positive and negative contributions to the charge canceling out.

We may encode the information about the value of the p -brane charge and the asymptotic orientation of an infinite p -brane's spatial worldvolume section by defining a p -form charge Q_p whose magnitude $|Q_p|$ is equal to the electric or magnetic charge $Q_{e/m}$ of the p -brane and which is proportional to the p -brane's asymptotic spatial section volume form. For example, if the p -brane is asymptotically oriented as above, $Q_p = Q_{e/m} dx^1 \wedge dx^2 \wedge \dots \wedge dx^p$. Note that it is always possible to rotate the asymptotically flat coordinate basis in such a way that the p -form charge Q_p is proportional to a single p -form basis element, which labels the asymptotic orientation of the p -brane.

We conclude this section by giving some examples of p -form charges in $D = 11$ supergravity and by recalling the rôles they play in determining the residual supersymmetries of a p -brane configuration. It follows from the field equation (Bianchi identity) for the 4-form field strength F_4 of $D = 11$ supergravity in the presence of electrically (magnetically) charged 2-brane (5-brane) sources that the canonical 2-brane electric charge Q_Σ (5-form magnetic charge $P_{\tilde{\Sigma}}$) are given in terms of F_4 by

$$Q_\Sigma = \int_{\partial\Sigma} (*F_4 - \frac{1}{2}A_3 \wedge F_4) \quad (2.5)$$

$$P_{\tilde{\Sigma}} = \int_{\partial\tilde{\Sigma}} F_4. \quad (2.6)$$

Note that the canonical charges (2.5) and (2.6) differ from the source charges in (2.4) by a factor of κ_{11}^2 , *i.e.* $Q_{\text{canonical}} = \kappa_{11}^2 Q_{\text{source}}$. This is because the 11-dimensional gravitational coupling constant κ_{11} ($\kappa_{11}^2 = 8\pi G$) only multiplies the $D = 11$ supergravity action (as $\frac{1}{2\kappa_{11}^2}$) but not the p -brane source action. Henceforth, all charges will be understood to be canonical, unless otherwise stated. Both the source and the canonical charges here are dimensionful; we shall later discuss charge lattices in terms of dimensionless charges in sections 8 and 9.

It is easy to verify, for the standard charge Q_e 2-brane solution [12] oriented along the intersection rules between different p -branes [11]. There does not seem to be any way to associate such 'small $\partial\Sigma$ ' charges with p -forms, however.

$\{x^1 x^2\}$ directions, that $Q_\Sigma = 0$ unless $\partial\Sigma$ is topologically equivalent to the 7-dimensional sphere at transverse spatial infinity surrounding the 2-brane. This implies that the only non-vanishing independent component of the 2-form electric charge Q_2 is $Q_{12} = Q_e$. Similarly, for the charge Q_m 5-brane [13] oriented along the $\{x^1 x^2 \dots x^5\}$ directions, one must have that $\partial\tilde{\Sigma}$ is the 4-dimensional sphere at transverse spatial infinity surrounding the 5-brane. We therefore get $P_{12345} = Q_m$ with all other independent components of P_5 equal to zero. This is of course exactly what was expected from the general analysis given above.

The relevance of the electric 2-form charge Q_2 and the magnetic 5-form charge P_5 for determining the supersymmetries of a p -brane configuration in $D = 11$ supergravity comes from their appearance in the ‘maximally extended’ $D = 11$ supersymmetry algebra (1.1) The algebra (1.1) can be derived using the Nester form of $D = 11$ supergravity [2] whence it becomes apparent that the *spatial* components of Z_2 and Y_5 are proportional to the components of Q_2 and P_5 , the explicit relations being $Z_2 = \kappa_{11}^{-\frac{14}{9}} Q_2$ and $Y_5 = \kappa_{11}^{-\frac{3}{9}} P_5$, the factors of κ_{11} being chosen to correct for dimensionality. The mixed time-space components of Z_2 and Y_5 are associated with charges in Kaluza-Klein vacua [14]. Ignoring these and setting also the spatial momentum P_M to zero, we can rewrite (1.1) in the Majorana representation, where $C = \Gamma^0$, as

$$\{Q, Q\} = \mathcal{M} + \frac{1}{2} \Gamma^{0M_1 M_2} Z_{M_1 M_2} + \frac{1}{5!} \Gamma^{0M_1 M_2 \dots M_5} Y_{M_1 M_2 \dots M_5}, \quad (2.7)$$

where \mathcal{M} is the ADM mass of the p -brane. This form of the $D = 11$ supersymmetry algebra is appropriate for a static p -brane configuration. It makes explicit the dependence of the algebra, and hence the supersymmetries of the p -brane, on the 2- and 5-form charges Q_2 and P_5 . In the next section, we shall formulate the Dirac quantisation condition for Q_2 and P_5 .

3 Dirac quantisation conditions for p -branes

The usual Dirac quantisation condition between electric and magnetic charges in four dimensions admits a straightforward generalisation to extended objects in higher dimensions. Specifically, if an electrically charged p -brane exists in the presence of its magnetically charged dual ($\tilde{p} = D - p - 4$)-brane, then by considering the phase of the wavefunction of the p -brane as it is transported around the \tilde{p} -brane, one can derive a quantisation condition⁶ between the electric and magnetic charges [3, 4], namely $Q_e Q_m = \text{integer} \times 2\pi\kappa^2$. Strictly

⁶See also [5] and references therein for more detailed discussions of Dirac quantisation conditions in string theory.

speaking, this is a quantisation condition for the *magnitudes* of the electric and magnetic charges only. However, as we have seen in the previous section, a ($p \neq 0$)-brane charge carries additional indices labeling the asymptotic directions along which the spatial section of the p -brane worldvolume is oriented. The magnitude of the charge together with the asymptotic spatial orientation of the p -brane are encoded in the p -form charge Q_p . It is always possible to find an asymptotically flat coordinate basis in which Q_p is proportional to a single basis p -form. This p -form points along the asymptotically flat spatial directions of the p -brane worldvolume. The magnitude of this p -form charge is equal to the electric or magnetic charge carried by the p -brane, *i.e.* $|Q_p| = Q_{e/m}$. The Dirac quantisation condition can then be rewritten in terms of the electric p -form charge Q_p and the magnetic \tilde{p} -form charge $P_{\tilde{p}}$ as $|Q_p| |P_{\tilde{p}}| = \text{integer} \times 2\pi\kappa^2$. Here, we want to generalise this condition by taking the asymptotic directions of the worldvolumes into account. To be precise, we shall show that there is a quantisation condition involving the magnitudes of Q_p and $P_{\tilde{p}}$ only if there is no overlap between the asymptotic orientations of the spatial sections of the electric p -brane and the magnetic \tilde{p} -brane worldvolumes.

To begin, let us recall how one arrives at the Dirac quantisation condition for p -branes [3, 4]. We take as an example the quantisation of the electric 2-brane charge and the magnetic 5-brane charge in $D = 11$. Suppose we bring a 2-brane ‘probe’ with charge $|Q_2| = Q_e$ into a 5-brane $D = 11$ supergravity background with charge $|P_5| = Q_m$. The 2-brane and the 5-brane share a common time direction, but for the moment we shall not make any assumptions about the relative orientation of their spatial worldvolume sections. We denote the 2-brane coordinates by $X^{\underline{M}} = (T, X^M)$ ($M = 1, 2, \dots, 10$); A_3 is the singular 3-form potential for the 4-form field strength F_4 in the presence of the 5-brane. Now consider deforming the 2-brane (*i.e.* a spatial section of the 2-brane worldvolume) through a spacelike path at a constant time, with identical initial and final 2-brane configurations. The chosen spatial section of the 2-brane worldvolume traces out a closed spacelike three-dimensional surface \mathcal{W} . Of course, the surface \mathcal{W} does not correspond to a physical motion of the 2-brane, but nevertheless it needs to be taken into account in a quantum-mechanical description (*e.g.* in a path integral formulation) of the 2-brane. As the 2-brane is taken around \mathcal{W} , the 2-brane’s wavefunction acquires a phase factor

$$\exp\left(\frac{iQ_e}{3!\kappa^2} \oint_{\mathcal{W}} A_{M_1 M_2 M_3} dX^{M_1} \wedge dX^{M_2} \wedge dX^{M_3}\right). \quad (3.1)$$

Here, the charge multiplying the integral is a canonically defined charge and hence we need the additional κ^{-2} factor (*c.f.* section 2).

Using Stokes' theorem, we can rewrite the integral of A_3 over \mathcal{W} as the integral of $F_4 = dA_3$ over any 'capping' surface \mathcal{M} whose boundary is \mathcal{W} , *i.e.* $\partial\mathcal{M} = \mathcal{W}$:

$$\begin{aligned} & \frac{Q_e}{3!\kappa^2} \oint_{\mathcal{W}} A_{M_1 M_2 M_3} dX^{M_1} \wedge dX^{M_2} \wedge dX^{M_3} \\ &= \frac{Q_e}{4!\kappa^2} \int_{\mathcal{M}} F_{M_1 M_2 M_3 M_4} dX^{M_1} \wedge dX^{M_2} \wedge dX^{M_3} \wedge dX^{M_4} \\ &= \frac{Q_e}{\kappa^2} \Phi_{\mathcal{M}}, \end{aligned} \tag{3.2}$$

where $\Phi_{\mathcal{M}}$ is the flux of F_4 through the cap \mathcal{M} . Taking two different choices \mathcal{M}_1 and \mathcal{M}_2 for the cap, one can end up with a discrepancy $\Phi_{\mathcal{M}_1} - \Phi_{\mathcal{M}_2}$ if the two caps taken together form a four-dimensional closed surface $\mathcal{M}_{\text{total}} = \mathcal{M}_1 \cup \mathcal{M}_2$ that captures the net flux from the 5-brane, which by Gauss' law is equal to the magnetic charge, *i.e.* $\Phi_{\text{total}} = Q_m$. This may be viewed as the discrepancy between zero and $\int_{\mathcal{M}_{\text{total}}} dA$, which would have vanished if the gauge potential A_3 for the 5-brane background had been everywhere non-singular. In the quantum-mechanical description, this discrepancy gives rise to a phase factor

$$\exp\left(i \frac{Q_e Q_m}{\kappa^2}\right) \tag{3.3}$$

in the wavefunction of the 2-brane probe. The Dirac quantisation condition arises from the requirement that this phase factor equal unity, *i.e.* that

$$Q_e Q_m = 2\pi\kappa^2 n, \quad n \in \mathbb{Z}. \tag{3.4}$$

The above derivation produces a Dirac quantisation condition involving the *magnitudes* of the 2-brane's electric 2-form charge Q_2 and the 5-brane's magnetic 5-form charge P_5 whenever the orientations of the 2-brane and the 5-brane allow one to construct a surface $\mathcal{M}_{\text{total}}$, with \mathcal{W} as the boundary separating \mathcal{M}_1 and \mathcal{M}_2 , that captures the 5-brane's total magnetic flux Φ_{total} .

