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T-DUALITY IN ARBITRARY STRING BACKGROUNDS

Mark Evans[a,c] **and Ioannis Giannakis**[a,b] †

[(a)] *Physics Department, Rockefeller University*
1230 York Avenue, New York, NY 10021-6399

[(b)] *Center for Theoretical Physics, Texas A&M University*
College Station, TX 77843-4242

[(c)] *Citicorp Securities, Inc.*
399 Park Avenue., New York, NY 10017.

Abstract

T-Duality is a poorly understood symmetry of the space-time fields of string theory that interchanges long and short distances. It is best understood in the context of toroidal compactification where, loosely speaking, radii of the torus are inverted. Even in this case, however, conventional techniques permit an understanding of the transformations only in the case where the metric on the torus is endowed with Abelian Killing symmetries. Attempting to apply these techniques to a general metric appears to yield a non-local world-sheet theory that would defy interpretation in terms of space-time fields. However, there is now available a simple but powerful general approach to understanding the symmetry transformations of string theory, which are generated by certain similarity transformations of the stress-tensors of the associated conformal field theories. We apply this method to the particular case of T-Duality and i) rederive the known transformations, ii) prove that the problem of non-locality is illusory, iii) give an explicit example of the transformation of a metric that lacks Killing symmetries and iv) derive a simple transformation rule for arbitrary string fields on tori.

† e-mail: evans@theory.rockefeller.edu, giannak@theory.rockefeller.edu

1. Introduction.

One of the severest tests for a theory of particle physics with any claim to completeness is the manner in which it deals with the ultra-violet problems of field theory. It would appear that string theory does this in a uniquely appealing manner. Rather than delicately adjusting itself to cancel the ultra-violet divergences, string theory appears simply to abolish the ultra-violet—something it can do because of the dynamical nature of space-time in a theory of gravity. It is fair to say that this phenomenon is understood only incompletely, but string theory, in a completely differentiable way, seems to deny the existence of distances significantly shorter than the Planck length. If we try to probe shorter and shorter distances, a reinterpretation of the degrees of freedom shows that we are deluding ourselves, and are, in fact, probing longer and longer distances: the ultra-violet eludes our grasp by turning out to be the infra-red in disguise. Such a fascinating phenomenon would have many attractive consequences, but one in particular stands out for its philosophical importance. For most theories of particle physics, a skeptic may always wonder whether it is simply an effective theory, derived from something more fundamental at shorter distances, and whether this process of reduction continues indefinitely to shorter and shorter distances. String theory, if it truly denies the existence of shorter distances, can perhaps lay some claim to finality.

Unfortunately, we understand this phenomenon only in examples. The simplest is the one-loop diagram in perturbation theory. Here, we are instructed to integrate over a single copy of the moduli-space of the torus, the famous fundamental region, which is usually defined in such a way that the modular parameter, τ, never approaches the origin that, naively, would describe arbitrarily small loops. More directly, gedanken-experiments at ultra-Planckian energies [1] show the need to adjust the uncertainty principle in such a way that there is a maximal spatial resolution, no matter how great the momentum.

However, the most studied example of this phenomenon is surely T-Duality of compactified string theory (For a recent review see [2]). It is not a totally new idea, being a strong-weak coupling duality of the world-sheet theory, of a type going back to Kramers and Wannier [3], but it made its first appearance in string theory in the work of Kikkawa and Yamasaki and of Sakai and Senda [4]. These authors considered a string moving on a space-time in which one spatial dimension is compactified—a circle of radius R (in suitable units of the Planck length). In such a background there are two noteworthy types of excitation: strings with (integer quantized) momenta in the compact direction, and strings winding around the compact dimension an integer number of times. The masses of the momentum excitations are of the form $\sim n/R$, with integral n, while those of the winding modes are $\sim mR$, where m is the number of times the string wraps around the compact dimension. Already, there is apparent a duality in which interchanging the rôle of the momentum and winding modes is equivalent to mapping $R \to 1/R$. This result was further generalised in [5]. Also suggestions have been made in the literature towards an understanding of T-duality as a canonical transformation [6], [7] and a duality symmetric formulation of string world-sheet dynamics was proposed in [8].

To see this duality in more detail, consider the simplest case of a string moving on a circle. The world-sheet stress tensor is

$$T(\sigma) = \tfrac{1}{4}R\left(\pi(\sigma)/R + X'(\sigma)\right)^2, \tag{1.1}$$

where $X(\sigma)$ is the coordinate of the string (of period 2π) and $\pi(\sigma)$ is its conjugate momentum:

$$[\pi(\sigma), X'(\sigma')] = i\delta'(\sigma - \sigma') \tag{1.2}$$

However, $\pi(\sigma)$ and $X'(\sigma)$ are algebraically indistinguishable in equation (1.2), which means that we could just as well interpret equations (1.1) and (1.2) as describing a string with coordinate $\tilde{X}(\sigma)$ and momentum $\tilde{\pi}(\sigma)$ given by,

$$\begin{aligned}\tilde{\pi}(\sigma) &= X'(\sigma)\\ \tilde{X}'(\sigma) &= \pi(\sigma).\end{aligned} \tag{1.3}$$

In terms of *these* variables, the stress tensor is

$$T(\sigma) = \tfrac{1}{4}R\left(\tilde{X}'(\sigma)/R + \tilde{\pi}(\sigma)\right)^2 = \tfrac{1}{4}1/R\left(R\tilde{\pi}(\sigma) + \tilde{X}'(\sigma)\right)^2, \tag{1.4}$$

which, comparing with equation (1.1), clearly describes a string moving on a circle of radius $1/R$.

In essence, this is as far as the problem is understood (we shall briefly mention various embellishments below), but this understanding is only partial. The algebra automorphism of equation (1.3) tells us how to map $X'(\sigma)$, but is silent on the transformation of $X(\sigma)$ itself, or, worse yet, would seem to imply a transformation which is nonlocal on the world-sheet:

$$\tilde{X}(\sigma) \sim \int^{\sigma} dx \pi(x). \tag{1.5}$$

If we wished to make a T-Duality transformation on a configuration where the stress-tensor depended on $X(\sigma)$ as well as $X'(\sigma)$, we would apparently end up with an unacceptable (and uninterpretable) non-local stress-tensor. This happens as soon as we consider, for example, field configurations where the space-time metric is not flat (We should mention that we shall not concern ourselves in this paper with locality of the theory in space-time, our use of the word local, will refer exclusively to the world-sheet. For work relevant to this issue see [9]).

To the best of our knowledge, this problem has been resolved only partially in the literature, and progress in understanding T-Duality has been limited to finding as many circumstances as possible where it need not be confronted.

The simplest such extension is to consider space-time compactified on a product of several circles endowed with a flat metric. This straightforward generalization of the one-circle case gives a separate duality for each circle, and this approach was pushed to its limit in the work of Buscher [10], who showed how to implement T-duality transformations on manifolds with Abelian isometries. In essence, the existence of n Killing fields with vanishing Lie-bracket for any pair means that a coordinate system may be found in which the space-time metric is independent of n coordinates. Thus it is not surprising that it is possible to implement a separate T-Duality for each such cyclic coordinate. The results for tori may also be extended to orbifolds, which are flat almost everywhere [11], WZW and coset models [12]. The technique most commonly used, gauging the isometries and then getting the two dual descriptions of the same system with different choices of fields to be integrated out, is not the one described above, but the results and scope are the same [13] . In particular, isometries may be gauged only if they exist, so this technique does *not* permit us to dualize configurations which depend in an essential way on all the coordinates. That world-sheet locality is preserved in these cases was proven by Hassan [14], who showed that it followed from conservation of the isometry current. Aspects of non-abelian dualities were addressed in [15].

How, then, are we to dualize string configurations where the action or stress-tensor of the CFT depend on $X(\sigma)$ in an essential way? Equation (1.5) might lead us to think that the problem is intractable or, worse yet, uninteresting. However, equation (1.5) is the result of an essentially *classical* world-sheet analysis. Our main purpose in this paper is to explain how to do a fully quantum mechanical analysis, and to demonstrate that, when this is done, *all world-sheet non-localities are cancelled!* We regard this as a string miracle of the first rank.

The crucial observation was made some time ago by Dine, Huet and Seiberg [16], who observed that T-duality is, in fact, a finite gauge transformation. The Kaluza-Klein mechanism tells us that if a generally covariant theory (such as string theory) is compactified on a circle, a $U(1)$ gauge symmetry will result. (For string theory it is actually $U(1)\times U(1)$, because string theory possesses two-form gauge invariance in addition to general covariance). It is a famous result of string theory that this unbroken gauge symmetry is enhanced to $SU(2)\times SU(2)$ when the radius of the circle has a critical value. The observation of Dine, Huet and Seiberg is that applying these additional gauge transformations with a parameter of the correct magnitude changes the sign of the Kaluza-Klein dilaton, inverting the radius of compactification. This observation seems more natural, although no less profound, when we recall that the extra gauge bosons are winding modes, while the Kaluza-Klein excitations are momentum modes. The enhanced gauge symmetry therefore mixes winding and momentum modes, just as T-Duality interchanges them.

In recent years our understanding of the gauge invariances of string theory has improved considerably [17], [18]. We understand (in principle, at least) how to implement gauge transformations on arbitrary backgrounds. Combining this knowledge with the DHS insight should therefore enable us to understand T-Duality applied to arbitrary field configurations. As we shall see, this hope is fully realized.

