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ON A NEW CLASS OF MODELS IN ELASTICITY

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Dedicated to Professor David P. Mason

Abstract- Recently, Rajagopal and co-workers have shown (see Rajagopal [1], Rajagopal and Srinivasa [2],[3], Bustamante and Rajagopal[4], Rajagopal and Saccomandi [5]) that if by an elastic body one means a body that is incapable of dissipation, then the class of such bodies is far larger than either Green elastic or for that matter Cauchy elastic bodies as one could model elastic bodies using implicit constitutive relations between the Cauchy stress and the deformation gradient or implicit constitutive relations that are rate equations involving the Piola-Kirchhoff stress and the Green-St.Venant Strain (see Rajagopal and Srinivasa [2]). Such a generalized framework allows one to develop models whose linearization with regard to the smallness of the displacement gradient allows one to obtain models that have limited linearized strains even while the stresses are very large. Such a possibility has important consequences to problems which, within the context of the classical linearized theory, leads to singularities. In this short paper, we illustrate the implications of such models by considering simple problems within the context of a specific model belonging to the general class, wherein the strains remain small as the stresses tend to very large values.

Keywords- Cauchy stress, Piola-Kirchhoff stress, Green-St.Venant Strain, Implicit constitutive equation, Linearized strain.

1. INTRODUCTION

Recently, Rajagopal [6] has studied a variety of simple deformations within the context of a model that belongs to a new class of models that have been developed to describe the elastic response of bodies. The novel feature about this model is the fact that even when the non-dimensionalized stresses are large, the linearized strain remains small, thereby making the use of the theory consistent for the study of problems wherein the non-dimensionalized stresses can become arbitrarily large while the strains remain small. This allows it to become a vehicle to describe problems wherein one runs into stress singularities such as the situation when one is confronted with a concentrated load or in the study of cracks. When such problems are addressed within the context of linearized elasticity, one runs into a problem. As the strain is linearly related to the stress, when the stress becomes singular, the strain becomes singular. More importantly, the strains become sufficiently large so that the theory is not applicable in a reasonably large area around a concentrated load or a crack tip. The problem in classical linearized elasticity stems from the linear relationship between the Cauchy stress and the linearized strain. As goes the stress, so goes the strain, the linearized strain does not grow at a slower rate than the stress. It would be interesting to construct models wherein the strain grows much slower than the stress or better still, the strains remain limited as the stress grows. This special model which Rajagopal [6] constructed does precisely that, it exhibits a limiting strain that can be fixed a priori, however large the stress may become. The particular model to be studied in this paper and studied earlier by Rajagopal [6], belongs to a sub-class of the general class of models that Rajagopal [6] introduced. This class which deserves some attention and analysis in virtue of the novel features that its members present, making them possible candidates to describe interesting phenomena that have hitherto been unexplained within the classical approaches, both within the context of small and large deformation theories.

The model studied here has a finite deformation counterpart, namely one wherein the nonlinear stretches remain finite (not necessarily small) as the stress becomes large. Rajagopal and Saccomandi [5] studied the response of such bodies. They showed that models with limiting chain extensibility fall into the class of implicit models of elasticity introduced by Rajagopal [7],[1].

Bustamante and Rajagopal [4] and Bustamante [8] have studied two dimensional problems,within the context of such large deformation theories, with a view towards extending their analysis to that of the problem of a crack by considering a body with an elliptic hole and allowing the aspect ratio of the ellipse to tend to zero. The model used in this paper stems from for the class of models for the response of elastic solids introduced by Rajagopal [7], wherein he considered implicit constitutive relations to describe the response of both solids and fluids. Later, in a paper titled Elasticity of Elasticity, Rajagopal [6] showed that Cauchy Elastic and Green Elastic bodies form a sub-set of Elastic bodies, if by an elastic body one means a body that is incapable of dissipation in any process that it undegoes. This work was subsequently extended by Rajagopal and Srinivasa [2],[3] who showed that a firm thermodynamic basis could be provided for such models. They showed that one could associate a stored energy with the body, but the Piola stress is not the derivative of this stored energy with respect to the deformation gradient. In the traditional approach one assumes that the stored energy for an elastic body depends only on the deformation gradient. However, if one allows the stored energy to depend on both the deformation gradient and the stress (see Rajagopal [7], Rajagopal and Srinivasa [2]), then it is possible to come up with a very large class of elastic bodies, in the sense that they are incapable of dissipation. The classical Cauchy elastic and Green elastic bodies are sub-classes of this more generalized class of materials. More recently, Rajagopal and Srinivasa [3] presented a method to describe elastic materials from a purely Eulerian perspective, that is, without introducing a reference configuration or the notion of a deformation gradient.

They also used such a framework to develop models for elastic solids that are neither Green elastic nor hypoelastic[1].

Such models have particular relevance to modeling the response of biological matter that grow and atrophy wherein one cannot use a Lagrangian approach for describing the response of a body, as a part of the body which exists currently might not have existed some time ago and a part of the body that did exist some time ago might have atrophied. Recently, Noll [12] has introduced a framework for elasticity that does not require the notion of a deformation gradient. Also, Tao and Rajagopal [6] have developed a framework for elasticity within the context of relative deformation gradient.

Rajagopal [6] documents two models (see equations (3.12) and (3.13) in what follows) in which the linearized strain bears a non-linear relation to the Cauchy stress. Both these models reduce to the classical linearized elascticity model when one requires a linear relationship between the linearized strain and the Cauchy stress. Rajagopal [6] goes on to discuss the model given by equation (3.12) in the paper within the context of uniaxial extension, shear, circumferential shear, telescopic shear and some combinations of these deformations. The model, in all the deformations considered presents a finite strain that could be fixed to be arbitrarily small a priori as the stress goes to infinity. In order to illustrate such models with limiting strain, Rajagopal [6] just considered a model that was restricted to the trace of the Cauchy stress tensor being non-negative. While the model that Rajagopal [6] considered does not exhibit limiting strain when the trace of the stress is negative,

[1]Truesdell [9] introduced the notion of a hypoelastic solid whose constitutive equation is given by $\dot{\mathbf{T}} = \mathbf{T}\mathbf{W}^T + \mathbf{W}\mathbf{T} + \mathbf{A}(\mathbf{T})\mathbf{D}$ where $\mathbf{A}$ is a fourth order tensor that depends on the Cauchy stress $\mathbf{T}$ and $\mathbf{D}$ and $\mathbf{W}$ are the symmetric and skew part of the velocity gradient (see equation (2.7) for the definitions of $\mathbf{D}$ and $\mathbf{W}$) and the dot denotes the material time derivative. Bernstein [10] realized that certain additional demands need to be made if the model is to be physically reasonable. Recently, Bernstein and Rajagopal [11] have studied hypoelastic materials from a thermodynamic point of view.

it can be easily modified to do so for all values of the trace of the stress. The model given by equation(3.12) does behave well in that the strains remains

bounded by an arbitrarily small value that can be fixed a priori, when the non-dimensionalized stresses are compressive or tensile. We shall consider a slight modification of this model and study a class of simple but instructive problems such as a uniaxial state of stress, state of pure shear stress, circumferential shearing and telescopic shearing.