Let us now take the asymptotic orientations of the 2-brane and 5-brane into account to assess exactly under what conditions one can construct $\mathcal{M}_{\text{total}}$. We can always choose an asymptotically flat coordinate system $x^{\underline{M}} = (t, x^M)$ such that the spatial section of the 5-brane worldvolume lies asymptotically along the $\{x^1, x^2, \dots, x^5\}$ directions. It follows from the general considerations in section 2 that the non-vanishing component of the magnetic 5-form charge P_5 is

$$P_{12345} = \int_{\partial\tilde{\Sigma}} F_4 = Q_m, \tag{3.5}$$

where $\partial\tilde{\Sigma}$ is topologically equivalent to the S^4 at transverse spatial infinity that surrounds the 5-brane. This identifies the closed surface that captures the total 5-brane flux as

$\mathcal{M}_{\text{total}} = S^4$. The two capping surfaces \mathcal{M}_1 and \mathcal{M}_2 for the 2-brane path \mathcal{W} must therefore correspond to the ‘northern’ and ‘southern’ hemispheres of the S^4 , with \mathcal{W} being the ‘equatorial’ S^3 . Note that $\mathcal{M}_{\text{total}}$ lies entirely within the asymptotic transverse space of the 5-brane.

Let’s now bring the 2-brane into play, which we assume to be asymptotically oriented along the $\{x^{M_1}, x^{M_2}\}$ directions, and see how we can generate a closed spacelike surface \mathcal{W} , by deforming the 2-brane configuration of fixed 2-form charge through a closed path \mathcal{W} which is topologically equivalent to an ‘equatorial’ S^3 at transverse spatial infinity.⁷ The asymptotic orientation of the 2-brane is part of its boundary conditions, and hence has to be maintained throughout the motion around the closed path.

Two distinctly different cases now arise, according to whether or not the asymptotic orientations of the spatial sections of the 2-brane and 5-brane worldvolumes partially coincide. Suppose first that they do coincide, *i.e.* that $M_1 \in \{1, 2, 3, 4, 5\}$ or $M_2 \in \{1, 2, 3, 4, 5\}$. Recall that this means that, as we approach spatial infinity on the 2-brane’s worldvolume, one (or both) of the tangent vectors to the 2-brane spatial worldvolume section starts pointing in directions along which the 5-brane is asymptotically oriented. It follows that *any* three-dimensional surface \mathcal{W} generated by taking the given 2-brane around a closed path shares the following property: there exists a subspace of \mathcal{W} with tangent vectors lying parallel to spatial infinity on the 2-brane worldvolume, for which one (or two) of these tangent vectors point along the directions of the asymptotic orientation of the 5-brane. Compare this to the ‘equatorial’ S^3 which we wish to generate. Its tangent vectors are everywhere linearly independent from those of the spatial worldvolume section of the 5-brane. To see this, it is useful to note that the ‘equatorial’ S^3 lies at transverse spatial infinity in the 5-brane spacetime, where the metric is *flat*, by the usual boundary conditions for the 5-brane $D = 11$ supergravity background. Alternatively, the geometry of the spacelike hypersurfaces of the 5-brane background at transverse spatial infinity has topology $\mathbb{R}^5 \times S^4$, where \mathbb{R}^5 corresponds to the flat spatial worldvolume of the 5-brane and S^4 to the four-sphere at infinity. Clearly, the ‘equatorial’ S^3 lies within the S^4 factor, which is everywhere transverse to the spatial worldvolume of the 5-brane. Thus, in order for the capping surface $\mathcal{M}_{\text{total}}$ to capture the 5-brane’s flux, the path \mathcal{W} must lie within the topological equivalence class of

⁷There are two ways to view the path \mathcal{W} . One is as the motion of a 2-brane held rigidly in the same shape as it is taken around in the 5-brane background, but compactifying the points at the 2-brane’s spatial infinity, so that $\mathbb{R}^2 \times S^1 \rightarrow S^3$. Another way to view \mathcal{W} is to keep the asymptotic configuration of the 2-brane fixed, but to deform its shape, expanding out a ‘bubble’ of 2-brane that then sweeps out an S^3 path surrounding the magnetic charge centre. The latter appears to be the point of view adopted in [3].

paths ‘surrounding’ the 5-brane as discussed in section 2, with tangent vectors everywhere independent of those of the 5-brane worldvolume. Any other path can be deformed into one for which the flux captured is zero. It follows that one cannot establish a quantisation condition for the electric 2-brane and magnetic 5-brane charges whenever the spatial worldvolumes of the 2-brane and 5-branes have an asymptotic coincidence of orientations. Such a 2-brane/5-brane configuration will be called Dirac-insensitive.

Whenever there is no asymptotic coincidence of orientations between the spatial sections of the 2-brane and the 5-brane worldvolumes, *i.e.* when $M_1 \notin \{1, 2, 3, 4, 5\}$ and $M_2 \notin \{1, 2, 3, 4, 5\}$, then the asymptotic directions along which the 2-brane is oriented will also lie in the asymptotic transverse space of the 5-brane. There will then be no topological considerations preventing the construction of $\mathcal{M}_{\text{total}}$ as a union of capping surfaces of \mathcal{W} . The above argument leading to the quantisation of the electric 2-brane and magnetic 5-brane charges then applies, and thus one obtains the quantisation condition (3.4) involving the magnitudes of the electric 2-form charge Q_2 and the magnetic 5-form charge P_5 .

One can verify the orientation dependence of the Dirac quantisation condition explicitly for a case in which the spatial worldvolumes of both the 2-brane and the 5-brane are strictly flat. A flat 5-brane oriented along the $\{x^1, x^2, \dots, x^5\}$ directions with charge Q_m is given by the following classical solution [13]

$$\begin{aligned} ds_{11}^2 &= H^{-\frac{1}{3}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{2}{3}} dy^m dy^m \\ F_4 &= *(d^6 x \wedge dH^{-1}), \end{aligned} \tag{3.6}$$

where H is a harmonic function of the $\{y^m\}$ with a pole strength such that $P_{12345} = Q_m$. Furthermore, let the flat 2-brane ‘probe’ be oriented along the $\{x^{M_1}, x^{M_2}\}$ directions and construct a closed spacelike surface \mathcal{W} as described above. The quantum mechanical phase factor (3.1) associated with \mathcal{W} now reduces to

$$\exp\left(\frac{iQ_e}{\kappa^2} \oint_{\mathcal{W}} \frac{\partial X^P}{\partial \sigma} A_{M_1 M_2 P}\right), \tag{3.7}$$

where the $\frac{\partial X^P}{\partial \sigma}$ vector points in the ‘third’ direction along the path \mathcal{W} , *i.e.* the one not lying along the spatial section of the 2-brane worldvolume; the coordinate volume element on \mathcal{W} has also been suppressed. Now note that since the non-vanishing components of F_4 in (3.6) all point in directions transverse to the 5-brane, one can always find a gauge in which the potential A_3 is entirely transverse as well. The phase factor (3.7) is then trivially equal to unity whenever M_1 or M_2 point along the spatial worldvolume of the 5-brane, because the integrand vanishes identically in such cases. This confirms that whenever the (strictly

flat) 2-brane and 5-brane overlap, one cannot establish a quantisation condition for their charges.

Let us now summarise the general situation for the $D = 11$ Dirac quantisation condition in $D = 11$ between the electric 2-form charge Q_2 and the magnetic 5-form charge P_5 . The requirement that there be no coincidence between the asymptotic orientations of the spatial worldvolume sections of the 2-brane and the 5-brane is formulated concisely as the condition that $Q_2 \wedge P_5 \neq 0$. In this case, the magnitudes $|Q_2| = Q_e$ and $|P_5| = Q_m$ obey a quantisation condition. Therefore, the Dirac quantisation condition, taking into account the asymptotic orientations of the 2-brane and the 5-brane, may be written in general

$$Q_2 \wedge P_5 = 2\pi\kappa^2 n \frac{Q_2 \wedge P_5}{|Q_2| |P_5|}, \quad n \in \mathbb{Z}. \quad (3.8)$$

Equivalently, one may express this condition as

$$(|Q_2| |P_5| - 2\pi\kappa^2 n) Q_2 \wedge P_5 = 0. \quad (3.9)$$

Note that (3.8) or (3.9) becomes vacuous whenever the 2-brane and the 5-brane asymptotically align, *i.e.* whenever $Q_2 \wedge P_5 = 0$. Such cases are precisely the Dirac-insensitive configurations. However, these configurations clearly form a subset of measure zero within the general configuration space of a 2-brane and a 5-brane. Since a small rotation is all that is necessary to change an insensitive configuration into one for which Dirac quantisation becomes applicable, generic 2-branes and 5-branes in $D = 11$ must all satisfy the Dirac condition.

It is nonetheless worth noting the existence of the Dirac insensitive configurations for several reasons. One is that this will be relevant for the comparison that we shall shortly make between higher and lower dimensional Dirac quantisation conditions in theories that are related by dimensional reduction. Another reason for taking note of the insensitive configurations is the observation that the insensitive configurations coincide with those of intersecting p -branes, which also possess zero-force properties related to the preservation of residual supersymmetry. Yet a third reason concerns the sharpness with which the p -form charges are defined in quantum mechanics.

One might consider that quantum fluctuations would smear out the orientation of a p -brane, so that the measure-zero set of Dirac-insensitive configurations might disappear at the quantum level. We shall not enter into a detailed discussion of the question, but shall be content to make some indicative observations as to why this insensitive set may nonetheless persist at the quantum level. We have seen that the conserved charges carried by p -branes

are naturally p -form objects, and these carry sharply-defined information about the object's asymptotic spatial orientation. Note that these p -form charges would have as conjugate variables the time-independent modes of the p -form gauge parameters Λ_p for the $(p+1)$ -form gauge potentials A_{p+1} ($\delta A_{p+1} = d\Lambda_p$). Since these gauge parameters do not contain physical degrees of freedom, one might expect there to be no inconsistency with having the p -form charges sharply defined at the quantum level. Another way to think of this is to recall that in order for a p -brane to carry a non-vanishing charge, it must either be infinite in extent or must be wrapped around a compact dimension, with the charge arising for essentially topological reasons whenever it is possible to 'capture' the p -brane with the boundary of the charge integration surface. Infinite p -branes have sharply-defined asymptotic orientations because their moments of inertia are infinite. Moreover, p -branes wrapped around compact dimensions have sharply-defined orientations when they are in their ground states, while their excited states have energies that tend to infinity as one shrinks the radius of the compact dimension. Accordingly, the Dirac-insensitive configurations may have a more persistent role than they might at first seem to have, even though they constitute only a subset of measure zero within the set of all p -brane configurations.

4 Dimensional reduction and Dirac quantisation conditions

Now let us consider the Dirac quantisation condition for p -branes in the context of dimensional reduction to $D = 4$. In this case, one has only to deal with electrically and magnetically charged particles (*i.e.* black holes). For this, one first needs to study how the charges of the higher-dimensional p -branes are related to those of their dimensionally reduced descendants.