How are gauge symmetries implemented? The simple answer is that a similarity transformation should be applied to the stress tensor of the conformal field theories:

$$T(\sigma) \longmapsto e^{ih} T(\sigma) e^{-ih}. \tag{1.6}$$

This yields a new stress tensor from which we may read off the transformed space-time fields. (the stress tensor is to string theory exactly what a superfield is to supergravity). What is the operator h that implements a gauge transformation? For each gauge symmetry there exists a corresponding current algebra on the world-sheet, generated by, say, $J^a(\sigma)$ (a spans the adjoint representation of the gauge algebra). For a gauge transformation with parameter $\Lambda^a(X)$, the generator, h, in equation (1.6) is just

$$h = \int d\sigma \, \Lambda^a(X(\sigma)) J^a(\sigma). \tag{1.7}$$

It really is that simple.

In the next section we shall explain this approach to the study of gauge symmetry in string theory in more detail, summarizing earlier work, while in section 3 we shall demonstrate the power and versatility of these methods by deriving most of the known results on T-Duality. Section 4 is devoted to a discussion of general backgrounds; we shall dualize a simple non-flat background that lacks isometries, and derive a general transformation rule for arbitrary background fields. As advertized, it is local. Section 5 is devoted to some concluding remarks.

2. Deformations of Conformal Field Theories and Symmetries.

In this section we shall review earlier work [17], [18] on deformations of conformal field theories and symmetries of string theory. For more details the reader is referred to the original papers, or the review contained in [19]. That rare reader already familiar with this work may skip this section without loss.

To study symmetries, we seek transformations of the space-time fields that take one solution of the classical equations of motion to another that is physically equivalent. Since, "Solutions of the classical equations of motion," are, for the case of string theory [20], two-dimensional conformal field theories [21], we are thus interested in physically equivalent conformal field theories.

Any quantum mechanical theory (including a CFT) is defined by three elements: i) an algebra of observables, $\mathcal{A}$ (determined by the degrees of freedom of the theory and their equal-time commutation relations), ii) a representation of that algebra and iii) a distinguished element of $\mathcal{A}$ that generates temporal evolution (the Hamiltonian). Note that for the same $\mathcal{A}$ we may have many choices of Hamiltonian, so that $\mathcal{A}$ should more properly be associated with a *deformation class* of theories than with one particular theory. For a CFT, we further want $\mathcal{A}$ to be generated by local fields, $\Phi(\sigma)$ (operator valued distributions on a circle parameterized by σ), and we require not just a single distinguished operator, but two distinguished fields, $T(\sigma)$ and $\overline{T}(\sigma)$. In terms of these fields the Hamiltonian, H, and generator of translations, P, may be written

$$H = \int d\sigma (T(\sigma) + \overline{T}(\sigma)) \tag{2.1}$$

$$P = \int d\sigma (T(\sigma) - \overline{T}(\sigma)) \tag{2.2}$$

and they must satisfy Virasoro$\times$Virasoro:

$$[T(\sigma), T(\sigma')] = \frac{-ic}{24\pi}\delta'''(\sigma-\sigma') + 2iT(\sigma')\delta'(\sigma-\sigma') - iT'(\sigma')\delta(\sigma-\sigma') \tag{2.3a}$$

$$[\overline{T}(\sigma), \overline{T}(\sigma')] = \frac{ic}{24\pi}\delta'''(\sigma-\sigma') - 2i\overline{T}(\sigma')\delta'(\sigma-\sigma') + i\overline{T}'(\sigma')\delta(\sigma-\sigma') \tag{2.3b}$$

$$[T(\sigma), \overline{T}(\sigma')] = 0. \tag{2.3c}$$

Except on σ, a prime denotes differentiation. T and $\overline{T}$ are the non-vanishing components of the stress-tensor, and must satisfy (2.3) if they are to generate conformal transformations. Also of interest are the so-called *primary fields* of dimension $(d, \overline{d})$, $\Phi(\sigma)$, defined by the conditions

$$\begin{aligned}
[T(\sigma), \Phi_{(d,\overline{d})}(\sigma')] &= id\Phi_{(d,\overline{d})}(\sigma')\delta'(\sigma-\sigma') - (i/\sqrt{2})\partial\Phi_{(d,\overline{d})}(\sigma')\delta(\sigma-\sigma') \\
[\overline{T}(\sigma), \Phi_{(d,\overline{d})}(\sigma')] &= -i\overline{d}\Phi_{(d,\overline{d})}(\sigma')\delta'(\sigma-\sigma') - (i/\sqrt{2})\overline{\partial}\Phi_{(d,\overline{d})}(\sigma')\delta(\sigma-\sigma')
\end{aligned} \tag{2.4}$$

Clearly, then, two CFTs will be physically equivalent if there is an isomorphism between the corresponding algebras of observables, $\mathcal{A}$, that maps stress tensor to stress tensor. (The mapping of primary to primary is then automatic). The simplest example of such an isomorphism is an inner automorphism, or similarity transformation

$$\Phi(\sigma) \mapsto e^{ih}\Phi(\sigma)e^{-ih}, \tag{2.5}$$

for any fixed operator h. *Thus the physics will be unchanged if we change a CFT's stress tensor by just such a similarity transformation.*

Now, the stress tensor is parameterized by the space-time fields of string theory. For example,

$$T_{G_{\mu\nu}}(\sigma) = \tfrac{1}{2} G_{\mu\nu}(X) \partial X^\mu \partial X^\nu \tag{2.6}$$

corresponds to the space-time metric $G_{\mu\nu}$, with all other fields vanishing. Thus an appropriate similarity transformation (2.5) applied to T will produce a change in T which corresponds to a change in the space-time fields, *without changing the physics.* This change in the space-time fields is therefore a *symmetry* transformation.

We may clarify the way in which the change in the stress tensor may be interpreted as a change in the space-time fields by first considering the more general problem of deforming a conformal field theory (we now consider deformations which, while they preserve conformal invariance, *need not be symmetries, e.g.* we may deform flat empty space so that a weak gravitational wave propagates through it). It is straightforward to show that, to first order, the Virasoro algebras (2.3) are preserved by deforming the choice of stress tensor by a so-called *canonical deformation* [17],

$$\delta T(\sigma) = \delta \overline{T}(\sigma) = \Phi_{(1,1)}(\sigma) \tag{2.7}$$

where $\Phi_{(1,1)}(\sigma)$ is a primary field of dimension (1,1) with respect to the stress tensor *. We reiterate: (2.7) does *not* in general correspond to a symmetry transformation, although it preserves conformal invariance. Since (1,1) primary fields are vertex operators for physical states, they are in natural correspondence with the space-time fields, and equation (2.7) makes the connection between changes of the stress tensor and changes of the space-time fields more transparent.

Returning now to the problem of symmetries, if we take the generator h in equation (2.5) to be the zero mode of an infinitesimal (1,0) or (0,1) primary field (a current), then it is straightforward to see that its action on the stress tensor is necessarily a canonical deformation, as in equation (2.7), and so may be translated easily into a change in the space-time fields (for examples, see [17]). It is well known that *conserved* currents generate symmetries [22], but within the formalism described here, conservation is *not* necessary, a fact that does not seem to have been widely appreciated. Indeed, it is not hard to see that a non-conserved current generates a symmetry that is spontaneously broken by the particular background being considered [17].

At this point it is convenient to assess the strengths and weaknesses of our analysis so far—canonical deformations and symmetries generated by the zero-modes of currents. We begin with the strengths:

- The fact that a current zero-mode generates a canonical deformation guarantees that we can translate the inner automorphism into a transformation on the physical space-time fields.

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* This is a straightforward consequence of the definition of a primary field, eq. (2.4) and the form of the Virasoro algebras, eq. (2.3).

- In this way, one may exhibit symmetries both familiar (general coordinate and two-form gauge transformations, regular non-abelian gauge transformations—including the Green-Schwarz modification—for the heterotic string) and unfamiliar (an infinite class of spontaneously broken, level-mixing gauge supersymmetries) [17].

However, there are also deficiencies:

- Explicit calculations are hard to perform for a general configuration of the space-time fields. It would appear that we need to know the precise form of the stress-tensor and its currents for that general configuration (recall that a current is only a current relative to a particular stress-tensor). This is a very hard problem, essentially requiring us to find the general solution to the equations of motion before we can discuss symmetries.
- It is hard to say anything about the symmetry algebra. This is just the algebra of the generators, but it is *not* in general true that the commutator of two current zero-modes is itself a current zero-mode: our symmetry "algebra" does not close!
- Finally, by considering a few examples, it is easy to see that the canonical deformation of equation (2.7) corresponds to turning on space-time fields *in a particular gauge* (something like Landau or harmonic gauge), and so symmetries generated by zero-modes of currents preserve this gauge condition, since they generate canonical deformations. We would like to understand the gauge principle behind string theory without imposing gauge conditions.

