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On Potential Energies and Constraints in the Dynamics of Rigid Bodies and Particles

OLIVER M. O'REILLY[a,*] and ARUN R. SRINIVASA[b]

[a]*Department of Mechanical Engineering, University of California at Berkeley, Berkeley, California 94720-1740, USA;* [b]*Department of Mechanical Engineering, Texas A&M University, College Station, Texas 77843-3123, USA*

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A new treatment of kinematical constraints and potential energies arising in the dynamics of systems of rigid bodies and particles is presented which is suited to Newtonian and Lagrangian formulations. Its novel feature is the imposing of invariance requirements on the constraint functions and potential energy functions. These requirements are extensively used in continuum mechanics and, in the present context, one finds certain generalizations of Newton's third law of motion and an elucidation of the nature of constraint forces and moments. One motivation for such a treatment can be found by considering approaches where invariance requirements are ignored. In contrast to the treatment presented in this paper, it is shown that this may lead to a difficulty in formulating the equations governing the motion of the system.

Key words: Constraints; Invariance; Rigid body dynamics

1 INTRODUCTION

There are two classical approaches to the formulation of the equations of motion of a system of rigid bodies and particles. One approach uses the balances of linear and angular momenta of each of the rigid bodies and the balance of linear momentum for each individual particle. These equations are supplemented by constraints, prescriptions of constraint forces and moments, and specifications of applied forces and moments to form a closed system of equations from which both the motion of the system and the constraint forces and moments can be determined. Another approach is to use Lagrange's equations of motion, which has the advantages of automatically incorporating several of the constraints and eliminating some of the equations which are identically satisfied by the constraint forces and moments.

Recently, Casey [1] clarified several issues concerning the equivalence of the two approaches discussed above. There is however an obstacle remaining and that concerns the constraint and potential forces and moments. In Lagrangian mechanics, these force and moments appear as generalized forces. In most texts, such as Gantmacher [2] and Rosenberg [3], general functional forms of the potential energies and constraints are postulated after some motivational examples. Indeed, although extensively used in Lagrangian mechanics, the issue of potential moments in the context of a single unconstrained rigid body was only fairly recently resolved by Antman [4] (cf. also Simmonds [5]). Recent discussions on issues concerning

* Corresponding author. Fax: 510-642-6163; E-mail: oreilly@me.berkeley.edu

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constraints and constrained mechanical systems can be found in, for example, Huston [6] and Udwadia and Kalaba [7].

In the present work, we examine the general functional forms of constraint functions and potential energy functions for these systems, and establish representations for the forces and moments associated with them. One of the novel features of our treatment is the extensive use of invariance requirements for both the constraint and potential energy functions. In continuum mechanics, these invariance requirements are commonly imposed on both the strain-energy function and the constraint function (cf., *e.g.*, [8]–[13]). We find that imposition of these requirements yields generalizations of Newton's third law of motion to systems of particles and rigid bodies. Further, it clarifies some issues pertaining to the specification of forces and moments associated with time-dependent potential energies.

Our work can be viewed as an extension of Antman's work [4] to the case of a system of rigid bodies and particles. Furthermore, the generalizations of Newton's third law we find were motivated by an earlier work of Noll [14]. He showed, in the context of two particles, how this law can be arrived at using invariance requirements. We also note that our work can be used in conjunction with the work of Casey [1, 15] to establish several of the alternative forms of Lagrange's equations of motion that have appeared in the literature.[1]

Primarily because of their computational advantages, there has been an increased interest in recent years in the use of Euler parameters to parameterize the rotation tensor of a rigid body (see, for example, [18]–[21]). To encompass all possible parameterizations, we present our results in a representation-free form. However, we also indicate how the associated derivatives can be calculated.

The paper is organized as follows. In the forthcoming section, several preliminary results on the motions of rigid bodies are presented. These results can be trivially interpreted for systems of particles and rigid bodies. In Section 3, the invariance requirements are introduced and some of their consequences presented. Then, in Section 4, the general functional forms of a kinematical constraint and a potential energy function which are compatible with these requirements are discussed. Subsequently, in Section 5, prescriptions for the constraint and conservative forces and moments are presented. The next section discusses how Newton's third law for constraint and conservative forces and moments naturally arises as a consequence of the invariance requirements. A problem associated with an approach where invariance requirements are not imposed is discussed in Section 7, and a resolution of this difficulty is also presented. An appendix presents a Proposition which is used in several sections of this paper.

2 PRELIMINARIES

In this paper, a system of rigid bodies and particles is considered. For ease of exposition, it suffices to consider a system of $N + 1$ rigid bodies $\mathcal{B}_\Theta$, $(\Theta = \infty, \ldots, \mathcal{N} + \infty)$. Our notation follows the works of Beatty [22] and Casey [1, 23]. A fixed reference configuration is defined for each body. This configuration uniquely identifies the position vector $\mathbf{X}_\Theta$ of a material point X_Θ of the body $\mathcal{B}_\Theta$.[2] The position vector of this material point in the present or current

[1]For example, the Routh-Voss equations and the Boltzmann–Hamel equations. Details on these equations can be found in Hamel [16] and Papastavridis [17].

[2]In this paper, lower-case Greek indices range from 1 to 2 and are not summed when repeated. Upper-case Greek indices range from either 1 to N or 1 to $N+1$ or, in the appendix, 1 to $6N+6$ and are not summed when repeated. Lower-case Latin indices range from 1 to 3 and are summed when repeated. We also employ a tensor notation discussed in [1,22,23].

configuration of the body is denoted by $\mathbf{x}_\Theta$. All of the aforementioned position vectors are defined relative to a fixed origin O in Euclidean three-space.