The D -dimensional bosonic Lagrangian resulting from the dimensional reduction of eleven-dimensional supergravity takes the form

$$\begin{aligned} \mathcal{L} = & eR - \frac{1}{2}e(\partial\vec{\phi})^2 - \frac{1}{48}e e^{\vec{a}\cdot\vec{\phi}} F_4^2 - \frac{1}{12}e \sum_i e^{\vec{a}_i\cdot\vec{\phi}} (F_3^{(i)})^2 - \frac{1}{4}e \sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}} (F_2^{(ij)})^2 \\ & - \frac{1}{4}e \sum_i e^{\vec{b}_i\cdot\vec{\phi}} (\mathcal{F}_2^{(i)})^2 - \frac{1}{2}e \sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}} (F_1^{(ijk)})^2 - \frac{1}{2}e \sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}} (\mathcal{F}_1^{(ij)})^2 + \mathcal{L}_{FFA} , \end{aligned} \quad (4.1)$$

where the 'dilaton vectors' \vec{a} , \vec{a}_i , \vec{a}_{ij} , \vec{a}_{ijk} , \vec{b}_i , \vec{b}_{ij} are constants that characterise the couplings of the dilatonic scalars $\vec{\phi}$ to the various gauge fields. They are given by [15]

$$\begin{array}{ll}
F_{MNPQ} & \text{vielbein} \\
4 - \text{form} : & \vec{a} = -\vec{g} , \\
3 - \text{forms} : & \vec{a}_i = \vec{f}_i - \vec{g} , \\
2 - \text{forms} : & \vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g} , \quad \vec{b}_i = -\vec{f}_i , \\
1 - \text{forms} : & \vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g} , \quad \vec{b}_{ij} = -\vec{f}_i + \vec{f}_j ,
\end{array} \tag{4.2}$$

where

$$\vec{g} \cdot \vec{g} = \frac{2(11-D)}{D-2}, \quad \vec{g} \cdot \vec{f}_i = \frac{6}{D-2}, \quad \vec{f}_i \cdot \vec{f}_j = 2\delta_{ij} + \frac{2}{D-2}. \tag{4.3}$$

The field strengths are associated with the gauge potentials in the obvious way; for example F_4 is the field strength for A_3 , $F_3^{(i)}$ is the field strength for $A_2^{(i)}$, *etc.* The complete expressions for the Kaluza-Klein modifications to the various field strengths are given in [15], as are the cubic Wess-Zumino terms \mathcal{L}_{FFA} coming from the $F_4 \wedge F_4 \wedge A_3$ term in the eleven-dimensional Lagrangian. The eleven-dimensional and D -dimensional metrics are related by [15, 16]

$$ds_{11}^2 = e^{\frac{1}{3}\vec{g} \cdot \vec{\phi}} ds_D^2 + \sum_i e^{2\vec{\gamma}_i \cdot \vec{\phi}} (h^i)^2, \tag{4.4}$$

where $\vec{\gamma}_i = \frac{1}{6}\vec{g} - \frac{1}{2}\vec{f}_i$, and

$$h^i = dz^i + \mathcal{A}_1^i + \mathcal{A}_{0j}^i dz^j. \tag{4.5}$$

We shall define the electric and magnetic charges for each field strength to be the canonical Noether charges

$$Q_e = \int (e^{\vec{c} \cdot \vec{\phi}} * F + K(A)), \quad Q_m = \int \tilde{F}, \tag{4.6}$$

where $F = \tilde{F} + \dots$ is the field strength, with the ellipses representing the Kaluza-Klein modifications, $\tilde{F} = dA$, and \vec{c} is the dilaton vector corresponding to F , as given in (4.2). The term $K(A)$ represents the contributions coming from the Wess-Zumino terms \mathcal{L}_{FFA} in the D -dimensional Lagrangian. The vector \vec{c} denotes the dilaton vector for F , as given above. Let us now see how these charges are related to charges in $D = 11$. We begin by considering the cases where the D -dimensional field strengths come from the dimensional reduction of \hat{F}_4 in $D = 11$. From this, the following fields can arise in D dimensions: F_4 , $F_3^{(i)}$, $F_2^{(ij)}$ or $F_1^{(ijk)}$. The expansion of the eleven-dimensional 4-form field strength \hat{F}_4 in terms of the D -dimensional fields is as follows [15]:

$$\hat{F}_4 = F_4 + F_3^{(i)} \wedge h^i + \frac{1}{2} F_2^{(ij)} \wedge h^i \wedge h^j + \frac{1}{6} F_1^{(ijk)} \wedge h^i \wedge h^j \wedge h^k. \tag{4.7}$$

It is easy to show from (4.4) that the eleven-dimensional Hodge dual $\hat{*}$ of \hat{F}_4 is related to the D -dimensional Hodge duals $*$ of the D -dimensional fields by

$$\hat{*}\hat{F}_4 = e^{\vec{a}\cdot\vec{\phi}} * F_4 \wedge v + e^{\vec{a}_i\cdot\vec{\phi}} * F_3^{(i)} \wedge v^i + \frac{1}{2} e^{\vec{a}_{ij}\cdot\vec{\phi}} * F_2^{(ij)} \wedge v^{ij} + \frac{1}{6} e^{\vec{a}_{ijk}\cdot\vec{\phi}} * F_1^{(ijk)} \wedge v^{ijk} , \quad (4.8)$$

where we have defined

$$v = \frac{1}{(11-D)!} \epsilon_{i_1 \dots i_{11-D}} h^{i_1} \wedge \dots \wedge h^{i_{11-D}} , \quad v_i = \frac{1}{(10-D)!} \epsilon_{ii_2 \dots i_{11-D}} h^{i_2} \wedge \dots \wedge h^{i_{11-D}} , \quad (4.9)$$

$$v_{ij} = \frac{1}{(9-D)!} \epsilon_{ij i_3 \dots i_{11-D}} h^{i_3} \wedge \dots \wedge h^{i_{11-D}} , \quad v_{ijk} = \frac{1}{(8-D)!} \epsilon_{ijk i_4 \dots i_{11-D}} h^{i_4} \wedge \dots \wedge h^{i_{11-D}} .$$

From (4.6), one may then obtain the expressions for the eleven-dimensional charges in terms of the D -dimensional ones. The results are given in Table 1 below.

	F_4	$F_3^{(i)}$	$F_2^{(ij)}$	$F_1^{(ijk)}$
Electric $Q_e^{11} =$	$Q_e^D V$	$Q_e^D \frac{V}{L_i}$	$Q_e^D \frac{V}{L_i L_j}$	$Q_e^D \frac{V}{L_i L_j L_k}$
Magnetic $Q_m^{11} =$	Q_m^D	$Q_m^D L_i$	$Q_m^D L_i L_j$	$Q_m^D L_i L_j L_k$

Table 1: Relations between Q^{11} and Q^D

In Table 1, L_i denotes the period of the compactification coordinate z^i , and the compactification volume is $V = \int d^{11-D} \vec{z} = \prod_{i=1}^{11-D} L_i$. Note that the expressions for the eleven-dimensional charges in terms of the D -dimensional canonically-defined charges do not depend on the scalar moduli of the D -dimensional theory.

From the results in Table 1, we see that any Dirac quantisation condition between a conjugate pair of electric and magnetic charges in D dimensions, namely

$$Q_e^D Q_m^D = 2\pi \kappa_D^2 n , \quad (4.10)$$

agrees precisely with the Dirac quantisation condition

$$Q_e^{11} Q_m^{11} = 2\pi \kappa_{11}^2 n \quad (4.11)$$

between the membrane and the 5-brane charges in $D = 11$, since the gravitational couplings in the two cases are clearly related by

$$\kappa_{11}^2 = V \kappa_D^2 . \quad (4.12)$$

The result (4.11) gives the quantisation condition on the magnitudes of the membrane and 5-brane charges. As we have discussed in the previous section, the true quantisation

condition must take account of the form structure of the charges for the extended objects. The form indices correspond to the worldvolume spatial-section indices of the objects. Let us consider the special case where one reduces to $D = 4$, so that the conjugate pairs of objects subject to Dirac quantisation conditions are simply electric and magnetic black holes, for which the charges carry no indices. Then, from the Dirac quantisation condition for any electric and magnetic charge carried by one given 2-form field strength in $D = 4$, we can deduce the corresponding quantisation condition on membranes and 5-branes in $D = 11$. In particular, a black hole supported by an electric charge for the field strength $F_2^{(ij)}$ in $D = 4$ will oxidise to a membrane in $D = 11$ with spatial worldvolume coordinates z^i and z^j . Conversely, a magnetic black hole supported by the same field strength will oxidise to a 5-brane with world-volume directions complementary to these, *i.e.* $\{z^{k_1}, \dots, z^{k_5}\}$, where i, j, k_1, \dots, k_5 are all different [18]. Thus we obtain a Dirac quantisation condition involving the membrane and 5-brane charges when the worldvolume spatial sections of the two objects share no common directions. This agrees precisely with the results obtained in the previous section.

In order for a Dirac quantisation condition in a lower dimension D , as expressed in (4.10), to be inherited from the original $D = 11$ condition (4.11), one requires at each step of dimensional reduction a mixed combination of a diagonal dimensional reduction for one p -brane and a vertical dimensional reduction for the other. Otherwise, the electric and magnetic brane solutions in the lower dimension will not be supported by the same field strength, and so no Dirac quantisation condition will arise. If one persists nonetheless in making other combinations of dimensional reduction, the orientation-sensitivity of the Dirac condition (3.8) will give a nil result again, fitting in precisely with the expected pattern of quantisation conditions in lower dimensions.

For example, consider making a diagonal dimensional reduction for both a 2-brane and a 5-brane in $D = 11$. This will produce in $D = 10$ a 1-brane (*i.e.* a string) supported by a 3-form field strength and a 4-brane supported by a 4-form field strength, so there should be no Dirac quantisation condition. But taking into account the orientation sensitivity expressed in (3.8), one sees that this is precisely what happens, because in order to make a diagonal dimensional reduction for both of the $D = 11$ branes, they must share a spatial worldvolume direction, in which case (3.8) gives zero, as required.

As another example, consider making vertical dimensional reductions for a 2-brane and a 5-brane. Vertical dimensional reduction works by first making a ‘stack’ of branes in the compactification direction [20], in order to generate a translational symmetry in a

direction transverse to the worldvolume. Both electric and magnetic charges now need to be interpreted as charge densities per unit length in this compactification direction, in addition to the natural p -brane interpretation of charge as a density per unit p volume. Letting the period in the compact direction be L , the total phase factor for a closed-circuit deformation as discussed in section 3 will be $e^{i\kappa_{11}^{-2}Q_e Q_m L^2}$, with one factor of L in the exponent coming from each stack of branes. In the compactification limit, one has $\lim_{L \rightarrow 0} \frac{L^2}{\kappa_{11}^2} = 0$, so no Dirac quantisation condition is inherited in the lower dimension. This case should be contrasted with that of a mixed diagonal/vertical reduction, where only one of the branes is stacked up and with the corresponding charge being given a density interpretation in the compactification dimension. For such a mixed reduction configuration, one has a phase factor $e^{i\kappa_{11}^{-2}Q_e Q_m L}$, and, noting the relation (4.12) between the gravitational couplings, one obtains precisely the lower-dimensional Dirac quantisation condition (4.10). Note also that the relative orientations of the two branes in the higher dimension do not in this case correspond to a Dirac-insensitive configuration.