It turns out that these three drawbacks are intimately related, and may be largely overcome by moving beyond canonical deformations, equation (2.7), and beyond the zero-modes of currents as symmetry generators. Equation (2.7) is *not* the most general infinitesimal deformation that preserves the Virasoro algebras (2.3). In [18] we showed that, for the massless degrees of freedom of the bosonic string in flat space, we could find a distinct deformation of the stress tensor for each solution of the linearized Brans-Dicke equations. This correspondence was found by considering the general translation invariant *ansatz* of naive dimension two for δT;

$$\begin{aligned}\delta T =& H^{\nu\lambda}(X)\partial X_\nu \overline{\partial} X_\lambda + A^{\nu\lambda}(X)\partial X_\nu \partial X_\lambda + \\ & B^{\nu\lambda}(X)\overline{\partial} X_\nu \overline{\partial} X_\lambda + C^{\nu}(X)\partial^2 X_\nu + D^{\lambda}(X)\overline{\partial}^2 X_\lambda,\end{aligned} \tag{2.8}$$

with a similar, totally independent *ansatz* for $\delta\overline{T}$. The fields $H^{\mu\nu}$ *etc.* turn out to be characterized in terms of solutions to the linearized Brans-Dicke equation when we demand that the deformation preserves (to first order) the Virasoro algebras (2.3).

By considering this more general *ansatz*, we get more than just covariant equations of motion—we also understand a larger set of symmetry generators, h. Indeed, any generator that preserves the form of the *ansatz* (2.8) must necessarily generate a change in the stress tensor that corresponds to a change in the space-time fields.

The condition that δT be of naive dimension two (with which we shall soon dispense) is preserved if h is of naive dimension zero. The condition of translation invariance is

$$[P, \delta T(\sigma)] = -i\delta T'(\sigma), \tag{2.9}$$

which may be preserved by demanding that h commute with P, the generator of translations, (2.2). (Equation (2.9) may also be thought of as a gauge condition, but not one that has any obvious interpretation in terms of the space-time fields). Taken together, these conditions characterize h as the zero-mode of a field of naive dimension one.

The lesson to be drawn from this massless example is clear: the way to introduce space-time fields unconstrained by gauge conditions is to consider an *arbitrary translation invariant ansatz* for the deformation of the stress tensor, and to ask only that it preserve the Virasoro algebras. To move beyond the massless level, we simply drop the requirement that the naive dimension be two. We argued in [18] that, as with the superfield formulation of supersymmetric theories, this was likely to introduce auxiliary fields beyond the massless level, but so be it. (Indeed, this whole formulation of string theory is completely parallel to a superspace approach, with T and $\overline{T}$ as superfields and derivatives of the world-sheet scalars playing the rôle of the odd coordinates of superspace).

Having dropped any requirement on the naive dimension of δT, we know that *any* operator h that commutes with P will generate a symmetry transformation on our space-time fields (possibly including the auxiliaries). This extension of the set of symmetry generators ameliorates each of the drawbacks mentioned above:

- Since the symmetry generators are no longer currents and the stress-tensor is generic, we no longer need explicit forms for the general CFT or its currents. To give explicit symmetry transformations on our expanded set of fields we need only calculate the commutator of the general zero-mode with the general field. Even this is problematic, however; see [23]
- While zero-modes of currents do not, in general, close under commutation, the set of *all* zero-modes *does* close. This algebra may be characterized as the centralizer of P, the generator of world-sheet translations, equation (2.2).
- It should also be emphasized that our symmetry algebra is now *background independent*. Recall that, naively at least, the algebra of observables, $\mathcal{A}$, is attached to a *deformation class* of CFTs, the elements of which differ only in their choice of stress-tensor. Our set of symmetry generators—all zero-modes—is manifestly independent of any particular choice of stress-tensor, and so is their algebra. (Formally, this follows from the fact that P can deform by at most a central element, so that its centralizer is invariant under deformation.)

For example, we may generate general coordinate and two-form gauge transformations (at least about flat space) by choosing

$$h = \int d\sigma [\xi^\mu(X)\partial X_\mu + \zeta^\mu(X)\overline{\partial} X_\mu] \tag{2.10}$$

with δT given by (2.8). In the next section we shall identify the operator h which implements T-duality in string theory.

3. Duality in spaces with abelian isometries.

In the previous section we have sketched how symmetries of string theory (*i.e.* symmetry transformations on the spacetime fields of string theory) are generated by inner automorphisms acting on the operator algebra of the theory. These inner automorphisms, *i.e.* just similarity transformations, generate the infinite dimensional symmetry algebra that underlies string theory. It includes both the unbroken symmetries (space-time diffeomorphisms, gauge transformations) and an infinite class of spontaneously gauge broken (super) symmetries (higher symmetries, dualities) which mix different mass levels. The symmetry generators (zero modes of operators which implement the inner automorphisms) are independent of any particular choice of stress-tensor since the algebra of observables is attached to a *deformation class* of CFT's. In this section we will identify the operator h which maps the operator algebra onto itself and in addition can be pulled back to space-time and be interpreted as a T-duality transformation on the spacetime fields. This will be achieved by fixing the operator algebra and constructing the operator h at the self-dual point. The effect of a T-duality transformation on space-time fields can be calculated then by applying the inner automorphism to an arbitrary stress-tensor in the *deformation class*.

For illustrative purposes we will initially consider a simple example, string propagation on a circle of radius R. Since the coordinate of the string $X(\sigma,\tau)$ parametrizes a circle S^1, it obeys a periodicity condition

$$X(\sigma+2\pi,\tau) = X(\sigma,\tau) + 2\pi n R. \tag{3.1}$$

The most general expression for $X(\sigma,\tau)$ satisfying the two-dimensional wave equation and consistent with the boundary condition, Eq. (3.1) then becomes

$$X(\sigma,\tau) = x + p\tau + nR\sigma + i\sum_{n\neq 0}\frac{1}{n}(a_n e^{-in(\tau-\sigma)} + \tilde{a}_n e^{-in(\tau+\sigma)}). \tag{3.2}$$

The periodicity condition of the string coordinate $X(\sigma,\tau)$ implies that the momentum p is quantized, $p=\frac{m}{R}$, while the second term in the boundary condition of $X(\sigma,\tau)$ describes winding string states. The integer n counts how many times the string wraps around the circle. The two components of the world-sheet stress-tensor which describe this particular conformal field theory are given by

$$T_R(\sigma) = \tfrac{1}{2} : \hat{\partial X}\hat{\partial X} : (\sigma) \qquad \overline{T}_R(\sigma) = \tfrac{1}{2} : \hat{\overline{\partial}X}\hat{\overline{\partial}X} : (\sigma) \tag{3.3}$$

where we have defined the stress-tensor through a point splitting regularization as follows:

$$T_R(\sigma) = \tfrac{1}{2} : \hat{\partial X}\hat{\partial X} : (\sigma) = \lim_{\epsilon\to 0}\tfrac{1}{2}\hat{\partial X}(\sigma)\hat{\partial X}(\sigma+\epsilon) + \tfrac{1}{4\pi\epsilon^2} \tag{3.4}$$

The subscript R indicates that the stress-tensor depends on the radius of compactification R which, since it may vary, becomes a spacetime field.

We should now address a minor technical point that can nonetheless be very confusing. The operator written as $\hat{\partial X}(\sigma)$ in equation (3.4) is not the same operator at different radii. This is apparent if we express it in terms of $\pi(\sigma)$ and $X(\sigma)$:

$$\hat{\partial X} = \frac{1}{\sqrt{2}}(\frac{\sqrt{\alpha'}}{R}\pi + \frac{R}{\sqrt{\alpha'}}X') \qquad \hat{\overline{\partial} X} = \frac{1}{\sqrt{2}}(\frac{\sqrt{\alpha'}}{R}\pi - \frac{R}{\sqrt{\alpha'}}X') \tag{3.5}$$

$\pi(\sigma)$ and $X(\sigma)$ have fixed, R-independent, commutation relations,

$$[\pi(\sigma), X(\sigma')] = i\delta(\sigma - \sigma') \tag{3.6}$$

while the operators $\hat{\partial X}(\sigma)$ do not. If we want to compare CFT's at different values of R, it is essential that we express the stress-tensors in terms of fixed, R-independent operators, such as $\pi(\sigma)$ and $X(\sigma)$. It is slightly more convenient, however, to work instead with the light-cone derivatives at the critical radius. Thus, by the symbols ∂X and $\overline{\partial} X$ (without the hat), we shall mean

$$\partial X = \frac{1}{\sqrt{2}}(\frac{\sqrt{\alpha'}}{R_{cr}}\pi(\sigma) + \frac{R_{cr}}{\sqrt{\alpha'}}X'(\sigma)) \qquad \overline{\partial} X = \frac{1}{\sqrt{2}}(\frac{\sqrt{\alpha'}}{R_{cr}}\pi(\sigma) - \frac{R_{cr}}{\sqrt{\alpha'}}X'(\sigma)), \tag{3.7}$$

neither more nor less. At the critical radius these are indeed the light-cone derivatives, but for other values of R they are not. Since we shall never actually be interested in taking light-cone derivatives, there is no danger of ambiguity.