2. PRELIMIARIES

Let $\mathbf{x}$ denote the position of a particle in the current configuration $\kappa_t(B)$ which is at $\mathbf{X}$ in the stress free reference configuration $\kappa_R(B)$. Let $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ denote the motion of a particle and let us denote by $\mathbf{u}$ and $\mathbf{F}$ the displacement and deformation gradient through

$$\mathbf{u} := \mathbf{x} - \mathbf{X}, \tag{2.1}$$

and

$$\mathbf{F} := \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}. \tag{2.2}$$

We shall assume $\boldsymbol{\chi}$ to be sufficiently differentiable to make all the operations meaningful. We also note that any quantity associated with the body can be described with respect to $(\mathbf{X}, t)$ or $(\mathbf{x}, t)$ and the representation that is implied should become obvious from the context.

We define the velocity $\mathbf{v}$ through

$$\mathbf{v} = \frac{\partial \boldsymbol{\chi}}{\partial t}. \tag{2.3}$$

We define the stretch tensors $\mathbf{B}$ and $\mathbf{C}$ through

$$\mathbf{B} := \mathbf{F}\mathbf{F}^T, \mathbf{C} := \mathbf{F}^T\mathbf{F}. \tag{2.4}$$

and Green-St.Venant strain $\mathbf{E}$ and the Almansi-Hamel strain $\mathbf{e}$ through

$$\mathbf{E} := \frac{1}{2}(\mathbf{C} - \mathbf{1}), \qquad \mathbf{e} := \frac{1}{2}(\mathbf{1} - \mathbf{B}^{-1}). \tag{2.5}$$

In the above definitions, the superscript T denotes the transpose operation. The velocity gradient $\mathbf{L}$ and the associated symmetric and skew tensors $\mathbf{D}$ and $\mathbf{W}$ are defined respectively through

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \tag{2.6}$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \qquad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) \tag{2.7}$$

Under the assumption

$$\max \|\nabla \mathbf{u}\| = \mathbf{0}(\delta), \delta \ll 1, \tag{2.8}$$

$$\mathbf{X} \in K_R(B), t \in \mathbb{R}$$

where $\| \bullet \|$ stands for the usual trace norm, we find

$$\mathbf{E} = \boldsymbol{\varepsilon} + \mathbf{0}(\delta^2), \qquad \mathbf{e} = \boldsymbol{\varepsilon} + \mathbf{0}(\delta^2), \tag{2.9}$$

where

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right]. \tag{2.10}$$

The above kinematical definitions suffice for the purpose of this paper. A more detailed discussion of the kinematics can be found in Truesdell [14].

3. CONSTITUTIVE THEORY

A body is said to be Cauchy elastic if the Cauchy stress in the body is given by the following constitutive equation:

$$\mathbf{T} = \mathbf{f}(\mathbf{F}). \tag{3.1}$$

If the body is inhomogeneous, the function $\mathbf{f}$ will depend on $\mathbf{X}$, otherwise it is said to be homogeneous. Also, we have suppressed the dependence of the stress on the density. We shall restrict our discussion to homogeneous bodies but the work can be easily extended to inhomogeneous bodies.

Instead of (2.8) as the starting point, Rajagopal ([7],[1]) considered the class of elastic models given by the implicit relation

$$\mathbf{f}(\mathbf{T}, \mathbf{F}) = \mathbf{0}. \tag{3.2}$$

A special sub-class of (3.2) is the constitutive equation

$$\mathbf{F} = \mathbf{f}(\mathbf{T}), \tag{3.3}$$

which is more in keeping with causality in that the stress is the cause and the deformation of the body is its effect (see Rajagopal [6] for a discussion of the relevant issues). In this paper, we will be discussing models that stem from the constitutive relation (3.2).

We shall be discussing isotropic bodies. Let us consider the implicit relation

$$\mathbf{f}(\mathbf{T}, \mathbf{B}) = \mathbf{0}. \tag{3.4}$$

We note that the above class includes the explicit models of the form:

$$\mathbf{T} = \beta_0 \mathbf{I} + \beta_1 \mathbf{B} + \beta_2 \mathbf{B}^2, \tag{3.5}$$

where the $\beta_i, i = 0, 1, 2$ depend on $\rho, tr\mathbf{B}, tr\mathbf{B}^2$ and $tr\mathbf{B}^3$ which is the representation for the stress for the most general isotropic compressible Cauchy elastic model (see Truesdell and Noll [15]). If the coefficients do not explicitly depend on the reference particle then the model is for a homogeneous body, otherwise it can represent an inhomogeneous body. It can be shown that a Cauchy elastic body that is not Green elastic can be an infinite source of energy. That this is indeed the case was pointed out by Green himself (Green (1839) and recently Carroll (2009) has shown a simple model wherein he shows that this is indeed the case. Hence, a Cauchy elastic body that is not Green elastic is not a viable possibility. Of course, the material coefficients that appear in (3.5) can be expressed in terms of the stored energy in which case the modified expression would represent a Green elastic body. We shall find it convenient to use the representation (3.5) but we should bear in mind that the material coefficients are given in terms of the stored energy of the material.

It would be appropriate at this juncture to point out a key difference between constitutive relations of the form (3.5) and (3.9) that we encounter later. In the classical theory involving a constitutive equation of the form (3.6), where an explicit expression is provided for the Cauchy stress in terms of the stretch tensor and thus the displacement gradient; one substitutes the expression for the stress into the balance of linear momentum and obtains a non-linear partial differential equation for the displacement. Thus, one has to contend with a partial differential equation for the displacement and the balance of mass, namely four coupled scalar partial differential equation. However, when we deal with the implicit constitutive relation of the form (3.5) (or for that matter equation (**??**) which we encounter later that provides an explicit expression for the stretch in terms of the stress) one does not have the luxury of substituting the expression for the stress into the constitutive equation; one is faced with the onerous task of solving the balance of linear momentum simultaneously with the constitutive equation (3.5) and the balance of mass; ten nonlinear coupled scalar partial differential equations. We immediately see that the problem on hand is a great deal more complicated than the usual classical model of non-linear elasticity.

If one linearizes the above model (3.5) under the assumption (2.9), then one obtains the classical linearized elastic solid model given by

$$\mathbf{T} = \lambda(tr\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}, \tag{3.6}$$

where λ and μ are the Lame constants. In the case of anisotropic bodies described by implicit theories, one has to start from an assumption different from (3.5). The important point to bear in mind is that irrespective of whether the body is isotropic or anisotropic, compressible or incompressible, homogeneous or inhomogeneous, if one starts with the assumption of Cauchy elasticity, that is (3.1) and linearizes by appealing to (2.9) one is inexorably led to a model in which the stress and strain bear a linear relation.