Thus we have seen that there is a complete accord between the form structure of the Dirac quantisation condition for p -branes (3.8) and the natural Dirac quantisation conditions occurring in lower dimensions, obtained from the higher-dimensional one by Kaluza-Klein dimensional reduction. The absence of Dirac conditions between descendant branes that are not duals of each other in the lower dimension does not mean, however, that there are no relations at all between the spectra of such branes. We shall see in sections 8 and 9 that there are ‘incidental’ relations between non-dual branes, which are implied by T duality together with the existence of certain ‘scale-setting’ p -branes.

5 Waves, NUTs and quantisation conditions

In the previous section, we showed how the Dirac quantisation rules for charges constructed using components of the $D = 11$ 4-form field strength are related under dimensional reduction. One should also consider the Dirac quantisation rules for charges constructed using the Kaluza-Klein vectors occurring in lower dimensions, and should investigate how these rules are related to the quantisation rules in $D = 11$. However, although objects carrying these charges in the lower dimension will be ordinary p -branes, *e.g.* D0-branes in $D = 10$, their oxidations to $D = 11$ will give either pp waves, in the case of electric charges, or Taub-NUT-like monopoles (*i.e.* NUTs), in the case of magnetic charges.

Let us begin by considering the reduction from $D = 11$ to $D = 10$. The metrics in the

two dimensions are related by

$$ds_{11}^2 = e^{\frac{1}{6}\phi} ds_{10}^2 + e^{-\frac{4}{3}\phi} (dz + \mathcal{A}_1)^2 . \quad (5.1)$$

The total canonically normalised momentum P_z carried by a wave moving in the z direction in eleven dimensions is given by [21]

$$P_z = \int \hat{*} d\hat{e}^z , \quad (5.2)$$

where the integral is restricted to the $11 - 2 = 9$ dimensional boundary of the spacelike hypersurface in $D = 11$ spacetime, $\hat{e}^z = e^{-\frac{2}{3}\phi} (dz + \mathcal{A}_1)$ is the vielbein for the eleventh direction, and $\hat{*}$ denotes the eleven-dimensional Hodge dual, as given earlier. Note that (5.2) defines the total momentum of a plane wave with translational invariance in the direction of propagation, which therefore would diverge in an uncompactified spacetime. At transverse spatial infinity, the dilaton ϕ tends to the constant value ϕ_0 , and so we find that the momentum (5.2) can be rewritten in a fashion ready for compactification as

$$P_z = e^{-\frac{5}{6}\phi_0} \int * \mathcal{F}_2 \wedge (dz + \mathcal{A}_1) , \quad (5.3)$$

where $*$ denotes the ten-dimensional Hodge dual. Since the electric charge in $D = 10$ is given by $Q_e = \int e^{-\frac{3}{2}\phi} * \mathcal{F}_2$, it follows that the total momentum of the pp wave in $D = 11$ is related to the electric charge of the Kaluza-Klein vector in $D = 10$ by

$$P_z = Q_e e^{\frac{2}{3}\phi_0} L , \quad (5.4)$$

where L is the period of the compactifying coordinate z . From (5.1) we see that the physical radius of the eleventh dimension is given by $R_{11} = e^{-\frac{2}{3}\phi_0} L / (2\pi)$, and so we have

$$P_z = \frac{Q_e L^2}{2\pi R_{11}} . \quad (5.5)$$

At the quantum level, the momentum of a wave must be quantised so that an integer number of wavelengths fit around the circle. Thus we must have that

$$P_z = \frac{n \kappa_{11}^2}{R_{11}} , \quad (5.6)$$

where the κ_{11}^2 factor compensates for the dimension (length)⁸ of the canonically normalised momentum (compare the relation between canonical and source charges given in section 2). Putting this together with (5.5), we see that the electric charge in $D = 10$ carried by the D0-brane obtained from the pp wave by dimensional reduction is quantised according to

$$Q_e^{10} = \frac{2\pi n \kappa_{11}^2}{L^2} = \frac{2\pi n \kappa_{10}^2}{L} . \quad (5.7)$$

Note that the quantisation of the D0-brane charge implied by these eleven-dimensional considerations is quite different from the usual kind of Dirac quantisation condition, in that it occurs without needing the existence of a conjugate magnetically-charged object. If, nevertheless, one does consider the magnetic charge at the same time, it becomes important to verify that the above quantisation is consistent with the Dirac quantisation condition. The magnetic dual of the D0-brane in $D = 10$ is a D6-brane. From the eleven-dimensional point of view, it is a NUT. The associated ‘NUT charge’ $Q_{\text{NUT}} = \int \mathcal{F}_2$ is classically quantised, in the sense that in order for the eleven-dimensional metric (5.1) to be non-singular, the period of z must satisfy $L = 1/k \int \mathcal{F}_2 = Q_{\text{NUT}}/k$, where k is any integer. Since the magnetic charge carried by the D6-brane in $D = 10$ is also given by $Q_{\text{m}} = \int \mathcal{F}_2 = Q_{\text{NUT}}$, it follows that we have a classical discretisation of the allowed magnetic charge values

$$Q_{\text{m}}^{10} = k L , \quad k \in \mathbb{Z} . \quad (5.8)$$

Thus, combining this with (5.7), we see indeed that one obtains a result that is consistent with the $D = 10$ Dirac quantisation condition, namely

$$Q_{\text{e}}^{10} Q_{\text{m}}^{10} = 2\pi \kappa_{10}^2 q , \quad (5.9)$$

where q is an integer.

An important distinction between the quantisations (5.7) and (5.8) for D0-branes and D6-branes in $D = 10$, as compared to the standard Dirac quantisation rules, is that an absolute scale is obtained for the individual electric and magnetic charges of the D0-branes and D6-branes. Usually, in a Dirac quantisation condition, there is an arbitrariness under which the unit of electric charge can be scaled up by any factor, with a simultaneous corresponding scaling down of the unit of magnetic charge. As we shall see in section 8, the absolute scale of the Kaluza-Klein pp wave quantisation will play an important rôle in giving all of the charges in type IIA theory the same scale, and this gives an important consistency condition for the conjectured existence of M-theory.

6 Dirac quantisation for dyonic and self-dual p -branes

Dyonic p -branes can arise in any even dimension, and self-dual p -branes can arise in $D = 4n + 2$ dimensions (in the case of Lorentzian spacetime signature). In fact, it is appropriate to divide the discussion of all dyonic p -branes into two classes corresponding to $D = 4n + 2$ and to $D = 4n$. We shall begin by considering dyons in $D = 4n$. First, we recall that in

$D = 4$, the electric and magnetic charges e and g carried by two dyons must satisfy the Dirac quantisation condition

$$e_1 g_2 - e_2 g_1 = 2\pi n . \quad (6.1)$$

An analogous result can also be obtained for dyonic membranes in $D = 8$. These membranes can be obtained from simple single-charge electric or magnetic membranes by making a transformation under the $SL(2, \mathbb{R})$ factor of the $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ global symmetry group of $D = 8$ maximal supergravity [22]. The Dirac quantisation rule for dyonic membranes can be derived simply by acting with an $SL(2, \mathbb{R})$ transformation on the standard $Q_e Q_m = 2\pi \kappa_8^2 n$ quantisation rule for a pure electric and a pure magnetic membrane. The result, which is $SL(2, \mathbb{R})$ invariant, is

$$Q_1^T \Omega Q_2 = 2\pi \kappa_8^2 n , \quad (6.2)$$

where $Q = (Q_e, Q_m)$, and Ω is the $SL(2, \mathbb{R})$ -invariant antisymmetric matrix

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \quad (6.3)$$

To be precise, as we have explained in section 2, the charges in this case really carry indices, corresponding to the world-volume directions for the two dyons. The quantisation condition (6.2) arises only in cases where the two individual dyonic membranes have non-aligned orientations of their spatial world-volumes, so that the wedge product of their charge forms is non-zero. It is worth recalling that, in both $D = 4$ and $D = 8$, the quantisation conditions become vacuous for the case of two dyons whose electric and magnetic charges are in the same ratio. One way in which this may be seen is by observing that a global symmetry transformation can be used to rotate both of the dyons simultaneously to a purely electric or purely magnetic form, for which there is no quantisation condition.

The situation is quite different in $D = 4n + 2$ dimensions. Let us first consider the $D = 6$ case. There are five 3-forms $F_3^{(i)}$ in $D = 6$ which, together with their duals, form a ten-dimensional irreducible vector representation under the global symmetry group⁸ $O(5, 5)$. The $O(5, 5)$ -invariant Dirac quantisation condition for dyonic strings with five electric charges \vec{Q} and five magnetic charges \vec{P} will then be

$$\frac{1}{2} (\vec{Q}_1 + \vec{P}_1 , \vec{Q}_1 - \vec{P}_1) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \vec{Q}_2 + \vec{P}_2 \\ \vec{Q}_2 - \vec{P}_2 \end{pmatrix} = 2\pi \kappa_6^2 n , \quad (6.4)$$

and so

$$\vec{Q}_1 \cdot \vec{P}_2 + \vec{Q}_2 \cdot \vec{P}_1 = 2\pi \kappa_6^2 n . \quad (6.5)$$

⁸See, for example, [16] for a detailed discussion of this symmetry. A general discussion of duality symmetries in dimensions $4k$ and $4k + 2$, including cases with matter coupling, may be found in [17].

If we specialise to the case where only one of the five field strengths carries the electric and magnetic charges, this result reduces to

$$Q_e^{(1)} Q_m^{(2)} + Q_e^{(2)} Q_m^{(1)} = 2\pi \kappa_6^2 n , \quad (6.6)$$

where $Q_e^{(1)}$ and $Q_m^{(1)}$ are the electric and magnetic charges of the first dyon, and $Q_e^{(2)}$ and $Q_m^{(2)}$ are the charges for the second dyon.

This *symmetric* quantisation condition may seem surprising, but one may also verify that the symmetric structure is correct for $(2k + 1)$ -forms in dimensions $D = 4k + 2$ by generalising the original quantum phase arguments of Ref. [23], noting that in $D = 4k + 2$ dimensions, one has (for Minkowski signature) $\tilde{F}_{2k+1} = F_{2k+1}$. In such cases, the phase acquired by taking one dyon with charges (Q_e, Q_m) around another with (electric, magnetic) sources (J_{2k}, \tilde{J}_{2k}) is $\kappa^{-2}(Q_e \int *J_{2k} + Q_m \int *\tilde{J}_{2k})$, confirming the symmetric structure.⁹

As a consequence of this symmetric structure, the quantisation condition for the six-dimensional dyonic strings survives even when both of the dyons have the same ratio of electric to magnetic charges.¹⁰ In particular when the electric and magnetic charges are equal, $Q = Q_e = Q_m$, so that the dyonic strings are self-dual, then the quantisation rule (6.6) becomes

$$Q^{(1)} Q^{(2)} = \pi \kappa_6^2 n . \quad (6.7)$$

From this, it follows that the charge of the self-dual string acquires an absolute scale, given by

$$Q = \kappa_6 \sqrt{\pi} m , \quad (6.8)$$

where m is an integer. As in the previous cases, the Dirac quantisation condition arises only when the two self-dual strings have non-aligned spatial world-volume directions.

