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Abelian Gauge Fluxes and Local Models in F-Theory

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Abstract

We analyze the Abelian gauge fluxes in local F-theory models with $G_S = SU(6)$ and $SO(10)$. For the case of $G_S = SO(10)$, there is a no-go theorem which states that for an exotic-free spectrum, there are no solutions for $U(1)^2$ gauge fluxes. We explicitly construct the $U(1)^2$ gauge fluxes with an exotic-free bulk spectrum for the case of $G_S = SU(6)$. We also analyze the conditions for the curves supporting the given field content and discuss non-minimal spectra of the MSSM with doublet-triplet splitting.

1 Introduction

String theory is so far the most promising candidate for a unified theory. Building realistic models of particle physics to answer fundamental questions is one of the challenges in string theory. One of the main issues to be addressed from particle physics is the unification of gauge couplings. The natural solution to this question is the framework of grand unified theory (GUT). One task for string theory is whether it can accommodate GUT models. String theory makes contact with four-dimensional physics through various compactifications. There are two procedures to realize GUTs in string theory compactifications. The first is the top-down procedure in which the full compactification is consistent with the global geometry of extra dimensions and then the spectrum is close to GUT after breaking some symmetries [1]. In the bottom-up procedure, the gauge breaking can be understood in the decoupling limit of gravity [2, 3], particularly in the framework that D-branes are introduced on the local regions within the extra dimensions in type IIB compactification [2–4]. In this case we can neglect the effects from the global geometry for the time being, which makes the procedure more flexible and efficient. In addition, the construction of the local models can reveal the requirements for the global geometry. Eventually the local models need to be embedded into some compact geometry for UV completion.

In $SU(5)$ GUTs, there are two important Yukawa couplings, $\mathbf{10105_H}$ and $\mathbf{10\bar{5}_M\bar{5}_H}$. It is well-known that $\mathbf{10105_H}$ is forbidden in perturbative type IIB theory. However, it was shown in [5, 6] that the Yukawa coupling $\mathbf{10105_H}$ can be achieved by introducing non-perturbative corrections. From this perspective, the non-perturbative property is intrinsic for GUT model building in type IIB theory. F-theory is a non-perturbative 12-d theory built on the type IIB framework with an auxiliary two-torus ([7], see [8] for review). The ordinary string extra dimensions are regarded as a base manifold and the two-torus is equivalent to an elliptic curve as a fiber on this base manifold. The modulus of the elliptic curve is identified as axion-dilaton in type IIB theory. Due to the SL(2,Z) monodromy of the modulus, F-theory is essentially non-perturbative in type IIB language. There are elegant correspondences between physical objects in type IIB and geometry in F-theory. The modular parameter of the elliptical fiber, identified with the axion-dilaton in type IIB, varies over the base. Singularities develop when the fibers degenerate. The loci of the singular fibers indicate the locations of the seven-branes in type IIB and the type of the singularity determines the gauge group of the world-volume theory of the seven-branes [9]. According to the classification of the singular fibration, there are singularities of types A, D, and E. The first two types have perturbative descriptions in Type IIB. More precisely, A-type and D-type singularities correspond to configurations of the $D7$-branes and $D7$-branes along O-planes, respectively [10]. For the singularity of type E, there is no perturbative description in type IIB, which means that F-theory captures a non-perturbative part of the type IIB theory. Under geometric assumptions, the full F-theory can decouple from gravity [11–13]. In this way, one can focus on the gauge theory descending from world-volume theory of the seven-branes supported by the

local geometry of the discriminant loci in the base manifold of a elliptically fibered Calabi-Yau fourfold. Recently some local supersymmetric GUT models have been built in this F-theory context [11–27], and some progress has been made in constructing global models [28–33]. Supersymmetry breaking has been discussed in [34–36], and the application to cosmology has been studied in [37]. It has become more clear that F-theory provides a very promising framework for model building of supersymmetric GUTs. To build local $SU(5)$ GUTs in F-theory, one can start with engineering a Calabi-Yau fold with an A_4 singularity. To decouple from gravity, it is required that the volume of S, which is a component of the discriminant locus and is wrapped by seven-branes is contractible to zero size.[1] We assume that S can contract to a point and thus possesses an ample canonical bundle K_S^{-1} [11–13]. In particular, we focus on the case that S is a del Pezzo surface wrapped by seven-branes, which engineers an eight-dimensional supersymmetric gauge theory with gauge group $G_S = SU(5)$ in $\mathbb{R}^{3,1} \times S$. Other components S_i' of the discriminant locus intersect S along the curves Σ_i. Due to the collision of the singularities, the gauge group G_S will be enhanced to G_{Σ_i} on Σ_i and the matter in the bi-fundamental representations will be localized on the curves [38]. It was shown in [11–13] that the spectrum is given by the bundle-valued cohomology groups. In [11–13], the minimal $SU(5)$ GUT has been studied. In that case, with non-trivial $U(1)_Y$ gauge flux, the GUT group is broken into $G_{\rm std} \equiv SU(3) \times SU(2) \times U(1)_Y$. Furthermore, one can obtain an exotic-free spectrum of the minimal supersymmetric Standard Model (MSSM) from those curves with doublet-triplet splitting but no rapid proton decay. The success of the minimal $SU(5)$ GUT model motivates us to pursue other local GUT models from higher rank gauge groups. The next simplest one is gauge group of rank five, namely $SO(10)$ and $SU(6)$. These two non-minimal $SU(5)$ GUTs have been studied in [24]. For the latter, one can get an exotic-free spectrum, but due to the lack of an extra $U(1)$ flux, the GUT group cannot be broken into $G_{\rm std}$. To avoid this difficulty, it is natural to study local F-theory models of $G_S = SU(6)$ and $G_S = SO(10)$ with supersymmetric $U(1)^2$ gauge fluxes, which consist of two supersymmetric $U(1)$ gauge fluxes and are associated with rank two polystable bundles over S. The aim of the present paper is to construct explicitly the supersymmetric $U(1)^2$ gauge fluxes in local F-theory models of $G_S = SU(6)$ and $SO(10)$ and study the matter spectrum of the MSSM.

For the case of $G_S = SO(10)$, there is a no-go theorem [12] which states that for an exotic-free spectrum, there are no solutions for $U(1)^2$ gauge fluxes. For the case of $G_S = SU(6)$, we can explicitly construct supersymmetric $U(1)^2$ gauge fluxes. It turns out that each flux configuration contains two fractional lines bundles. One of the gauge fluxes is universal and has the same $U(1)_Y$ hypercharge flux as the minimal $SU(5)$ GUT [11–13]. The second one varies along with the configurations of the bulk zero modes. With suitable supersymmetric $U(1)^2$ gauge fluxes, the bulk spectrum will be exotic-free and the chiral matter will come from the curves. The restriction of these $U(1)^2$ fluxes to the curves induce $U(1)$ fluxes over the curves, which breaks

[1] There are two ways in which we could take $V_S \to 0$. The first way is by requiring S to contract to a point, and the second is by requiring S to contract to a curve of singularities. See [29, 30] for the details.

the enhanced gauge group G_Σ into $G_{\rm std} \times U(1)$. In this case, the Higgs fields can be localized on the curves $\Sigma_{SU(7)}$ and $\Sigma_{SO(12)}$. On the $\Sigma_{SU(7)}$, non-trivial induced fluxes break $SU(7)$ into $G_{\rm std} \times U(1)$. With suitable fluxes, doublet-triplet splitting can be achieved. However, the situations become more complicated on the curves with $G_\Sigma = SO(12)$. Since the dimension of the adjoint representation of $SO(12)$ is higher than $SU(7)$, one gets more constraints to solve for given field configurations, which results in difficulties for doublet-triplet splitting. By explicitly solving the allowed field configurations, one can find that there are still a few solutions with doublet-triplet splitting. From the analysis, it is clear that if one engineers the Higgs fields on the curve $\Sigma_{SU(7)}$ instead of $\Sigma_{SO(12)}$, this is the case. To obtain a complete matter spectrum of the MSSM, we analyze the case of Σ_{E_6} in addition to $\Sigma_{SU(7)}$ and $\Sigma_{SO(12)}$. It is extremely difficult to obtain the minimal spectrum of the MSSM without exotic fields. However, we found that in some cases, the exotic fields can form trilinear couplings with the doublets or triplets on the curves with $G_\Sigma = SU(7)$. When these fields get vacuum expectation values (vevs), the exotic fields will be decoupled from the low-energy spectrum. A way to do this is that we introduce extra curves supporting the doublets or triplets, which intersect the curves hosting the exotic fields to form the couplings. With the help of these doublets or triplets, it turns out that the non-minimal spectrum of the MSSM without doublet-triplet splitting problem can be achieved by local F-theory model of $G_S = SU(6)$ with supersymmetric $U(1)^2$ gauge fluxes.

The organization of the rest of the paper is as follows: in section 2, we briefly review the construction of local F-theory model and local geometry, in particular the geometry of the del Pezzo surfaces. In section 3, we include a brief review of the $SU(5)$ GUTs with $G_S = SU(5)$, $SO(10)$, and $SU(6)$. We also introduce the notion of stability of the vector bundle, in particular, that of the polystable bundle of rank two in section 4. In section 5, we review a no-go theorem for the case of $G_S = SO(10)$ and construct explicitly supersymmetric $U(1)^2$ gauge fluxes for the case of $G_S = SU(6)$. We also give examples for non-minimal spectra of the MSSM with doublet-triplet splitting. We conclude in section 6.

2 F-theory and Local Geometry

In this section we shall review some important ingredients of the local F-theory models and local geometry, and in particular the geometry of the del Pezzo surfaces.

2.1 F-theory

Consider F-theory compactified on an elliptically fibred Calabi-Yau fourfold, $T^2 \to X \to B$ with sections, which can be realized in the Weierstrass form,

$$y^2 = x^3 + fx + g, \tag{2.1}$$

where x and y are the complex coordinates on the fiber, f and g are sections of the suitable line bundles over the base manifold B. The degrees of f and g are determined by the Calabi-Yau condition, $c_1(X) = 0$. The degenerate locus of fibers is given by the discriminant $\Delta = 4f^3 + 27g^2 = 0$, which is in general a codimension one reducible subvariety in the base B. For local models, we focus on one component S of the discriminant locus $\Delta = 0$, which will be wrapped by a stack of the seven-branes and supports the GUT model. In order to decouple from the gravitational sector, the anti-canonical bundle K_S^{-1} of the surface S is assumed to be ample. According to the classification theorem of algebraic surfaces, the surface S is a del Pezzo surface and birational to the complex projective plane $\mathbb{P}^2$. There are ten del Pezzo surfaces: $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, and $dP_k,\ k = 1, 2, ..., 8$, which are blow-ups of k generic points on $\mathbb{P}^2$. In this paper we shall focus on the case of $S = dP_k,\ 2 \leqslant k \leqslant 8$ with (-2) 2-cycles[2]. In the vicinity of S, the geometry of X may be regarded approximately an ALE fibration over S. The singularity of the ALE fiberation determines the gauge group G_S of 8d $\mathcal{N} = 1$ super-Yang-Mills theory. After compacifying on S and partially twisting, the resulting effective theory is 4d $\mathcal{N} = 1$ super-Yang-Mills theory whose gauge group is the commutant of structure group of the vector bundle over S in G_S [11–13]. Let V be a holomorphic vector bundle over S. The unbroken gauge group in 4d is the commutant Γ_S of H_S in G_S, where H_S is the structure group of the bundle V. In order to preserve supersymmetry, the bundle V has to admit a hermitian connection A satisfying the Donaldson-Uhlenbeck-Yau (DUY) equation [1]

$$F_{mn} = F_{\bar{m}\bar{n}} = 0, \quad g^{m\bar{n}} F_{m\bar{n}} = 0, \tag{2.2}$$

where $g_{m\bar{n}}$ is a Kähler metric on S, and F is the curvature of the connection A. It was shown in [39, 40] that a bundle admitting a hermitian connection solving Eq. (2.2) is equivalent to a (semi) stable bundle, which is guaranteed by the Donaldson-Uhlebecker-Yau theorem. We shall in the next section define the stability of vector bundles and briefly review some facts about the equivalence. The spectrum from the bulk is given by the bundle-valued cohomology groups $H^i_{\bar{\partial}}(S, R_k)$ and their duals, where $R_k = V,\ \wedge^k V$, or EndV. The spectrum of the bulk transforms in the adjoint representation of G_S. The decomposition of adG_S into representations of $\Gamma_S \times H_S$ is

$$\mathrm{ad}G_S = \bigoplus_k \rho_k \otimes \mathcal{R}_k, \tag{2.3}$$

where ρ_k and $\mathcal{R}_k$ are representations of Γ_S and H_S, respectively. The matter fields are determined by the zero modes of the Dirac operator on S. It was shown in [12,13] that

[2] A (-2) 2-cycle is a 2-cycle with self-intersection number -2.

the chiral and anti-chiral spectrum is determined by the bundle-valued cohomology groups

$$H^0_{\bar\partial}(S, R_k^\vee)^\vee \oplus H^1_{\bar\partial}(S, R_k) \oplus H^2_{\bar\partial}(S, R_k^\vee)^\vee \tag{2.4}$$

and

$$H^0_{\bar\partial}(S, R_k) \oplus H^1_{\bar\partial}(S, R_k^\vee)^\vee \oplus H^2_{\bar\partial}(S, R_k) \tag{2.5}$$

respectively, where $\vee$ stands for the dual bundle and R_k is the vector bundle on S whose sections transform in the representation $\mathcal{R}_k$ of the structure group H_S. By the vanishing theorem of del Pezzo surfaces [12], the number of chiral fields ρ_k and anti-chiral fields ρ_k^* can be calculated by

$$N_{\rho_k} = -\chi(S, R_k) \tag{2.6}$$

and

$$N_{\rho_k^*} = -\chi(S, R_k^\vee), \tag{2.7}$$

respectively. In particular, when $V = L_1 \oplus L_2$ with structure group $U(1) \times U(1)$, according to Eq. (2.6), the chiral spectrum of $\rho_{r,s}$ is determined by

$$N_{\rho_{r,s}} = -\chi(S, {L_1}^r \otimes L_2^s), \tag{2.8}$$

where r and s correspond respectively to the $U(1)_1$ and $U(1)_2$ charges of the representations in the group theory decomposition. In order to preserve supersymmetry, the gauge bundle V has to obey the DUY equation (2.2), which is equivalent to the polystability conditions, namely

$$J_S \wedge c_1(L_1) = J_S \wedge c_1(L_2) = 0, \tag{2.9}$$

where J_S is the Kähler form on S. We will discuss the polystability conditions in more detail in section 4.