Rajagopal [1] showed that if one starts with (2.8), and then appeals to the linearization for the kinematics, namely (2.5), then one can obtain a model in which the linearized strain can bear a non-linear relationship to the stress. To see this, we note that if $\mathbf{f}$ is an isotropic function, then it follows that (see Spencer [16]):

$$\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{B} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{B}^2 + \alpha_5(\mathbf{TB} + \mathbf{BT}) + \alpha_6(\mathbf{T}^2\mathbf{B} + \mathbf{BT}^2)$$
$$+\alpha_7(\mathbf{TB}^2 + \mathbf{B}^2\mathbf{T}) + \alpha_8(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2) = \mathbf{0}, \tag{3.7}$$

where the material moduli $\alpha_i, i = 0, ..., 8$ depend on

$$\rho, tr\mathbf{T}, tr\mathbf{B}, tr\mathbf{T}^2, tr\mathbf{B}^2, tr\mathbf{T}^3, tr\mathbf{B}^3, tr\left(\mathbf{TB}\right), tr\left(\mathbf{T}^2\mathbf{B}\right), tr\left(\mathbf{B}^2\mathbf{T}\right), tr\left(\mathbf{T}^2\mathbf{B}^2\right). \tag{3.8}$$

To show our intended result it is unnecessary to work with the full model (2.9). Let us consider the far simpler sub-class given by

$$\mathbf{B} = \hat{\alpha}_0\mathbf{1} + \hat{\alpha}_1\mathbf{T} + \hat{\alpha}_2\mathbf{T}^2, \tag{3.9}$$

where $\hat{\alpha}_i, i = 1, 2, 3$ depend on $\rho, tr\mathbf{T}, tr\mathbf{T}^2,$, and $tr\mathbf{T}^3$.

Equation (3.9) will form the starting point for an approximation which leads to a different small displacement gradient theory that allows us to have a non-linear relationship between the linearized strain and the stress. We shall use exactly the same small displacement gradient approximation that leads to the classical linearized theory of elasticity, but now within the context of model (3.9). We note that under the approximation (2.9), the model (3.9) reduces to

$$\boldsymbol{\varepsilon} = \overline{\alpha_0}\mathbf{1} + \overline{\alpha_1}\mathbf{T} + \overline{\alpha_2}\mathbf{T}^2, \tag{3.10}$$

since

$$\mathbf{B} = \mathbf{1} + 2\boldsymbol{\varepsilon} + \mathbf{0}(\delta^2). \tag{3.11}$$

Since the strain is dimensionless, the material moduli $\overline{\alpha_1}$ and $\overline{\alpha_2}$ need to have dimensions those are the inverse of the stress and the square of the stress, respectively. For the sake of simplicity, we shall drop the bar above the material moduli that appear in equation (3.10). With regard to the relation (3.10), while $\boldsymbol{\varepsilon}$ is required to be small there are no such demands on the stress and it can be arbitrarily large.

We will consider a special simple model that belongs to the class defined by (3.10), namely the model [2]

$$\boldsymbol{\varepsilon} = \alpha \left\{ \left[1 - \exp\left(\frac{-\beta\,(tr\mathbf{T})}{1 + \delta\left(tr\mathbf{T}^2\right)^{1/2}} \right) \right] \mathbf{1} + \frac{\gamma}{1 + \left(tr\mathbf{T}^2\right)^{1/2}} \mathbf{T} \right\}, \tag{3.12}$$

which is not an implicit model but provides an explicit relationship for the linearized strain in terms of the stress. We note that when $\mathbf{T} = \mathbf{0}$, $\boldsymbol{\varepsilon} = \mathbf{0}$. In order to indicate the models ability to exhibit limiting stress, Rajagopal [6] set δ to be zero in the model (3.12). While it does serve the purpose of exhibiting limiting strain, this very special case has an inherent defect. Unfortunately, the deficiency does not show up in the various examples that Rajagopal [6] considered, namely uniaxial extension, shear, circumferential shear, telescopic shear, etc., because in the states of stress that were considered the mean normal stress was non-negative. However, we note that if the state of stress is compressive, or in general if the mean normal stress is negative, then if it is sufficiently large, we will once again violate the assumption of small strain due to the exponential term in the equation. However, this defect is not reflected in the full model (3.12).The other model suggested by Rajagopal [6] in the same paper was [3]

$$\boldsymbol{\varepsilon} = \alpha \left[1 - \frac{1}{1 + \frac{tr\mathbf{T}}{1+\delta\left(tr\mathbf{T}^2\right)^{1/2}}} \right] \mathbf{1} + \beta \left[1 + \frac{1}{1 + \gamma\left(tr\mathbf{T}^2\right)} \right]^n \mathbf{T}, \tag{3.13}$$

and it has a similar drawback with respect to compressive strains.

[2] There is a typographical error in the paper by Rajagopal [6] in that the square root that appears in the exponential term in equation (3.13) is missing.

[3] There is a misprint in the paper by Rajagopal [6], the square root that appears in the term that is multiplied by δ is missing.

In this paper we shall consider a different model than the ones above, which does not present such problems.

We shall consider the model

$$\boldsymbol{\varepsilon} = \alpha \left[1 - \exp\left(\frac{-\lambda\,(tr\mathbf{T})}{1 + \left(tr\mathbf{T}^2\right)^{1/2}} \right) \right] \mathbf{1} + \beta \frac{1}{\left(1 + \gamma \left(tr\mathbf{T}^2\right)^{n/2}\right)^{1/n}} \mathbf{T}. \tag{3.14}$$

In the above equation (3.12),α, λ, β and n are constants. A few remarks concerning (3.15) are called for. Let us recall the one dimensional form of the classical Hooke's Law, namely

$$\sigma = E\varepsilon \tag{3.15}$$

which is in fact better expressed as

$$\varepsilon = \frac{\sigma}{E} \tag{3.16}$$

since it is in conformity with the "effect" being expressed in terms of the cause. In the above equation E is the Youngs modulus. The important observation to make is that if the linearized model is to be meaningful, the right hand side has to be appropriately small. If (3.14) is to make sense in that the left hand side remains appropriately small, then the constants that appear in the right hand side have to be of appropriate value. We note that when the stress is zero, the linearized strain is zero. We shall see that, for the class of problems considered, the structure of the model is such that the strain on the left hand side of (3.14) remains small, fixed a priori, even when the non-dimensional stress becomes infinite. Also, if we linearize the right hand side by requiring that $\lambda\,(tr\mathbf{T})$ and $\gamma\left(tr\mathbf{T}^2\right)^{1/2}$ are appropriately small, then the model (3.14) leads to the classical linearized elastic model. It then follows that

$$\alpha\lambda = -\frac{\nu}{E}, \qquad \beta = \frac{1+\nu}{E}. \tag{3.17}$$

where ν is the Poissons ratio in the linearized elastic body. It follows that α , is negative, while λ and β are positive. We also recognize that whenever the stress is traceless, the linearized strain for the model defined by (3.14) is traceless, a property that the model shares with the classical linearized elastic model. However, the general model (3.10) does not share this feature. The second term on the right hand side of the model (3.14) can be viewed as the counterpart of the generalized Neo-Hookean model (see Knowles [17]) in that it provides a power-law relation for the linearized strain in terms of the stress. Such a model can stress soften or stress stiffen. In the model considered by Knowles where the stress is related to the stretch tensor $\mathbf{B}$ through a power law, the governing equations lose ellipticity in the anti-plane problem that he considered depending on the value of the power-law exponent.