We saw in equation (1.3) that T-duality involves the interchange of $\pi(\sigma)$ and $X'(\sigma)$, and so we seek an operator h that achieves this; we need

$$e^{i\pi h}\frac{\sqrt{\alpha'}}{R_{cr}}\pi(\sigma)e^{-i\pi h} = -\frac{R_{cr}}{\sqrt{\alpha'}}X'(\sigma) \qquad e^{i\pi h}\frac{R_{cr}}{\sqrt{\alpha'}}X'(\sigma)e^{-i\pi h} = -\frac{\sqrt{\alpha'}}{R_{cr}}\pi(\sigma), \tag{3.8}$$

which from the definition (3.7) is equivalent to,

$$e^{i\pi h}\partial X(\sigma)e^{-i\pi h} = -\partial X(\sigma) \qquad e^{i\pi h}\overline{\partial} X(\sigma)e^{-i\pi h} = \overline{\partial} X(\sigma). \tag{3.9}$$

To find this operator, h, we draw on the famous result that, at the critical radius, *i.e.* $R = R_{cr} = \sqrt{2}$, the $U(1)_L \times U(1)_R$ gauge symmetry of the theory extends to $SU(2)_L \times SU(2)_R$. This symmetry enhancement is due to the appearance of extra $(1,0)$ and $(0,1)$ operators $e^{\pm i\sqrt{2}X_L}(\sigma)$, $e^{\pm i\sqrt{2}X_R}(\sigma)$. Subsequently the operators $(\sqrt{2}i\partial X(\sigma)$, $e^{\pm i\sqrt{2}X_L}(\sigma))$

form an $SU(2)_L$ current algebra. It is straightforward now to construct the operator h if we recall the familiar formula for the generators of $SU(2)$

$$e^{i\pi J_2} J_3 e^{-i\pi J_2} = -J_3 \tag{3.10}$$

as follows

$$h = \frac{1}{2i}\int d\sigma (e^{i\sqrt{2}X_L} - e^{-i\sqrt{2}X_L})(\sigma) \tag{3.11}$$

We can verify that this particular choice of h satisfies Eq. (3.9) by writing

$$e^{i\pi h}\partial X(\sigma)e^{-i\pi h} = \partial X(\sigma) + i\pi[h, \partial X(\sigma)] + \frac{1}{2}(i\pi)^2[h,[h,\partial X(\sigma)],] + \cdots \tag{3.12}$$

and calculating the commutators explicitly. This construction of h is simply a consequence of combining the insight of Dine, Seiberg and Huet [16], that T-duality is an enhanced gauge transformation, with our understanding that gauge transformations are implemented through inner automorphisms generated by the corresponding world-sheet current algebra, as explained in equation (1.7).

Having constructed the operator h which implements the inner automorphism of the operator algebra, we proceed to calculate its effect on the stress-tensor of the theory, Eq. (3.4), at a general value of R. As was explained above, we must first express the stress tensor in terms of an R-independent basis of the operator algebra. Thus we need to express $\hat{\partial X}(\sigma) = \frac{1}{\sqrt{2}}(\frac{\sqrt{\alpha'}}{R}\pi(\sigma) + \frac{R}{\sqrt{\alpha'}}X'(\sigma))$ in terms of $\partial X(\sigma) = \frac{1}{\sqrt{2}}(\frac{\sqrt{\alpha'}}{R_{cr}}\pi(\sigma) + \frac{R_{cr}}{\sqrt{\alpha'}}X'(\sigma))$ and $\overline{\partial}X(\sigma) = \frac{1}{\sqrt{2}}(\frac{\sqrt{\alpha'}}{R_{cr}}\pi(\sigma) - \frac{R_{cr}}{\sqrt{\alpha'}}X'(\sigma))$,

$$\hat{\partial X}(\sigma) = \frac{1}{2}[\frac{R_{cr}^2}{R}(\partial X + \overline{\partial}X) + R(\partial X - \overline{\partial}X)](\sigma) \tag{3.13}$$

Substituting into Eq. (3.4) we obtain the following expression for the stress-tensor at radius R

$$\begin{aligned} T_R = \lim_{\epsilon\to 0}\frac{1}{8}[(\frac{{R_{cr}}^2}{R} + R)^2\partial X(\sigma)\partial X(\sigma+\epsilon) + (\frac{{R_{cr}}^2}{R} - R)^2\overline{\partial}X(\sigma)\overline{\partial}X(\sigma+\epsilon) \\ + (\frac{{R_{cr}}^4}{R^2} - R^2)(\partial X(\sigma)\overline{\partial}X(\sigma+\epsilon) + \overline{\partial}X(\sigma)\partial X(\sigma+\epsilon)) + \frac{1}{4\pi\epsilon^2}] \end{aligned} \tag{3.14}$$

Since the inner automorphism generated by h changes the sign of ∂X, its effect on T_R is simply obtained:

$$\begin{aligned} e^{i\pi h}T_R(\sigma)e^{-i\pi h} &= \lim_{\epsilon\to 0}\frac{1}{8}[(\frac{{R_{cr}}^2}{R} + R)^2\partial X(\sigma)\partial X(\sigma+\epsilon) + (\frac{{R_{cr}}^2}{R} - R)^2\overline{\partial}X(\sigma)\overline{\partial}X(\sigma+\epsilon) \\ &+ (-\frac{{R_{cr}}^4}{R^2} + R^2)(\partial X(\sigma)\overline{\partial}X(\sigma+\epsilon) + \overline{\partial}X(\sigma)\partial X(\sigma+\epsilon)) + \frac{1}{4\pi\epsilon^2}] \\ &= \lim_{\epsilon\to 0}\frac{1}{8}[(\frac{{R_{cr}}^2}{\tilde{R}} + \tilde{R})^2\partial X(\sigma)\partial X(\sigma+\epsilon) + (\frac{{R_{cr}}^2}{\tilde{R}} - \tilde{R})^2\overline{\partial}X(\sigma)\overline{\partial}X(\sigma+\epsilon) \\ &+ (\frac{{R_{cr}}^4}{\tilde{R}^2} - \tilde{R}^2)(\partial X(\sigma)\overline{\partial}X(\sigma+\epsilon) + \overline{\partial}X(\sigma)\partial X(\sigma+\epsilon)) + \frac{1}{4\pi\epsilon^2}] = T_{\tilde{R}=\frac{R_{cr}^2}{R}}(\sigma) \end{aligned} \tag{3.15}$$

Thus the inner automorphism generated by h maps the world-sheet stress-tensor onto a different one. The resulting conformal field theory is isomorphic to the original one. But this particular automorphism can be interpreted as a transformation on the spacetime field R, the radius of the circle. Hence the transformation $R \to \tilde{R} = \frac{R_{cr}^2}{R}$ is a symmetry of string theory.

Next we turn our attention to a more generalized setting: string propagation on a D-dimensional flat torus in the presence of a constant antisymmetric background field $b_{\mu\nu}$. In one of the symmetric points of the deformation class the gauge symmetry $[U(1)_L]^D \times [U(1)_R]^D$ of a generic point is enhanced to $[SU(2)_L]^D \times [SU(2)_R]^D$. The stress-tensor of this particular conformal field theory, represented by the symmetric point under consideration, is given by

$$T_G(\sigma) = \frac{1}{2} G^{\mu\nu} : \partial X_\mu \partial X_\nu : (\sigma) \qquad \overline{T}_G(\sigma) = \frac{1}{2} G^{\mu\nu} : \overline{\partial} X_\mu \overline{\partial} X_\nu : (\sigma) \tag{3.16}$$

where $G_{\mu\nu}$ is a constant diagonal metric (the identity!) and the antisymmetric background field has been set to zero. As in the previous example, the symbols $\partial X_\mu(\sigma)$ and $\overline{\partial} X_\mu(\sigma)$ should be thought of merely as the combinations

$$\partial X_\mu(\sigma) = \frac{1}{\sqrt{2}}(\pi_\mu + G_{\mu\nu} X'^\nu)(\sigma), \qquad \overline{\partial} X_\mu(\sigma) = \frac{1}{\sqrt{2}}(\pi_\mu - G_{\mu\nu} X'^\nu)(\sigma) \tag{3.17}$$

There are D independent inner automorphisms in this case, one for each dimension of the torus. Since at the symmetric point the symmetry group is enhanced to $[SU(2)_L]^D \times [SU(2)_R]^D$, they are generated by the J_2 generators of the several $SU(2)$'s. The corresponding operators $h^{(i)}$ which implement the inner automorphisms are thus given by

$$h^{(i)} = \frac{1}{2i} \int d\sigma (e^{i\sqrt{2} k_\mu^{(i)} X^\mu} - e^{-i\sqrt{2} k_\mu^{(i)} X^\mu}), \tag{3.18}$$

where $k_\mu^{(i)}$ is a suitable basis of Killing forms on the torus. For a D-dimensional flat torus, there are D of them $(i = 1, \cdots, D)$ and we have chosen a particular basis where $k^{(i)} = (1, 0, 0, \cdots), (0, 1, 0, \cdots), (0, 0, 1, \cdots), \ldots$.

As we deform our conformal field theory away from the symmetric point the stress-tensor changes and at a generic point of the deformation class takes the form

$$T_{g,b}(\sigma) = \frac{1}{2} g^{\mu\nu} : \partial \hat{X}_\mu \partial \hat{X}_\nu : (\sigma) \qquad \overline{T}_{g,b}(\sigma) = \frac{1}{2} g^{\mu\nu} : \overline{\partial} \hat{X}_\mu \overline{\partial} \hat{X}_\nu : (\sigma), \tag{3.19}$$

where

$$\partial \hat{X}_\mu(\sigma) = \frac{1}{\sqrt{2}}(\pi_\mu + (g_{\mu\nu} + b_{\mu\nu}) X'^\nu)(\sigma), \qquad \overline{\partial} \hat{X}_\mu(\sigma) = \frac{1}{\sqrt{2}}(\pi_\mu - (g_{\mu\nu} + b_{\mu\nu}) X'^\nu)(\sigma). \tag{3.20}$$

Note that g represents a generic (constant) metric, while G is the metric at the point of enhanced symmetry.