Another way to obtain chiral matter is from intersecting seven-branes along a curve, which is a Riemann surface. Let S and S' be two components of the discriminant locus Δ with gauge groups G_S and $G_{S'}$, respectively. The gauge group on the curve Σ will be enhanced to G_Σ, where $G_\Sigma \supset G_S \times G_{S'}$. Therefore, chiral matter appears as the bi-fundamental representations in the decomposition of $\mathrm{ad}G_\Sigma$

$$\mathrm{ad}G_\Sigma = \mathrm{ad}G_S \oplus \mathrm{ad}G_{S'} \oplus_k (\mathcal{U}_k \otimes \mathcal{U}'_k). \tag{2.10}$$

As mentioned above, the presence of H_S and $H_{S'}$ will break $G_S \times G_{S'}$ to the commutant subgroup when non-trivial gauge bundles on S and S' with structure groups H_S and $H_{S'}$ are turned on. Let $\Gamma = \Gamma_S \times \Gamma_{S'}$ and $H = H_S \times H_{S'}$, the decomposition of $\mathcal{U} \otimes \mathcal{U}'$ into irreducible representation is

$$\mathcal{U} \otimes \mathcal{U}' = \bigoplus\nolimits_k (v_k, \mathcal{V}_k), \tag{2.11}$$

where v_k and $\mathcal{V}_k$ are representations of Γ and H, respectively. The light chiral fermions in the representation v_k are determined by the zero modes of the Dirac operator on

Σ. It is shown in [12, 13] that the net number of chiral fields v_k and anti-chiral fields v_k^* is given by

$$N_{v_k} - N_{v_k^*} = \chi(\Sigma, K_\Sigma^{1/2} \otimes V_k), \tag{2.12}$$

where V_k is the vector bundle whose sections transform in the representation $\mathcal{V}_k$ of the structure group H. In particular, if H_S and $H_{S'}$ are $U(1) \times U(1)$ and $U(1)$, respectively, G_Σ can be broken into $G_M \times U(1) \times U(1) \times U(1) \subset G_S \times U(1)$. In this case, the bi-fundamental representations in Eq. (2.10) will be decomposed into

$$\bigoplus_j (\sigma_j)_{r_j, s_j, r'_j}, \tag{2.13}$$

where r_j, s_j and r'_j correspond to the $U(1)$ charges of the representations in the group theory decomposition and σ_j are representations in G_M. The representations $(\sigma_j)_{r_j,s_j,r'_j}$ are localized on Σ [12,13,38] and as shown in [12,13], the generation number of the representations $(\sigma_j)_{r_j,s_j,r'_j}$ and $(\bar{\sigma}_j)_{-r_j,-s_j,-r'_j}$ can be calculated by

$$N_{(\sigma_j)_{r_j,s_j,r'_j}} = h^0(\Sigma, K_\Sigma^{1/2} \otimes L_{1\Sigma}^{r_j} \otimes L_{2\Sigma}^{s_j} \otimes L'^{r'_j}_\Sigma) \tag{2.14}$$

and

$$N_{(\bar{\sigma}_j)_{-r_j,-s_j,-r'_j}} = h^0(\Sigma, K_\Sigma^{1/2} \otimes L_{1\Sigma}^{-r_j} \otimes L_{2\Sigma}^{-s_j} \otimes L'^{-r'_j}_\Sigma), \tag{2.15}$$

where $L_{1\Sigma} \equiv L_1|_\Sigma$, $L_{2\Sigma} \equiv L_2|_\Sigma$, and $L'_\Sigma \equiv L'|_\Sigma$ are the restrictions of the line bundles L_1, L_2 and L' to the curve Σ, respectively. Note that from Eq. (2.20) below, if $c_1(L_{1\Sigma}^{r_j} \otimes L_{2\Sigma}^{s_j} \otimes L'^{r'_j}_\Sigma) = 0$, then $N_{(\sigma_j)_{r_j,s_j,r'_j}} = N_{(\bar{\sigma}_j)_{-r_j,-s_j,-r'_j}} = 0$. If $c_1(L_{1\Sigma}^{r_j} \otimes L_{2\Sigma}^{s_j} \otimes L'^{r'_j}_\Sigma) \neq 0$, then only one of them is non-vanishing. Using these properties, we can solve the doublet-triplet splitting problem with suitable line bundles. In addition to the analysis of the spectrum, the pattern of Yukawa couplings also has been studied [11–13, 32]. By the vanishing theorem of del Pezzo surfaces [12,13], Yukawa couplings can form in two different ways. In the first way, the coupling comes from the interaction between two fields on the curves and one field on the bulk S. In the second way, all three fields are localized on the curves which intersect at a point where the gauge group G_p is further enhanced by two ranks. Recently, flavor physics in F-theory models has been studied in [15–20, 26, 27, 32, 33]. When one turns on bulk three-form fluxes, the structure of the Yukawa couplings will be distorted and non-commutative geometry will emerge [27]. The case of rk(V) = 1 and minimal $SU(5)$ GUT model has been studied in [11–13]. In this article, we shall focus on the case that V is a polystable bundle of rank two. We will study non-minimal cases, namely $G_S = SU(6)$ and $SO(10)$ with these rank two polystable bundles and the spectrum of the MSSM.

2.2 Local Geometry

To make the present paper self-contained, in this section we include a brief review of the geometry of the del Pezzo surfaces, curves on the surfaces and some useful formulae.

2.2.1 Del Pezzo Surfaces

As mentioned in the previous section, in local models we require that the anti-canonical bundle K_S^{-1} of the surface S wrapped by the seven-branes be ample. An algebraic surface with ample anti-canonical bundle is called a del Pezzo surface. It was shown that there are ten families of del Pezzo surfaces: $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ and the blow-ups of $\mathbb{P}^2$ at k generical points, where $1 \leqslant k \leqslant 8$. [42, 43]. In what follows, we shall briefly review the geometry of the del Pezzo surfaces.

The del Pezzo surface S is an algebraic surface with ample anti-canonical bundle, namely $K_S^{-1} > 0$. It follows that $h^1(S, \mathcal{O}_S) = h^2(S, \mathcal{O}_S) = 0$[3] and that $\chi(S, \mathcal{O}_S) = \sum_{i=0}^{2}(-1)^i h^i(S, \mathcal{O}_S) = 1$. According to the classification theorem of algebraic surfaces, these surfaces are birational to the complex projective plane $\mathbb{P}^2$. It was shown in [11–13] that to obtain an exotic-free bulk spectrum, the gauge fluxes have to correspond to the dual of (-2) 2-cycle in S. The Picard group of $\mathbb{P}^2$ is generated by hyperplane divisor H with intersection number $H \cdot H = 1$. Thus there is no (-2) 2-cycle in $\mathbb{P}^2$. Let us turn to the case of dP_k. The Picard group of dP_k is generated by the hyperplane divisor H, which is inherited from $\mathbb{P}^2$ and the exceptional divisors $E_i,\ i = 1, 2, .., k$ from the blow-ups with intersection numbers $H \cdot H = 1$, $H \cdot E_i = 0$, and $E_i \cdot E_j = -\delta_{ij},\ \forall\ i, j$. It is easy to see that dP_1 contains no (-2) 2-cycles. It follows that the candidates of the del Pezzo surfaces containing (-2) 2-cycles are dP_k with $2 \leqslant k \leqslant 8$. In what follows, I shall focus on the del Pezzo surfaces dP_k with $2 \leqslant k \leqslant 8$. The canonical divisor of dP_k is $K_S = -3H + E_1 +, ..., +E_k$. The first term comes from $K_{\mathbb{P}^2} = -3H$ and the rest comes from the blow-ups, which lead to the exceptional divisors $E_1,\ E_2, ...,\ E_k$. For local models in F-theory, the curves supporting matter fields are required to be effective. Next we shall define effective curves and the Mori cone. Consider a complex surface Y and its homology group $H_2(Y, \mathbb{Z})$. Let C be a holomorphic curve in Y. Then $[C] \in H_2(Y, \mathbb{Z})$ is called an effective class if $[C]$ is equivalent to C. The Mori cone $\overline{\mathrm{NE}}(Y)$ is spanned by a countable number of generators of the effective classes [44, 45]. The Mori cones $\overline{\mathrm{NE}}(dP_k)$ of the del Pezzo surfaces dP_k are all finitely generated [42]. To be concrete, we list the generators of the Mori cones of $dP_k,\ 2 \leqslant k \leqslant 8$ in Table 1.

With the Mori cone, one can easily check that the anti canonical divisor $-K_S$ is ample.[4] The dual of the Mori cone is the ample cone, denoted by $\mathrm{Amp}(dP_k)$, which is defined by $\mathrm{Amp}(dP_k) = \{\omega \in H_2(dP_k, \mathbb{R}) |\ \omega \cdot \zeta > 0,\ \forall \zeta \in \overline{\mathrm{NE}}(dP_k)\}$. Each ample divisor ω in the ample cone is associated with a Kähler class J_S. In this article we choose "large volume polarization", namely $\omega = AH - \sum_{i=1}^{k} a_k E_k$ with $A \gg a_k > 0$ [11,12]. It is easy to check that ω is ample. For the del Pezzo surfaces S and a line bundle $\mathcal{L}$ over S, there are two useful theorems. One is the Riemann-Roch

[3] It can be easily seen by the Kodaira vanishing theorem which states that for any ample line bundle $\mathcal{L}$, $h^i(S, K_S \otimes \mathcal{L}) = 0,\ \forall i > 0$.

[4] Here we can apply the Nakai-Moishezon criterion which states that for any divisor D, D is ample if and only if $D \cdot D > 0$ and $D \cdot C_\alpha > 0$, where C_α are generators of the Mori cone.

Mori Cone	Generators	Number
$\overline{\mathrm{NE}}(dP_2)$	$E_i,\ H-E_1-E_2$	3
$\overline{\mathrm{NE}}(dP_3)$	$E_i,\ H-\sum_{m=1}^{2}E_{i_m}$	6
$\overline{\mathrm{NE}}(dP_4)$	$E_i,\ H-\sum_{m=1}^{2}E_{i_m}$	10
$\overline{\mathrm{NE}}(dP_5)$	$E_i,\ H-\sum_{m=1}^{2}E_{i_m},\ 2H-\sum_{n=1}^{5}E_{i_n}$	16
$\overline{\mathrm{NE}}(dP_6)$	$E_i,\ H-\sum_{m=1}^{2}E_{i_m},\ 2H-\sum_{n=1}^{5}E_{i_n}$	27
$\overline{\mathrm{NE}}(dP_7)$	$E_i,\ H-\sum_{m=1}^{2}E_{i_m},\ 2H-\sum_{n=1}^{5}E_{i_n},\ 3H-2E_i-\sum_{p=1}^{6}E_{i_p}$	56
$\overline{\mathrm{NE}}(dP_8)$	$E_i,\ H-\sum_{m=1}^{2}E_{i_m},\ 2H-\sum_{n=1}^{5}E_{i_n},\ 3H-2E_i-\sum_{p=1}^{6}E_{i_p},$ $4H-2\sum_{q=1}^{3}E_{i_q}-\sum_{r=1}^{5}E_{i_r},\ 5H-2\sum_{l=1}^{6}E_{i_l}-E_r-E_s,$ $6H-3E_i-2\sum_{m=1}^{7}E_{i_m}$	240

Table 1: The generators of the Mori cone $\overline{\mathrm{NE}}(dP_k)$ for $k=2,...8$, where all indices are distinct.

theorem [44, 45], which says that

$$\chi(S,\mathcal{L})=1+\frac{1}{2}c_1(\mathcal{L})^2-\frac{1}{2}c_1(\mathcal{L})\cdot K_S. \tag{2.16}$$

Another one is the vanishing theorem ([11], also see [46]), which states that for a non-trivial holomorphic vector bundle $\mathcal{V}$ over S satisfying the DUY equation (2.2),

$$H^0_{\bar\partial}(S,\mathcal{V})=H^2_{\bar\partial}(S,\mathcal{V})=0. \tag{2.17}$$

These two theorems simplify the calculation of the spectrum. Note that the vanishing theorem (2.17) holds when $\mathcal{V}$ is a line bundle. It follows from Eq. (2.16) and Eq. (2.17) that $h^1(S,\mathcal{L})=-\chi(S,\mathcal{L})=-(1-\frac{1}{2}c_1(\mathcal{L})\cdot K_S+\frac{1}{2}c_1(\mathcal{L})^2)$. The number of zero modes will be determined by the intersection numbers $c_1(\mathcal{L})\cdot K_S$ and $c_1(\mathcal{L})^2$.