We shall not consider the general model but we will only consider a special case, namely that corresponding to $n = 2$ as our intent is merely to illustrate the interesting features that the class of models predict, namely limiting strain even while the stress becomes unbounded. The problems that we consider are semi-inverse problems. We shall assume a form for the stress and determine the solution for the displacement corresponding to the specific assumption, that satisfies the boundary condition. It is possible that the full system of equations might have solutions other than those that follow from the semi-inverse assumption that is being made.

4. SPECIAL BOUNDARY VALUE PROBLEMS

4.1. Uniaxial tensile loading

Let us consider the problem of uniaxial tensile loading wherein we assume that the state of stress $\mathbf{T}$ takes the form

$$\mathbf{T} = T\left(\mathbf{e}_x \otimes \mathbf{e}_x\right). \tag{4.1}$$

Where T is a constant and $\mathbf{e}_x$ is the unit vector in the x-coordinate direction. First, let us consider what the implications of the assumption (4.1) with regard to the general model (3.10). It immediately follows from

(3.10) that only non-zero components of the linearized strain are

$$\begin{aligned}\varepsilon_{xx} &= \alpha_0\left(T, T^2, T^3\right) + \left[\alpha_1\left(T, T^2, T^3\right)\right]T + \left[\alpha_2\left(T, T^2, T^3\right)\right]T^2 \\ \varepsilon_{yy} &= \varepsilon_{zz} = \alpha_0\left(T, T^2, T^3\right).\end{aligned} \tag{4.2}$$

Depending on the specific structure of the material functions we can determine the exact manner in which the strain varies with the applied stress. We notice that

$$tr\boldsymbol{\varepsilon} = 3\alpha_0\left(T, T^2, T^3\right) + \left[\alpha_1\left(T, T^2, T^3\right)\right]T + \left[\alpha_2\left(T, T^2, T^3\right)\right]T^2, \tag{4.3}$$

which in general is not zero. Let us consider the model (3.14) subject to the same state of stress; it is easy to show that in this case the strains have the following form:

$$\varepsilon_{xx} = \alpha\left[1 - \exp\left(\frac{-\lambda T}{\left(1+T\right)^2}\right)\right] + \beta\left[\frac{T}{\left(1+\gamma T^2\right)^{1/2}}\right], \tag{4.4}$$

and

$$\varepsilon_{yy} = \varepsilon_{zz} = \alpha\left[1 - \exp\left(\frac{-\lambda T}{(1+T)^2}\right)\right]. \tag{4.5}$$

The other components of the linearized strain are zero. Even in the limit as $T \to \infty$ the normal strain in the x-coordinate direction,$\varepsilon_{xx} \to \alpha\left[1 - \exp(-\lambda)\right] + \frac{\beta}{\sqrt{\gamma}}$, while the normal strains in the y and z- coordinate directions $\varepsilon_{yy}, \varepsilon_{zz} \to \alpha\left[1 - \exp(-\lambda)\right]$ and the Frobenius norm of the linearized strain

$$\|\varepsilon\| \to \left\{\left[\alpha\left(1 - \exp(-\lambda)\right) + \frac{\beta}{\sqrt{\gamma}}\right]^2 + 2\left[\alpha\left(1 - \exp(-\lambda)\right)\right]^2\right\}^{1/2}$$

and is thus bounded. Appropriate choices for α ,β ,γ , and λ can make it as small as one wishes. In marked contrast, the linearized model (3.4) is such that the Frobenius norm of the linearized strain increases as the stress increases and blows up as stress tends to infinity, violating the basic assumption that the strain is small.

4.2. Simple Shear

Next, we will consider the counterpart to the problem of the state of pure shear, that is we will consider the case when the Cauchy stress tensor takes the form

$$\mathbf{T} = T\left(\mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x\right), \tag{4.6}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors in the x and y -coordinate direction, respectively, and T is a constant. In the case of (3.12) we find that the only non-zero components of the strain are

$$\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{yy} = \alpha_0\left(0, 2T^2, 0\right) + \alpha_2\left(0, 2T^2, 0\right)T^2, \\
\varepsilon_{zz} &= \alpha_0\left(0, 2T^2, 0\right), \\
\varepsilon_{xy} &= \varepsilon_{yx} = \alpha_1\left(0, 2T^2, 0\right)T.
\end{aligned} \tag{4.7}$$

All other components of the strain are zero. We notice that a simple shear stress produces strains in the normal directions and is akin to what happens in the case of the classical nonlinear elastic model (3.6) when it is subject to a state of shear strain, namely the development of normal stresses perpendicular to the plane of shear. Also, we note that in marked contrast to the classical linearlized elastic model we find that

$$tr\boldsymbol{\varepsilon} = 3\alpha_0\left(0, 2T^2, 0)\right) + 2\alpha_2\left(0, 2T^2, 0\right) \tag{4.8}$$

which is not usually zero. In fact, for it to be zero, the material moduli have to meet a very special condition, namely that

$$3\alpha_0\left(0, 2T^2, 0\right) + 2\alpha_2\left(0, 2T^2, 0\right) = 0. \tag{4.9}$$

In the case of the special model (3.14) we find that no normal strains are introduced and the only components of the strain that are non-zero are given by

$$\varepsilon_{xy} = \varepsilon_{yx} = \beta \frac{T}{(1 + 2\gamma T^2)^{1/2}}. \tag{4.10}$$

When $T \to \infty$, $\varepsilon_{xy} \to \frac{\beta}{\sqrt{2\gamma}}$ and thus, the shear strain reaches a critical value as the shear stress tends to infinity. In the linearized theory $\varepsilon_{xy} \to \infty$ as $T \to \infty$ contradicting the original assumption of the theory that the displacement gradient and hence the strain is very small. We see that there is no such contradiction in the case of the model (3.14).

4.3. Torsion

Let us next suppose that the stress has the form

$$\mathbf{T} = T\left(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta\right), \tag{4.11}$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are unit vectors along the θ and z directions and T is a constant. Once again, the equations of equilibrium are met automatically.

In the case of the general model (3.10) a very simple calculation leads to

$$\boldsymbol{\varepsilon} = \alpha_0\left(0, 2T^2, 0\right)\mathbf{1} + \alpha_1\left(0, 2T^2, 0\right)\mathbf{T} + \alpha_2\left(0, 2T^2, 0\right)\mathbf{T}^2, \tag{4.12}$$

$$\begin{aligned} &\varepsilon_{rr} = \alpha_0\left(0, 2T^2, 0\right), \varepsilon_{\theta\theta} = \alpha_0\left(0, 2T^2, 0\right) + \alpha_2\left(0, 2T^2, 0\right)2T^2, \\ &\varepsilon_{zz} = \alpha_0\left(0, 2T^2, 0\right) + \alpha_2\left(0, 2T^2, 0\right)T^2. \end{aligned} \tag{4.13}$$

Notice that $\varepsilon_{zz} \neq 0$ and $\varepsilon_{rr} \neq 0$. Thus, in general, the cylinder will become longer or shorter and will undergo radial expansion or compression,essentially the counterpart of POYNTING effect. We also find that

$$tr\boldsymbol{\varepsilon} = 3\alpha_0\left(0, T^2,)\right) + 2\alpha_2\left(0, T^2, 0\right), \tag{4.14}$$

which is generally not equal to zero.