The motion of the rigid body $\mathcal{B}_\Theta$ has a particular form:

$$\mathbf{x}_\Theta = \mathbf{Q}_\Theta \mathbf{X}_\Theta + \mathbf{q}_\Theta, \tag{1}$$

where $\mathbf{q}_\Theta = \mathbf{q}_\Theta(t)$ is a vector-valued function of time and $\mathbf{Q}_\Theta = \mathbf{Q}_\Theta(t)$ is a proper-orthogonal (rotation) tensor.[3] It is convenient to define the position vectors of the centers of mass of the rigid bodies in their present and reference configurations. These are, respectively, $\bar{\mathbf{x}}_\Theta$ and $\bar{\mathbf{X}}_\Theta$. As the center of mass of a rigid body moves as if it were a material point of the rigid body, $\bar{\mathbf{x}}_\Theta = \mathbf{Q}_\Theta \bar{\mathbf{X}}_\Theta + \mathbf{q}_\Theta$, we find from (1) that

$$\mathbf{x}_\Theta = \mathbf{Q}_\Theta(\mathbf{X}_\Theta - \bar{\mathbf{X}}_\Theta) + \bar{\mathbf{x}}_\Theta. \tag{2}$$

To consider a system of n rigid bodies and m particles, we would set $N = n + m - 1$, identify the position vector of the ith particle by $\bar{\mathbf{x}}_{n+i}$, and suppress any functional dependency on $\mathbf{Q}_{n+1}, \ldots, \mathbf{Q}_{n+m}$.

The angular velocity tensors $\mathbf{\Omega}_\Theta$ and their associated angular velocity vectors ω_Θ are defined in the usual manner:

$$\mathbf{\Omega}_\Theta = \dot{\mathbf{Q}}_\Theta \mathbf{Q}_\Theta^{\mathrm{T}}, \quad \omega_\Theta = -\frac{1}{2}\varepsilon[\mathbf{\Omega}_\Theta], \tag{3}$$

where ε is a third-order tensor which is known as the alternator, and the superposed dot denotes the time derivative.

In several sections of this paper, we will calculate the derivative of a function with respect to a rotation tensor. To illuminate this matter, consider a function $\Sigma(\mathbf{R})$. We define the operators $\sigma_{\mathbf{R}}$ and $\Sigma_{\mathbf{R}}$ by

$$\Sigma_{\mathbf{R}} = \frac{1}{2}\left(\frac{\partial \Sigma}{\partial \mathbf{R}}\mathbf{R}^{\mathrm{T}} - \mathbf{R}\left(\frac{\partial \Sigma}{\partial \mathbf{R}}\right)^{\mathrm{T}}\right), \quad \sigma_{\mathbf{R}} = -\varepsilon[\Sigma_{\mathbf{R}}]. \tag{4}$$

Suppose that $\mathbf{R}$ is parameterized using 3 parameters $\{\gamma^i\}$, for instance three Euler angles or the (three components of the) Rodrigues vector: $\mathbf{R} = \bar{\mathbf{R}}(\gamma^1, \gamma^2, \gamma^3)$. Then one can define the sets of basis vectors $\{\mathbf{g}^j\}$ and $\{\mathbf{g}_i\}$:[4]

$$\mathbf{g}_i = -\frac{1}{2}\varepsilon\left[\frac{\partial \bar{\mathbf{R}}}{\partial \gamma^i}\mathbf{R}^{\mathrm{T}}\right], \quad \mathbf{g}^j \cdot \mathbf{g}_i = \delta_i^j, \tag{5}$$

where δ_i^j is the Kronecker delta. With some manipulations using the chain-rule, one finds the representations

$$\Sigma_{\mathbf{R}} = -\frac{1}{2}\frac{\partial \bar{\Sigma}}{\partial \gamma^i}\varepsilon \mathbf{g}^i, \quad \sigma_{\mathbf{R}} = \frac{\partial \bar{\Sigma}}{\partial \gamma^i}\mathbf{g}^i, \tag{6}$$

where $\Sigma = \Sigma(\mathbf{R}) = \bar{\Sigma}(\gamma^i)$. Related representations can be obtained if the rotation tensor is parameterized using a 4-parameter representation, such as Euler parameters, but we do not pause to discuss them.

[3] A tensor $\mathbf{R}$ is a rotation tensor if $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = 1$, where $\mathbf{I}$ is the identity tensor and the superscript T denotes transpose. For an extensive review on the various methods, such as Euler angles and Euler parameters, of parameterizing the rotation tensor, the reader is referred to Shuster [24].

[4] If γ^i are Euler angles, then the set $\{\mathbf{g}_i\}$ is often known as the Euler basis.

3 INVARIANCE CONSIDERATIONS AND RESULTS

Here, we discuss a scalar-valued function Ψ of the motion of the $N+1$ bodies:

$$\Psi = \bar{\Psi}(\bar{\mathbf{x}}_\Delta, \bar{\mathbf{x}}_{N+1}, \mathbf{Q}_\Delta, \mathbf{Q}_{N+1}, t), \quad (\Delta = 1, \ldots, N). \tag{7}$$

We seek a functional form of Ψ which is invariant under a superposed rigid body motion of the entire system of rigid bodies. A function which is invariant in this respect is termed properly invariant in this paper. If, in addition, we were to assume that the function was form-invariant under a change of Euclidean observer, then it follows from a result of Svendsen and Bertram [12] that the function will also be material frame – indifferent in the sense defined by Truesdell and Noll [13].