For local models, we require that all curves be effective. That is, the homological classes of the curves in $H_2(S,\mathbb{Z})$ can be written as non-negative integral combinations of the generators of the Mori cone, namely $\Sigma=\sum_\beta n_\beta\mathcal{C}_\beta$ with $n_\beta\in\mathbb{Z}_{\geqslant 0}$[5]. To calculate the genus of the curve, we can apply the adjunction formula, which says that for a smooth, irreducible curve of genus g, the following equation holds

$$\Sigma\cdot(\Sigma+K_S)=2g-2. \tag{2.18}$$

In the present paper, we shall choose genus zero curves to support the matter in the GUTs or MSSM, which means that all matter curves satisfy the equation $\Sigma\cdot(\Sigma+K_S)=-2$. To calculate the spectrum from the curves, we also need the Rieman-Roch

[5]By abuse of notation, we use Σ to denote the homological class of the curve Σ.

theorem [44, 45] for the algebraic curves. For the case of the algebraic curve Σ, the Rieman-Roch theorem states that for a line bundle $\mathcal{L}$ over Σ,

$$h^0(\Sigma, \mathcal{L}) - h^1(\Sigma, \mathcal{L}) = 1 - g + c_1.(\mathcal{L}). \tag{2.19}$$

In particular, for the case of $g = 0$, we have

$$h^0(\Sigma, K_\Sigma^{1/2} \otimes \mathcal{L}) = \begin{cases} c_1(\mathcal{L}), & \text{if } c_1(\mathcal{L}) \geqslant 0 \\ 0, & \text{if } c_1(\mathcal{L}) < 0, \end{cases} \tag{2.20}$$

where $K_\Sigma^{1/2}$ is the spin bundle of Σ and the Serre duality [44, 45] has been used. Eq. (2.20) will be useful to calculate the spectrum from the curves.

3 $U(1)$ Gauge Fluxes

In this section we briefly review some ingredients of $SU(5)$ GUT Models with $G_S = SU(5)$, $SU(10)$ and $SU(6)$. In these models, we introduce a non-trivial $U(1)$ gauge flux to break gauge group G_S. We are primarily interested in doublet-triple splitting and an exotic-free spectrum of the MSSM. From now on, unless otherwise stated, the del Pezzo surface S is assumed to be dP_8.

3.1 $G_S = SU(5)$

Before discussing the case of $G_S = SO(10),\ SU(6)$, let us review the case of $G_S = SU(5)$ [11–13]. On the bulk, we consider the following breaking pattern [41]:

$$\begin{array}{rcl} SU(5) & \to & SU(3) \times SU(2) \times U(1)_S \\ \mathbf{24} & \to & (\mathbf{8}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3})_0 + (\mathbf{3}, \mathbf{2})_{-5} + (\bar{\mathbf{3}}, \mathbf{2})_5 + (\mathbf{1}, \mathbf{1})_0. \end{array} \tag{3.1}$$

The bulk zero modes are given by

$$(\mathbf{3}, \mathbf{2})_{-5} \in H^0_{\bar\partial}(S, L^5)^\vee \oplus H^1_{\bar\partial}(S, L^{-5}) \oplus H^2_{\bar\partial}(S, L^5)^\vee \tag{3.2}$$

$$(\bar{\mathbf{3}}, \mathbf{2})_5 \in H^0_{\bar\partial}(S, L^{-5})^\vee \oplus H^1_{\bar\partial}(S, L^5) \oplus H^2_{\bar\partial}(S, L^{-5})^\vee, \tag{3.3}$$

where $\vee$ stands for the dual and L is the supersymmetric line bundle associated with $U(1)_S$. Let $N_{(\mathbf{A},\mathbf{B})_c}$ be the number of the fields in the representation $(\mathbf{A}, \mathbf{B})_c$ under $SU(3) \times SU(2) \times U(1)_S$, where c is the charge of $U(1)_S$. Note that $(\mathbf{3}, \mathbf{2})_{-5}$ and $(\bar{\mathbf{3}}, \mathbf{2})_5$ are exotic fields in the MSSM. In order to eliminate the exotic fields $(\mathbf{3}, \mathbf{2})_{-5}$ and $(\bar{\mathbf{3}}, \mathbf{2})_5$, it is required that $\chi(S, L^{\pm 5}) = 0$. It follows from the Riemann-Roch theorem (2.16) that $c_1(L^{\pm 5})^2 = -2$ and $c_1(L^{\pm 5})$ correspond to a root of E_8, $E_i - E_j$, $i \neq j$, which leads to a fractional line bundle[6] $L = \mathcal{O}_S(E_i - E_j)^{\pm 1/5}$ [11–13]. In this case,

[6] From now on, all indices appearing in the divisors will be assumed to be distinct unless otherwise stated.

all matter fields must come from the curves. Now we turn to the spectrum from the curves. In general, the gauge groups on the curves will be enhanced at least by one rank. With $G_S = SU(5)$, the gauge groups on the curves G_Σ can be enhanced to $SU(6)$ or $SO(10)$ [38]. We first focus on the curves supporting the matter fields in an $SU(5)$ GUT. To obtain complete matter multiples of $SU(5)$ GUT, it is required that $L_\Sigma = \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$, where L' is a line bundle associated with $U(1)'$. Consider the following breaking patterns:

$$\begin{array}{rcl} SU(6) & \to & SU(5) \times U(1)' \\ \mathbf{35} & \to & \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{5}_6 + \bar{\mathbf{5}}_{-6} \end{array} \tag{3.4}$$

$$\begin{array}{rcl} SO(10) & \to & SU(5) \times U(1)' \\ \mathbf{45} & \to & \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{10}_4 + \overline{\mathbf{10}}_{-4}. \end{array} \tag{3.5}$$

From the patterns (3.4) and (3.5), it can be seen by counting the dimension of the adjoint representations that matter fields $\mathbf{5}_6$ and $\bar{\mathbf{5}}_{-6}$ are localized on the curves with $G_\Sigma = SU(6)$ while $\mathbf{10}_4$ and $\overline{\mathbf{10}}_{-4}$ are localized on the curve with $G_\Sigma = SO(10)$. The Higgs fields localize on the curves with $G_\Sigma = SU(6)$ as well. Since on the matter curves L_Σ is required to be trivial, the only line bundle used to determine the spectrum is L'_Σ. With non-trivial L'_Σ, it is not difficult to engineer three copies of the matter fields, $3 \times \mathbf{5}_6$, $3 \times \bar{\mathbf{5}}_{-6}$, and $3 \times \mathbf{10}_4$. In order to get doublet-triplet splitting, it is required that $L_\Sigma \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$. With non-trivial L_Σ and L'_Σ, G_Σ will be broken into $G_{\rm std} \times U(1)'$. Consider the following breaking patterns,

$$\begin{array}{rcl} SU(6) & \to & SU(3) \times SU(2) \times U(1)_S \times U(1)' \\ \mathbf{35} & \to & (\mathbf{8},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{3})_{0,0} + (\mathbf{3},\mathbf{2})_{-5,0} + (\bar{\mathbf{3}},\mathbf{2})_{5,0} + (\mathbf{1},\mathbf{1})_{0,0} \\ & & +(\mathbf{1},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{2})_{3,6} + (\mathbf{3},\mathbf{1})_{-2,6} + (\mathbf{1},\bar{\mathbf{2}})_{-3,-6} + (\bar{\mathbf{3}},\mathbf{1})_{2,-6} \end{array} \tag{3.6}$$

$$\begin{array}{rcl} SO(10) & \to & SU(3) \times SU(2) \times U(1)_S \times U(1)' \\ \mathbf{45} & \to & (\mathbf{8},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{3})_{0,0} + (\mathbf{3},\mathbf{2})_{-5,0} + (\bar{\mathbf{3}},\mathbf{2})_{5,0} + (\mathbf{1},\mathbf{1})_{0,0} \\ & & +(\mathbf{1},\mathbf{1})_{0,0} + [(\mathbf{3},\mathbf{2})_{1,4} + (\bar{\mathbf{3}},\mathbf{1})_{-4,4} + (\mathbf{1},\mathbf{1})_{6,4} + c.c]. \end{array} \tag{3.7}$$

From the patterns (3.6) and (3.7), the field content of the MSSM is identified as shown in Table 2.

Q_L	u_R	d_R	e_R	L_L	H_u	H_d
$(\mathbf{3},\mathbf{2})_{1,4}$	$(\bar{\mathbf{3}},\mathbf{1})_{-4,4}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-6}$	$(\mathbf{1},\mathbf{1})_{6,4}$	$(\mathbf{1},\bar{\mathbf{2}})_{-3,-6}$	$(\mathbf{1},\mathbf{2})_{3,6}$	$(\mathbf{1},\bar{\mathbf{2}})_{-3,-6}$

Table 2: Field content of the MSSM from $G_S = SU(5)$.

The superpotential is as follows:

$$\mathcal{W}_{\rm MSSM} \supset Q_L u_R H_u + Q_L d_R H_d + L_L e_R H_d + \cdots . \tag{3.8}$$

Note that the $U(1)_S$ in the patterns is consistent with $U(1)_Y$ in the MSSM and that this is the only way to consistently identify the fields in the patterns (3.6) and (3.7)

with the MSSM. Now we are going to analyze the conditions for the curves to support the field content in Table 2. We choose the curve $\Sigma_{SU(6)}$ to be a genus zero curve and let $(m_1, m_2) = (N_{(\bar{\mathbf{3}},\mathbf{1})_{2,-6}}, N_{(\mathbf{1},\bar{\mathbf{2}})_{-3,-6}})$, where $N_{(\mathbf{A},\mathbf{B})_{a,b}}$ is the number of the fields in the representation $(\mathbf{A},\mathbf{B})_{a,b}$ under $SU(3)\times SU(2)\times U(1)_S\times U(1)'$, and a, b are the charges of $U(1)_S$ and $U(1)'$, respectively. Note that $(\mathbf{3},\mathbf{1})_{-2,6}$ is exotic in the MSSM. To avoid the exotic, we require that $m_1 \in \mathbb{Z}_{\geqslant 0}$. Given (m_1, m_2), the homological class of the curve $\Sigma_{SU(6)}$ has to satisfy the following equation:[7]

$$(E_i - E_j)\cdot\Sigma_{SU(6)} = m_2 - m_1, \tag{3.9}$$

where $L = \mathcal{O}_S(E_j - E_i)^{1/5}$ has been used. By Eq. (3.9), we can engineer three copies of d_R, three copies of L_L, one copy of H_d, and one copy of H_u on the individual curves as shown in Table 3.

Multiplet	(m_1, m_2)	Conditions	Σ
$3\times d_R$	(3, 0)	$(E_i - E_j)\cdot\Sigma = -3$	$5H - 4E_j - E_i$
$3\times L_L$	(0, 3)	$(E_i - E_j)\cdot\Sigma = 3$	$4H + 2E_j - E_i$
$1\times H_d$	(0, 1)	$(E_i - E_j)\cdot\Sigma = 1$	$H - E_i - E_l$
$1\times H_u$	(0,-1)	$(E_i - E_j)\cdot\Sigma = -1$	$H - E_j - E_s$

Table 3: Field content of the $SU(6)$ Curve from $G_S = SU(5)$.

Note that all field configurations in Table 3 obey the conditions, $L_\Sigma \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$. In local models, the curves are required to be effective. With Table 1, it is not difficult to check that all curves in Table 3 are effective. The results in Table 3 show that the triplet and double states in $\mathbf{5}_6$ or $\bar{\mathbf{5}}_{-6}$ of $SU(5)$ can be separated by the restrictions of the supersymmetric line bundles to the curves. Next let us turn to the curve with $G_\Sigma = SO(10)$. Set $(l_1, l_2, l_3) = (N_{(\mathbf{3},\mathbf{2})_{1,4}}, N_{(\bar{\mathbf{3}},\mathbf{1})_{-4,4}}, N_{(\mathbf{1},\mathbf{1})_{6,4}})$. To avoid exotics in the MSSM, it is required that $l_k \in \mathbb{Z}_{\geqslant 0},\ k = 1, 2, 3$. Given (l_1, l_2, l_3), the curve $\Sigma_{SO(10)}$ has to satisfy the following equations:[8]

$$\begin{cases} (E_i - E_j)\cdot\Sigma_{SO(10)} = l_2 - l_1 \\ l_3 = 2l_1 - l_2. \end{cases} \tag{3.10}$$

To obtain the minimal spectrum of the MSSM, we require that $l_1, l_2 \leqslant 3$. Taking the conditions, $L_\Sigma \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$ into account, we have the following configurations:

$$(l_1, l_2, l_3) = \left\{\ (1,2,0), (1,0,2), (2,1,3), (2,3,1)\ \right\}. \tag{3.11}$$

From the configurations in (3.11), it is clear that unlike with $G_\Sigma = SU(6)$, it is impossible to engineer the matter fields $3\times Q_L$, $3\times u_R$, and $3\times e_R$ on the individual

[7] $L_{\Sigma_{SU(6)}} = \mathcal{O}_{\Sigma_{SU(6)}}(\frac{(m_1-m_2)}{5})$ and $L'_{\Sigma_{SU(6)}} = \mathcal{O}_{\Sigma_{SU(6)}}(-\frac{(3m_1+2m_2)}{30})$

[8] $L_{\Sigma_{SO(10)}} = \mathcal{O}_{\Sigma_{SO(10)}}(\frac{(l_1-l_2)}{5})$ and $L'_{\Sigma_{SO(10)}} = \mathcal{O}_{\Sigma_{SO(10)}}(\frac{(4l_1+l_2)}{20})$

curves with $G_\Sigma = SO(10)$, which correspond to $(l_1, l_2, l_3) = (3, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 3)$, respectively, without extra matter fields. Fortunately, in this case all Higgs fields come from $\Sigma_{SU(6)}$ instead of $\Sigma_{SO(10)}$. Although the field content on $\Sigma_{SO(10)}$ is more complicated than that on $\Sigma_{SU(6)}$, we can engineer the spectrum of the MSSM as shown in Table 4.