In the case of the special model (3.14), we find that the only non-zero components of the linearized strain are

$$\varepsilon_{\theta z} = \varepsilon_{z\theta} = \beta \frac{T}{(1 + 2\gamma T^2)^{1/2}}. \tag{4.15}$$

We find that $\varepsilon_{\theta z} \to \frac{\beta}{\sqrt{2\gamma}}$ as $T \to \infty$, i.e., we once again have a limiting value for the strain. For the rest of the paper, we shall only consider the special model (3.14).

4.4. Circumferential shear of the annular region between two cylinders

We shall consider the circumferential shearing of an annular cylinder of inner radius R_i and outer radius R_0 so that in a cylindrical polar co-ordinate system the stress has the form

$$\mathbf{T} = T(r)\left(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r\right). \tag{4.16}$$

It follows from the equations of equilibrium that

$$\frac{dT}{dr} + \frac{2T}{r} = 0, \tag{4.17}$$

and thus

$$T(r) = T(R) = \frac{C}{R^2}. \tag{4.18}$$

Next,

$$\varepsilon_{r\theta} = \frac{1}{2} R \frac{dg}{dR}, \tag{4.19}$$

and by (3.14)

$$\varepsilon_{r\theta} = \frac{1}{2} R \frac{dg}{dr} = \frac{\beta T(R)}{\left[1 + 2\gamma T^2(R)\right]^{1/2}}. \tag{4.20}$$

We first note that the maximum value that $\varepsilon_{r\theta}$ can take is $\frac{\beta}{(2\gamma)^{1/2}}$. It also follows from (4.20) that

$$\frac{dg}{dR} = \frac{2\beta C}{R\left(R^4 + 2\gamma C^2\right)^{1/2}}. \tag{4.21}$$

The solution for $g(R)$ is obtained by integrating (4.21) and then enforcing the boundary conditions

$$g(R_0) = \Omega, \tag{4.22}$$

and

$$g(R_i) = 0. \tag{4.23}$$

We could also interchange the boundary conditions by interchanging the conditions at the inner and outer radius. This might lead to the structure of possible boundary layer developments adjacent to the boundaries to be different. One can integrate the differential equation (4.21), enforce the boundary conditions (4.22) and (4.23) to obtain

$$g(R) = -\frac{\beta}{\sqrt{2\gamma}} \ln \left(\frac{4\gamma C^2 + 2C\sqrt{2\gamma(R^4 + 2\gamma C^2)}}{DR^2} \right) \tag{4.24}$$

where D, the constant of integration, is found from the boundary conditions along with C as:

$$\begin{aligned} D &= \frac{4\gamma C^2 + 2C\sqrt{2\gamma\left(R_i^4 + 2\gamma C^2\right)}}{R_i}, \\ C &= \frac{R_i^4\left(A^2-1\right)}{2\sqrt{2\gamma\left[A^2R_i^4 + A^2\frac{R_i^8}{R_0^4} - \frac{R_i^6}{R_0^2}A\left(A^2-1\right)\right]}}, \\ A &= \exp\left(\frac{\Omega\sqrt{2\gamma}}{\beta}\right). \end{aligned} \tag{4.25}$$

Figures 1 and 2 show how the angular displacement g varies with the radius R in the case of a thick walled and thin walled cylinders, respectively. In the case of a thick walled cylinder, we see that while the variation of g is linear for small values of the ration , $\frac{\Omega}{\beta}$, the variation of g with the radius is non-linear for larger values of the ratio. On the other hand, in the case of a thin walled cylinder the variation of g with respect to the radius is linear even for larger values of the ratio $\frac{\Omega}{\beta}$. This is to be expected as one can approximate the thin annulus as a shell over which the strains can be averaged.

Instead of the boundary conditions (4.22), (4.23) one could also prescribe for instance the displacement at the inner radius to be zero and the shear stress at the outer radius. In this case, we would have to solve the differential equation (4.21), subject to (4.22) and the constant C is determined to be

$$C = T_0 R_0^2, \tag{4.26}$$

where T_0 is the shear stress prescribed at the outer boundary.

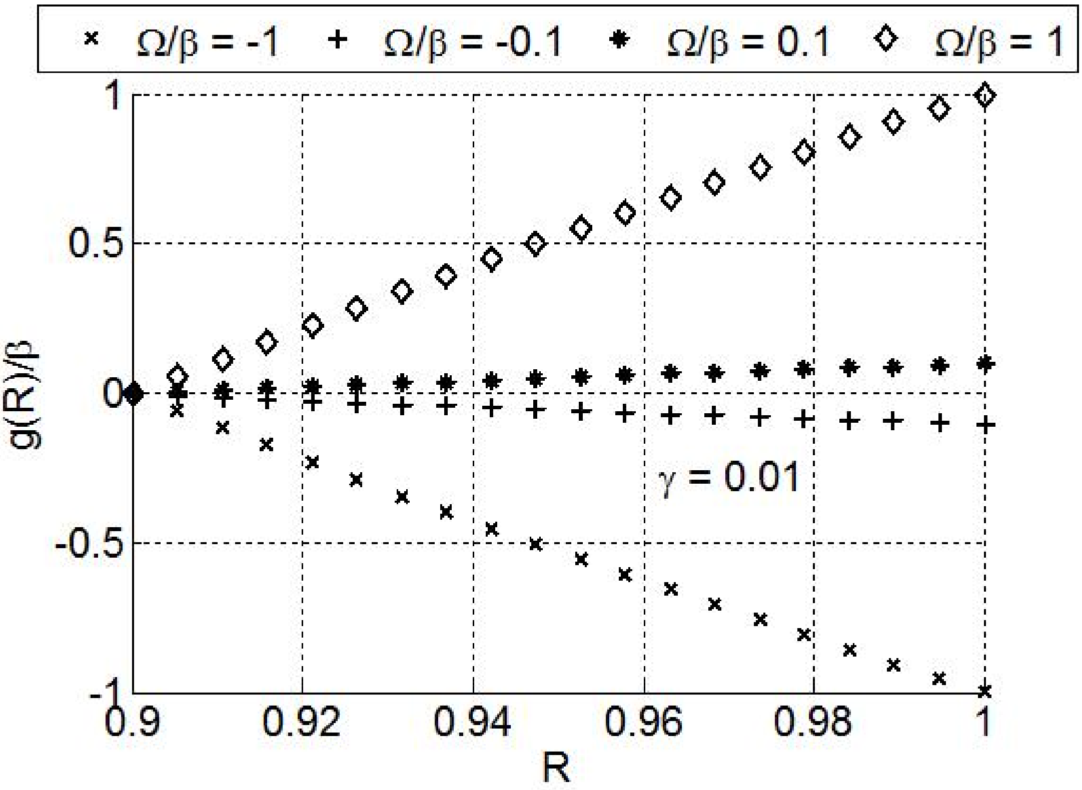

Figure 1: Variation of g(R) with R for a thick walled cylinder when $\gamma = 0.01$ for various values of Ω