We recall that a superposed rigid body motion induces the following transformations on the motion of a rigid body $\mathcal{B}_\Theta$:

$$\mathbf{x}_\Theta^+(t^+) = \mathbf{S}\mathbf{x}_\Theta(t) + \mathbf{s}, \quad \mathbf{Q}_\Theta^+(t^+) = \mathbf{S}\mathbf{Q}_\Theta(t), \quad (\Theta = 1, \ldots, N+1). \tag{8}$$

Here, the $+$ denotes the transformed quantity, $\mathbf{S} = \mathbf{S}(t)$ is a rotation tensor, $\mathbf{s} = \mathbf{s}(t)$ is a vector, and $t^+ = t + a$, where a is a constant. The transformations (8) imply, from (2), that the velocity vectors $\bar{\mathbf{v}}_\Theta = \dot{\bar{\mathbf{x}}}_\Theta$ and angular velocity tensors transform as

$$\bar{\mathbf{v}}_\Theta^+(t^+) = \dot{\mathbf{S}}\bar{\mathbf{x}}_\Theta(t) + \mathbf{S}\bar{\mathbf{v}}_\Theta(t) + \dot{\mathbf{s}}, \quad \mathbf{\Omega}_\Theta^+(t^+) = \dot{\mathbf{S}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}\mathbf{\Omega}_\Theta(t)\mathbf{S}^{\mathrm{T}}. \tag{9}$$

The corresponding transformation of the angular velocity vectors is easily inferred.

For the function $\bar{\Psi}$, we find, with the assistance of (8), that

$$\Psi^+ = \bar{\Psi}(\mathbf{S}\bar{\mathbf{x}}_\Delta + \mathbf{s}, \mathbf{S}\bar{\mathbf{x}}_{N+1} + \mathbf{s}, \mathbf{S}\mathbf{Q}_\Delta, \mathbf{S}\mathbf{Q}_{N+1}, t + a). \tag{10}$$

We require Ψ to be properly invariant for all possible values of a, $\mathbf{s}$, and $\mathbf{S}$: $\bar{\Psi} = \bar{\Psi}^+$. Consequently, $\bar{\Psi}$ cannot be an explicit function of time. By choosing $\mathbf{S} = \mathbf{Q}_{N+1}^{\mathrm{T}}$ and $\mathbf{s} = -\mathbf{Q}_{N+1}^{\mathrm{T}}\bar{\mathbf{x}}_{N+1}(t)$, we can define a functional form which is properly invariant:

$$\Psi = \hat{\Psi}(\mathbf{Y}_\Delta, \mathbf{y}_\Delta), \tag{11}$$

where

$$\mathbf{y}_\Delta = \mathbf{Q}_{N+1}^{\mathrm{T}}(\bar{\mathbf{x}}_\Delta - \bar{\mathbf{x}}_{N+1}), \quad \mathbf{Y}_\Delta = \mathbf{Q}_{N+1}^{\mathrm{T}}\mathbf{Q}_\Delta. \tag{12}$$

We emphasize that $\mathbf{y}_\Delta^+ = \mathbf{y}_\Delta$ and $\mathbf{Y}_\Delta^+ = \mathbf{Y}_\Delta$. It should be clear from our present arguments that one could define alternatives to $\hat{\Psi}$ using the rotation tensor and position vector of the center of mass of another rigid body instead of the rigid body $\mathcal{B}_{N+\infty}$. We also remark that the construction of the properly invariant functional form of Ψ has some similarities to the pivot point selection used by Casey and Naghdi [25] in their construction of properly invariant approximate theories of deformable bodies.

We now consider the time derivative of $\hat{\Psi}$:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \sum_{\Delta=1}^{N} \frac{\partial\hat{\Psi}}{\partial\mathbf{y}_\Delta} \cdot \dot{\mathbf{y}}_\Delta + \sum_{\Delta=1}^{N} \mathrm{tr}\left(\frac{\partial\hat{\Psi}}{\partial\mathbf{Y}_\Delta}\dot{\mathbf{Y}}_\Delta^{\mathrm{T}}\right), \tag{13}$$

where tr denotes the trace of a tensor. We can rewrite this expression by substituting for $\mathbf{y}_\Delta$, $\mathbf{Y}_\Delta$ and their time derivatives:

$$\begin{aligned}\frac{\mathrm{d}\Psi}{\mathrm{d}t} &= \sum_{\Delta=1}^{N}\left(\mathbf{Q}_{N+1}\frac{\partial\hat{\Psi}}{\partial\mathbf{y}_\Delta}\right)\cdot(\bar{\mathbf{v}}_\Delta - \bar{\mathbf{v}}_{N+1}) + \sum_{\Delta=1}^{N}(\mathbf{Q}_{N+1}\hat{\psi}_{\mathbf{Y}_\Delta})\cdot(\omega_\Delta - \omega_{N+1}) \\ &\quad + \sum_{\Delta=1}^{N}(\bar{\mathbf{x}}_\Delta - \bar{\mathbf{x}}_{N+1})\times\left(\mathbf{Q}_{N+1}\frac{\partial\hat{\Psi}}{\partial\mathbf{y}_\Delta}\right)\cdot\omega_{N+1},\end{aligned} \tag{14}$$

where $\hat{\psi}_{\mathbf{Y}_\Delta}$ is defined using (4). It should be noted that, because $\mathbf{y}_\Delta$ and $\mathbf{Y}_\Delta$ are properly invariant, so too are $\partial\hat{\Psi}/\partial\mathbf{Y}_\Delta$, $\hat{\psi}_{\mathbf{Y}_\Delta}$, and $\mathrm{d}\hat{\Psi}/\mathrm{d}t$.

All of the aforementioned results can be extended to the case of several rigid bodies and particles. For example, if the system of interest consists of a particle and a rigid body, then consider a function $\Omega = \bar{\Omega}(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \mathbf{Q}_1, t)$, where $\bar{\mathbf{x}}_2$ denotes the position vector of the particle. Adapting the invariance requirements discussed previously, we find that the properly invariant functional form of Ω is $\hat{\Omega}(\mathbf{y})$, where $\mathbf{y} = \mathbf{Q}_1^{\mathrm{T}}(\bar{\mathbf{x}}_2 - \bar{\mathbf{x}}_1)$.