Multiplet	Curve	Σ	g_Σ	L_Σ	L'_Σ
$1 \times Q_L + 2 \times u_R$	$\Sigma^1_{SO(10)}$	$2H - E_2 - E_3$	0	$\mathcal{O}_{\Sigma^1_{SO(10)}}(-1)^{1/5}$	$\mathcal{O}_{\Sigma^1_{SO(10)}}(1)^{3/10}$
$2 \times Q_L + 1 \times u_R$ $+3 \times e_R$	$\Sigma^2_{SO(10)}$	$2H - E_1 - E_4$	0	$\mathcal{O}_{\Sigma^2_{SO(10)}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^2_{SO(10)}}(1)^{9/20}$
$3 \times d_R$	$\Sigma^1_{SU(6)}$	$5H - 4E_1 - E_2$	0	$\mathcal{O}_{\Sigma^1_{SU(6)}}(1)^{3/5}$	$\mathcal{O}_{\Sigma^1_{SU(6)}}(-1)^{3/10}$
$3 \times L_L$	$\Sigma^2_{SU(6)}$	$4H + 2E_1 - E_2$	0	$\mathcal{O}_{\Sigma^2_{SU(6)}}(-1)^{3/5}$	$\mathcal{O}_{\Sigma^2_{SU(6)}}(-1)^{1/5}$
$1 \times H_d$	$\Sigma^d_{SU(6)}$	$2H - E_2 - E_4$	0	$\mathcal{O}_{\Sigma^d_{SU(6)}}(-1)^{1/5}$	$\mathcal{O}_{\Sigma^d_{SU(6)}}(-1)^{1/15}$
$1 \times H_u$	$\Sigma^u_{SU(6)}$	$H - E_1 - E_3$	0	$\mathcal{O}_{\Sigma^u_{SU(6)}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^u_{SU(6)}}(1)^{1/15}$

Table 4: A minimal spectrum of the MSSM from $G_S = SU(5)$, where $L = \mathcal{O}_S(E_1 - E_2)^{1/5}$.

From Table 4, we find that for the case of $G_S = SU(5)$, we can get an exotic-free, minimal spectrum of the MSSM with doublet-triplet splitting. In addition, by arranging H_u and H_d on different curves, rapid proton decay can be avoided [11–13].

3.2 $G_S = SO(10)$

For the case of $G_S = SO(10)$ [24], we first look at the spectrum from the bulk. Consider the following breaking pattern,

$$\begin{array}{rcl} SO(10) & \to & SU(5) \times U(1)_S \\ \mathbf{45} & \to & \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{10}_4 + \overline{\mathbf{10}}_{-4}. \end{array} \tag{3.12}$$

The bulk zero modes are determined by

$$\mathbf{10}_4 \in H^0_{\bar\partial}(S, L^{-4})^\vee \oplus H^1_{\bar\partial}(S, L^4) \oplus H^2_{\bar\partial}(S, L^{-4})^\vee \tag{3.13}$$

$$\overline{\mathbf{10}}_{-4} \in H^0_{\bar\partial}(S, L^4)^\vee \oplus H^1_{\bar\partial}(S, L^{-4}) \oplus H^2_{\bar\partial}(S, L^4)^\vee. \tag{3.14}$$

To eliminate $\mathbf{10}_4$ and $\overline{\mathbf{10}}_{-4}$, it is required that $\chi(S, L^{\pm 4}) = 0$, which give rise to the fractional line bundles $L = \mathcal{O}_S(E_i - E_j)^{\pm 1/4}$. In this case, all chiral fields must come from the curves. Let us turn to the spectrum from the curves. With $G_S = SO(10)$, the gauge groups on the curve can be enhanced to $G_\Sigma = SO(12)$ or $G_\Sigma = E_6$. The

breaking chains and matter content from the enhanced adjoints of the curves are

$$\begin{array}{ccccc} SO(12) & \to & SO(10)\times U(1)' & \to & SU(5)\times U(1)'\times U(1)_S \\ \mathbf{66} & \to & \mathbf{45}_0+\mathbf{1}_0 & \to & \mathbf{24}_{0,0}+\mathbf{1}_{0,0}+\mathbf{10}_{0,4}+\overline{\mathbf{10}}_{0,-4}+\mathbf{1}_{0,0} \\ & & +\mathbf{10}_2+\overline{\mathbf{10}}_{-2} & & +\mathbf{5}_{2,2}+\bar{\mathbf{5}}_{2,-2}+\bar{\mathbf{5}}_{-2,-2}+\mathbf{5}_{-2,2} \end{array} \tag{3.15}$$

$$\begin{array}{ccccc} E_6 & \to & SO(10)\times U(1)' & \to & SU(5)\times U(1)'\times U(1)_S \\ \mathbf{78} & \to & \mathbf{45}_0+\mathbf{1}_0 & \to & \mathbf{24}_{0,0}+\mathbf{1}_{0,0}+\mathbf{10}_{0,4}+\overline{\mathbf{10}}_{0,-4}+\mathbf{1}_{0,0} \\ & & +\mathbf{16}_{-3}+\overline{\mathbf{16}}_3 & & +(\mathbf{10}_{-3,-1}+\bar{\mathbf{5}}_{-3,3}+\mathbf{1}_{-3,-5}+c.c.). \end{array} \tag{3.16}$$

Note that the $U(1)_S$ charges of the fields localized on the curves should be conserved in each Yukawa coupling. The superpotential is as follows:

$$\mathcal{W} \supset \mathbf{10}_{-3,-1}\mathbf{10}_{-3,-1}\mathbf{5}_{-2,2}+\mathbf{10}_{-3,-1}\bar{\mathbf{5}}_{-3,3}\bar{\mathbf{5}}_{2,-2}+\cdots. \tag{3.17}$$

In order to get complete matter multiplets in $SU(5)$ GUT, we require that L_Σ and L'_Σ are both non-trivial. With non-trivial L_Σ and L'_Σ, we can engineer field content with minimal singlets as shown in Table 5 [24].

Multiplet	Curve	Σ	g_Σ	L_Σ	L'_Σ
$3\times\mathbf{10}_{-3,-1}$*	$\Sigma^1_{E_6}$	$4H+2E_1-E_2$	0	$\mathcal{O}_{\Sigma^1_{E_6}}(-1)^{3/4}$	$\mathcal{O}_{\Sigma^1_{E_6}}(-1)^{3/4}$
$3\times\bar{\mathbf{5}}_{-3,3}$†	$\Sigma^2_{E_6}$	$5H+3E_2-E_5$	0	$\mathcal{O}_{\Sigma^2_{E_6}}(1)^{3/4}$	$\mathcal{O}_{\Sigma^2_{E_6}}(-1)^{1/4}$
$1\times\mathbf{5}_{-2,2}$	$\Sigma^1_{SO(12)}$	$3H+E_3-E_1$	0	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)^{1/4}$	$\mathcal{O}_{\Sigma^1_{SO(12)}}(-1)^{1/4}$
$1\times\bar{\mathbf{5}}_{2,-2}$	$\Sigma^2_{SO(12)}$	$H-E_2-E_3$	0	$\mathcal{O}_{\Sigma^2_{SO(12)}}(-1)^{1/4}$	$\mathcal{O}_{\Sigma^2_{SO(12)}}(1)^{1/4}$

Table 5: An $SU(5)$ GUT model from $G_S=SO(10)$, where $L=\mathcal{O}_S(E_1-E_2)^{1/4}$.

However, because of the lack of extra $U(1)$ gauge fluxes or Wilson lines, the doublet-triplet splitting is not achievable in the present case. This motivates us to consider supersymmetric $U(1)^2$ fluxes.

3.3 $G_S=SU(6)$

To look at the spectrum from the bulk , we consider the following breaking pattern,

$$\begin{array}{ccc} SU(6) & \to & SU(5)\times U(1)_S \\ \mathbf{45} & \to & \mathbf{24}_0+\mathbf{1}_0+\mathbf{5}_6+\bar{\mathbf{5}}_{-6}. \end{array} \tag{3.18}$$

The bulk zero modes are given by

$$\mathbf{5}_6\in H^0_{\bar\partial}(S,L^{-6})^\vee\oplus H^1_{\bar\partial}(S,L^6)\oplus H^2_{\bar\partial}(S,L^{-6})^\vee \tag{3.19}$$

*With six additional singlets

†With three additional singlets

$$\bar{\mathbf{5}}_{-6} \in H^0_{\bar\partial}(S, L^6)^\vee \oplus H^1_{\bar\partial}(S, L^{-6}) \oplus H^2_{\bar\partial}(S, L^6)^\vee. \tag{3.20}$$

To eliminate $\mathbf{5}_6$ and $\overline{\mathbf{5}}_{-6}$, it is required that $\chi(S, L^{\pm 6}) = 0$, which gives rise to the fractional line bundles $L = \mathcal{O}_S(E_i - E_j)^{\pm 1/6}$ [24]. In this case, all chiral fields must come from the curves. Let us turn to the spectrum from the curves. With $G_S = SU(6)$, the gauge groups on the curve can be enhanced to $G_\Sigma = SU(7)$, $G_\Sigma = SO(12)$ or $G_\Sigma = E_6$. The breaking chains and matter content from the enhanced adjoints of the curves are

$$\begin{array}{rcll} SU(7) & \to & SU(6)\times U(1)' & \to \; SU(5)\times U(1)'\times U(1)_S \\ \mathbf{48} & \to & \mathbf{35}_0 + \mathbf{1}_0 + \mathbf{6}_{-7} + \bar{\mathbf{6}}_7 & \to \; \mathbf{24}_{0,0} + \mathbf{1}_{0,0} + \mathbf{5}_{0,6} + \bar{\mathbf{5}}_{0,-6} + \mathbf{1}_{0,0} \\ & & & \quad + \mathbf{5}_{-7,1} + \mathbf{1}_{-7,-5} + \bar{\mathbf{5}}_{7,-1} + \mathbf{1}_{7,5} \end{array} \tag{3.21}$$

$$\begin{array}{rcll} SO(12) & \to & SU(6)\times U(1)' & \to \; SU(5)\times U(1)'\times U(1)_S \\ \mathbf{66} & \to & \mathbf{35}_0 + \mathbf{1}_0 + \mathbf{15}_2 + \overline{\mathbf{15}}_{-2} & \to \; \mathbf{24}_{0,0} + \mathbf{1}_{0,0} + \mathbf{5}_{0,6} + \bar{\mathbf{5}}_{0,-6} + \mathbf{1}_{0,0} \\ & & & \quad + \mathbf{10}_{2,2} + \mathbf{5}_{2,-4} + \overline{\mathbf{10}}_{-2,-2} + \bar{\mathbf{5}}_{-2,4} \end{array} \tag{3.22}$$

$$\begin{array}{rcll} E_6 & \to & SU(6)\times U(1)' & \to \; SU(5)\times U(1)'\times U(1)_S \\ \mathbf{78} & \to & \mathbf{35}_0 + \mathbf{1}_0 + \mathbf{1}_{\pm 2} & \to \; \mathbf{24}_{0,0} + 2\times\mathbf{1}_{0,0} + \mathbf{5}_{0,6} + \bar{\mathbf{5}}_{0,-6} + \mathbf{1}_{\pm 2,0} \\ & & \quad + \mathbf{20}_1 + \mathbf{20}_{-1} & \quad + \mathbf{10}_{1,-3} + \overline{\mathbf{10}}_{1,3} + \mathbf{10}_{-1,-3} + \overline{\mathbf{10}}_{-1,3}. \end{array} \tag{3.23}$$

In this case, the $U(1)_S$ charges of the fields localized on the curves should be conserved in each Yukawa coupling. The superpotential is:

$$\mathcal{W} \;\supset\; \mathbf{10}_{2,2}\mathbf{10}_{2,2}\mathbf{5}_{2,-4} + \mathbf{10}_{2,2}\bar{\mathbf{5}}_{7,-1}\bar{\mathbf{5}}_{7,-1} + \cdots . \tag{3.24}$$

With non-trivial L_Σ and L'_Σ, we can engineer configurations of the curves with desired field content but without any exotic fields as shown in Table 6 [24].