In this case, the solution takes the form

$$g(R) = -\frac{\beta}{\sqrt{2\gamma}} \ln \left(\frac{4\gamma T_0^2 R_0^4 + 2T_0 R_0^2 \sqrt{2\gamma(R^4 + 2\gamma T_0^2 R_0^4)}}{DR^2} \right) \tag{4.27}$$

where

$$D = \frac{4\gamma T_0^2 R_0^4 + 2T_0 R_0^2 \sqrt{2\gamma \left(R_i^4 + 2\gamma T_0^2 R_0^4\right)}}{R_i^2} \tag{4.28}$$

The solutions to this particular specification of boundary conditions is portrayed in Figures 3 and 4 for the thick walled and thin walled cases. Once again we find that the function g varies nonlinearly in the case of the thick walled cylinder, for larger values of $\frac{\Omega}{\beta}$ while the variation is linear in the thin walled case, as is to be expected.

4.5. Telescopic Shearing

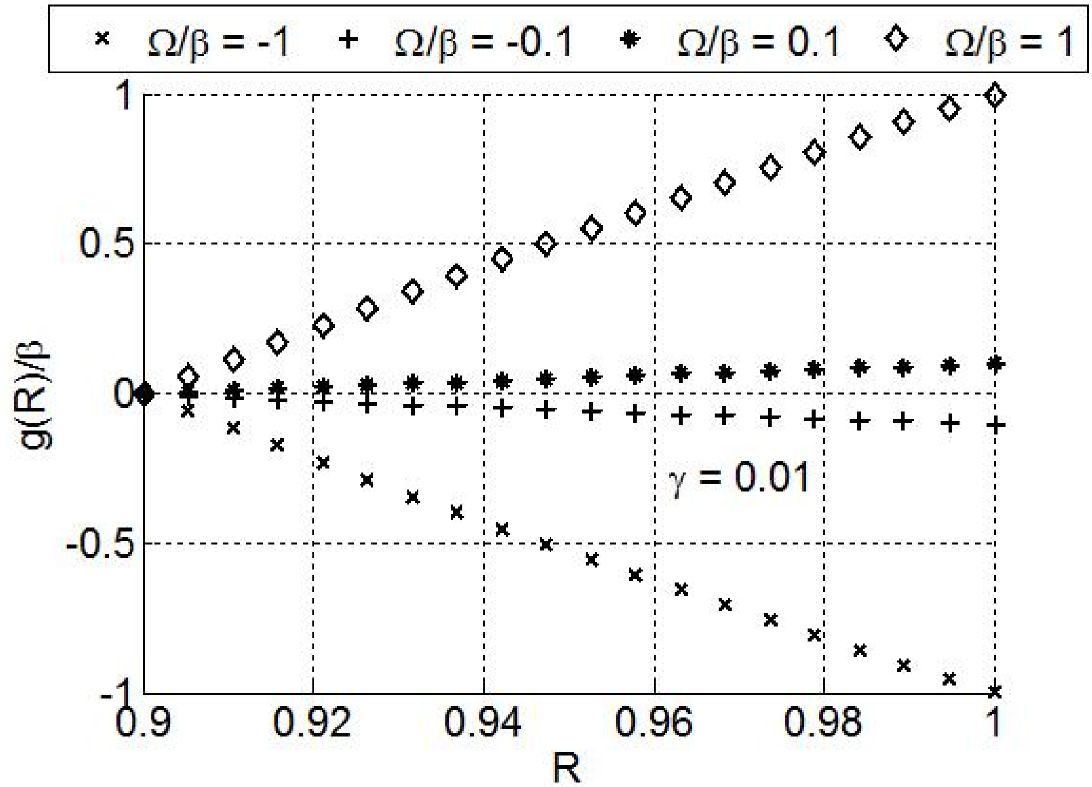

Figure 2: Variation of g(R) with R for a thin walled cylinder when $\gamma = 0.01$ for various values of Ω

Let us next consider the deformation from $(R, \Theta, Z) \mapsto (r, \theta, z)$, in a cylindrical polar coordinate system by applying a stress field of the form

$$\mathbf{T} = T(R)\left(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r\right). \tag{4.29}$$

We will assume a deformation of the form

$$r = R, \theta = \Theta, z = Z + f(R). \tag{4.30}$$

A trivial calculation shows that

$$\boldsymbol{\varepsilon} = \frac{1}{2}\begin{pmatrix} 0 & 0 & f'(R) \\ 0 & 0 & 0 \\ f'(R) & 0 & 0 \end{pmatrix}. \tag{4.31}$$

It immediately follows from (3.15) that

$$\varepsilon_{rz} = \frac{1}{2}f'(R) = \frac{\beta T(R)}{\left[1 + 2\gamma\left(T(R)\right)^2\right]^{1/2}}, \tag{4.32}$$

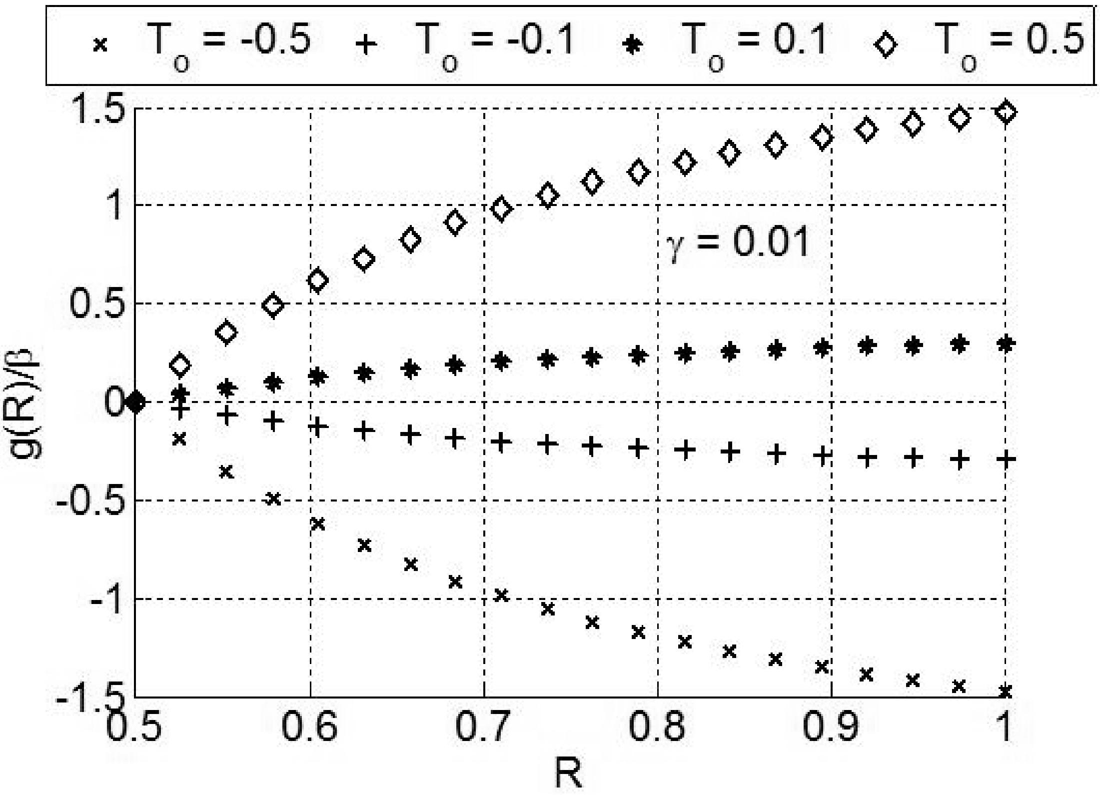

Figure 3: Variation of g(R) with R for a thick walled cylinder when $\gamma = 0.01$ for various values of T_0