The case of a system of $N+1$ particles deserves further comment. Here, starting from a function Φ of the form (7), where the dependency on the rotation tensors is suppressed, it can be shown that if this function is properly invariant, then it can only depend on the relative position vectors of the particles. Furthermore, $\Phi = \hat{\Phi}(\mathbf{S}(\bar{\mathbf{x}}_\Delta - \bar{\mathbf{x}}_{N+1})) = \hat{\Phi}(\bar{\mathbf{x}}_\Delta - \bar{\mathbf{x}}_{N+1})$ for all rotation tensors $\mathbf{S}$. One can then use Cauchy's representation theorem for hemitropic functions (see [13]) to show that Φ must be a function of the inner products and scalar triple products of $\bar{\mathbf{x}}_\Delta - \bar{\mathbf{x}}_{N+1}$.

4 POTENTIAL ENERGIES AND KINEMATICAL CONSTRAINTS

In classical dynamics, the most common forms of potential energies are associated with gravitational forces, constant forces and spring forces and moments. Using the results established in the previous section, we now postulate the general functional form of a potential energy U associated with a system of $N+1$ rigid bodies which satisfies the invariance requirement $U = U^+$:

$$U = \hat{U}(\mathbf{Y}_\Delta, \mathbf{y}_\Delta), \tag{15}$$

where $\mathbf{Y}_\Delta$ and $\mathbf{y}_\Delta$ are defined by (12). We shall shortly address the forces and moments associated with this potential energy. We also note that our remarks on the function $\hat{\Psi}$ and $\bar{\Psi}$ in the previous section clearly pertain to U.

A kinematical constraint is an assumed restriction on the motions of the rigid bodies and particles. By considering a wide variety of constraints in classical dynamics, it is possible to postulate the general form of a kinematical constraint. Here, we suppose that there are $K \leq 6(N+1)$ constraints on the motions of the $N+1$ rigid bodies, and postulate their general forms:

$$\Pi_J = \sum_{\Theta=1}^{N+1}\mathbf{f}_\Theta^J\cdot\bar{\mathbf{v}}_\Theta + \sum_{\Theta=1}^{N+1}\mathbf{g}_\Theta^J\cdot\omega_\Theta = 0, \quad (J = 1, \dots, K), \tag{16}$$

where

$$\mathbf{f}_\Theta^J = \mathbf{f}_\Theta^J(\mathbf{y}_\Delta, \mathbf{Y}_\Delta), \quad \mathbf{g}_\Theta^J = \mathbf{g}_\Theta^J(\mathbf{y}_\Delta, \mathbf{Y}_\Delta), \quad (\Delta = 1, \dots, N). \tag{17}$$

The K constraints are assumed to be independent. A constraint $\Pi_J = 0$ is said to be integrable if there exists a function $\pi_J = 0$ of the form (11) such that $\mathrm{d}\pi_J/\mathrm{d}t = \Pi_J$. To establish the possible existence of such a function, one parameterizes $\bar{\mathbf{x}}_\Theta$ and $\mathbf{Q}_\Theta$ using curvilinear coordinates and Euler angles, respectively, and then invokes standard integrability criteria.[5]

We require the constraints (16) to be invariant under all transformations of the form (8): $\Pi_J = \Pi_J^+$. Imposing this restriction and using (8) and (9), we find that

$$\begin{aligned}\sum_{\Theta=1}^{N+1}\{\mathbf{S}^{\mathrm{T}}(\mathbf{f}_\Theta^J)^+ - \mathbf{f}_\Theta^J\}\cdot\bar{\mathbf{v}}_\Theta + \sum_{\Theta=1}^{N+1}\{\mathbf{S}^{\mathrm{T}}(\mathbf{g}_\Theta^J)^+ - \mathbf{g}_\Theta^J\}\cdot\omega_\Theta + \left\{\sum_{\Theta=1}^{N+1}\mathbf{f}_\Theta^J\right\}\cdot\dot{\mathbf{s}} \\ + \left\{\sum_{\Theta=1}^{N+1}\mathbf{S}^{\mathrm{T}}(\mathbf{g}_\Theta^J)^+ + \bar{\mathbf{x}}_\Theta\times\mathbf{S}^{\mathrm{T}}(\mathbf{f}_\Theta^J)^+\right\}\cdot -\frac{1}{2}\varepsilon[\dot{\mathbf{S}}\mathbf{S}^{\mathrm{T}}] = 0.\end{aligned} \tag{18}$$

As (18) is required to hold for all motions of the rigid bodies and all transformations (8), we find that it is necessary for all the terms in the curly brackets to vanish:

$$(\mathbf{f}_\Theta^J)^+ = \mathbf{S}\mathbf{f}_\Theta^J, \quad (\mathbf{g}_\Theta^J)^+ = \mathbf{S}\mathbf{g}_\Theta^J, \quad \sum_{\Theta=1}^{N+1}\mathbf{f}_\Theta^J = \mathbf{0}, \quad \sum_{\Theta=1}^{N+1}\mathbf{g}_\Theta^J + \sum_{\Theta=1}^{N+1}\bar{\mathbf{x}}_\Theta\times\mathbf{f}_\Theta^J = \mathbf{0}. \tag{19}$$

These conditions are easily shown to be sufficient for $\Pi_J = \Pi_J^+$.

The constraints (16) that we consider are far less general than those normally considered in treatments of rigid body dynamics. First, we exclude rheonomic constraints because they are not properly invariant. Secondly, the restrictions $\Pi_J = \Pi_J^+$ and (19) are not normally imposed. Our motivation for imposing these requirements arose when considering how constraints presented themselves: they appear from the interaction of a rigid body or particle with another rigid body or particle. It is when the motion of this other body is prescribed that the invariance requirements are violated and the constraints can then become rheonomic. Related comments apply to the potential energy function.

Finally, we note that the Euler parameter constraint discussed in [18]–[21] is not a constraint in the sense defined here. It arises when one uses 4 Euler parameters to parameterize the rotation tensor. As such, it is a restriction on the parameterization of the rotation tensor, but not a physical restriction on the motion of the rigid body.