As in our previous simple example, we first must express $\partial\hat{X}_\mu$ in terms of ∂X_μ and $\overline{\partial}X_\mu$ as follows:

$$\partial\hat{X}_\mu = \frac{1}{2}[(\partial X_\mu + \overline{\partial}X_\mu) + (g_{\mu\rho} + b_{\mu\rho})G^{\rho\nu}(\partial X_\nu - \overline{\partial}X_\nu)] \tag{3.21}$$

Substituting into Eq. (3.19) we get

$$T_{g,b}(\sigma) = \frac{1}{8}[(g^{\mu\nu} + g^{\mu\rho}(g_{\rho\sigma} + b_{\rho\sigma})G^{\sigma\nu} + g^{\rho\nu}(g_{\rho\sigma} + b_{\rho\sigma})G^{\mu\sigma} + g^{\rho\sigma}(g_{\rho\kappa} + b_{\rho\kappa})(g_{\sigma\lambda} + b_{\sigma\lambda}) G^{\mu\kappa}G^{\lambda\nu})\partial X_\mu\partial X_\nu + (g^{\mu\nu} - g^{\mu\rho}(g_{\rho\sigma} + b_{\rho\sigma})G^{\sigma\nu} - g^{\rho\nu}(g_{\rho\sigma} + b_{\rho\sigma})G^{\mu\sigma} + g^{\rho\sigma}(g_{\rho\kappa} + b_{\rho\kappa}) (g_{\sigma\lambda} + b_{\sigma\lambda})G^{\mu\kappa}G^{\lambda\nu})\overline{\partial}X_\mu\overline{\partial}X_\nu + (g^{\mu\nu} - g^{\rho\sigma}(g_{\rho\kappa} + b_{\rho\kappa})(g_{\sigma\lambda} + b_{\sigma\lambda})G^{\mu\kappa}G^{\lambda\nu})(\partial X_{(\mu}\overline{\partial}X_{\nu)})] \tag{3.22}$$

As we remarked above, there are now D separate T-dualities. We shall consider them separately below, but for now we shall follow the historical route, and consider just the product of all D of them. The corresponding transformation on the stress tensor is, therefore,

$$e^{i\pi h^{(1)}}\cdots e^{i\pi h^{(n)}}T_{g,b}e^{-i\pi h^{(1)}}\cdots e^{-i\pi h^{(n)}}(\sigma) = T_{\tilde{g},\tilde{b}}, \tag{3.23}$$

where $\tilde{g}^{\alpha\beta}$ and $\tilde{b}^{\nu\sigma}$ satisfy

$$\tilde{g}^{\alpha\beta} = g^{\mu\nu}(g_{\mu\rho} + b_{\mu\rho})(g_{\nu\sigma} + b_{\nu\sigma})G^{\rho\alpha}G^{\sigma\beta} \qquad \tilde{g}^{\alpha\nu}(\tilde{g}_{\nu\sigma} + \tilde{b}_{\nu\sigma}) = g^{\beta\nu}(g_{\nu\rho} + b_{\nu\rho})G^{\rho\alpha}G_{\sigma\beta}. \tag{3.24}$$

These two relations can be summarized as

$$\tilde{g}^{\mu\nu} + \tilde{b}^{\mu\nu} = (g_{\kappa\lambda} + b_{\kappa\lambda})G^{\kappa\mu}G^{\lambda\nu}. \tag{3.25}$$

The separate T-duality transformations generated by the individual $h^{(i)}$, Eq. (3.18), are part of the $O(d,d,Z)$ group of dualities of toroidal compactifications discovered in [6], and correspond to the so-called factorized dualities. The remaining $O(d,d,Z)$ duality transformations are implemented by different inner automorphisms of the operator algebra [24].

We conclude this section by considering the effects of T-duality on string backgrounds with abelian isometries. The existence of an abelian isometry implies that we can choose our coordinates $X^\mu = (\theta, X^i)$ in such a way that the spacetime metric is independent of θ. This implies that this particular string solution (string propagating on a target-space admitting an abelian isometry) is represented by a conformal field theory whose stress tensor is of the form

$$T_{g,b}(\sigma) = \frac{1}{2}g^{\theta\theta}(X) : \hat{\partial}\theta\hat{\partial}\theta : (\sigma) + : g^{i\theta}(X)\partial\hat{X}_i\hat{\partial}\theta : (\sigma) + \frac{1}{2} : g^{ij}(X)\partial\hat{X}_i\partial\hat{X}_j : (\sigma). \tag{3.26}$$

As before, this CFT may be deformed to a point with an enhanced symmetry,

$$T_G(\sigma) = \frac{1}{2} G^{\theta\theta}(X) : \partial\theta\partial\theta : (\sigma) + \frac{1}{2} : G^{ij}(X) \partial X_i \partial X_j : (\sigma) \tag{3.27}$$

and a similar expression for $\overline{T}_G(\sigma)$, where

$$\begin{aligned} \partial\theta &= \frac{1}{\sqrt{2}}(\pi_\theta + G_{\theta\theta}\theta') & \overline{\partial}\theta &= \frac{1}{\sqrt{2}}(\pi_\theta - G_{\theta\theta}\theta') \\ \partial X_i &= \frac{1}{\sqrt{2}}(\pi_i + G_{ij}X^{j\prime}) & \overline{\partial} X_i &= \frac{1}{\sqrt{2}}(\pi_i + G_{ij}X^{j\prime}) \end{aligned} \tag{3.28}$$

The first term in Eq. (3.27) has been defined through a point-splitting regularization while for the second term we will assume that an adequate prescription exists in order to define this composite operator so that it commutes with operators constructed from θ. Note that Eq. (3.27) is the direct product of two CFT's which do not interact with one another—the fields θ are governed by a free, toroidal field theory, and they do not interact with the fields X^i, which may have much more general interactions among themselves.

As should by now be familiar, we must first express $\hat{\partial\theta}$ and $\partial\hat{X}_i$ in terms of $\partial\theta$, ∂X_i, $\overline{\partial}\theta$ and $\overline{\partial}X_i$, the light-cone derivatives of the theory with enhanced symmetry.

$$\begin{aligned} \hat{\partial\theta} &= \frac{1}{2}[(\partial\theta + \overline{\partial}\theta) + g_{\theta\theta}G^{\theta\theta}(\partial\theta - \overline{\partial}\theta) + (g_{\theta i} + b_{\theta i})G^{ij}(\partial X_j - \overline{\partial}X_j)] \\ \partial\hat{X}_i &= \frac{1}{2}[(\partial X_i + \overline{\partial}X_i) + (g_{i\theta} + b_{i\theta})G^{\theta\theta}(\partial\theta - \overline{\partial}\theta) + g_{i\kappa}G^{\kappa j}(\partial X_j - \overline{\partial}X_j)] \end{aligned} \tag{3.29}$$

Then the stress-tensor, Eq. (3.19), takes the unnecessarily intimidating form,

$$
\begin{aligned}
&T_{g,b}(\sigma) = \frac{1}{8}[(g^{\theta\theta} + 2G^{\theta\theta} + g_{\theta\theta}G^{\theta\theta}G^{\theta\theta} + 2g^{\theta i}b_{i\theta}G^{\theta\theta} + g^{ij}b_{i\theta}b_{j\theta}G^{\theta\theta})\partial\theta\partial\theta + \\
&(g^{\theta\theta} - 2G^{\theta\theta} + g_{\theta\theta}G^{\theta\theta}G^{\theta\theta} - 2g^{\theta i}b_{i\theta}G^{\theta\theta} + 2g^{ij}b_{i\theta}b_{j\theta}G^{\theta\theta})\overline{\partial}\theta\overline{\partial}\theta + \\
&(2g^{\theta\theta} - 2g_{\theta\theta}G^{\theta\theta}G^{\theta\theta} - g^{ij}b_{i\theta}b_{j\theta}G^{\theta\theta})\partial\theta\overline{\partial}\theta + \\
&(2g^{\theta\theta}b_{\theta j}G^{ij} + 2g^{\theta i} + 2g_{j\theta}G^{\theta\theta}G^{ij} + 2g^{ij}b_{j\theta}G^{\theta\theta} \\
&\qquad\qquad + 2g^{\theta j}b_{j\theta}b_{\theta\kappa}G^{\theta\theta}G^{j\kappa} + 2g^{\kappa j}b_{j\rho}b_{\kappa\theta}G^{\rho i}G^{\theta\theta})\partial\theta\partial X_i + \\
&(-2g^{\theta\theta}b_{\theta j}G^{ij} + 2g^{\theta i} - 2g_{j\theta}G^{\theta\theta}G^{ij} + 2g^{ij}b_{j\theta}G^{\theta\theta} \\
&\qquad\qquad - 2g^{\theta j}b_{j\theta}b_{\theta\kappa}G^{\theta\theta}G^{j\kappa} - 2g^{\kappa j}b_{j\rho}b_{\kappa\theta}G^{\rho i}G^{\theta\theta})\partial\theta\overline{\partial}X_i + \\
&(2g^{\theta\theta}b_{\theta j}G^{ij} + 2g^{\theta i} - 2g_{j\theta}G^{\theta\theta}G^{ij} - 2g^{ij}b_{j\theta}G^{\theta\theta} \\
&\qquad\qquad - 2g^{\theta j}b_{j\theta}b_{\theta\kappa}G^{\theta\theta}G^{j\kappa} - 2g^{\kappa j}b_{j\rho}b_{\kappa\theta}G^{\rho i}G^{\theta\theta})\overline{\partial}\theta\partial X_i + \\
&(-2g^{\theta\theta}b_{\theta j}G^{ij} + 2g^{\theta i} + 2g_{j\theta}G^{\theta\theta}G^{ij} - 2g^{ij}b_{j\theta}G^{\theta\theta} \\
&\qquad\qquad + 2g^{\theta j}b_{j\theta}b_{\theta\kappa}G^{\theta\theta}G^{j\kappa} + 2g^{\kappa j}b_{j\rho}b_{\kappa\theta}G^{\rho i}G^{\theta\theta})\overline{\partial}\theta\overline{\partial}X_i + \\
&(g^{ij} + g_{\kappa\lambda}G^{\kappa i}G^{\lambda j} + g^{\theta\theta}b_{\theta\kappa}b_{\theta\lambda}G^{\kappa i}G^{\lambda j} + g^{\lambda\sigma}b_{\lambda\kappa}b_{\sigma\rho}G^{\kappa i}G^{\rho j} \\
&\qquad\qquad + 2G^{ij} + 2g^{\theta i}b_{\theta\kappa}G^{\kappa j} + 2g^{\lambda j}b_{\lambda\kappa}G^{\kappa i})\partial X_i\partial X_j + \\
&(g^{ij} + g_{\kappa\lambda}G^{\kappa i}G^{\lambda j} + g^{\theta\theta}b_{\theta\kappa}b_{\theta\lambda}G^{\kappa i}G^{\lambda j} + g^{\lambda\sigma}b_{\lambda\kappa}b_{\sigma\rho}G^{\kappa i}G^{\rho j} \\
&\qquad\qquad - 2G^{ij} - 2g^{\theta i}b_{\theta\kappa}G^{\kappa j} - 2g^{\lambda j}b_{\lambda\kappa}G^{\kappa i})\overline{\partial}X_i\overline{\partial}X_j + \\
&(g^{ij} - g^{\theta\theta}b_{\theta\kappa}b_{\theta\lambda}G^{\kappa i}G^{\lambda j} - g_{\kappa\lambda}G^{\kappa i}G^{\lambda j} - g^{\lambda\sigma}b_{\lambda\kappa}b_{\sigma\rho}G^{\kappa i}G^{\rho j} \\
&\qquad\qquad\qquad - g^{\lambda j}b_{\lambda\kappa}G^{\kappa i})(\partial X_i\overline{\partial}X_j + \overline{\partial}X_i\partial X_j)]
\end{aligned}
\quad . \tag{3.30}
$$