Multiplet	Curve	Σ	g_Σ	L_Σ	L'_Σ
$3\times\mathbf{10}_{2,2}$	$\Sigma^1_{SO(12)}$	$4H + 2E_2 - E_1$	0	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)^{1/2}$	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)$
$3\times\bar{\mathbf{5}}_{7,-1}$	$\Sigma^1_{SU(7)}$	$5H + 3E_1 - E_6$	0	$\mathcal{O}_{\Sigma^1_{SU(7)}}(-1)^{1/2}$	$\mathcal{O}_{\Sigma^1_{SU(7)}}(1)^{5/14}$
$1\times\mathbf{5}_{2,-4}$	$\Sigma^2_{SO(12)}$	$3H + E_1 - E_3$	0	$\mathcal{O}_{\Sigma^2_{SO(12)}}(-1)^{1/6}$	$\mathcal{O}_{\Sigma^2_{SO(12)}}(1)^{1/6}$
$1\times\bar{\mathbf{5}}_{7,-1}$	$\Sigma^2_{SU(7)}$	$H - E_2 - E_3$	0	$\mathcal{O}_{\Sigma^2_{SU(7)}}(-1)^{1/6}$	$\mathcal{O}_{\Sigma^2_{SU(7)}}(1)^{5/42}$

Table 6: An $SU(5)$ GUT model from $G_S = SU(6)$, where $L = \mathcal{O}_S(E_1 - E_2)^{1/6}$.

Although in this case one can obtain an exotic-free spectrum in an $SU(5)$ GUT, the doublet-triplet splitting can not be achieved, similar to the case of $G_S = SO(10)$. Again this motivates us to consider supersymmetric $U(1)^2$ gauge fluxes. On the other hand, to get the spectrum of the MSSM, we also need some mechanisms to break $SU(5) \subset G_\Sigma$ into $SU(3)\times SU(2)\times U(1)_Y$. One possible way is to consider supersymmetric $U(1)^2$ gauge fluxes instead of U(1) fluxes. These supersymmetric $U(1)^2$ gauge fluxes correspond to polystable bundles of rank two with structure group $U(1)^2$. In the next section we shall discuss polystable bundles of rank two.

4 Gauge Bundles

In this section we shall briefly review the notion of stability of the vector bundle and the relation between (semi) stable bundles and the DUY equation. In addition, we also discuss the semi-stable bundles of rank two, in particular, polystable bundles over S.

4.1 Stability

Let E be a holomorphic vector bundle over a projective surface S and J_S be a Kähler form on S. The slope $\mu(E)$ is defined by

$$\mu(E) = \frac{\int_S c_1(E) \wedge J_S}{\mathrm{rk}(E)}. \tag{4.1}$$

The vector bundle E is (semi)stable if for every subbundle or subsheaf $\mathcal{E}$ with $\mathrm{rk}(\mathcal{E}) < \mathrm{rk}(E)$, the following inequality holds

$$\mu(\mathcal{E}) < (\leqslant)\mu(E). \tag{4.2}$$

Assume that $E = \oplus_i^k \mathcal{E}_i$, then E is polystable if each $\mathcal{E}_i$ is a stable bundle with $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2) = ... = \mu(\mathcal{E}_k)$ [39, 40]. It is clear that every line bundle is stable and polystable bundle is a type of semistable bundle. The Donaldson-Uhlenbeck-Yau theorem [39, 40] states that a (split) irreducible holomorphic bundle E admits a hermitian connection satisfying Eq. (2.2) if and only if E is (poly)stable. As mentioned in section 2.1, to preserve supersymmetry, the connection of the bundle has to obey the DUY equation (2.2), which is equivalent to the (poly) stable bundle. In particular, when the bundle is split, supersymmetry requires that the bundle is polystable. In the next section we primarily focus on polystable bundles of rank two over S.

4.2 Rank Two Polystable Bundle

Here we are interested in the case $S = dP_k$. Consider the case of $V = L_1 \oplus L_2$, where L_1 and L_2 are line bundles over S and set $L_i = \mathcal{O}_S(D_i),\ i = 1, 2$, where D_i are divisors in S. Before writing down a more explicit expression for the bundle V, we first consider the stability condition of the polystable bundle. Recall that the bundle V is polystable if $\mu(L_1) = \mu(L_2)$ where μ is slope defined by Eq. (4.1). To solve the DUY equation Eq. (2.2), it is required that $\mu(L_1) = \mu(L_2) = 0$. It follows that $c_1(L_1) \wedge J_S = c_1(L_2) \wedge J_S = 0$ or equivalently,

$$D_1 \cdot \omega = D_2 \cdot \omega = 0, \tag{4.3}$$

where ω is the dual ample divisor of Kähler form J_S in the Kähler cone. In particular, in this case we choose "large volume polarization", namely $\omega = AH - \sum_{i=1}^{k} a_i E_i,\ A \gg a_i > 0$ [11,12]. Note that Eq. (4.3) is exactly the BPS equations, $c_1(L_i) \wedge J_S = 0,\ i = 1, 2$ for supersymmetric line bundles. So the polystable bundle V is a direct sum of the supersymmetric line bundles L_1 and L_2. In section 5.2 we shall apply physical constraints to the polystable bundle that satisfies the Eq. (4.3) and derive the explicit expression of the $U(1)^2$ gauge fluxes L_1 and L_2.

4.3 Supersymmetric $U(1)^2$ Gauge Fluxes

Each supersymmetric $U(1)^2$ gauge flux configuration contains two fractional line bundles, which may not be well-defined themselves. It is natural to ask whether it makes sense for these configurations to be polystable vector bundles of rank two. In what follows, we shall show that supersymmetric $U(1)^2$ gauge fluxes can be associated with polystable vector bundles of rank two. Let us consider the case of $G_S = SU(6)$ and the breaking pattern through $SU(6) \to SU(5) \times U(1) \to SU(3) \times SU(2) \times U(1)_1 \times U(1)_2$. Let L_1 and L_2 be two supersymmetric line bundles, which associate to $U(1)_1$ and $U(1)_2$, respectively. Write $L_i = \mathcal{O}_S(D_i),\ i = 1, 2$, where D_i are in general "$\mathbb{Q}$-divisors" which means that D_i are the linear combinations of the divisors in S with rational coefficients. Now we consider the rotation of the $U(1)$ charges, $U(1)_1$ and $U(1)_2$, given by

$$\widetilde{\mathbb{U}} = \mathrm{M}\mathbb{U} \tag{4.4}$$

with $\mathbb{U} = (U(1)_1, U(1)_2)^t$, $\widetilde{\mathbb{U}} = (\widetilde{U(1)}_1, \widetilde{U(1)}_2)^t$, and $\mathrm{M} \in GL(2, \mathbb{Q})$, where t represents the transpose. We define $\widetilde{L}_1$ and $\widetilde{L}_2$ to be two line bundles which associate to $\widetilde{U(1)}_1$ and $\widetilde{U(1)}_2$, respectively and write $\widetilde{L}_i = \mathcal{O}_S(\widetilde{D}_i),\ i = 1, 2$. Let $(\mathbf{A}, \mathbf{B})_{c,d}$ and $(\mathbf{A}, \mathbf{B})_{\tilde{c},\tilde{d}}$ be representations in the breaking patten $SU(6) \to SU(3) \times SU(2) \times U(1)_1 \times U(1)_2$ and $SU(6) \to SU(3) \times SU(2) \times \widetilde{U(1)}_1 \times \widetilde{U(1)}_2$, respectively. Up to a linear combination of $U(1)$ charges, we have $N_{(\mathbf{A},\mathbf{B})_{c,d}} = N_{(\mathbf{A},\mathbf{B})_{\tilde{c},\tilde{d}}}$, which requires that the corresponding divisors be transferred as follows:

$$\widetilde{\mathbb{D}} = (\mathrm{M}^{-1})^t \mathbb{D}, \tag{4.5}$$

where $\mathbb{D} = (D_1, D_2)^t$, $\widetilde{\mathbb{D}} = (\widetilde{D}_1, \widetilde{D}_2)^t$. In general, $\widetilde{D}_i$ are $\mathbb{Q}$-divisors via the rotation (4.5). However, it is possible to get integral divisors $\widetilde{D}_i$ by a suitable choice of the matrix $\mathrm{M} = \mathrm{M}_*$. Once this is done, we obtain two corresponding line bundles, $\widetilde{L}_1$ and $\widetilde{L}_2$ since $\widetilde{D}_i \in H_2(S, \mathbb{Z}),\ i = 1, 2$. Moreover, if $\mu(\widetilde{L}_1) = \mu(\widetilde{L}_2) = 0$, we can construct the polystable bundle $V = \widetilde{L}_1 \oplus \widetilde{L}_2$. Note that when L_i are supersymmetric, which means that they satisfy the BPS condition (4.3), by the transformation (4.5) we have $\mu(\widetilde{L}_1) = \mu(\widetilde{L}_2) = 0$. As a result, each supersymmetric $U(1)^2$ gauge fluxes is associated with a polystable vector bundle of rank two if the suitable matrix M_* exists. To be concrete, let us consider the case of $G_S = SU(6)$. The breaking pattern via $G_{\mathrm{std}} \times U(1)$

is as follows:

$$\begin{aligned} SU(6) &\to SU(3)\times SU(2)\times U(1)_1\times U(1)_2 \\ \mathbf{35} &\to (\mathbf{8},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{3})_{0,0}+(\mathbf{3},\mathbf{2})_{-5,0}+(\bar{\mathbf{3}},\mathbf{2})_{5,0}+(\mathbf{1},\mathbf{1})_{0,0} \\ &\quad +(\mathbf{1},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{2})_{3,6}+(\mathbf{3},\mathbf{1})_{-2,6}+(\mathbf{1},\bar{\mathbf{2}})_{-3,-6}+(\bar{\mathbf{3}},\mathbf{1})_{2,-6}. \end{aligned} \tag{4.6}$$

Let L_1 and L_2 be the supersymmetric line bundles associated to $U(1)_1$ and $U(1)_2$, respectively. Note that $U(1)_1$ can be identified as $U(1)_Y$ in the MSSM. The exotic-free spectrum from the bulk requires that L_1 and L_2 are fractional line bundles. The details could be found in section 5.2. Now consider the rotation

$$\mathrm{M}=\begin{pmatrix} -\frac{1}{5} & \frac{1}{10} \\ 0 & \frac{1}{6} \end{pmatrix}. \tag{4.7}$$

Then we obtain

$$\begin{aligned} SU(6) &\to SU(3)\times SU(2)\times U(1)_1\times U(1)_2 \\ \mathbf{35} &\to (\mathbf{8},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{3})_{0,0}+(\mathbf{3},\mathbf{2})_{1,0}+(\bar{\mathbf{3}},\mathbf{2})_{-1,0}+(\mathbf{1},\mathbf{1})_{0,0} \\ &\quad +(\mathbf{1},\mathbf{1})_{0,0}+(\mathbf{1},\mathbf{2})_{0,1}+(\mathbf{3},\mathbf{1})_{1,1}+(\mathbf{1},\bar{\mathbf{2}})_{0,-1}+(\bar{\mathbf{3}},\mathbf{1})_{-1,-1} \end{aligned} \tag{4.8}$$

with $\widetilde{L}_1 = L_1^{-5}$ and $\widetilde{L}_2 = L_1^3\otimes L_2^6$. It is clear that $N_{(\mathbf{A},\mathbf{B})_{c,d}} = N_{(\mathbf{A},\mathbf{B})_{\bar{c},\bar{d}}}$ with respect to (4.6) and (4.8). It turns out that $\widetilde{L}_1$ and $\widetilde{L}_2$ are truly line bundles. Furthermore, one can show that BPS condition (4.3) for (L_1, L_2) is equivalent to the stability conditions of the polystable bundle $V = \widetilde{L}_1 \oplus \widetilde{L}_2$ by the transformation (4.5). In this case, we know that supersymmetric $U(1)^2$ gauge fluxes are associated with polystable bundles of rank two with the same number of zero modes charged under $U(1)^2$. With this correspondence, we can avoid talking about the gauge bundle defined by the direct sum of two fractional line bundles. In other words, a supersymmetric $U(1)^2$ gauge flux (L_1, L_2) is well-defined in the sense that it can be associated with a well-defined polystable bundle of rank two. Form now on, we shall simply use the phrase $U(1)^2$ gauge fluxes in stead of polystable bundle in the following sections.

5 $U(1)^2$ Gauge Fluxes

In this section we consider $U(1)^2$ gauge fluxes in local F-theory models, in particular we focus on the case of $G_S = SO(10)$ and $SU(6)$. With the gauge fluxes, G_S can be broken into $G_{\text{std}} \times U(1)$. For the case of $G_S = SO(10)$, there is a no-go theorem which states that there do not exist $U(1)^2$ gauge fluxes such that the spectrum is exotic-free. This result was first shown in [12]. We review the case in section 5.1 for completeness. For the case of $G_S = SU(6)$, with appropriate physical conditions, we shall show that there are finitely many supersymmetric $U(1)^2$ gauge fluxes with an exotic-free bulk spectrum and we obtain the explicit expression of these gauge fluxes as well. With these explicit flux configurations, we study doublet-triplet splitting and the spectrum of the MSSM. The details can be found in section 5.2 and 5.3.