Now let us consider the type IIB theory in $D = 10$, where there exists a self-dual 5-form field strength, which can support a self-dual 3-brane. The Dirac quantisation condition for

⁹Although not widely commented upon in the literature, this symmetric structure has been recognised in a number of contexts, including in particular the basic case of $D = 2$, and in the $O(5, 5)$ -invariant six-dimensional supergravity [24]. A clear treatment of the dyonic quantisation condition in $D = 4k + 2$ dimensions based upon the requirement that Dirac strings be unobservable has been given in [25]. This argument fixes the overall coefficients in the conditions (6.4–6.6). We shall see in Section 8 that this coefficient is also essential for consistency with the ordinary Dirac condition obtained in one less dimension by dimensional reduction from (6.4–6.6).

¹⁰Naively, one might think that dyons of equal charge-ratio could be rotated to purely electric or purely magnetic complexions, just as can be done in $4n$ dimensions. However this is not possible here, since there is no duality symmetry between electric and magnetic strings coupled to the same field strength (see the Appendix).

the self-dual 3-brane is once again symmetric, and analogously to the case of the self-dual string in $D = 6$, one finds that the charge Q satisfies the condition

$$Q^{(1)} Q^{(2)} = \pi \kappa_{10}^2 n . \quad (6.9)$$

This implies that there is again an absolute scale for the charge of the self-dual 3-brane, namely

$$Q = \kappa_{10} \sqrt{\pi} m , \quad (6.10)$$

where m is an integer.

One way to understand the origin of the quantisation condition (6.9) for self-dual 3-branes is by studying the Dirac quantisation condition for black holes in $D = 4$ that arise from 3-branes wrapping around the internal coordinates. The 2-form field strengths $F_2^{(\alpha\beta)}$ in $D = 4$, where the usual $i, j \dots$ indices have been split as $i = (1, 2, \alpha)$, *etc.*, can also be obtained by dimensional reduction from the self-dual 5-form in the type IIB theory. In particular, this implies that both electric and magnetic black holes using a given $F_2^{(\alpha\beta)}$ can be oxidised to give rise to self-dual 3-branes in $D = 10$. Furthermore, in $D = 4$ only the electric and magnetic charges for the *same* field strength $F_2^{(\alpha\beta)}$ suffer a Dirac quantisation condition. As we saw in section 3, this pair of black holes will oxidise to give a pair of 3-branes in $D = 10$ with no overlapping spatial worldvolume coordinates, in accordance with the discussion in section 2 showing that Dirac quantisation conditions arise only in cases with no spatial orientation coincidences. Thus, the quantisation condition (6.9) for the self-dual 3-brane should be obtained as a consequence of the ordinary $D = 4$ quantisation condition for black holes.

In order to make the relation between quantisation conditions in the various dimensions more precise, one needs to be careful with certain factors that arise in the dimensional reduction process for self-dual fields. The dimensional reduction of the $D = 10$ self-dual 5-form yields both a 4-form H_4 and a 5-form H_5' in $D = 9$; the $D = 10$ self-duality condition then yields the $D = 9$ condition that the 5-form is in fact the dual of the 4-form: $H_5' = \tilde{H}_4$. Upon eliminating H_5' from the formalism, one obtains a theory involving only a 4-form H_4 , but this form does not yet have the canonical normalisation. In order to correct for its non-standard normalisation, one needs to rescale it: $H_4^{\text{canonical}} = \sqrt{2}H_4$. Accordingly, charges defined using $H_4^{\text{canonical}}$ and its further dimensional descendants need to be scaled up by $\sqrt{2}$ in order to be compared with the charge scale set for the $D = 10$ self-dual 3-brane by (6.10); for example, for the $D = 9$ electric 2-brane and the dual magnetic 3-brane, both

supported by $H_4^{\text{canonical}}$, one obtains the quantisation condition

$$Q_e^9 Q_m^9 = 2\pi \kappa_9^2 n , \quad (6.11)$$

as expected. Further dimensional reductions of the pure electric and pure magnetic solutions in $D = 4$ proceed as discussed earlier, with the relations between the charges as given in Table 1. Accordingly, the $D = 10$ quantisation condition (6.9) both implies and is implied by the standard $D = 4$ black-hole Dirac quantisation conditions.

A similar argument to the above, in which a self-dual string in $D = 6$ is obtained by oxidation from a dual pair of 2-charge black-holes in $D = 4$, gives the quantisation condition (6.7). To see this, we note that a self-dual string can be first diagonally and then vertically reduced to an electric black hole in $D = 4$. A second self-dual string with no spatial orientation coincidences relative to the first can first be vertically and then diagonally reduced, giving the magnetic dual black hole in $D = 4$ that is dual to the electric one first obtained. The Dirac quantisation condition for the electric/magnetic black holes in $D = 4$ then implies the quantisation rule for non-aligned self-dual strings in $D = 6$, after taking into account a factor of $1/\sqrt{2}$ rescaling analogous to that discussed above for the $D = 10$ 3-brane case.¹¹ By taking the electric and magnetic charges on the black holes in $D = 4$ to be independent, one can similarly derive the Dirac quantisation rule (6.6) for dyonic strings in $D = 6$.

7 Quantisation conditions, intersections and supersymmetry

In the previous sections, we have seen that there are several subtleties that arise in the Dirac quantisation conditions for extended objects. In particular, the relative orientation of a conjugate pair of electric and magnetic objects is important for determining whether or not a quantisation condition between them arises. Relative orientations are also important in the discussion of intersecting p -branes. In fact, the fraction of preserved supersymmetry is determined by the relative orientation. This suggests that there is a close relationship in M-theory or in type II string theory between the preservation of supersymmetry, intersections and the Dirac quantisation conditions for extended objects.

Intersecting p -branes can be viewed as the higher-dimensional oxidations of multi-charge p -branes in a lower dimension. We shall concentrate in particular on intersecting p -branes that reduce to multi black hole solutions in $D = 4$. Since supersymmetry is preserved under

¹¹The details of the dimensional reduction of self-dual strings in $D = 6$ have been given in Ref. [26].

dimensional reduction, it follows that the relationships that we have mentioned above can equally well be addressed in a four-dimensional framework. In $D = 4$, there are a total of 28 2-form field strengths. Thus there can be in total 56 charges, comprised of 28 electric and 28 magnetic charges. Any pair of these can be used to construct a 2-charge black hole solution. However, there are three quite distinct kinds of 2-charge solution that can arise, which can be distinguished by the fraction of supersymmetry that they preserve. Let us suppose that the dilaton vectors for the two field strengths are \vec{c}_1 and \vec{c}_2 . We may then define the quantity $\tilde{\Delta}$ by

$$\tilde{\Delta} = \frac{1}{4}(\epsilon_1 \vec{c}_1 + \epsilon_2 \vec{c}_2)^2 + 1, \quad (7.1)$$

where ϵ is $+1$ if the field carries an electric charge, and -1 if it carries a magnetic charge. The three possible kinds of 2-charge black-hole can then be characterised by $\tilde{\Delta}$, as given in Table 2 below [18].

	Mass	Bogomol'nyi eigenvalues	Supersymmetry
$\tilde{\Delta} = 3$	$m = \sqrt{Q_1^2 + Q_2^2}$	$\mu = m \pm \sqrt{Q_1^2 + Q_2^2}$	$\frac{1}{2}$
$\tilde{\Delta} = 2$	$m = Q_1 + Q_2 $	$\mu = m \pm Q_1 \pm Q_2 $	$\frac{1}{4}$
$\tilde{\Delta} = 1$	$m = (Q_1^{2/3} + Q_2^{2/3})^{3/2}$	$\mu = m \pm \sqrt{Q_1^2 + Q_2^2}$	0

Table 2: The three kinds of 2-charge black hole

The $\tilde{\Delta} = 3$ solution is nothing but a transformation of a single-charge black hole under an $SL(2, \mathbb{R})$ subgroup of the E_7 U-duality group, and so it preserves $\frac{1}{2}$ of the supersymmetry. The $\tilde{\Delta} = 2$ solution is a ‘genuine’ 2-charge solution, which cannot be reduced to a single-charge solution by any duality transformation. As usual for such solutions, it preserves $\frac{1}{4}$ of the supersymmetry. The $\tilde{\Delta} = 1$ solution is a dyonic black hole [27] where the electric and magnetic charges are carried by the same field strength. This solution preserves no supersymmetry. In all cases the fractions of preserved supersymmetry can be read off from the eigenvalues of the Bogomol’nyi matrix, which is constructed as the anticommutator of the eleven-dimensional supercharges [28, 15]. Specifically, each of its zero eigenvalues corresponds to an unbroken component of supersymmetry. The $\tilde{\Delta} = 3, 2$ and 1 solutions are described in terms of 1, 2 and 0 harmonic functions respectively.

As we have discussed previously, the Dirac quantisation conditions in four dimensions arise only between a pair of electric and magnetic charges that are carried by the same field strength. As we have seen above, such a pair of charges arise in the $\tilde{\Delta} = 1$ dyonic black hole

solution. Oxidising the three kinds of 2-charge black holes back to $D = 11$, they all become intersections of M-branes, waves or NUTs. In particular, a membrane and a 5-brane can have three possible kinds of intersection, with three kinds of relative orientation, depending on whether they originate from $\tilde{\Delta} = 3, 2$ or 1 black holes in four dimensions. These can be called non-harmonic, harmonic and non-supersymmetric intersections respectively [18]. Specifically, the non-harmonic intersections occur when the membrane and 5-brane share two common spatial world-volume coordinates; in harmonic intersections they share one such coordinate; and in the non-supersymmetric intersections there are no common spatial world-volume coordinates. Thus we see that whenever the relative orientations of the membrane and 5-brane are such that they give supersymmetric intersections, there is no Dirac quantisation condition between them.

Note that although we have taken the case of a membrane and a 5-brane as an example, a similar conclusion applies to all the other possible intersections in $D \leq 11$, namely: brane intersections that preserve any degree of supersymmetry are not subject to Dirac quantisation conditions. While we shall not give a formal proof of the correspondence between supersymmetric and Dirac-insensitive configurations, one could most likely be worked out by exploiting the relation between the Dirac quantisation conditions and the quantisation of angular momentum in the presence of the field corrections to the angular momentum of dual pairs of electric and magnetic objects, generalising the classic discussion for dyonic point particles [19].