5 PRESCRIPTIONS FOR THE CONSERVATIVE FORCES AND MOMENTS AND THE CONSTRAINT FORCES AND MOMENTS

In this section, we discuss prescriptions for the constraint forces $\mathbf{F}_{c\Theta}$ and moments $\mathbf{M}_{c\Theta}$ and conservative forces $\mathbf{F}_{p\Theta}$ and moments $\mathbf{M}_{p\Theta}$ associated with a system of $N+1$ rigid bodies which are subject to the constraints (16) where the system has a potential energy U (cf. (15)).

Any prescriptions of $\mathbf{F}_{c\Theta}$ and $\mathbf{M}_{c\Theta}$ must be physically meaningful and also result in a closed system of equations from which the motions of the various constituents can be determined. Here, following Casey [1, 15], we prescribe these forces and moments using the normality prescription:

$$\mathbf{F}_{c\Theta} = \sum_{J=1}^{K}\lambda_J\mathbf{f}_\Theta^J, \quad \mathbf{M}_{c\Theta} = \sum_{J=1}^{K}\lambda_J\mathbf{g}_\Theta^J, \quad (\Theta = 1,\ldots,N+1). \tag{20}$$

[5] Rosenberg [3] and Papastavridis [17] present reviews of these criteria.

where λ_J is a Lagrange multiplier,[6] $\mathbf{M}_{c\Theta}$ is the constraint moment exerted on $\mathcal{B}_\Theta$ relative to $\bar{\mathbf{x}}_\Theta$, and $\mathbf{F}_{c\Theta}$ is the constraint force exerted on $\mathcal{B}_\Theta$.

It is well-known for constraints of the form (16), that the normality prescription is identical to a prescription, which is credited to Lagrange, that the combined virtual work of the constraint forces and moments is zero (cf., *e.g.*, [3, 15, 17]). Furthermore, using (16), it is easy to see that the combined mechanical power $\mathcal{P}$ of $\mathbf{F}_{c\Theta}$ and $\mathbf{M}_{c\Theta}$ is zero, where

$$\mathcal{P} = \sum_{\Theta=1}^{N+1} \mathbf{F}_{c\Theta} \cdot \bar{\mathbf{v}}_\Theta + \sum_{\Theta=1}^{N+1} \mathbf{M}_{c\Theta} \cdot \omega_\Theta. \tag{21}$$

Indeed, using the Proposition discussed in the Appendix,[7] one could solve the equation $\mathcal{P} = 0$ subject to (16) for $\mathbf{F}_{c\Theta}$ and $\mathbf{M}_{c\Theta}$ to find the solution (20). In other words, the normality prescription in this case is equivalent to a work-less prescription. It should be clear that the normality prescription is not universal. For instance, for the case of two rigid bodies sliding on each other, this prescription would not be physically realistic if the contact involved friction.

Turning to a prescription of $\mathbf{F}_{p\Theta}$ and $\mathbf{M}_{p\Theta}$, we first assume that

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -\sum_{\Theta=1}^{N+1} \mathbf{F}_{p\Theta} \cdot \bar{\mathbf{v}}_\Theta - \sum_{\Theta=1}^{N+1} \mathbf{M}_{p\Theta} \cdot \omega_\Theta. \tag{22}$$

We now seek the solution $\mathbf{F}_{p\Theta}$ and $\mathbf{M}_{p\Theta}$ of (22) for all motions of the rigid bodies which are compatible with the constraints (16). Invoking the Proposition discussed in the Appendix, we find that

$$\begin{aligned}
\mathbf{F}_{p\Delta} &= -\mathbf{Q}_{N+1}\frac{\partial \hat{U}}{\partial \mathbf{y}_\Delta} + \sum_{J=1}^{K} \mu_J \mathbf{f}_\Delta^J, \\
\mathbf{M}_{p\Delta} &= -\mathbf{Q}_{N+1}\hat{u}_{\mathbf{y}_\Delta} + \sum_{J=1}^{K} \mu_J \mathbf{g}_\Delta^J, \\
\mathbf{F}_{p(N+1)} &= \sum_{\Delta=1}^{N} \mathbf{Q}_{N+1}\frac{\partial \hat{U}}{\partial \mathbf{y}_\Delta} - \sum_{\Delta=1}^{N}\sum_{J=1}^{K} \mu_J \mathbf{f}_\Delta^J, \\
\mathbf{M}_{p(N+1)} &= \sum_{\Delta=1}^{N}\sum_{J=1}^{K}\left((\bar{\mathbf{x}}_\Delta - \bar{\mathbf{x}}_{N+1}) \times \left(\mathbf{Q}_{N+1}\frac{\partial \hat{U}}{\partial \mathbf{y}_\Delta} - \mu_J \mathbf{f}_\Theta^J\right)\right) \\
&\quad + \sum_{\Delta=1}^{N}\sum_{J=1}^{K}(\mathbf{Q}_{N+1}\hat{u}_{\mathbf{y}_\Delta} - \mu_J \mathbf{g}_\Delta^J).
\end{aligned} \tag{23}$$

Here, μ_J are Lagrange multipliers and $\hat{u}_{\mathbf{y}_\Delta}$ is defined using (4) in conjunction with $\hat{U}$. In writing (23), we have also used fact that $\dot{U}$ is properly invariant and, consequently, restrictions of the form $(19)_{3,4}$ can be imposed. If the constraint forces and moments are specified using the normality prescription, then the multipliers in (23) can be subsumed into $\mathbf{F}_{c\Theta}$ and $\mathbf{M}_{c\Theta}$.

[6]Following Casey and Carroll [26], we do not assume that $\lambda_J^+ = \lambda_J$.

[7]Implicit in the use of this Proposition is the assumption that $\mathbf{F}_{c\Theta}$ and $\mathbf{M}_{c\Theta}$ are independent of $\bar{\mathbf{v}}_\Theta$ and ω_Θ.