5.1 $G_S = SO(10)$

5.1.1 $U(1)^2$ Gauge Flux Configurations

The maximal subgroups of $SO(10)$ which contain $G_{\rm std}$ and the consistent MSSM spectrum are as follows [12]:

$$SO(10) \supset SU(5) \times U(1) \supset G_{\rm std} \times U(1) \tag{5.1}$$

$$SO(10) \supset SU(2) \times SU(2) \times SU(4) \supset G_{\rm std} \times U(1) \tag{5.2}$$

For the latter, one of $SU(2)$ groups needs to be broken into $U(1) \times U(1)$ to get the consistent $U(1)_Y$ charge in the MSSM. It follows from the patterns (5.1) and (5.2) that up to linear combinations of the $U(1)$ charges in the breaking patterns, it is enough to analyze the case of $U(1)^2$ gauge fluxes which breaks $SO(10)$ via the sequence $SO(10) \to SU(5) \times U(1) \to G_{\rm std} \times U(1)$. The breaking pattern is as follows:

$$\begin{array}{rcl} SO(10) & \to & SU(3) \times SU(2) \times U(1)_1 \times U(1)_2 \\ \mathbf{45} & \to & (\mathbf{8},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{3})_{0,0} + (\mathbf{3},\mathbf{2})_{-5,0} + (\overline{\mathbf{3}},\mathbf{2})_{5,0} + (\mathbf{1},\mathbf{1})_{0,0} \\ & & +(\mathbf{1},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{1})_{6,4} + (\overline{\mathbf{3}},\mathbf{1})_{-4,4} + (\mathbf{3},\mathbf{2})_{1,4} + (\mathbf{1},\mathbf{1})_{-6,-4} \\ & & +(\mathbf{3},\mathbf{1})_{4,-4} + (\overline{\mathbf{3}},\mathbf{2})_{-1,-4}. \end{array} \tag{5.3}$$

Note that $U(1)_1$ can be identified with $U(1)_Y$ in the MSSM. Let $\widetilde{L}_3$ and $\widetilde{L}_4$ be non-trivial supersymmetric line bundles associated with $U(1)_1$ and $U(1)_2$, respectively, in the breaking pattern (5.3). The bulk zero modes are given by

$$(\mathbf{3},\mathbf{2})_{-5,0} \in H^0_{\bar\partial}(S, \widetilde{L}_3^5)^\vee \oplus H^1_{\bar\partial}(S, \widetilde{L}_3^{-5}) \oplus H^2_{\bar\partial}(S, \widetilde{L}_3^5)^\vee \tag{5.4}$$

$$(\overline{\mathbf{3}},\mathbf{2})_{5,0} \in H^0_{\bar\partial}(S, \widetilde{L}_3^{-5})^\vee \oplus H^1_{\bar\partial}(S, \widetilde{L}_3^5) \oplus H^2_{\bar\partial}(S, \widetilde{L}_3^{-5})^\vee \tag{5.5}$$

$$(\mathbf{3},\mathbf{2})_{1,4} \in H^0_{\bar\partial}(S, \widetilde{L}_3^{-1} \otimes \widetilde{L}_4^{-4})^\vee \oplus H^1_{\bar\partial}(S, \widetilde{L}_3^{1} \otimes \widetilde{L}_4^{4}) \oplus H^2_{\bar\partial}(S, \widetilde{L}_3^{-1} \otimes \widetilde{L}_4^{-4})^\vee \tag{5.6}$$

$$(\overline{\mathbf{3}},\mathbf{2})_{-1,-4} \in H^0_{\bar\partial}(S, \widetilde{L}_3^{1} \otimes \widetilde{L}_4^{4})^\vee \oplus H^1_{\bar\partial}(S, \widetilde{L}_3^{-1} \otimes \widetilde{L}_4^{-4}) \oplus H^2_{\bar\partial}(S, \widetilde{L}_3^{1} \otimes \widetilde{L}_4^{4})^\vee \tag{5.7}$$

$$(\mathbf{3},\mathbf{1})_{4,-4} \in H^0_{\bar\partial}(S, \widetilde{L}_3^{-4} \otimes \widetilde{L}_4^{4})^\vee \oplus H^1_{\bar\partial}(S, \widetilde{L}_3^{4} \otimes \widetilde{L}_4^{-4}) \oplus H^2_{\bar\partial}(S, \widetilde{L}_3^{-4} \otimes \widetilde{L}_4^{4})^\vee \tag{5.8}$$

$$(\overline{\mathbf{3}},\mathbf{1})_{-4,4} \in H^0_{\bar\partial}(S, \widetilde{L}_3^{4} \otimes \widetilde{L}_4^{-4})^\vee \oplus H^1_{\bar\partial}(S, \widetilde{L}_3^{-4} \otimes \widetilde{L}_4^{4}) \oplus H^2_{\bar\partial}(S, \widetilde{L}_3^{4} \otimes \widetilde{L}_4^{-4})^\vee, \tag{5.9}$$

$$(\mathbf{1},\mathbf{1})_{6,4} \in H^0_{\bar\partial}(S, \widetilde{L}_3^{-6} \otimes \widetilde{L}_4^{-4})^\vee \oplus H^1_{\bar\partial}(S, \widetilde{L}_3^{6} \otimes \widetilde{L}_4^{4}) \oplus H^2_{\bar\partial}(S, \widetilde{L}_3^{-6} \otimes \widetilde{L}_4^{-4})^\vee \tag{5.10}$$

$$(\mathbf{1},\mathbf{1})_{-6,-4} \in H^0_{\bar\partial}(S, \widetilde{L}_3^{6} \otimes \widetilde{L}_4^{4})^\vee \oplus H^1_{\bar\partial}(S, \widetilde{L}_3^{-6} \otimes \widetilde{L}_4^{-4}) \oplus H^2_{\bar\partial}(S, \widetilde{L}_3^{6} \otimes \widetilde{L}_4^{4})^\vee. \tag{5.11}$$

To avoid exotics, it is clear that the line bundles $\widetilde{L}_3^5$, $\widetilde{L}_3^1 \otimes \widetilde{L}_4^4$, $\widetilde{L}_3^4 \otimes \widetilde{L}_4^{-4}$, and $\widetilde{L}_3^6 \otimes \widetilde{L}_4^4$ cannot be trivial. Let $N_{(\mathbf{A},\mathbf{B})_{a,b}}$ be the number of the fields in the representation $(\mathbf{A},\mathbf{B})_{a,b}$ under $SU(3) \times SU(2) \times U(1)_1 \times U(1)_2$, where a and b are the charges of $U(1)_1$ and $U(1)_2$, respectively. By the vanishing theorem (2.17), the exotic-free spectrum requires that

$$N_{(\mathbf{3},\mathbf{2})_{-5,0}} = -\chi(S,E) = 0 \tag{5.12}$$

$$N_{(\bar{\mathbf{3}},\mathbf{2})_{5,0}} = -\chi(S, E^{-1}) = 0 \tag{5.13}$$

$$N_{(\bar{\mathbf{3}},\mathbf{2})_{-1,-4}} = -\chi(S, F^{-1}) = 0 \tag{5.14}$$

$$N_{(\mathbf{3},\mathbf{1})_{4,-4}} = -\chi(S, E^{-1} \otimes F^{-1}) = 0 \tag{5.15}$$

$$N_{(\mathbf{1},\mathbf{1})_{-6,-4}} = -\chi(S, E \otimes F^{-1}) = 0. \tag{5.16}$$

We define

$$N_{(\mathbf{3},\mathbf{2})_{1,4}} = -\chi(S, F) \equiv \beta_1, \tag{5.17}$$

$$N_{(\bar{\mathbf{3}},\mathbf{1})_{-4,4}} = -\chi(S, E \otimes F) \equiv \beta_2 \tag{5.18}$$

$$N_{(\mathbf{1},\mathbf{1})_{6,4}} = -\chi(S, E^{-1} \otimes F) \equiv \beta_3, \tag{5.19}$$

where $E = \widetilde{L}_3^{-5}$, $F = \widetilde{L}_3^1 \otimes \widetilde{L}_4^4$ and $\beta_i \in \mathbb{Z}_{\geqslant 0},\ i = 1, 2, 3$. By Eqs. (5.12)-(5.14), and Eq. (5.17), we obtain the following equations

$$\begin{cases} c_1(E)^2 = -2 \\ c_1(F)^2 = -\beta_1 - 2 \\ c_1(E) \cdot K_S = 0 \\ c_1(F) \cdot K_S = \beta_1. \end{cases} \tag{5.20}$$

Then by Eq. (5.20) and Eq. (5.15), we obtain

$$c_1(E) \cdot c_1(F) = 1. \tag{5.21}$$

On the other hand, using Eq. (5.20) and Eq. (5.16), we have

$$c_1(E) \cdot c_1(F) = -1, \tag{5.22}$$

which leads to a contradiction. Therefore, there do not exist solutions for given $\beta_i \in \mathbb{Z}_{\geqslant 0},\ i = 1, 2, 3$ such that Eqs. (5.12)-(5.19) hold. This is a no-go theorem shown in [12]. Due to this no-go theorem, we are not going to study this case further. In the next section we turn to the case of $G_S = SU(6)$.

5.2 $G_S = SU(6)$

5.2.1 $U(1)^2$ Gauge Flux Configurations

The maximal subgroups of $SU(6)$ which contain G_{std} and the consistent MSSM spectrum are as follows [12]:

$$SU(6) \supset SU(5) \times U(1) \supset G_{\text{std}} \times U(1) \tag{5.23}$$

$$SU(6) \supset SU(2) \times SU(4) \times U(1) \supset G_{\text{std}} \times U(1) \tag{5.24}$$

$$SU(6) \supset SU(3) \times SU(3) \times U(1) \supset G_{\text{std}} \times U(1). \tag{5.25}$$

It follows from Eqs. (5.23)-(5.25) that up to linear combinations of the $U(1)$ charges in the breaking patterns, it is enough to analyze the case of $U(1)^2$ gauge fluxes which break $SU(6)$ via the sequence $SU(6) \to SU(5) \times U(1) \to G_{\rm std} \times U(1)$. The breaking pattern is as follows:

$$\begin{array}{rcl} SU(6) & \to & SU(3) \times SU(2) \times U(1)_1 \times U(1)_2 \\ \mathbf{35} & \to & (\mathbf{8},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{3})_{0,0} + (\mathbf{3},\mathbf{2})_{-5,0} + (\overline{\mathbf{3}},\mathbf{2})_{5,0} + (\mathbf{1},\mathbf{1})_{0,0} \\ & & +(\mathbf{1},\mathbf{1})_{0,0} + (\mathbf{1},\mathbf{2})_{3,6} + (\mathbf{3},\mathbf{1})_{-2,6} + (\mathbf{1},\overline{\mathbf{2}})_{-3,-6} + (\overline{\mathbf{3}},\mathbf{1})_{2,-6}. \end{array} \tag{5.26}$$

Note that $U(1)_1$ is consistent with $U(1)_Y$ in the MSSM. Let L_1 and L_2 be non-trivial supersymmetric line bundles associated with $U(1)_1$ and $U(1)_2$, respectively, in the breaking pattern (5.26). The bulk zero modes are given by

$$(\mathbf{3},\mathbf{2})_{-5,0} \in H^0_{\bar\partial}(S, L_1^5)^\vee \oplus H^1_{\bar\partial}(S, L_1^{-5}) \oplus H^2_{\bar\partial}(S, L_1^5)^\vee \tag{5.27}$$

$$(\overline{\mathbf{3}},\mathbf{2})_{5,0} \in H^0_{\bar\partial}(S, L_1^{-5})^\vee \oplus H^1_{\bar\partial}(S, L_1^5) \oplus H^2_{\bar\partial}(S, L_1^{-5})^\vee \tag{5.28}$$

$$(\mathbf{1},\mathbf{2})_{3,6} \in H^0_{\bar\partial}(S, L_1^{-3} \otimes L_2^{-6})^\vee \oplus H^1_{\bar\partial}(S, L_1^3 \otimes L_2^6) \oplus H^2_{\bar\partial}(S, L_1^{-3} \otimes L_2^{-6})^\vee \tag{5.29}$$

$$(\mathbf{1},\overline{\mathbf{2}})_{-3,-6} \in H^0_{\bar\partial}(S, L_1^3 \otimes L_2^6)^\vee \oplus H^1_{\bar\partial}(S, L_1^{-3} \otimes L_2^{-6}) \oplus H^2_{\bar\partial}(S, L_1^3 \otimes L_2^6)^\vee \tag{5.30}$$

$$(\mathbf{3},\mathbf{1})_{-2,6} \in H^0_{\bar\partial}(S, L_1^2 \otimes L_2^{-6})^\vee \oplus H^1_{\bar\partial}(S, L_1^{-2} \otimes L_2^6) \oplus H^2_{\bar\partial}(S, L_1^2 \otimes L_2^{-6})^\vee \tag{5.31}$$

$$(\overline{\mathbf{3}},\mathbf{1})_{2,-6} \in H^0_{\bar\partial}(S, L_1^{-2} \otimes L_2^6)^\vee \oplus H^1_{\bar\partial}(S, L_1^2 \otimes L_2^{-6}) \oplus H^2_{\bar\partial}(S, L_1^{-2} \otimes L_2^6)^\vee. \tag{5.32}$$

Note that $(\mathbf{3},\mathbf{2})_{-5,0}$, $(\overline{\mathbf{3}},\mathbf{2})_{5,0}$, and $(\mathbf{3},\mathbf{1})_{-2,6}$ are exotic fields in the MSSM. To avoid these exotics, L_1^5 and $L_1^{-2} \otimes L_2^6$ need to be non-trivial line bundles. If $L_1^3 \otimes L_2^6$ is trivial, it follows from Eq. (5.29) and Eq. (5.30) that $N_{(\mathbf{1},\mathbf{2})_{3,6}} = N_{(\mathbf{1},\overline{\mathbf{2}})_{-3,-6}} = 1$. By the vanishing theorem (2.17), no exotic fields requires that

$$N_{(\mathbf{3},\mathbf{2})_{-5,0}} = -\chi(S, L_1^{-5}) = 0 \tag{5.33}$$

$$N_{(\overline{\mathbf{3}},\mathbf{2})_{5,0}} = -\chi(S, L_1^{5}) = 0 \tag{5.34}$$

$$N_{(\mathbf{3},\mathbf{1})_{-2,6}} = -\chi(S, L_1^{-2} \otimes L_2^{6}) = 0. \tag{5.35}$$

We define

$$N_{(\overline{\mathbf{3}},\mathbf{1})_{2,-6}} = -\chi(S, L_1^{2} \otimes L_2^{-6}) \equiv \alpha_3, \tag{5.36}$$

where $\alpha_3 \in \mathbb{Z}_{\geqslant 0}$. Note that since $L_1^3 \otimes L_2^6$ is trivial, then $L_1^2 \otimes L_2^{-6} \cong L_1^5$. It follows from Eq. (5.34) that $\alpha_3 = 0$[9]. Therefore, the non-trivial conditions are (5.33) and (5.34), namely $\chi(S, L_1^{\pm 5}) = 0$, which imply that $c_1({L_1}^{\pm 5})^2 = -2$ and $c_1(L_1^{\pm 5}) \cdot K_S = 0$. Note that $c_1({L_1}^{\pm 5}) \in H_2(S, \mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{H, E_i,\ i = 1, 2, 3, ...8\}$, where H and E_i are the hyperplane divisor and exceptional divisors in $S = dP_8$. Immediately we get a fractional line bundle[10] $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ and then $L_2 = \mathcal{O}_S(E_i - E_j)^{1/10}$. It is clear that L_1 and L_2 satisfy the BPS condition (4.3). As a result, (L_1, L_2) is a

[9]This case will be denoted by $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)^*$ later.