8 Dirac quantisation conditions and the M-theory conjecture

So far we have discussed the quantisation conditions for a variety of different charges. For some, like those of Kaluza-Klein waves and NUTs, the electric and magnetic charges are separately discretised in their own right, independent of (but consistent with) any Dirac quantisation condition. In other words, the absolute units of electric and magnetic charge are separately determined in these cases. In the case of self-dual p -branes, the absolute unit of charge is also determined, by virtue of the Dirac quantisation condition and its symmetric form in $D = 4k + 2$ dimensions. For more general p -branes, however, the charges satisfy only the ordinary Dirac quantisation condition, and this fixes only the *product* of the electric and magnetic charges associated with each field strength, leaving the absolute units for each of the charges undetermined. Thus in D -dimensions, the most general minimum-charge

solution to the Dirac quantisation condition $Q_e Q_m = 2\pi\kappa_D^2$ is given by¹²

$$Q_e = (2\pi)^\gamma \kappa_D^{2(D-n-1)/(D-2)}, \quad Q_m = (2\pi)^{1-\gamma} \kappa_D^{2(n-1)/(D-2)}. \quad (8.1)$$

where γ is a free parameter. Here we have made use of the fact that the charges and κ_D have the following engineering dimensions

$$[Q_e] = L^{D-n-1}, \quad [Q_m] = L^{n-1}, \quad [\kappa_D] = L^{D/2-1}. \quad (8.2)$$

The degree of freedom parameterised by γ is obviously undesirable in a theory that is believed to be fundamental. For example, in the type IIA string theory there are three field strengths in all, namely F_4 , $F_3^{(1)}$ and $F_2^{(1)}$, whose electric and magnetic charges are each subject to Dirac quantisation conditions. There are therefore three as-yet undetermined parameters γ_n associated with the absolute charge scales of the three pairs of minimum electric and magnetic charges, namely

$$Q_{e(n)}^A = (2\pi)^{\gamma_n} \kappa_A^{(9-n)/4}, \quad Q_{m(n)}^A = (2\pi)^{1-\gamma_n} \kappa_A^{(n-1)/4} \quad (8.3)$$

for the field strengths of degree $n = 4, 3$ and 2 , where the A superscripts indicate charges in the type IIA theory. On the other hand, string theory is supposed to have only one free parameter, namely the string tension. Purely within the string theory itself, it is hard to see how these extra parameters in the spectrum of states can be fixed.

It has been conjectured that the strong-coupling limit of type II string theory is described by a theory in eleven dimensions, known as M-theory. Its low-energy limit is described by the usual eleven-dimensional supergravity Lagrangian. The membrane in $D = 11$ can be double-dimensionally reduced to give the perturbative NS-NS string of the type IIA theory in $D = 10$. It can alternatively be vertically reduced to give an R-R membrane in $D = 10$. Similarly, the 5-brane in $D = 11$ can be vertically reduced to an NS-NS 5-brane in $D = 10$, or diagonally reduced to an R-R 4-brane. The M-theory conjecture implies that the string and membrane charges in $D = 10$ have scale sizes related by the compactification period L_1 , since, as we showed in section 3, dimensional reduction yields canonically defined charges related as shown in Table 1. Similarly, the 5-brane and 4-brane charges in $D = 10$ should have related scale sizes. Furthermore, the D0-brane and D6-brane in type IIA theory now are interpreted as the dimensional reductions of the Kaluza-Klein waves and NUTs, whose charge discretisations are, as we have seen in section 5, already absolutely determined. Thus

¹²Note that in the following discussion, we shall always present only the minimum charge units. The allowed charges are then any integer multiples of these minimum units.

with the introduction of M-theory the number of undetermined charge parameters in the type IIA string is reduced from three to only one, namely

$$Q_{e(4)}^M = (2\pi)^\alpha \kappa_{11}^{4/3} , \quad Q_{m(4)}^M = (2\pi)^{1-\alpha} \kappa_{11}^{2/3} . \quad (8.4)$$

Here the superscript M indicates M-theory charges, and the numerical subscripts on the charges indicate the degree of the associated field strength. The duality between the type IIA string and M-theory implies that $\kappa_{11}^2 = \kappa_A^2 L_1$, where L_1 is the period of the compactifying coordinate z_1 . Furthermore, we can express the three parameters γ_n in (8.3) in terms of the single parameter α , by equating the type IIA charges to those coming from the dimensional reduction of M-branes, waves and NUTs in $D = 11$. Thus we have

$$(2\pi)^{1-\gamma_1} = \left(\frac{\kappa_{11}^2}{L_1^9}\right)^{-1/8} , \quad (2\pi)^{\alpha-\gamma_2} = \left(\frac{\kappa_{11}^2}{L_1^9}\right)^{1/12} , \quad (2\pi)^{\gamma_3-\alpha} = \left(\frac{\kappa_{11}^2}{L_1^9}\right)^{1/24} . \quad (8.5)$$

Although M-theory reduces the three free parameters in the spectrum of the type IIA string to one, there still seems to be no mechanism within the theory itself for determining the absolute quantised values of the M-brane charges. The situation changes, however, if one also takes into account the T-duality that relates the type IIA and type IIB strings when they are compactified on a circle. In fact this perturbative T-duality enables us to fix the absolute scales of all the charges. To see this, recall that the type IIB theory has the following field content in $D = 10$: the NS-NS sector comprises the metric, the dilaton and a 2-form potential, while the R-R sector comprises an axion, another 2-form potential and a 4-form potential whose field strength is self-dual. Compactification of the type IIB theory on a circle gives a theory that can be related to the compactification of the type IIA theory on a circle of inverse radius. The relations between the gauge potentials of these two nine-dimensional theories (including the axions) are summarised in Table 3.

	IIA		T-duality	IIB	
	$D = 10$	$D = 9$		$D = 9$	$D = 10$
R-R fields	A_3	A_3	\longleftrightarrow	A_3	\mathbf{B}_4
		$A_2^{(1)}$	\longleftrightarrow	$A_2^{\mathbf{R}}$	$A_2^{\mathbf{R}}$
	$\mathcal{A}_1^{(1)}$	$\mathcal{A}_1^{(1)}$	\longleftrightarrow	$A_1^{\mathbf{R}}$	
		$\mathcal{A}_0^{(12)}$	\longleftrightarrow	χ	χ
NS-NS fields	$G_{\mu\nu}$	$\mathcal{A}_1^{(2)}$	\longleftrightarrow	A_1^{NS}	A_2^{NS}
		$A_2^{(1)}$	\longleftrightarrow	A_2^{NS}	
	$A_1^{(12)}$	\longleftrightarrow	\mathcal{A}_1	$G_{\mu\nu}$	

Table 3: Gauge potentials of type II theories in $D = 10$ and $D = 9$

The relation between the dilatonic scalars of the two nine-dimensional theories is given by

$$\begin{pmatrix} \phi \\ \varphi \end{pmatrix}_{IIA} = \begin{pmatrix} \frac{3}{4} & -\frac{\sqrt{7}}{4} \\ -\frac{\sqrt{7}}{4} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} \phi \\ \varphi \end{pmatrix}_{IIB}. \quad (8.6)$$

The dimensional reduction of the ten-dimensional string metric to $D = 9$ is given by

$$\begin{aligned} ds_{\text{str}}^2 &= e^{-\frac{1}{2}\phi} ds_{10}^2 \\ &= e^{-\frac{1}{2}\phi} (e^{\varphi/(2\sqrt{7})} ds_9^2 + e^{-\sqrt{7}\varphi/2} (dz_2 + \mathcal{A})^2), \end{aligned} \quad (8.7)$$

where ds_{10}^2 and ds_9^2 are the Einstein-frame metrics in $D = 10$ and $D = 9$. The radius of the compactifying circle, measured using the ten-dimensional string metric, is therefore given by $R = e^{-\frac{1}{4}\phi - \sqrt{7}\varphi/4}$. It follows from (8.6) that the radii R_{IIA} and R_{IIB} of the compactifying circles, measured using their respective ten-dimensional string metrics, are related by $R_{IIA} = 1/R_{IIB}$.

Now as we discussed in section 6, there is an absolute charge quantisation for the self-dual 3-brane supported by the self-dual 5-form field strength in type IIB. On other hand, the charges for the NS-NS and R-R 3-form field strengths each have an as-yet undetermined scale parameter. Thus the type IIB charges in $D = 10$ are given by

$$\begin{aligned} Q_{(5)}^B &= \sqrt{\pi} \kappa_B, \\ Q_{e(3)}^i &= (2\pi)^{\beta_i} \kappa_B^{3/2}, \quad Q_{m(3)}^i = (2\pi)^{1-\beta_i} \kappa_B^{1/2}. \end{aligned} \quad (8.8)$$

where $i = 1$ and 2 label the charges of the NS-NS and the R-R 3-form field strengths respectively. Note that we have omitted the charge for the axion χ , which we shall discuss

later. Thus before applying the T-duality relation, the charges in M-theory and in type IIB have a total of three parameters, namely α , β_1 and β_2 .

The dimensional reductions of M-theory and the type IIB theory to $D = 9$ give rise to a total of six field strengths with degrees ≥ 2 in each case, as presented in Table 3. The T-duality between the two theories implies that the electric charges and the magnetic charges for the related pairs of IIA and IIB fields, as indicated in Table 3, should be set equal. This gives rise to a total of twelve equations of constraint. However, the product of the electric and magnetic charges for each field strength is the same, simply giving rise to a constraint between the Newton constants of the theories:

$$\frac{\kappa_{11}^2}{L_1 L_2} = \frac{\kappa_A^2}{L_2} = \frac{\kappa_B^2}{L_B} . \quad (8.9)$$

(We are denoting by L_1 and L_2 the periods of the compactifying coordinates z_1 and z_2 in the descent from M-theory, and by L_B the period of the compactifying coordinate in the descent from the type IIB theory.) This leaves us with six equations still to consider. The dimensional reduction for canonically-defined charges was given in Table 1. Without loss of generality, we shall present the equations relating just the electric charges of the two theories, following the order presented in Table 3. Thus for the R-R sector we have

$$\begin{aligned} \frac{(2\pi)^\alpha \kappa_{11}^{4/3}}{L_1 L_2} &= \sqrt{2\pi} \kappa_B , \\ \frac{(2\pi)^\alpha \kappa_{11}^{4/3}}{L_1} &= \frac{(2\pi)^{\beta_2} \kappa_B^{3/2}}{L_B} , \\ \frac{2\pi \kappa_{11}^2}{L_1^2 L_2} &= (2\pi)^{\beta_2} \kappa_B^{3/2} , \end{aligned} \quad (8.10)$$

and for NS-NS sector we have

$$\begin{aligned} \frac{2\pi \kappa_{11}^2}{L_1 L_2^2} &= (2\pi)^{\beta_1} \kappa_B^{3/2} , \\ \frac{(2\pi)^\alpha \kappa_{11}^{4/3}}{L_2} &= \frac{(2\pi)^{\beta_1} \kappa_B^{3/2}}{L_B} , \\ (2\pi)^\alpha \kappa_{11}^{4/3} &= \frac{2\pi \kappa_B^2}{L_B^2} . \end{aligned} \quad (8.11)$$