Since θ lives on a circle and does not interact with the other fields, we already know from our previous examples an inner automorphism that yields a T-duality. It is generated by

$$h = \frac{1}{2i}\int d\sigma(e^{i\sqrt{2}\theta_L} - e^{-i\sqrt{2}\theta_L}). \tag{3.31}$$

It's effect on the monstrous Eq. (3.30) is, once again, simply to change the sign of $\partial\theta$. With a certain amount of tedious algebra, this transformed stress tensor can be seen to be of the same form as Eq. (3.30), but with transformed *space-time* fields, *i.e.*

$$e^{i\pi h}T_{g,b}(\sigma)e^{-i\pi h} = T_{\tilde{g},\tilde{b}}, \tag{3.32}$$

where,

$$
\begin{aligned}
&\tilde{g}_{\theta\theta} = \frac{1}{g_{\theta\theta}}G^{\theta\theta}G^{\theta\theta} \qquad \tilde{g}_{ij} = g_{ij} - \frac{1}{g_{\theta\theta}}(g_{\theta i}g_{\theta j} - b_{\theta i}b_{\theta j}) \qquad \tilde{g}_{\theta i} = -\frac{b_{\theta i}}{g_{\theta\theta}}G^{\theta\theta} \\
&\tilde{b}_{\theta i} = \frac{g_{i\theta}}{g_{\theta\theta}}G^{\theta\theta} \qquad \tilde{b}_{ij} = b_{ij} - \frac{1}{g_{\theta\theta}}(g_{\theta i}b_{\theta j} - b_{\theta i}g_{\theta j}).
\end{aligned}
\tag{3.33}
$$

These transformations were first derived by Buscher, [10]. In the particular case where g^{ij} is flat, the transformation in Eq. (3.33) is one of the factorized dualities referred to in our earlier discussion of the flat torus.

4. Duality in spaces without isometries.

In section 3 we used our general understanding of symmetries to rederive much of what is currently known about T-duality. In so doing, we hope we persuaded the reader both of the correctness and the power and simplicity of these techniques. In this section we shall venture into *terra incognita* and demonstrate that our techniques are directly applicable to general field configurations. We shall show this both through general arguments and by working out an explicit example. To the best of our knowledge, no other technique is capable of dealing with configurations in which the space-time fields are unrepentantly dependent on the coordinates, X.

Recall that conventional techniques can deal quite effectively with world-sheet actions that contain only derivatives of $X(\sigma)$. At its core, T-duality simply interchanges the momentum $\pi(\sigma)$ and $X'(\sigma)$. Unfortunately, this appears to give a non-local transformation of $X(\sigma)$ itself

$$X(\sigma) \to \int^{\sigma} dx \pi(x).$$

On the other hand our technique implements T-duality as an inner automorphism of the operator algebra

$$e^{i\pi h}\frac{\sqrt{\alpha}'}{R_{cr}}\pi(\sigma)e^{-i\pi h} = -\frac{R_{cr}}{\sqrt{\alpha'}}X'(\sigma) \qquad e^{i\pi h}\frac{R_{cr}}{\sqrt{\alpha'}}X'(\sigma)e^{-i\pi h} = -\frac{\sqrt{\alpha}'}{R_{cr}}\pi(\sigma). \tag{4.1}$$

For exactly the same reasons, any function of $X(\sigma)$ transforms in the same way,

$$f(X(\sigma)) \to e^{i\pi h} f(X(\sigma))e^{-i\pi h} = f(X(\sigma)) + i\pi[h, f(X(\sigma))] + \frac{1}{2}(i\pi)^2[h,[h, f(X(\sigma))]] + \cdots \tag{4.2}$$

Recall from section 3 that the generator that implements T-duality is

$$h = \frac{1}{2i}\int d\sigma(e^{i\sqrt{2}X_L} - e^{-i\sqrt{2}X_L})(\sigma), \tag{4.3}$$

independent of the background fields. Since any reasonable function may be expanded in Fourier modes, it is sufficient to consider the effect of h only on such waves. When we do so, we immediately see that the non-localities of (4.2) appear to manifest themselves in this language also, since

$$: e^{\pm i\sqrt{2}X_L(z)} :: e^{ipX(w,\overline{w})} := (z-w)^{\pm\sqrt{2}p} : e^{\pm i\sqrt{2}X_L(z) + ipX(w,\overline{w})} : \tag{4.4}$$

For general p, the RHS of (4.4) has a cut in the complex plane. When the operator product is converted into a commutator via the usual deformation of the contour of integration, the discontinuity of (4.4) across the cut gives rise to a non-local commutator. Alternatively, it is not hard to show that the commutator of $X_L(\sigma)$ with itself is non-local,

$$[X_L(\sigma), X_L(\sigma')] \sim \Theta(\sigma - \sigma') \tag{4.5}$$

leading to similar concerns.

It would appear that nothing short of a miracle can save us, but this is string theory, and so, of course, a miracle occurs. Observe that the cut is absent, replaced by a pole when $\sqrt{2}p$ is an integer. The absence of a cut implies, in turn, a purely local commutator. But now recall that we are compactified on a circle, and so all functions of $X(\sigma)$ must be periodic under shifts of $2\pi R = 2\sqrt{2}\pi$. *This periodicity implies precisely the required quantization condition on* $p = \frac{n}{\sqrt{2}}$. (A reminder to the reader: the full operator algebra is the same for all radii, and we parameterize it using a form natural at R_{cr}, hence the periodicity condition above. This periodicity is the same for all radii; We have chosen to change radii by varying *metrics*, *not* by varying periodicities).

Note that the form of equation (4.4) is a consequence of summing an infinite number of contractions. As such it is an intrinsically non-perturbative, world-sheet quantum mechanical result. The miracle occurs only in the full result, and not in any classical or semi-classical approximation. (In such approximations, the factor $(z-w)^{-n}$ is replaced by a polynomial in $\ln(z-w)$, which unavoidably has cuts).