6 COMMENTS ON NEWTON'S THIRD LAW

In the context of a system of particles, Noll [14] recently showed how Newton's third law arose as a consequence of invariance requirements. Motivated by his work, we now show that a related result holds in the context of the system of interest in this paper.

Considering the representations for the forces and moments (20) and (23) and invoking the identities $(19)_{3,4}$ which arise because Π_J and U are assumed to be properly invariant, we find that

$$\mathbf{F}_{q(N+1)} = -\sum_{\Delta=1}^{N} \mathbf{F}_{p\Delta},$$
$$\mathbf{M}_{q(N+1)} = -\sum_{\Delta=1}^{N}((\bar{\mathbf{x}}_\Delta - \bar{\mathbf{x}}_{N+1}) \times \mathbf{F}_{q\Delta} + \mathbf{M}_{q\Delta}), \tag{24}$$

where $q = p$ or c. These equations represent the generalization of Newton's third law to a system of rigid bodies.

To illuminate (24), consider the case of 2 rigid bodies. Then, $N = 1$ and these equations reduce to

$$\mathbf{F}_{q2} = -\mathbf{F}_{q1}, \quad \mathbf{M}_{q2} = -\mathbf{M}_{q1} - (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) \times \mathbf{F}_{q1}. \tag{25}$$

Clearly, the reaction forces are equal and opposite, but the same cannot be said for the moments.[8] A simple illustration of this arises when one considers two rigid bodies which are connected by a ball and socket joint. The joint supports forces which are equal and opposite, however, because the centers of mass of the bodies are distinct, the constraint moments (relative to the center of mass of the bodies) due to these forces are not necessarily equal and opposite.

7 RELATED APPROACHES

Partially to motivate our extensive use of invariance requirements, we now discuss a treatment where these requirements are not imposed. As will shortly become apparent, such a treatment encounters a serious obstacle in one particular case. If a virtual work prescription is used, then this obstacle will be absent. We also show how it can be removed using the treatment presented in Sections 5 and 6 of the present paper.

7.1 A Treatment Without Invariance Requirements

The procedure outline in Section 5 depends critically on the constraint functions and potential energy functions being properly invariant. Hence, special care must be taken when dealing with time-dependent constraint functions and potential energy functions.

Consider a system of two rigid bodies which are subject to $K \leq 12$ constraints:

$$\Xi_J = \sum_{\alpha=1}^{2} \mathbf{a}_\alpha^J \cdot \bar{\mathbf{v}}_\alpha + \sum_{\alpha=1}^{2} \mathbf{b}_\alpha^J \cdot \omega_\alpha + c^J = 0, \quad (J = 1, \dots, K), \tag{26}$$

where $\mathbf{a}_\alpha^J = \mathbf{a}_\alpha^J(\bar{\mathbf{x}}_\beta, \mathbf{Q}_\beta, t)$, $\mathbf{b}_\alpha^J = \mathbf{b}_\alpha^J(\bar{\mathbf{x}}_\beta, \mathbf{Q}_\beta, t)$ and $c^J = c^J(\bar{\mathbf{x}}_\beta, \mathbf{Q}_\beta, t)$. We also assume the existence of a potential energy function $W = W(\bar{\mathbf{x}}_\beta, \mathbf{Q}_\beta, t)$. We emphasize that neither Ξ_J nor W are presumed to be properly invariant.

[8] A related result, in the context of the theory of a Cosserat point, can be seen in Eq. (37) of O'Reilly and Varadi [27].

To specify the constraint forces $\mathbf{F}_{c\alpha}$ and moments $\mathbf{M}_{c\alpha}$ required to ensure the satisfaction of (26), we again use the normality prescription:

$$\mathbf{F}_{c\alpha} = \sum_{J=1}^{K} \lambda_J \mathbf{a}_\alpha^J, \quad \mathbf{M}_{c\alpha} = \sum_{J=1}^{K} \lambda_J \mathbf{b}_\alpha^J, \tag{27}$$

where λ_J are Lagrange multipliers. Paralleling the establishment of (21), it is easy to see, with the assistance of (26), that this system of constraint forces and moments has a mechanical power $\mathcal{P} = -\sum_{J=1}^{K} \lambda_J c^J$. Furthermore, the combined virtual work of the system (27) is zero.

To calculate the conservative forces $\mathbf{F}_{t\alpha}$ and moments $\mathbf{M}_{t\alpha}$ associated with W, we first equate $-\mathrm{d}W/\mathrm{d}t$ to the combined mechanical power of $\mathbf{F}_{t\alpha}$ and $\mathbf{M}_{t\alpha}$. After some rearranging the following equation is obtained:

$$\sum_{\alpha=1}^{2} \left(\frac{\partial W}{\partial \bar{\mathbf{x}}_\alpha} + \mathbf{F}_{t\alpha} \right) \cdot \bar{\mathbf{v}}_\alpha + \sum_{\alpha=1}^{2} (w_{\mathbf{Q}_\alpha} + \mathbf{M}_{t\alpha}) \cdot \omega_\alpha + \frac{\partial W}{\partial t} = 0, \tag{28}$$

where $w_{\mathbf{Q}_\alpha}$ are defined with the assistance of (4). We seek a solution of (28) for all motions of the rigid bodies which are compatible with the constraints (26). Invoking the Proposition discussed in the Appendix, we find, apart from one case, that

$$\mathbf{F}_{t\alpha} = -\frac{\partial W}{\partial \bar{\mathbf{x}}_\alpha} + \sum_{J=1}^{K} \mu_J \mathbf{a}_\alpha^J, \quad \mathbf{M}_{t\alpha} = -w_{\mathbf{Q}_\alpha} + \sum_{J=1}^{K} \mu_J \mathbf{b}_\alpha^J, \tag{29}$$

where μ_J are Lagrange multipliers. As mentioned in the Appendix, these multipliers are not necessarily independent, however, they can be subsumed into the expressions for $\mathbf{F}_{c\alpha}$ and $\mathbf{M}_{c\alpha}$ above. For a single rigid body in the absence of constraints, $(29)_2$ agrees with Antman [4] and Simmonds [5].