The solution to the equations (8.9), (8.10) and (8.11) is given by

$$\begin{aligned} \kappa_{11}^2 &= \frac{1}{2\pi} (L_1 L_2 L_B)^3 , \quad \kappa_A^2 = \frac{1}{2\pi} L_1^2 (L_2 L_B)^3 , \quad \kappa_B^2 = \frac{1}{2\pi} (L_1 L_2)^2 L_B^4 , \\ \alpha &= -\frac{2}{3} , \quad (2\pi)^{3-4\beta_1} = \left(\frac{L_2}{L_1}\right)^2 , \quad (2\pi)^{3-4\beta_2} = \left(\frac{L_1}{L_2}\right)^2 . \end{aligned} \quad (8.12)$$

Thus we see that all three of the originally-free charge-scale parameters α , β_1 and β_2 are determined in terms of parameters within the theories, namely the periods of the compactifying coordinates. (Note that the three Newton constants are also expressed in terms of

the periods.) In particular, this has the consequence that the 2π factors for all the electric and magnetic charges cancel out, and thus all the minimum charge units can be expressed purely in terms of products of certain powers of the three periods L_1 , L_2 and L_B . For example, in M-theory we have

$$Q_{e(4)}^M = (L_1 L_2 L_B)^2, \quad Q_{m(4)}^M = L_1 L_2 L_B, \quad (8.13)$$

and in the type IIB string the NS-NS and R-R 3-form charges are given by

$$\begin{aligned} Q_{e(3)}^1 &= L_2 L_1^2 L_B^3, & Q_{m(3)}^1 &= L_2 L_B, \\ Q_{e(3)}^2 &= L_1 L_2^2 L_B^3, & Q_{m(3)}^2 &= L_1 L_B. \end{aligned} \quad (8.14)$$

It is easy to work out the analogous expressions for all the charges in the type IIA and type IIB theories, as functions of the three periods.

Note that the L periods are neither dynamical quantities nor moduli, and they are fixed independently of any specific solutions to the lower-dimensional equations of motion. Since they arise in combination with exponentials of the dilatonic scalars in (4.4), it follows that their values can be adjusted by field redefinitions in which the dilatonic scalars are shifted by constants. Since the periods have dimensions of length, they cannot be absorbed completely into the dilaton exponentials, but one can however fix their values in any convenient fashion, without loss of generality. A convenient choice is to take them all to be equal, and then it follows from (8.12) that we have

$$L_i = L_B \equiv L = (2\pi \kappa_{11}^2)^{1/9}. \quad (8.15)$$

This eliminates all the free parameters, with all charge units now being expressed purely in terms of the eleven-dimensional Newton constant κ_{11} . Thus in this convention, we find that the above analysis of the T-duality relation between the type IIA and IIB theories implies that all the canonical charges are absolutely determined, and their minimum units are given by

$$Q_e = L^{D-n-1}, \quad Q_m = L^{n-1}. \quad (8.16)$$

The T-duality relationship for the quantised charges also provides supporting evidence for the $SL(2, \mathbf{Z})$ U-duality of the type IIB theory. In particular, as we have shown, in the convention of (8.15) the units for the NS-NS and R-R charges are not independent parameters, rather they are equal. In this convention, the $SL(2, \mathbf{Z})$ group elements, under which the NS-NS and R-R charges form a doublet, are integers. On other hand, the 7-brane in the type IIB theory (supported by the axion χ) also (independently) discretises

the continuous $SL(2, \mathbb{R})$ global symmetry group to $SL(2, \mathbb{Z})$, owing to the topological structure of the solution. Indeed, in the latter case the group elements are also pure integers, since it is easy to verify that the allowed magnetic charges for the axion χ are now purely integral, owing to the T-duality. This is necessary for consistency with the conjectured non-perturbative $SL(2, \mathbb{Z})$ symmetry of the type IIB theory.

A similar, but somewhat different, discussion of the relations between the charges (or tensions) of the D-branes and the M-branes has been given in Refs [30, 6, 7]. This argument also makes use of T-duality to relate all the Dp -branes in the type IIA and type IIB theories, starting from M theory (*i.e.* from $D = 11$ supergravity). Specifically, the self-dual D3-brane in $D = 10$ type IIB theory can be dimensionally reduced either diagonally to a D2-brane or vertically to a D3-brane in $D = 9$. Mapping these over to the type IIA theory, and then oxidising them back to $D = 10$, they become a dual D2-brane and D4-brane pair. This give rise to another condition on the charges, namely on their quotient, in addition to the Dirac condition on their product. In fact, using the T-duality that relates a Dp -brane to a $D(p + 1)$ -brane, one can deduce from this that all of the R-R charges are integer multiples of absolutely-determined fundamental units.

One may next relate the NS-NS and R-R charges using the $SL(2, \mathbb{Z})$ duality symmetry of type IIB theory, under which the NS-NS and R-R 2-form potentials form a doublet. Another way to do this is to consider M-theory, in which the D2-brane and the D4-brane can be oxidised up to $D = 11$, where they become a membrane and a 5-brane. Consequently, the two M-branes are both related to the type IIB self-dual D3-brane, from which one acquires an additional constraint [30, 6]. The M-branes can also be dimensionally reduced to give rise to an NS-NS string and a 5-brane in $D = 10$, which implies that the NS-NS and R-R charges should be related.

In this paper we made use of additional considerations, namely that the charges associated with Kaluza-Klein vectors are absolutely determined by mechanisms that go beyond Dirac quantisation. In this case not only the product, but also the absolute units of each of the electric and magnetic charges, are separately fixed. This, together with the absolute quantisation of the self-dual charges, implies that all of the charges in the theory are absolutely determined, after applying the T-duality relation. This is to be expected, since in a fundamental theory there should be no further free parameters governing the charge spectrum, other than the coupling constant of the theory. It is important to note that M-theory alone is not enough to fix the charge lattice completely, and hence M-theory and the type IIB string are both necessary parts of the full theory in which all of the charges

are fixed.

It is interesting, by way of comparison, to analyse the situation in the case where M-theory is not introduced. Then, the type IIA charge units are parameterised by the three quantities γ_n , with $n = 2, 3$ and 4 for the charges (8.3) associated with the field strengths of degree n . In this case, the type IIA and type IIB theories have, *a priori*, a total of five parameters characterising the charge lattices. Applying the T-duality relation between the type IIA and type IIB theories leaves us with two free parameters, one in the NS-NS sector and the other in the R-R sector. Invoking the $SL(2, Z)$ symmetry of the type IIB theory enables us relate the NS-NS and R-R charges, which removes one more free parameter. Thus we see that, purely from a string perspective, the charge spectrum of the theory is not uniquely fixed, since one free parameter remains. It is interesting to note that M-theory by itself also has one free parameter that sets the scales in *its* charge lattices. However, when the duality symmetries relating M-theory, the type IIA string and the type IIB string are all utilised, the quantised units of all charges become absolutely determined.

9 Charge lattices and quantisation conditions

Throughout this paper, we have been considering the quantisation conditions for canonically defined charges descended from the $D = 11$ electric and magnetic charges (4.6) together with the wave/NUT and $D = 10$ 3-brane charge. These conditions have the effect of restricting the classically allowed families of p -brane solutions to a discrete set at the quantum level, with the allowed charges forming a charge lattice. In $D = 11$, there are conjugate electric and magnetic charge lattices, which are effectively one-dimensional since the quantisation condition (3.9) involves only the magnitudes of the electric and magnetic charge forms (for Dirac-sensitive orientations). We have seen that the unit of the one-dimensional electric charge lattice (and also the magnetic charge lattice) can be determined using T-duality between the type IIA and type IIB theories.

By the arguments given in the last section, all of the lower-dimensional charge lattices for branes of differing dimension p also prove to be integers times a fundamental unit given by (8.16). It should be emphasised, however, that if one considers a specific lower-dimensional maximal supergravity in isolation, without regard to its higher-dimensional origin, then the charge lattices for non-dual pairs of p -branes will not necessarily be related. This is because the Dirac quantisation condition is invariant under the global symmetry group of the theory. Consequently, there is no unique charge-lattice solution to the Dirac quantisation condition

alone; from any given Dirac-allowed lattice, others can be constructed by the action of the global symmetry group. In particular, there would seem to be no *a priori* reason even why the same Dirac-allowed lattice must arise at different points in the modulus space of the scalar fields. On the other hand, as we showed in the previous section, if the lower dimensional theory is viewed as a dimensional reduction from eleven dimensions, then the charge lattice in the lower dimension is fixed uniquely, and turns out to be independent of the values of the scalar moduli. Indeed, the lattices are all pure integers times a fundamental unit (8.16). In such a situation, where the Dirac-allowed charge lattices are independent of the moduli, the continuous global symmetry group is broken down to a discrete subgroup, namely the U-duality group [31]. This modulus-independent charge lattice, which cannot be derived purely from the Dirac quantisation condition in D -dimensional supergravity taken in isolation, plays a crucial rôle in the discretisation of the classical global symmetry group to the U-duality group [32].

The reduction of the supergravity symmetry group to a discrete subgroup can also be seen in the structure of the perturbative counterterms to supergravity, whether they occur with infinite coefficients in ordinary attempts at quantising (nonrenormalisable) supergravities, or with finite coefficients as determined in the effective supergravity theories obtained from superstring theory. For example, one may consider the simple case of $D = 4$, $N = 2$ supergravity coupled to $N = 2$ matter, which gives rise to non-renormalisable counterterms already at the 1 loop level [33]. In these one-loop counterterms, the Maxwell field occurs through the stress tensor $T^{\mu\nu}$, which is invariant under duality transformations. This good duality behaviour is actually in excess of the classical expectations, since the classical Maxwell action itself is not in fact duality invariant, but instead transforms by an overall phase. This homogeneous classical transformation behaviour of the action is sufficient to yield a duality symmetry of the field equations, but when the classical action is combined with the 1-loop counterterms, the homogeneity property of the total action's transformation behaviour is lost. The only transformations for which the phases of the classical action and the counterterms 'align' are those of the quantum Z_2 duality group, reproducing the conclusions independently obtained by considering the Dirac quantisation condition. Analogous arguments, taking into account the indications that the first non-trivial ('3-loop') extended supergravity counterterms [34, 35] are similarly invariant [35] under the full Cremmer-Julia duality groups, imply that the U-duality discretisations of these groups may also be viewed as arising from the differences in transformation behaviour between homogeneously transforming actions and the invariance of the counterterms. Again, the phases only align after

restriction to the discretised $G(\mathbb{Z})$ duality groups.

The existence of an unbroken U-duality group at the quantum level depends on the ‘registration’ between the units of charge sublattices for the various p -branes that are not directly related by the Dirac quantisation condition. If there were not such a registration, a U-duality transformation on one field type, mapping between allowed points on that field’s sublattice, would not at the same time map between allowed points on another field’s sublattice. In this sense, requiring the persistence of an unbroken U-duality at the quantum level may also be taken as a logical basis for requiring the different charge units to be brought into registration with each other.