So far, our discussion in this section has been rather general. We therefore turn now to a simple illustration of these ideas. For the example to be a sufficiently stringent test of our claims we should consider a solution corresponding to a space-time possessing curvature, but lacking isometries. Since curvature is always absent in one dimension, we consider the bosonic string compactified on a 2-torus. Thus space-time is $M^{24} \times T^2$. With a flat metric at the $SU(2)_L^2 \times SU(2)_R^2$ self-dual point on the torus, the piece of the stress-tensor corresponding to the torus is

$$T(\sigma) = \frac{1}{2}\delta_{\mu\nu}\partial X^\mu \partial X^\nu \qquad \overline{T}(\sigma) = \frac{1}{2}\delta_{\mu\nu}\overline{\partial} X^\mu \overline{\partial} X^\nu \qquad \mu,\nu = 1,2 \tag{4.6}$$

and the two independent duality transformations are generated by

$$h_1 = \frac{1}{2i}\int d\sigma (e^{i\sqrt{2}X_L{}^1} - e^{-i\sqrt{2}X_L{}^1}) \qquad h_2 = \frac{1}{2i}\int d\sigma (e^{i\sqrt{2}X_L{}^2} - e^{-i\sqrt{2}X_L{}^2}) \tag{4.7}$$

We proceed to turn on curvature by infinitesimally deforming this particular solution, adding the perturbation

$$\delta T(\sigma) = \delta\overline{T}(\sigma) = h_{\mu\nu}(X)\partial X^\mu \overline{\partial} X^\nu \tag{4.8}$$

with

$$h_{\mu\nu}(X) = \epsilon_{\mu\nu} e^{i\sqrt{2}X^1} + f_{\mu\nu} e^{i\sqrt{2}X^2} + c.c. \tag{4.9}$$

$\epsilon_{\mu\nu}$, $f_{\mu\nu}$ are polarization vectors and $X^{1,2} = {X_L}^{1,2} + {X_R}^{1,2}$. On its own, the deformed field theory is no longer conformally invariant. However it is straightforward to restore conformal invariance by making a similar deformation to the part of the field theory describing the uncompactified M^{24}. The resulting conformal field theory describes the propagation of a massive Kaluza-Klein excitation in M^{24}. The Riemann tensor for the curved metric $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}(X)$ in the linearized approximation is given by

$$R_{1212} = -[\epsilon_{22} e^{i\sqrt{2}X^1} + f_{11} e^{i\sqrt{2}X^2} + c.c.] \tag{4.10}$$

In two-dimensions R_{1212} encodes all the information concerning the curvature of the space. The Ricci scalar is given in terms of R_{1212}

$$R = 2\frac{R_{1212}}{g} = -2[\epsilon_{22} e^{i\sqrt{2}X^1} + f_{11} e^{i\sqrt{2}X^2} + c.c.] \tag{4.11}$$

Since the scalar curvature is not constant this two-dimensional curved space can have at most one isometry. In the Appendix we demonstrate that the metric $g_{\mu\nu}(X) = \delta_{\mu\nu} + h_{\mu\nu}(X)$ actually has no isometries in general.

To perform a T-duality transformation, we need only compute the effect of a similarity transformation by the operators in equation (4.7) on the stress-tensor. There are of course, two factorized dualities, but for illustrative purposes, we shall consider only the combined duality which is their product.

$$\begin{aligned} e^{i\pi h_2} e^{i\pi h_1} (T + \delta T)(\sigma) e^{-i\pi h_1} e^{-i\pi h_2} = {} & \frac{1}{2}\delta_{\mu\nu} e^{i\pi h_2} e^{i\pi h_1} : \partial X^\mu \partial X^\nu : e^{-i\pi h_1} e^{-i\pi h_2} \\ & + e^{i\pi h_2} e^{i\pi h_1} : h_{\mu\nu}(X) \partial X^\mu \overline{\partial} X^\nu : e^{-i\pi h_1} e^{-i\pi h_2} \end{aligned} \tag{4.12}$$

Since $h_{\mu\nu}(X)$ is given by Eq. (4.9) we must define what we mean by normal-ordered operators in the expression above

$$: e^{i\sqrt{2}X_L} \partial X(\sigma) := \lim_{\epsilon \to 0} [: e^{i\sqrt{2}X_L}(\sigma) : \partial X(\sigma + \epsilon) - \frac{2i : e^{i\sqrt{2}X_L(\sigma)} :}{4\pi\epsilon}] \tag{4.13}$$

and a similar expression for the anti-holomorphic part which commutes with the holomorphic one in the normal-ordered expression. In order to derive the effect of the inner automorphism on the stress-tensor we need to know how the automorphisms act on ∂X^μ, $\overline{\partial} X^\mu$, $e^{i\sqrt{2}X_L^\mu}$ and $e^{i\sqrt{2}X_R^\mu}$. The effect on the first two is known from the previous section

$$e^{i\pi h} \partial X^\mu(\sigma) e^{-i\pi h} = -\partial X^\mu(\sigma) \qquad e^{i\pi h} \overline{\partial} X^\mu(\sigma) e^{-i\pi h} = \overline{\partial} X^\mu(\sigma) \tag{4.14}$$

while the effect on the last two can be computed by writing

$$\begin{aligned} e^{i\pi h_1} e^{i\sqrt{2}(X_L^1 + X_R^1)(\sigma)} e^{-i\pi h_1} = {} & e^{i\sqrt{2}X_R^1}(\sigma)(e^{i\sqrt{2}X_L^1}(\sigma) + i\pi[h_1, e^{i\sqrt{2}X_L^1}(\sigma)] \\ & + \frac{1}{2}(i\pi)^2 [h_1, [h_1, e^{i\sqrt{2}X_L^1}(\sigma)]] + \cdots) \end{aligned} \tag{4.15}$$

and computing the commutators. This can be achieved by going from the cylinder to the complex plane, calculating the corresponding operator product expansions and computing the residue at the pole. The result then is given by

$$e^{i\pi h_1} e^{i\sqrt{2}(X_L^1+X_R^1)(\sigma)} e^{-i\pi h_1} = -e^{-i\sqrt{2}(X_L^1-X_R^1)}(\sigma) \tag{4.16}$$

With Equations (4.14) and (4.16) at our disposal we can now calculate the effect of the inner automorphism (duality transformation) on the stress-tensor. This reads

$$\begin{aligned}
& e^{i\pi h_2} e^{i\pi h_1}(T+\delta T)(\sigma) e^{-i\pi h_1} e^{-i\pi h_2} \\
&= \frac{1}{2}\delta_{\mu\nu} e^{i\pi h_2} e^{i\pi h_1} \lim_{\epsilon\to 0}[\delta_{\mu\nu}\partial X^\mu(\sigma)\partial X^\nu(\sigma+\epsilon) + \frac{1}{2\pi\epsilon^2}] e^{-i\pi h_1} e^{-i\pi h_2} \\
&+ e^{i\pi h_1} e^{i\pi h_2} \lim_{\epsilon\to 0}[\epsilon_{\mu\nu} : e^{i\sqrt{2}X_L^1} : (\sigma)\partial X^\mu(\sigma+\epsilon) \\
&\qquad\qquad - \frac{2i : e^{i\sqrt{2}X_L^1(\sigma)} :}{4\pi\epsilon}] : e^{i\sqrt{2}X_R^1}\overline{\partial} X^\nu : (\sigma) e^{-i\pi h_1} e^{-i\pi h_2} \\
&+ e^{i\pi h_1} e^{i\pi h_2} \lim_{\epsilon\to 0}[f_{\mu\nu} : e^{i\sqrt{2}X_L^2} : (\sigma)\partial X^\mu(\sigma+\epsilon) \\
&\qquad\qquad - \frac{2i : e^{i\sqrt{2}X_L^2(\sigma)} :}{4\pi\epsilon}] : e^{i\sqrt{2}X_R^2}\overline{\partial} X^\nu : (\sigma) e^{-i\pi h_1} e^{-i\pi h_2} \\
&= \frac{1}{2}\delta_{\mu\nu} : \partial X^\mu \partial X^\nu : + : (\epsilon_{\mu\nu} e^{-i\sqrt{2}(X_L^1 - X_R^1)} + f_{\mu\nu} e^{-i\sqrt{2}(X_L^2 - X_R^2)})\partial X^\mu \overline{\partial} X^\nu :
\end{aligned} \tag{4.17}$$

Equation (4.17) is our result. Where the starting stress-tensor described a massive Kaluza-Klein excitation in M^{24}, the transformed stress-tensor in equation (4.17) describes a winding excitation. The physical interpretation therefore is both transparent and unsurprising. We observe that up to a sign the result of the duality transformation is to replace the $X_+(\sigma) = X_L(\sigma) + X_R(\sigma)$ dependence of the metric with its dual coordinate $X_- = X_L(\sigma) - X_R(\sigma)$. On its own, the dual coordinate X_- is a very sick non-local operator, but it is a marvelous and well-known fact that all the operators in (4.17) are local. The deformation of the original conformal field theory corresponds to turning on a Kaluza-Klein graviton $h_{\mu\nu}(X)$, a momentum excitation of the string spectrum. The effect of T-duality is to transform the KK graviton into a winding mode excitation and subsequently the two conformal field theories which result from perturbing the original theory by sending weak gravitational KK waves and weak winding waves should be identified. In the following paragraphs we will derive similar results for arbitrary field configurations on tori.

Two more comments on equation (4.17) are appropriate. First, the reader may be wondering what happened to $R \to \frac{1}{R}$. The answer is that we considered a non-constant deformation of the theory at the critical radius. T-Duality turned the metric excitation into a winding excitation, but inverting the metric produced no visible effect precisely because we started at the critical radius. We could, however, have started just as easily at any radius; the constant piece of the metric would transform just as it did in section 3, and the constant piece of the metric would have been inverted. Indeed, each Fourier mode transforms independently as a simple consequence of the linearity of the similarity transformation ; we chose not to repeat the calculation of section 3, solely to avoid cluttering our formulas.

Our second comment relates to the size of our non-constant deformation. In the above analysis we chose to consider weak perturbations; $\epsilon_{\mu\nu}$ and $f_{\mu\nu}$ were taken to be infinitesimal. We did so only because this makes the analysis of the conditions for conformal invariance straightforward [17]. However, these conditions actually have no effect on the form of the T-duality transformation. Conformal field theories may be constructed by writing down general translation invariant *ansätze* for $T(\sigma)$ and $\overline{T}(\sigma)$, and then demanding that they satisfy Virasoro $\times$ Virasoro [18], which imposes conditions (called equation of motion!) on the space-time fields. These are Popeye conditions—they are what they are—but they do not affect in any way symmetry transformations on those fields; they just come along for the ride. (Since imposing Virasoro $\times$ Virasoro is an *algebraic* condition, it is of course preserved by similarity transformations). Thus our analysis of T-duality transformations is immediately extendable to finite deformations.