The situation where (29) does not hold arises when $\partial W/\partial t \neq 0$ and $c^1 = \cdots = c^K = 0$. In fact, according to the Proposition it is not possible to determine $\mathbf{F}_{t\alpha}$ and $\mathbf{M}_{t\alpha}$ in this case. An alternative viewpoint would be to attribute the lack of a solution in this case to the assumption that $\mathrm{d}W/\mathrm{d}t$ is the negative of the combined mechanical powers of $\mathbf{F}_{t\alpha}$ and $\mathbf{M}_{t\alpha}$ (cf. (22) and (26)).

7.2 Prescriptions Based on Virtual Displacements and Work

The difficulty above does not present itself in treatments of constraint and conservative forces and moments which are based on virtual work prescriptions. As mentioned in Section 7.1, the prescriptions of the constraint forces and moments are identical to those recorded by (27). Furthermore, as in Section 8 of Gantmacher [2] and Section 9.9 of Rosenberg [3], one equates the negative of the first variation of W to the combined virtual work of $\mathbf{F}_{t\alpha}$ and $\mathbf{M}_{t\alpha}$:[9]

$$\sum_{\alpha=1}^{2} \left(\frac{\partial W}{\partial \bar{\mathbf{x}}_\alpha} + \mathbf{F}_{c\alpha} \right) \cdot \delta \bar{\mathbf{x}}_\alpha + \sum_{\alpha=1}^{2} (w_{\mathbf{Q}_\alpha} + \mathbf{M}_{c\alpha}) \cdot \delta \boldsymbol{\theta}_\alpha = 0, \tag{30}$$

[9]Strictly speaking, Gantmacher [2] uses an equation of the form (30) to define the function W. Furthermore, neither he nor Rosenberg [3] mentions the role of (31) and the multipliers μ_J they introduce.

where $\delta\bar{\mathbf{x}}_\alpha$ and $\delta\boldsymbol{\theta}_\alpha = -(1/2)\boldsymbol{\varepsilon}[\delta\mathbf{Q}_\alpha\mathbf{Q}_\alpha^{\mathrm{T}}]$ are first variations (virtual displacements). One then seeks a solution (30) for all $\delta\bar{\mathbf{x}}_\alpha$ and $\delta\boldsymbol{\theta}_\alpha$ which are compatible with, from (26),

$$\Xi_J = \sum_{\alpha=1}^{2} \mathbf{a}_\alpha^J \cdot \delta\bar{\mathbf{x}}_\alpha + \mathbf{b}_\alpha^J \cdot \delta\boldsymbol{\theta}_\alpha + 0, \quad (J = 1, \ldots, K). \tag{31}$$

Using the Proposition, it is easy to see that because the problematic term $\partial W/\partial t$ is absent from (30) one would then obtain prescriptions identical to (29) in all cases.

7.3 Resolution

The difficulty noted above, when $\partial W/\partial t \neq 0$ and $c^1 = \cdots = c^K = 0$, can also be removed using the treatment presented in this paper. Specifically, one assumes that the explicit time dependency of the potential energy function W and constraints are due to the fact that the motion of a third rigid body is a prescribed function of time:

$$\bar{\mathbf{x}}_3 = \mathbf{p}(t), \quad \mathbf{Q}_3 = \mathbf{P}(t). \tag{32}$$

To proceed, one assumes, prior to (32) being imposed, that the system of 3 rigid bodies is subject to a set of K properly invariant constraints of the form (16). In addition, the system is assumed to have a potential energy function $U = \hat{U}(\mathbf{y}_\alpha, \mathbf{Y}_\alpha)$, where, from (12), $\mathbf{y}_\alpha = \mathbf{Q}_3^{\mathrm{T}}(\bar{\mathbf{x}}_\alpha - \bar{\mathbf{x}}_3)$ and $\mathbf{Y}_\alpha = \mathbf{Q}_3^{\mathrm{T}}\mathbf{Q}_\alpha$. In what follows, the constraint and conservative forces and moments are obtained using the treatment discussed in Section 5. After these prescriptions, (32) are then imposed. One of the consequences of this procedure is an illuminating interpretation of $\partial W/\partial t$.

Using the developments of Section 5 and imposing the invariance requirements, we find, from (20) and (23),

$$\begin{aligned}
&\mathbf{F}_{ci} = \sum_{J=1}^{K} \lambda_J \mathbf{f}_i^J, \quad \mathbf{M}_{ci} = \sum_{J=1}^{K} \lambda_J \mathbf{g}_i^J, \quad (i = 1, \ldots, 3),\\
&\mathbf{F}_{p\alpha} = -\mathbf{Q}_3 \frac{\partial \hat{U}}{\partial \mathbf{y}_\alpha} + \sum_{J=1}^{K} \mu_J \mathbf{f}_\alpha^J, \quad \mathbf{M}_{p\alpha} = -\mathbf{Q}_3 \hat{u}_{\mathbf{y}_\delta} + \sum_{J=1}^{K} \mu_J \mathbf{g}_\alpha^J,\\
&\mathbf{F}_{p3} = -\sum_{\alpha=1}^{2} \mathbf{F}_{p\alpha}, \quad \mathbf{M}_{p3} = -\sum_{\alpha=1}^{2} ((\bar{\mathbf{x}}_\alpha - \bar{\mathbf{x}}_3) \times \mathbf{F}_{q\alpha} + \mathbf{M}_{q\alpha}).
\end{aligned} \tag{33}$$

To simplify the resulting expressions, we have also used (24).