We have seen that dimensional reduction not only leads to a charge lattice in the lower dimension that is consistent with the Dirac quantisation conditions, but that it furthermore selects a unique lattice, which is independent of the values of the scalar moduli. This additional condition can be seen from Table 1, where ‘incidental’ quantisation relations between the various canonically-defined charge vectors arise even in cases where these are not directly required by the D -dimensional Dirac quantisation conditions (4.10) between conjugate pairs of electric and magnetic charges. For example, in $D = 10$ there are conjugate electric strings and magnetic 5-branes supported by the same 3-form field strength, for which a quantisation condition of the form (4.10) naturally occurs (provided the orientations do not fall into the measure-zero Dirac-insensitive set). From the point of view of the parent $D = 11$ supergravity theory, however, the 3-form field strength in $D = 10$ is just one of the two descendants of the 4-form field strength of the $D = 11$ theory. The other descendant is the 4-form field strength in $D = 10$. This 4-form field strength supports an electric 2-brane solution in $D = 10$, which is the vertical dimensional reduction [20] of the 2-brane in $D = 11$ [12]. Although this ten-dimensional 2-brane is supported by a different field strength from the ten-dimensional 5-brane, there is nonetheless an ‘incidental’ quantisation relation between the 2-brane and the 5-brane charges in $D = 10$, since their lattice units are related as shown in Table 1.

We see therefore that the $D = 11$ Dirac quantisation condition (3.9) is the parent of four $D = 10$ relations: two genuine $D = 10$ Dirac quantisation conditions, between the string and the 5-brane supported by the 2-form in $D = 10$, and between the 2-brane and the 4-brane supported by the 3-form in $D = 10$; plus the two incidental quantisation relations discussed above. As we discussed in section 4, the genuine Dirac quantisation conditions are characterised by the two possible sets of mixed vertical/diagonal dimensional reductions of the $D = 11$ 2-brane and 5-brane, which maintain the same supporting field strength for

their $D = 10$ descendants. The incidental quantisation relations, on the other hand, are characterised by the use of two vertical or two diagonal reductions, which give $D = 10$ solutions supported by two different field strengths. As we saw in section 4, configurations of branes in $D = 11$ corresponding to these cases do not yield true Dirac quantisation conditions. The distinction between the genuine and incidental cases is clearly seen in the period-dependent factors in Table 1: only for the genuine $D = 10$ Dirac quantisation conditions between conjugate solutions do the period factors L_i cancel out. The incidental quantisation conditions involve both T duality and the ‘absolute’ scale-setting properties of IIB self-dual 3-branes and D0-brane/ $(D - 4)$ -brane pairs in IIA theory, as we have seen.

From the standpoint of supergravity theories, the canonical charges, with their universal charge lattices, are the most natural ones to consider. For comparison with results in the literature, however, it is appropriate to relate the above canonical-charge results to those corresponding to different basic definitions of the charges. A well-known result in the context of $D = 4$ $U(1)$ gauge theories with symmetry breaking *via* a Higgs sector is the dependence of the charge lattice on the vacuum angle θ of the vacuum as well as the standard dependence on the unit e_0 of electric charge [29]. The ‘standard’ charge lattice to which these results pertain makes use charges defined differently from the canonical charges that we have been using. The canonical charges are obtained by an $SL(2, \mathbb{R})$ symmetry transformation (whose supergravity generalisations are the G supergravity duality symmetries) that precisely removes the (e_0, θ) modulus dependence of the standard charge lattice. If the (ϕ, χ) scalar sector is described by an $SL(2, \mathbb{R})$ matrix

$$V = \begin{pmatrix} e^{-\frac{\phi}{2}} & e^{\frac{\phi}{2}} \chi \\ 0 & e^{\frac{\phi}{2}} \end{pmatrix} \quad (9.1)$$

belonging to the Borel subgroup of $SL(2, \mathbb{R})$ [32], then the ‘physical’ charge lattice with the coupling constants restored is given by $Q_{\text{phys}} = V_0^{-1}Q$, where V_0 is the matrix V evaluated for the asymptotic scalar values, *i.e.* the moduli, with $e_0 = e^{\phi_0/2}$, $\theta = \chi_0$. One then finds that Q_{phys} scales up as e_0 is increased, and that the dependence of the electric and magnetic components of the charges on the vacuum angle θ agrees precisely [36] with that of Ref. [29].

Note that the lattice of physical charges $Q_{\text{phys}} = V_0^{-1}Q$ is still invariant under a quantum duality group, but that this is no longer simply the group of integer-valued $G(\mathbb{Z}) = SL(2, \mathbb{Z})$ matrices that maps the canonical Q charge lattice into itself. Instead, the Q_{phys} lattice is mapped into itself by the conjugated¹³ group of matrices $\hat{G}(\mathbb{Z}) = V_0^{-1}G(\mathbb{Z})V_0$.

¹³Note also that the sense of this conjugation is opposite to that for the vacuum stability group $H = SO(2)$ which leaves the scalar moduli (ϕ_0, χ_0) invariant; the latter is conjugated to $\hat{H} = V_0 H V_0^{-1}$. The

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Appendix: Dirac quantisation conditions in even dimensions

In this appendix, we shall discuss further the Dirac quantisation conditions for charges carried by field strengths of degree $n = \frac{1}{2}D$ in even dimensions D . For simplicity, we shall consider a single such field strength, coupled to gravity. We shall consider both Lorentzian spacetime signature and Euclidean spacetime signature.

Consider a Lagrangian of the form

$$\mathcal{L} = e R - \frac{1}{2n!} e F^2 , \quad (9.2)$$

where F is an n -form field strength. The energy-momentum tensor is given by

$$T_{\mu\nu} = F_{\mu\sigma_2\cdots\sigma_n} F_{\nu}{}^{\sigma_2\cdots\sigma_n} - \frac{1}{2n} F^2 g_{\mu\nu} . \quad (9.3)$$

We now define the Hodge dual of F by

$$\tilde{F}_{\mu_1\cdots\mu_n} = \frac{1}{n!} \epsilon_{\mu_1\cdots\mu_n}{}^{\nu_1\cdots\nu_n} F_{\nu_1\cdots\nu_n} , \quad (9.4)$$

The dualisations have the following properties, governed by dimension and signature:

	Lorentzian	Euclidean	
$D = 4k:$	$\tilde{\tilde{F}} = -F$	$\tilde{\tilde{F}} = F$	(9.5)
$D = 4k + 2:$	$\tilde{\tilde{F}} = F$	$\tilde{\tilde{F}} = -F$	

opposite conjugation properties of these two groups gives rise to the existence of the special ‘self-dual’ point $(\phi_0, \chi_0) = (0, 0)$ in modulus space, at which point the intersection group $\hat{G}(\mathbf{Z}) \cap \hat{H}$ is maximal. The action of this maximal group agrees with that of the Weyl group of the duality group G . This occurrence of the Weyl group of G as the maximal $\hat{G}(\mathbf{Z}) \cap \hat{H}$ intersection persists for the complete list of Cremmer-Julia supergravity symmetry groups G [37, 20].

It is straightforward to see that the energy-momentum tensor (9.3) can be rewritten using \tilde{F} in the form

$$\text{Lorentzian:} \quad T_{\mu\nu} = \frac{1}{2}(F_{\mu\sigma_2\cdots\sigma_n} F_{\nu}^{\sigma_2\cdots\sigma_n} + \tilde{F}_{\mu\sigma_2\cdots\sigma_n} \tilde{F}_{\nu}^{\sigma_2\cdots\sigma_n}) , \quad (9.6)$$

$$\text{Euclidean:} \quad T_{\mu\nu} = \frac{1}{2}(F_{\mu\sigma_2\cdots\sigma_n} F_{\nu}^{\sigma_2\cdots\sigma_n} - \tilde{F}_{\mu\sigma_2\cdots\sigma_n} \tilde{F}_{\nu}^{\sigma_2\cdots\sigma_n}) . \quad (9.7)$$

From these expressions, it is clear that when $D = 4k$, there is a continuous symmetry of the energy-momentum tensor under transformations

$$F_{\mu_1\cdots\mu_n} \longrightarrow c F_{\mu_1\cdots\mu_n} + s \tilde{F}_{\mu_1\cdots\mu_n} , \quad (9.8)$$

where the parameters c and s satisfy $c^2 + s^2 = 1$ in the Lorentzian case, and $c^2 - s^2 = 1$ in the Euclidean case. Thus in the Lorentzian case we may take $c = \cos \theta$, $s = \sin \theta$, while in the Euclidean case we may instead take $c = \cosh \theta$, $s = \sinh \theta$. On the other hand, in $D = 4k + 2$ dimensions there is no such continuous symmetry of the energy-momentum tensor, in either the Lorentzian or the Euclidean signature. Thus we have the following duality symmetries in the various cases, restricting attention to situations involving a single n -form field strength:

$$\begin{array}{l} \text{Lorentzian:} \\ \text{Euclidean:} \end{array} \quad \begin{cases} D = 4k : & SO(2) \\ D = 4k + 2 : & - \\ D = 4k : & SO(1, 1) \\ D = 4k + 2 : & - \end{cases} \quad (9.9)$$

In $D = 4k$ dimensions we can derive the Dirac quantisation condition for dyons from the quantisation condition for purely electric and magnetic charges, by acting on the latter using the duality symmetry. Thus, starting from the result that (dimensionful) charges $(e_1, 0)$ and (e_2, g_2) satisfy the condition $e_1 g_2 = 2\pi\kappa^2 n$, one can deduce that

$$e_1 g_2 - e_2 g_1 = 2\pi\kappa^2 n \quad (9.10)$$

for dyons in $D = 4k$ dimensions, regardless of the spacetime signature. By contrast, one cannot deduce the quantisation condition for dyons in $D = 4k + 2$ dimensions by any analogous calculation. In this case the electric and magnetic charges of the single field strength F cannot form a doublet under any duality symmetry.¹⁴

¹⁴The absence of a continuous duality symmetry in the case of a single $(2k + 1)$ -form field strength in $D = 4k + 2$ dimensions has been discussed from the point of view of the impossibility of realising such a symmetry as a canonical transformation in Ref. [25].

Of course an enlarged system with more field strengths can have a duality symmetry, such as in the case of $D = 6$ supergravity, where there is an $O(5, 5)$ symmetry. Using this $O(5, 5)$ symmetry, one can derive the dyonic quantisation condition

$$e_1 g_2 + e_2 g_1 = 2\pi\kappa^2 n \tag{9.11}$$

as we showed in section 5. We saw in section 6 that this result holds for dyons in all dimensions $D = 4k + 2$, using a generalisation of the arguments of Ref. [23]. Moreover, in section 5, we saw that the quantisation condition (9.11) is consistent with dimensional reduction. In section 5, we have exploited the fact that the Lorentzian $D = 4k + 2$ quantisation condition (9.11) gives rise to a Dirac quantisation condition for self-dual charges in $D = 4k + 2$ Lorentzian spacetimes. Note, however, that there is no such condition for self-dual charges in Euclidean $D = 4k$ dimensional spacetimes, since here the dyonic quantisation condition (9.10) becomes trivial in this case.

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