So far we have discussed space-time fields which are constant or which have the lowest mode on the torus excited. We end this section with a derivation of the transformation of higher modes. The first thing we need to understand is the transformation of pure functions of X; that is, we need to compute $e^{i\pi h}e^{i\sqrt{2}n(X_L+X_R)(\sigma)}e^{-i\pi h}$, where $n \in \mathbb{Z}$. With this result, we shall then be able to compute the transformation of arbitrary periodic functions of $X(\sigma)$ multiplied by arbitrary light-cone derivatives of $X(\sigma)$ by the same point-splitting techniques we used in (4.17). To compute this quantity we could in principle use the same technique we used above to compute the transformation of the lowest mode, expanding the exponential in powers of h and computing the multiple commutators. However, for the higher mode such an approach becomes increasingly more involved as n increases and, with it, the order of poles from contractions. Rather, it is easier to use the result for $n = 1$ and induction. First, we write

$$:e^{i\sqrt{2}nX_L(\sigma)} := \lim_{\epsilon\to 0} :e^{i\sqrt{2}X_L(\sigma+\epsilon)}::e^{i\sqrt{2}(n-1)X_L(\sigma)}:(\frac{1}{\epsilon})^{\frac{(n-1)}{\pi}} \tag{4.18}$$

It then follows that

$$e^{i\pi h}:e^{i\sqrt{2}nX_L(\sigma)}:e^{-i\pi h} = \lim_{\epsilon\to 0} e^{i\pi h}:e^{i\sqrt{2}X_L(\sigma+\epsilon)}:e^{-i\pi h}e^{i\pi h}:e^{i\sqrt{2}(n-1)X_L(\sigma)}:e^{-i\pi h}(\frac{1}{\epsilon})^{\frac{(n-1)}{\pi}} \tag{4.19}$$

and we may use equation (4.16) and induction to compute the result for arbitrary n. The result is

$$e^{i\pi h}:e^{i\sqrt{2}nX_L(\sigma)}:e^{-i\pi h} = (-1)^n :e^{-i\sqrt{2}nX_L(\sigma)}: \tag{4.20}$$

Since $X_L(\sigma)$ and $X_R(\sigma)$ are constructed from mutually commuting sets of creation and annihilation operators it follows that

$$e^{i\pi h} : e^{i\sqrt{2}n(X_L+X_R)(\sigma)} : e^{-i\pi h} = (-1)^n : e^{-i\sqrt{2}n(X_L-X_R)(\sigma)} : \tag{4.21}$$

Thus, modulo additional terms coming from contraction with light-cone derivatives, the n-th momentum mode is interchanged with the n-th winding mode.

We now have all the pieces we need in order to compute the transformation on any field. A general term in the stress tensor will be of the, "weighted tensor," type [23],

$$\phi_{\mu\nu\cdots\rho}(X)\partial^{w_1}X^{\mu}\partial^{w_2}X^{\nu}\cdots\partial^{w_n}X^{\kappa}\overline{\partial}^{v_1}X^{\lambda}\cdots\overline{\partial}^{v_m}X^{\rho} \tag{4.22}$$

Again, we may decompose such a term into Fourier modes, and consider the transformation of each separately under T-Duality. Using equations (4.21) and (4.14), we might expect the transformation to be,

$$\begin{aligned} &e^{i\sqrt{2}p(X_L+X_R)}\partial^{w_1}X^{\mu}\partial^{w_2}X^{\nu}\cdots\partial^{w_n}X^{\kappa}\overline{\partial}^{v_1}X^{\lambda}\cdots\overline{\partial}^{v_m}X^{\rho}(\sigma) \rightarrow \\ &\rightarrow (-1)^{n+p}e^{-i\sqrt{2}p(X_L-X_R)(\sigma)}\partial^{w_1}X^{\mu}\partial^{w_2}X^{\nu}\cdots\partial^{w_n}X^{\kappa}\overline{\partial}^{v_1}X^{\lambda}\cdots\overline{\partial}^{v_m}X^{\rho}(\sigma) \end{aligned} \tag{4.23}$$

This result is, in fact, correct, but it is not quite trivial to demonstrate it because of the normal-ordering present in such terms. However, the point-splitting method we have used repeatedly throughout this paper is applicable in this general case, and an inductive proof of equation (4.23) is not hard to construct. We shall not give the details here (they are straightforward, if a little tedious), but the first step is to prove the result for terms involving any mode and a single n'th derivative of X, and then to inductively increase the number of derivatives of X. This result implies that the transformation on *any* field ϕ, with n indices contracting with holomorphic derivatives of X is simply

$$\phi(X_+) \longleftrightarrow (-1)^n\phi(-X_- + \pi/\sqrt{2}). \tag{4.24}$$

Equation (4.24) gives us the transformation of arbitrary space-time fields, and is our final result.

5. Discussion.

What have we done in this paper? We have taken a general approach to understanding the symmetries of string theory and applied it to T-duality. We have rederived known results when background fields are constant (or can be made so), and we have shown how a string miracle enables us to deal with general field configurations on space-time tori. We worked out an explicit example of such a case, and derived a simple general formula for the transformation of arbitrary background fields. In our opinion, T-duality on tori is now fully understood (or, at least, understandable).

What have we not done? We have worked only on tori, and have ignored other topologies. Unfortunately, at our present level of understanding, each space-time topology must be studied separately, at least when studying T-duality. The reason is that the algebra of observables associated with a conformal field theory, while the same for all CFTs in a deformation class, appears to change when we change to another deformation class (*i.e.* another space-time topology). How does T-duality manifest itself on other topologies? Is there a relationship between T-duality and mirror symmetry [25]? Does some avatar of T-duality remain to influence string theory in uncompactified space-time? Does there exist some Ur-theory that would enable us to handle all space-time topologies simultaneously?

Similarly, we may wonder about the relationship of the present work to the currently fashionable S-duality. At first sight, our techniques would appear to be inapplicable to S-duality. After all, while our methods are fully quantum-mechanical on the world sheet, we have worked only on S^2, *i.e.* only at string tree-level. S-duality, being a strong-weak coupling duality would appear to require a fully quantum mechanical understanding of string theory. However, recent very interesting work [26] seems to suggest an intimate connection between S and T-dualities, and to do so at an entirely classical level of analysis. Perhaps the techniques of this paper will be able to throw some light on on S-duality after all.

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7. Appendix.

In this appendix we shall demonstrate that the metric $g_{\mu\nu}(X) = \delta_{\mu\nu} + h_{\mu\nu}(X)$ with

$$h_{\mu\nu}(X) = \epsilon_{\mu\nu} e^{i\sqrt{2}X^1} + f_{\mu\nu} e^{i\sqrt{2}X^2} + c.c. \tag{7.1}$$

does not possess any isometries. The problem of determining all infinitesimal isometries of the metric $g_{\mu\nu}(X)$ is reduced to the problem of determining all Killing vectors of the metric. The linearized Ricci scalar for $g_{\mu\nu}(X)$ is given by

$$R = \partial_\alpha \partial^\alpha h^\mu_\mu - \partial^\mu \partial^\nu h_{\mu\nu} = (-p^2 \epsilon^\mu_\mu + p^\mu p^\nu \epsilon_{\mu\nu}) e^{ipX} + (-q^2 f^\mu_\mu + q^\mu q^\nu f_{\mu\nu}) e^{iqX} + c.c \quad (7.2)$$

with $p^\mu = \sqrt{2}(1,0)$ and $q^\mu = \sqrt{2}(0,1)$. If V^μ is a Killing vector then $V^\mu \partial_\mu R = 0$. Since R is not constant then V^μ must be tangent to curves of constant R. Thus there can be at most one Killing vector field. A solution to the relation $V^\mu \partial_\mu R = 0$ is provided by

$$V^\lambda = p^\lambda (-q^2 f^\mu_\mu + q^\mu q^\nu f_{\mu\nu}) e^{iqX} - q^\lambda (-p^2 \epsilon^\mu_\mu + p^\mu p^\nu \epsilon_{\mu\nu}) e^{ipX} + c.c \quad (7.3)$$

By finding the curves of constant R one may characterize possible Killing vectors up to a scalar function $\phi(X)$. It suffices then to prove that there is no non-trivial scalar function $\phi(X)$ that satisfies as a result of the Killings equation the relation

$$\partial_{(\kappa}(e^\phi V_{\lambda)}) = e^\phi (V_\lambda \partial_\kappa \phi + V_\kappa \partial_\lambda \phi + \partial_\kappa V_\lambda + \partial_\lambda V_\kappa) = 0 \quad (7.4)$$

Contracting the indices of (7.4) with p^λ, p^κ (q^λ, q^κ) and taking into account the form of V^μ (7.3) we find

$$p^\mu \partial_\mu \phi = 0 \qquad q^\mu \partial_\mu \phi = 0 \quad (7.5)$$

The only solution to the above equations is that of a constant function which fails to yield a solution to Killing's equation for generic polarization vectors $f_{\mu\nu}$, $\epsilon_{\mu\nu}$. Thus the metric $g_{\mu\nu}(X)$ in general possess no isometries.