[10]Note that with $\alpha_3 = 0$, there is a symmetry $(L_1, L_2) \leftrightarrow (L_1^{-1}, L_2^{-1})$ in Eq. (5.33)-(5.36). Without loss of generality, we choose $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$.

supersymmetric $U(1)^2$ gauge flux configuration on the bulk. If $L_1^3 \otimes L_2^6$ is non-trivial, by the vanishing theorem (2.17), an exotic-free bulk spectrum requires that

$$N_{(\mathbf{3},\mathbf{2})_{-5,0}} = -\chi(S, L_1^{-5}) = 0 \tag{5.37}$$

$$N_{(\bar{\mathbf{3}},\mathbf{2})_{5,0}} = -\chi(S, L_1^{5}) = 0 \tag{5.38}$$

$$N_{(\mathbf{3},\mathbf{1})_{-2,6}} = -\chi(S, L_1^{-2} \otimes L_2^{6}) = 0. \tag{5.39}$$

We define

$$N_{(\mathbf{1},\mathbf{2})_{3,6}} = -\chi(S, L_1^{3} \otimes L_2^{6}) \equiv \alpha_1 \tag{5.40}$$

$$N_{(\mathbf{1},\bar{\mathbf{2}})_{-3,-6}} = -\chi(S, L_1^{-3} \otimes L_2^{-6}) \equiv \alpha_2 \tag{5.41}$$

$$N_{(\bar{\mathbf{3}},\mathbf{1})_{2,-6}} = -\chi(S, L_1^{2} \otimes L_2^{-6}) \equiv \alpha_3, \tag{5.42}$$

where $\alpha_i \in \mathbb{Z}_{\geqslant 0},\ i = 1, 2, 3$. To simplify the notation, we define $C = L_1^{-5}$, and $D = L_1^3 \otimes L_2^6$. By Eqs. (5.37)-(5.42) and the Riemann-Roch theorem (2.16), we obtain the following equations:

$$\begin{cases} c_1(C)^2 = -2 \\ c_1(D)^2 = -\alpha_1 - \alpha_2 - 2 \\ c_1(C) \cdot c_1(D) = 1 + \frac{1}{2}(\alpha_1 + \alpha_2 - \alpha_3) \\ \alpha_3 = \alpha_2 - \alpha_1 \\ c_1(C) \cdot K_S = 0 \\ c_1(D) \cdot K_S = \alpha_1 - \alpha_2. \end{cases} \tag{5.43}$$

Note that C and D are required to be honest line bundles, in other words, $c_1(C)$, $c_1(D) \in H_2(S, \mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{H, E_i,\ i = 1, 2, 3, ...8\}$. Note that $(\bar{\mathbf{3}}, \mathbf{1})_{2,-6}$ is a candidate for a matter field in the MSSM. Therefore, we shall restrict to the case of $\alpha_3 \leqslant 3$. In what follows, we shall demonstrate how to derive explicit expressions for $U(1)^2$ gauge fluxes from Eq. (5.43). For the case of $\alpha_3 = 0$, by the constraints in Eq. (5.43), we may assume $(\alpha_1, \alpha_2, \alpha_3) = (k, k, 0)$ with $k \in \mathbb{Z}_{\geqslant 0}$. We shall show that there is no solution for $k \geqslant 4$. Note that in this case, Eq. (5.43) reduces to

$$c_1(C)^2 = -2,\ \ c_1(D)^2 = -2k - 2,\ \ c_1(C) \cdot c_1(D) = 1 + k, \tag{5.44}$$

with $c_1(C) \cdot K_S = c_1(D) \cdot K_S = 0$. From the conditions $c_1(C)^2 = -2$, $c_1(C) \cdot K_S = 0$, and BPS condition (4.3), it follows that $C = \mathcal{O}_S(E_i - E_j)$, which is the universal line bundle in the case of $G_S = SU(6)$ since these two conditions are independent of $\alpha_i,\ i = 1, 2, 3$ and always appear in Eq. (5.43). Actually, the corresponding fractional line bundle L_1 of C is the $U(1)_Y$ hypercharge flux in the minimal $SU(5)$ GUT [11–13]. In what follows, we shall focus on the solutions for the line bundle D. By Eq. (5.44), we can obtain the upper bound of k. Write $D = \mathcal{O}_S(c_i E_i + c_j E_j + \tilde{D})$,[11] where $\tilde{D}$ is a integral divisor containing no H, E_i, and E_j. Note that the repeat indices are not a summation, and $c_i,\ c_j \in \mathbb{Z}$. By Eq. (5.44), we get $-c_i + c_j = k + 1$ and

[11] Due to the BPS condition (4.3), D contains no component H.

$c_1^2 + c_2^2 - \tilde{D}^2 = 2k + 2$. Note that $\tilde{D}^2 \leqslant 0$ by the construction. Using the inequality[12] $c_1^2 + c_2^2 \geqslant \frac{1}{2}(c_1 - c_2)^2$ and the condition $k \in \mathbb{Z}_{\geqslant 0}$, we obtain $0 \leqslant k \leqslant 3$, which implies that there is no solution D for $k \geqslant 4$. Next we shall explicitly solve the configurations (L_1, L_2) satisfying Eq. (5.43) for the case of $(\alpha_1, \alpha_2, \alpha_3) = (k, k, 0)$ with $0 \leqslant k \leqslant 3$.

Let us start with the simplification of Eq. (5.43). Note that in Eq. (5.43), there are two conditions that are independent of α_i, namely,

$$c_1(C)^2 = -2, \;\; c_1(C) \cdot K_S = 0, \tag{5.45}$$

which gives rise to the universal line bundle, $C = \mathcal{O}_S(E_i - E_j)$, as mentioned earlier. The remaining conditions are

$$\begin{cases} c_1(D)^2 = -\alpha_1 - \alpha_2 - 2 \\ c_1(C) \cdot c_1(D) = 1 + \frac{1}{2}(\alpha_1 + \alpha_2 - \alpha_3) \\ \alpha_3 = \alpha_2 - \alpha_1 \\ c_1(D) \cdot K_S = \alpha_1 - \alpha_2. \end{cases} \tag{5.46}$$

Since C is universal, all we have to do is to solve the line bundles D in Eq. (5.46) for a given $(\alpha_1, \alpha_2, \alpha_3)$ and $C = \mathcal{O}_S(E_i - E_j)$. When $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$, Eq. (5.46) reduces to

$$c_1(D)^2 = -2, \;\; c_1(C) \cdot c_1(D) = 1, \tag{5.47}$$

with $c_1(D) \cdot K_S = 0$. By Eq. (5.47), we have $D = \mathcal{O}_S(\pm E_l - E_i)$ or $\mathcal{O}_S(\pm E_l + E_j)$. The former gives rise to fractional line bundles $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ and $L_2 = \mathcal{O}_S(\pm 5E_l - 2E_i - 3E_j)^{1/30}$. For the latter, we have $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ and $L_2 = \mathcal{O}_S(\pm 5E_l + 3E_i + 2E_j)^{1/30}$. Recall that $K_S = -3H + \sum_{k=1}^{8} E_k$. To solve the condition $c_1(D) \cdot K_S = 0$, it is clear that D has to be $\mathcal{O}_S(E_l - E_i)$ or $\mathcal{O}_S(-E_l + E_j)$. The corresponding fractional line bundle is $\mathcal{O}_S(5E_l - 2E_i - 3E_j)^{1/30}$ or $\mathcal{O}_S(-5E_l + 3E_i + 2E_j)^{1/30}$. In addition to Eq. (5.47), these fractional line bundles need to satisfy the BPS condition (4.3). More precisely, for the case of $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ and $L_2 = \mathcal{O}_S(5E_l - 2E_i - 3E_j)^{1/30}$, BPS equation (4.3) reduces to

$$(E_i - E_j) \cdot \omega = 0, \;\; (5E_l - 2E_i - 3E_j) \cdot \omega = 0. \tag{5.48}$$

It is not difficult to see that[13] $\omega = AH - (E_i + E_j + E_l + ...)$ solves Eq. (5.48). Similarly, for the case of $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ and $L_2 = \mathcal{O}_S(-5E_l + 3E_i + 2E_j)^{1/30}$, L_1 and L_2 are also supersymmetric with respect to $\omega = AH - (E_i + E_j + E_l + ...)$. As a result, for the case of $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$, we find two supersymmetric $U(1)^2$ gauge flux configurations (L_1, L_2).

When $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)$, Eq. (5.46) reduces to

$$c_1(D)^2 = -4, \;\; c_1(C) \cdot c_1(D) = 2, \tag{5.49}$$

[12] In general, $(c_1(C)^2)(c_1(D)^2) \geqslant (c_1(C) \cdot c_1(D))^2$.

[13] "..." in ω always stands for non-relevant terms for checking the BPS condition Eq. (4.3). Of course, those terms are relevant for the ampleness of ω and note that the choice of the polarizations is not unique.

with $c_1(D)\cdot K_S=0$. By Eq. (5.49), D can be $\mathcal{O}_S(2E_j)$, $\mathcal{O}_S(-2E_i)$ or $\mathcal{O}_S([E_l,E_m]-E_i+E_j)$, where the bracket is defined by $[A_1,A_2,..A_k]=\{\pm A_1\pm A_2...\pm A_k\}$. For later use, we also define $[A_1,A_2,..A_k]'=\{\pm A_1\pm A_2...\pm A_k\}\setminus(+A_1+A_2+...+A_k)$, $[A_1,A_2,..A_k]''=\{\pm A_1\pm A_2...\pm A_k\}\setminus\{(+A_1+A_2+...+A_k),(-A_1-A_2-...-A_k)\}$, and $[A_1,A_2,..A_k]'''=\{(A_1+A_2...+A_{k-1}-A_k),(A_1+A_2...-A_{k-1}+A_k),...,(-A_1+A_2...+A_{k-1}+A_k)\}$. Note that $\mathcal{O}_S(2E_j)$, $\mathcal{O}_S(-2E_i)$, $\mathcal{O}_S(E_l+E_m-E_i+E_j)$, and $\mathcal{O}_S(-E_l-E_m-E_i+E_j)$ cannot solve the equation $c_1(D)\cdot K_S=0$. As a result, $D=\mathcal{O}_S([E_l,E_m]''-E_i+E_j)$, which correspond to the fractional bundles $L_2=\mathcal{O}_S(5[E_l,E_m]''-2E_i+2E_j)^{1/30}$. Clearly L_1 and L_2 satisfy Eq. (4.3) with $\omega=AH-(E_i+E_j+E_l+E_m+...)$.

For the case of $(\alpha_1,\alpha_2,\alpha_3)=(2,2,0)$, Eq. (5.46) becomes

$$c_1(D)^2=-6,\ \ c_1(C)\cdot c_1(D)=3, \tag{5.50}$$

with $c_1(D)\cdot K_S=0$. By Eq. (5.50), D can be $\mathcal{O}_S([E_l]-E_i+2E_j)$ or $\mathcal{O}_S([E_l]-2E_i+E_j)$. For the former, it is clear that $\mathcal{O}_S(E_l-E_i+2E_j)$ does not satisfy the condition $c_1(D)\cdot K_S=0$. Similarly, for the latter, $\mathcal{O}_S(-E_l-2E_i+E_j)$ is not a solution as well. In this case, the solutions are $L_2=\mathcal{O}_S(-5E_l-2E_i+7E_j)^{1/30}$ or $L_2=\mathcal{O}_S(5E_l-7E_i+2E_j)^{1/30}$. It is easy to see that the solutions also satisfy the BPS condition (4.3). Note that for the case of $\alpha_3=0$, taking $\omega=AH-(\sum_{k=1}^8 E_k)=(-K_S)+(A-3)H$, the conditions $c_1(C)\cdot K_S=c_1(D)\cdot K_S=0$ are equivalent to Eq. (4.3). Therefore, the solutions of Eq. (5.43) are all supersymmetric for the case of $\alpha_3=0$.