Evaluating the properly invariant constraints on the subset (32) yields constraints of the form (26) where

$$\begin{aligned}
&\mathbf{a}_\alpha^J(\bar{\mathbf{x}}_\beta, \mathbf{Q}_\beta, t) = \mathbf{f}_\alpha^J(\mathbf{P}^{\mathrm{T}}(\bar{\mathbf{x}}_\beta - \mathbf{p}), \mathbf{P}^{\mathrm{T}}\mathbf{Q}_\beta),\\
&\mathbf{b}_\alpha^J(\bar{\mathbf{x}}_\beta, \mathbf{Q}_\beta, t) = \mathbf{g}_\alpha^J(\mathbf{P}^{\mathrm{T}}(\bar{\mathbf{x}}_\beta - \mathbf{p}), \mathbf{P}^{\mathrm{T}}\mathbf{Q}_\beta),\\
&c^J(\bar{\mathbf{x}}_\beta, \mathbf{Q}_\beta, t) = \mathbf{f}_3^J(\mathbf{P}^{\mathrm{T}}(\bar{\mathbf{x}}_\beta - \mathbf{p}), \mathbf{P}^{\mathrm{T}}\mathbf{Q}_\beta) \cdot \dot{\mathbf{p}} + \mathbf{g}_3^J(\mathbf{P}^{\mathrm{T}}(\bar{\mathbf{x}}_\beta - \mathbf{p}), \mathbf{P}^{\mathrm{T}}\mathbf{Q}_\beta) \cdot \boldsymbol{\omega}_p,
\end{aligned} \tag{34}$$

where $\boldsymbol{\omega}_p = -(1/2)\boldsymbol{\varepsilon}[\dot{\mathbf{P}}\mathbf{P}^{\mathrm{T}}]$. It follows that after (34) have been imposed, $(33)_{1,2}$ will be identical to (27).

In addition, we identify

$$W = W(\bar{\mathbf{x}}_\beta, \mathbf{Q}_\beta, t) = \hat{U}(\mathbf{P}^{\mathrm{T}}(\bar{\mathbf{x}}_\beta - \mathbf{p}), \mathbf{P}^{\mathrm{T}}\mathbf{Q}_\beta). \tag{35}$$

Omitting details, it can be shown, using the chain-rule, that

$$\frac{\partial W}{\partial \bar{\mathbf{x}}_\beta} = \mathbf{P}\frac{\partial \hat{U}}{\partial \bar{\mathbf{y}}_\beta}, \quad w_{\mathbf{Q}_\alpha} = \mathbf{P}\hat{u}_{\mathbf{Y}_\alpha}. \tag{36}$$

It follows that the conservative forces and moments, $(33)_{3,4}$, reduce to

$$\mathbf{F}_{p\alpha} = -\frac{\partial W}{\partial \bar{\mathbf{x}}_\alpha} + \sum_{J=1}^{K} \mu_J \mathbf{a}_\alpha^J, \quad \mathbf{M}_{p\alpha} = -w_{\mathbf{Q}_\alpha} + \sum_{J=1}^{K} \mu_J \mathbf{b}_\alpha^J. \tag{37}$$

Clearly, (37) is equivalent to (29). However, (37) is valid regardless of the value of $\partial W/\partial t$.

Finally, by equating $\dot{W}$ to $\dot{U}$, we can also identify

$$\frac{\partial W}{\partial t} = -\sum_{\alpha=1}^{2}\left(\mathbf{P}\frac{\partial \hat{U}}{\partial \bar{\mathbf{y}}_\beta}\right)\cdot \dot{\mathbf{p}} - \sum_{\alpha=1}^{2}\left((\bar{\mathbf{x}}_\alpha - \bar{\mathbf{x}}_3) \times \left(\mathbf{P}\frac{\partial \hat{U}}{\partial \bar{\mathbf{y}}_\beta}\right) + \mathbf{P}\hat{u}_{\mathbf{y}_\alpha}\right)\cdot \omega_p. \tag{38}$$

Thus, $-\partial W/\partial t$ represents the combined mechanical power of the force $\mathbf{F}_{c3}$ and the moment $\mathbf{M}_{c3}$.

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APPENDIX

In several sections of this paper, we required the solution of a system of algebraic equations which are subject to linear constraints. The solutions are obtained from a Proposition that we record here. The proof is very straightforward and is consequently omitted.

PROPOSITION *The solution* $(\mathbf{a}_\Theta, \mathbf{b}_\Theta)$ *of*

$$\sum_{\Theta=1}^{N+1} \mathbf{a}_\Theta \cdot \bar{\mathbf{v}}_\Theta + \sum_{\Theta=1}^{N+1} \mathbf{b}_\Theta \cdot \omega_\Theta = f, \tag{39}$$

for all $(\bar{\mathbf{v}}_\Theta, \omega_\Theta)$ *which satisfy the K independent equations*

$$\sum_{\Theta=1}^{N+1} \mathbf{c}_\Theta \cdot \bar{\mathbf{v}}_\Theta + \sum_{\Theta=1}^{N+1} \mathbf{d}_\Theta^J \cdot \omega_\Theta = g^J, \quad (J = 1, \ldots, K), \tag{40}$$

can be divided into three classes. First, if $f \neq 0$ *and* $g^1 = \cdots = g^K = 0$, *then* (39) *does not have a solution. Otherwise, the solution is of the form*

$$\mathbf{a}_\Theta = \sum_{J=1}^{K} \lambda_J \mathbf{c}_\Theta^J, \quad \mathbf{b}_\Theta = \sum_{J=1}^{K} \lambda_J \mathbf{d}_\Theta^J. \tag{41}$$

Suppose one g^J, *say* $g^1, \neq 0$, *then* λ_J *for* $(J = 2, \ldots, K)$ *are independent and*

$$\lambda_1 = \frac{1}{g^1}\left(f - \sum_{J=2}^{K} \lambda_J g^J\right). \tag{42}$$

Finally, if $g^1 = \cdots = g^K = 0$ *and* $f = 0$, *then all of the* λ_J*'s are independent.*