and all other components of the strain are zero. As before, the maximum strain possible is $\frac{\beta}{\sqrt{2\gamma}}$. The equations of equilibrium reduce to

$$\frac{dT_{rz}}{dr} + \frac{T_{rz}}{r} = 0. \tag{4.33}$$

Thus,

$$T_{rz} = \frac{C}{R}, \tag{4.34}$$

and it follows from (4.32) and (4.34) that

$$\frac{df}{dR} = \frac{\beta C}{(R^2 + 2\gamma C^2)^{1/2}}, \tag{4.35}$$

which can be integrated to yield

$$f(R) = 2\beta C \ln\left(D\left[R + \sqrt{R^2 + 2\gamma C^2}\right]\right) \tag{4.36}$$

where D is a constant of integration.

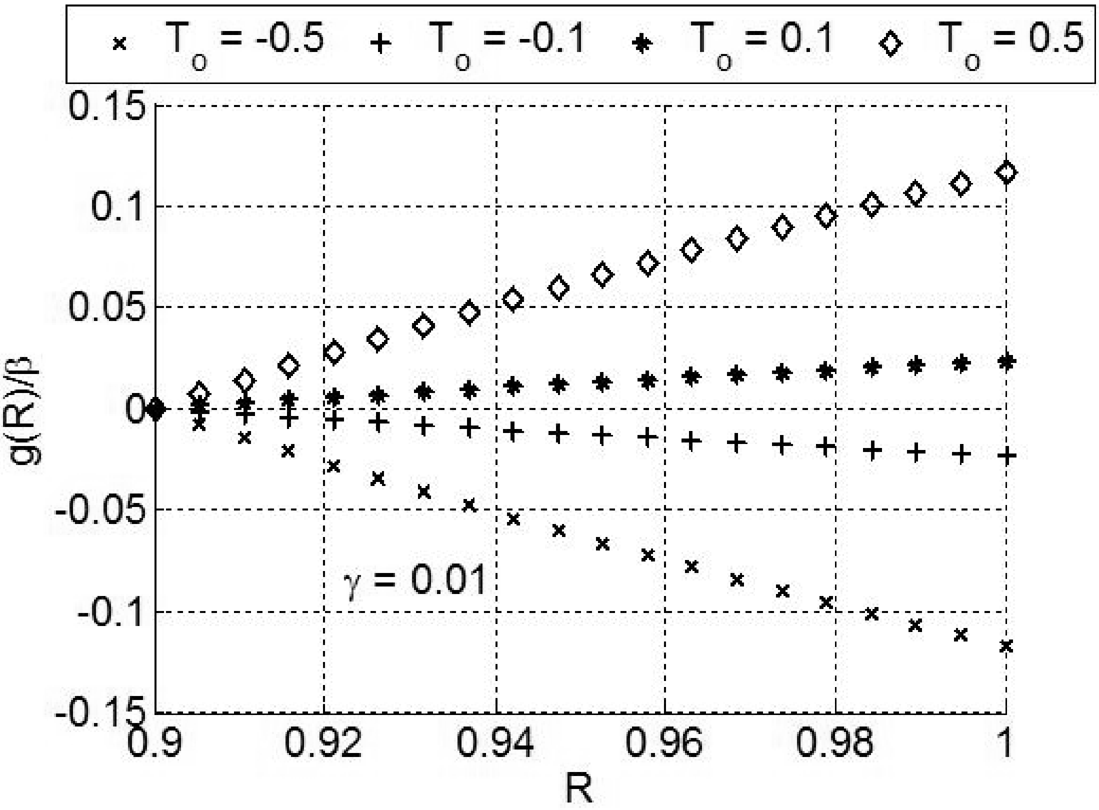

Figure 4: Variation of g(R) with R for thin walled cylinder when $\gamma = 0.01$ for various values of T_0

We shall enforce the boundary conditions

$$f(R_i) = U, \tag{4.37}$$

$$f(R_0) = 0, \tag{4.38}$$

and the solution for $f(R)$ can be obtained by solving (4.36), (4.37) and (4.38) where U is the displacement at the inner radius. It follows that D is given by

$$D = \frac{1}{R_0 + \sqrt{R_0^2 + 2\gamma C^2}} \tag{4.39}$$

and C is obtained by solving the nonlinear equation:

$$2C \ln\left(\frac{R_i + \sqrt{R_i^2 + 2\gamma C^2}}{R_0 + \sqrt{R_0^2 + 2\gamma C^2}}\right) = \frac{U}{\beta} \tag{4.40}$$

Here the nonlinear equations are solved by using built-in MATLAB function `fzero`.

The solution $f(R)$ is plotted in Figures 5 and 6. Interestingly, unlike the previous case of circumferential shearing, the axial displacements have the same qualitative features for both the thick and thin walled cylinders. While one can see a slight non-linearity in the solution for the thick walled case when $\frac{U}{\beta}$ is large, it is not significantly different from the linear solution that one obtains for the thin walled case.

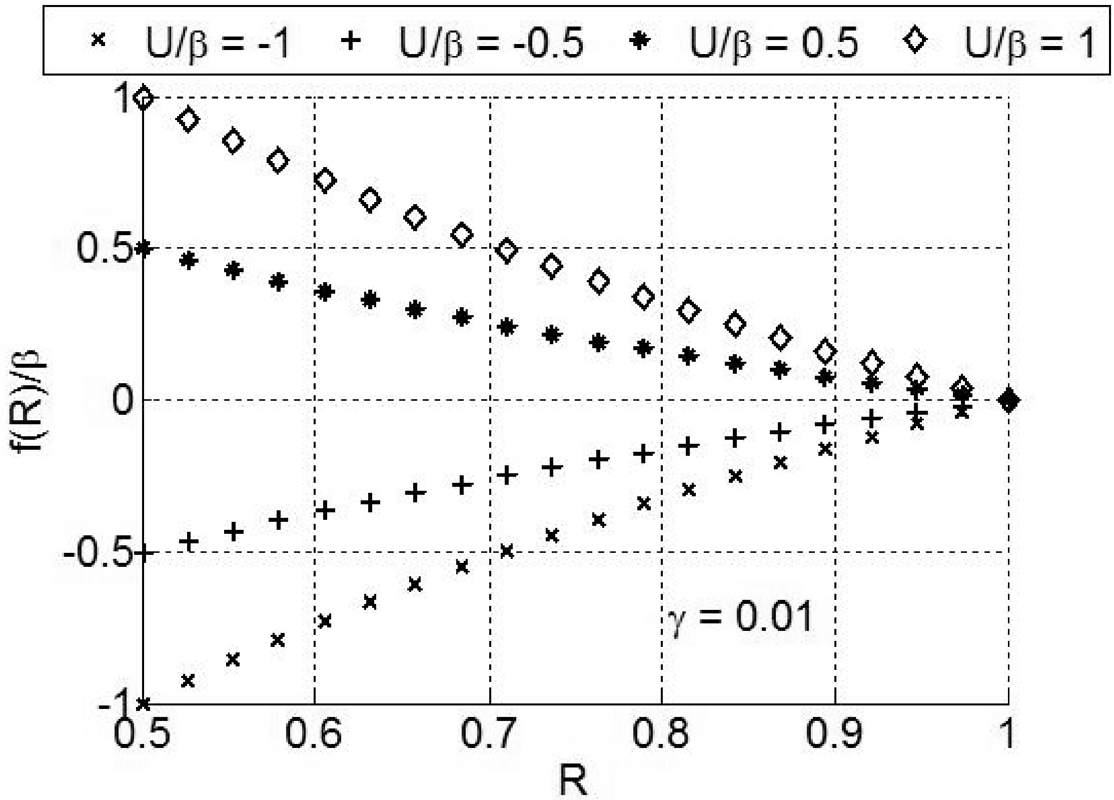

Figure 5: Variation of g(R) with R for a thick walled cylinder when $\gamma = 0.01$ for various values of U

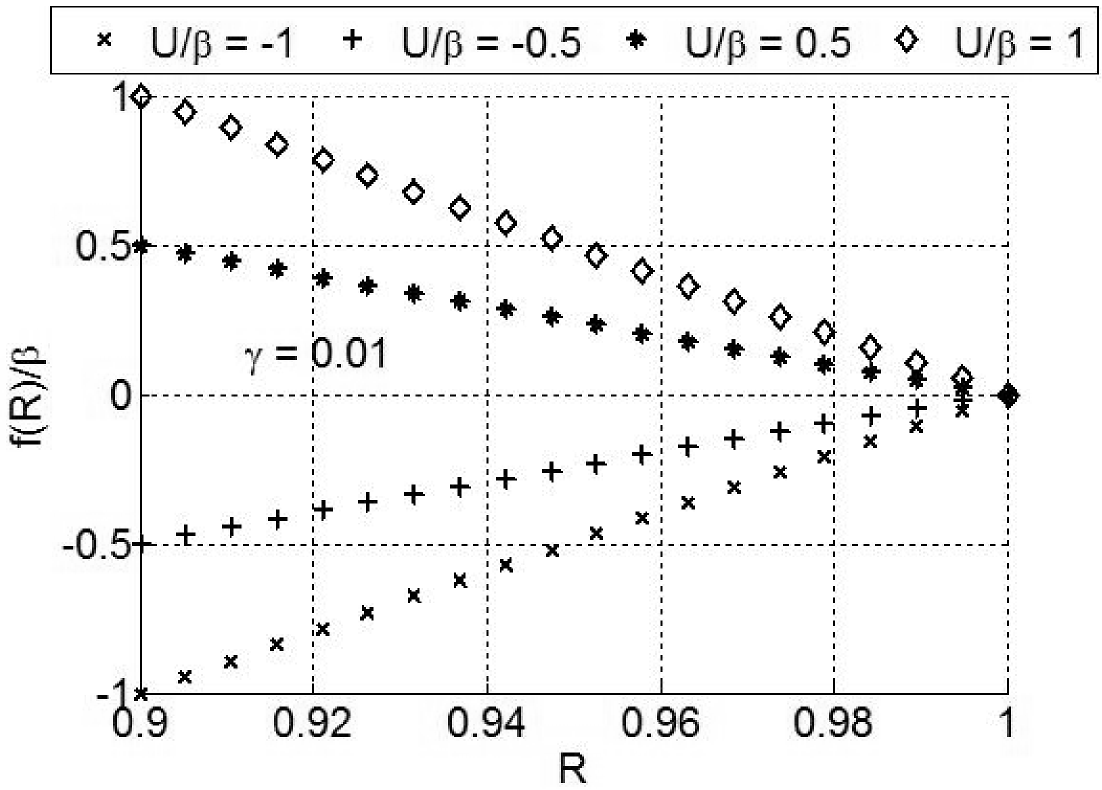

Figure 6: Variation of g(R) with R for a thin walled cylinder when $\gamma = 0.01$ for various values of U

5. CONCLUDING REMARKS

We have considered the response of a new class of elastic materials wherein the linearized strain bears a non-linear relationship to the Cauchy stress. Such a class of models do not belong to either Cauchy elastic or Green elastic bodies but is a special sub-class of bodies whose response is given by implicit constitutive equation for such bodies. The class of models that are studied are such that the strains continue to be small, well within the requirements when the linearization is supposed to hold, even as the stresses blow up, for all the problems that have been considered. This is not the case in the classical linearized theory of elasticity, for all the problems considered. In the classical theory, the linearized strain blows up as the stress blows up, thereby contradicting the starting point that requires the linearized strain to be small. Models such as the one that has been considered here will have important implications for problems such as the propagation of cracks, as well as problems that lead to singularities in the stress and hence singularities in the strain as when a concentrated load is applied. The model considered is but one of infinity of candidates and it seems that it would be worthwhile to consider the class of such models and also more complicated implicit models involving the Cauchy stress and the linearized strain. It would also be worthwhile to study the response of the model (3.7) which relates the Cauchy-Green stretch to the stress and consider some appropriate sub-classes to study the finite deformation response of elastic solids.

The fully general implicit models of the form (3.7) and (3.8) are too complicated to be of practical use as it would be well nigh impossible to outline an experimental program, wherein the material moduli, which are functions of the principal invariants of the Cauchy stress, that characterize the body, through which they could be measured. However, one can establish general results for such models such as the development of normal stresses due to shear and the counterparts of the Poynting and Kelvin effects. We can also establish universal relations for the class of such models. Thus, the models belonging to the class (3.7) and (3.8) allow us to pick meaningful subclasses wherein the material moduli would be constant, thereby allowing us reasonably simple models to work with.In this context, the model (3.12) is one such model that allows the linearized strain to be arbitrarily small even though the stresses might be large.

It would be interesting to find if one finds non-uniqueness of a cube that is subject to a shear stress, which would be the counterpart to the non-uniqueness that Rivlin [18]observed within the classical theory of non-linear elasticity, and also two dimensional problems studied by Kearsley [19] and MacSithigh [20].

The models considered in this paper are that for isotropic compressible elastic solids. Another generalization of the implicit model, that has served as the starting point of our analysis, would be to consider implicit models to
describe the anisotropy of the response of elastic bodies that have limiting strain.

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