Next we consider the case of $(\alpha_1,\alpha_2,\alpha_3)=(3,3,0)$. In this case, the line bundle D satisfies the following equations:

$$c_1(D)^2=-8,\ \ c_1(C)\cdot c_1(D)=4, \tag{5.51}$$

with $c_1(D)\cdot K_S=0$. By Eq. (5.51), we obtain $D=\mathcal{O}_S(2E_j-2E_i)$. The corresponding fractional line bundle is $L_2=\mathcal{O}_S(E_j-E_i)^{7/30}$. Obviously, L_2 satisfies the condition $c_1(D)\cdot K_S=0$, and Eq. (4.3) for $\omega=AH-(E_i+E_j+...)$.

Next we shall consider the case of $\alpha_3=1$. By the constraints of Eq. (5.46), we may assume that $(\alpha_1,\alpha_2,\alpha_3)=(m,m+1,1)$, where $m\in\mathbb{Z}_{\geqslant 0}$. Then Eq. (5.46) becomes

$$c_1(D)^2=-2m-3,\ \ c_1(C)\cdot c_1(D)=1+m, \tag{5.52}$$

with $c_1(D)\cdot K_S=-1$. Again the first thing we need to do is to get the upper bound of m. Eq. (5.52) implies that $1-\sqrt{6}\leqslant m\leqslant 1+\sqrt{6}$. Since $m\in\mathbb{Z}_{\geqslant 0}$, we obtain $0\leqslant m\leqslant 3$. Therefore, the possible configurations are $(\alpha_1,\alpha_2,\alpha_3)=(0,1,1),(1,2,1),(2,3,1)$ or $(3,4,1)$.

Let us look at the case of $(\alpha_1,\alpha_2,\alpha_3)=(0,1,1)$. In this case, Eq. (5.52) reduces to the following equations

$$c_1(D)^2=-3,\ \ c_1(C)\cdot c_1(D)=1. \tag{5.53}$$

It is easy to see that D can be $\mathcal{O}_S([E_l, E_m] - E_i)$ or $\mathcal{O}_S([E_l, E_m] + E_j)$. Note that $\mathcal{O}_S([E_l, E_m]'' - E_i)$, $\mathcal{O}_S(-E_l - E_m - E_i)$, $\mathcal{O}_S(E_l + E_m + E_j)$, and $\mathcal{O}_S(-E_l - E_m + E_j)$ do not satisfy the equation $c_1(D) \cdot K_S = -1$, so we have to eliminate these cases. It turns out that the resulting fractional line bundles are $\mathcal{O}_S(5(E_l+E_m)-2E_i-3E_j)^{1/30}$ and $\mathcal{O}_S(5[E_l, E_m]''+3E_i+2E_j)^{1/30}$. In order to preserve supersymmetry, the solutions need to solve Eq. (4.3). For the case of $L_2 = \mathcal{O}_S(5(E_l + E_m) - 2E_i - 3E_j)^{1/30}$, Eq. (4.3) reduces to

$$(E_i - E_j) \cdot \omega = 0, \quad [(E_l + E_m) - E_i] \cdot \omega = 0. \tag{5.54}$$

For another fractional line bundle $L_2 = \mathcal{O}_S(5[E_l, E_m]'' + 3E_i + 2E_j)^{1/30}$, Eq. (4.3) becomes

$$(E_i - E_j) \cdot \omega = 0, \quad ([E_l, E_m]'' + E_i) \cdot \omega = 0 \tag{5.55}$$

It is clear that $\omega = AH - (E_l + E_m + 2E_i + 2E_j + ...)$ solves Eq. (5.54) and $\omega = AH - (2E_l + E_m + E_i + E_j + ...)$ solves Eq. (5.55) if $[E_l, E_m]'' = -E_l + E_m$. For the case of $[E_l, E_m]'' = E_l - E_m$, $\omega = AH - (E_l + 2E_m + E_i + E_j + ...)$ is a solution of Eq. (5.55). Therefore, $\mathcal{O}_S(5(E_l+E_m)-2E_i-3E_j)^{1/30}$ and $\mathcal{O}_S(5[E_l, E_m]''+3E_i+2E_j)^{1/30}$ are supersymmetric. In this case, the solutions of Eq. (5.53) and the equations, $c_1(C) \cdot K_S = 0$, $c_1(D) \cdot K_S = -1$ satisfy Eq. (4.3). It seems that for the case $\alpha_3 = 1$, the condition $c_1(C) \cdot K_S = 0$, $c_1(D) \cdot K_S = -1$ is stronger than BPS condition (4.3). For example, $D = \mathcal{O}_S(E_l - E_m - E_i)$ with corresponding fractional line bundle $L_2 = \mathcal{O}_S(5E_l - 5E_m - 2E_i - 3E_j)^{1/30}$ is supersymmetric but does not satisfy the condition $c_1(D) \cdot K_S = -1$. Actually, we shall see that this is not the case in the next examples.

Let us turn to the case of $(\alpha_1, \alpha_2, \alpha_3) = (3, 4, 1)$. In this case, Eq. (5.52) reduces to

$$c_1(D)^2 = -9, \quad c_1(C) \cdot c_1(D) = 4. \tag{5.56}$$

It is not difficult to find that the solutions are $D = \mathcal{O}_S([E_l] - 2E_i + 2E_j)$ and the corresponding fractional line bundle are $L_2 = \mathcal{O}_S(5[E_l]-7E_i+7E_j)^{1/30}$. Note that only $D = \mathcal{O}_S(E_l - 2E_i + 2E_j)$ satisfies the condition $c_1(D) \cdot K_S = -1$. However, it is clear that it does not satisfy the BPS condition (4.3), which means that no configuration (L_1, L_2) for an exotic-free spectrum exists in this case. From this example, we know that for the case of $\alpha_3 = 1$, the solutions of Eq. (5.46) are not guaranteed to be supersymmetric and vice versa. Therefore, in general we need to check these two conditions for each solution in the case of $\alpha_3 \in \mathbb{Z}_{>0}$. Following a similar procedure, one can obtain all configurations (L_1, L_2) for the cases of $\alpha_3 = 1$. We summarize the results of $\alpha_3 = 0, 1$ in Table 7 in which all L_1 and L_2 satisfy the BPS condition (4.3) for suitable polarizations ω and the conditions $L_1^5 \neq \mathcal{O}_S$, $L_1^{-2} \otimes L_2^6 \neq \mathcal{O}_S$ and $L_1^3 \otimes L_1^6 \neq \mathcal{O}_S$.

Next we consider the case of $\alpha_3 = 2$. By the last constraint of Eq. (5.43), we may assume $(\alpha_1, \alpha_2, \alpha_3) = (l, l + 2, 2)$, where $l \in \mathbb{Z}_{\geqslant 0}$. One can show that the necessary condition for existence of the solutions of Eq. (5.43) is $0 \leqslant l \leqslant 3$. Therefore, $(\alpha_1, \alpha_2, \alpha_3)$ can be $(0, 2, 2)$, $(1, 3, 2)$, $(2, 4, 2)$ or $(3, 5, 2)$. Following the previous procedure, one can obtain all configurations (L_1, L_2) for the case of $\alpha_3 = 2$.

	$(\alpha_1, \alpha_2, \alpha_3)$	L_2
1	$(1,1,0)^*$	$\mathcal{O}_S(E_i - E_j)^{1/10}$
2	$(0,0,0)$	$\mathcal{O}_S(5E_l - 2E_i - 3E_j)^{1/30}$ $\mathcal{O}_S(-5E_l + 3E_i + 2E_j)^{1/30}$
3	$(1,1,0)$	$\mathcal{O}_S(5[E_l, E_m]'' - 2E_i + 2E_j)^{1/30}$
4	$(2,2,0)$	$\mathcal{O}_S(-5E_l - 2E_i + 7E_j)^{1/30}$ $\mathcal{O}_S(5E_l - 7E_i + 2E_j)^{1/30}$
5	$(3,3,0)$	$\mathcal{O}_S(E_j - E_i)^{7/30}$
6	$(0,1,1)$	$\mathcal{O}_S(5[E_l, E_m]'' + 3E_i + 2E_j)^{1/30}$ $\mathcal{O}_S(5(E_l + E_m) - 2E_i - 3E_j)^{1/30}$
7	$(1,2,1)$	$\mathcal{O}_S(-5E_l + 3E_i + 7E_j)^{1/30}$ $\mathcal{O}_S(5[E_l, E_m, E_k]''' - 2E_i + 2E_j)^{1/30}$
8	$(2,3,1)$	$\mathcal{O}_S(5[E_l, E_m]'' - 2E_i + 7E_j)^{1/30}$ $\mathcal{O}_S(5(E_l + E_m) - 7E_i + 2E_j)^{1/30}$
9	$(3,4,1)$	No Solution

Table 7: Flux configurations for $G_S = SU(6)$ with $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ and $\alpha_3 = 0, 1$.

For the case of $\alpha_3 = 3$, we may assume that $(\alpha_1, \alpha_2, \alpha_3) = (n, n+3, 3)$ with $n \in \mathbb{Z}_{\geqslant 0}$. The necessary condition for existence of the solutions of Eq. (5.43) is $0 \leqslant n \leqslant 4$, which implies that $(\alpha_1, \alpha_2, \alpha_3) = (0,3,3), (1,4,3), (2,5,3), (3,6,3)$, or $(4,7,3)$. Following the previous procedure, one can obtain all configurations (L_1, L_2) for the case of $\alpha_3 = 3$. Let us look at the case of $(\alpha_1, \alpha_2, \alpha_3) = (3,6,3)$. In this case, Eq. (5.46) reduces to

$$c_1(D)^2 = -11, \quad c_1(C) \cdot c_1(D) = 4, \tag{5.57}$$

with $c_1(D) \cdot K_S = -3$. It follows from Eq. (5.57) that D can be $\mathcal{O}_S([E_l] - E_i + 3E_j)$, $\mathcal{O}_S([E_l] - 3E_i + E_j)$, or $\mathcal{O}_S([E_l, E_m, E_n] - 2E_i + 2E_j)$. When one takes the condition $c_1(D) \cdot K_S = -3$ into account, there are only two solutions, $D = \mathcal{O}_S(E_l - E_i + 3E_j)$ or $\mathcal{O}_S((E_l + E_m + E_n) - 2E_i + 2E_j)$, which corresponds to the fractional line bundles $\mathcal{O}_S(5E_l - 2E_i + 12E_j)^{1/30}$ and $\mathcal{O}_S(5(E_l + E_m + E_n) - 7E_i + 7E_j)^{1/30}$, respectively. However, these two solutions cannot satisfy Eq. (4.3). Therefore, in this case there do not exist any $U(1)^2$ gauge fluxes for an exotic-free spectrum. A similar situation occurs in the case of $(\alpha_1, \alpha_2, \alpha_3) = (4,7,3)$. In this case, D can be $\mathcal{O}_S(-3E_i + 2E_j)$ or $\mathcal{O}_S(-2E_i + 3E_j)$ by Eq. (5.46). However, they neither solve Eq. (4.3) nor satisfy the condition $c_1(D) \cdot K_S = -3$. As a result, there are no $U(1)^2$ gauge fluxes without producing exotics in this case. We summarize the results of $\alpha_3 = 2, 3$ in Table 8 in which all L_1 and L_2 satisfy the BPS condition (4.3) for suitable polarizations ω and the conditions $L_1^5 \neq \mathcal{O}_S$, $L_1^{-2} \otimes L_2^6 \neq \mathcal{O}_S$ and $L_1^3 \otimes L_1^6 \neq \mathcal{O}_S$.

	$(\alpha_1, \alpha_2, \alpha_3)$	L_2
1	$(0, 2, 2)$	$\mathcal{O}_S(5(E_l + E_m + E_k) - 2E_i - 3E_j)^{1/30}$ $\mathcal{O}_S(5[E_l, E_m, E_k]''' + 3E_i + 2E_j)^{1/30}$
2	$(1, 3, 2)$	$\mathcal{O}_S(5[E_l, E_m]'' + 3E_i + 7E_j)^{1/30}$ $\mathcal{O}_S(5[E_l, E_m, E_n, E_k]''' - 2E_i + 2E_j)^{1/30}$
3	$(2, 4, 2)$	$\mathcal{O}_S(5[E_l, E_m, E_k]''' - 2E_i + 7E_j)^{1/30}$ $\mathcal{O}_S(5(E_l + E_m + E_k) - 7E_i + 2E_j)^{1/30}$
4	$(3, 5, 2)$	No Solution
5	$(0, 3, 3)$	$\mathcal{O}_S(5(E_l + E_m + E_n + E_k) - 2E_i - 3E_j)^{1/30}$ $\mathcal{O}_S(5[E_l, E_m, E_n, E_k]''' + 3E_i + 2E_j)^{1/30}$
6	$(1, 4, 3)$	$\mathcal{O}_S(5[E_l, E_m, E_k]''' + 3E_i + 7E_j)^{1/30}$ $\mathcal{O}_S(5[E_l, E_m, E_n, E_k, E_p]''' - 2E_i + 2E_j)^{1/30}$
7	$(2, 5, 3)$	$\mathcal{O}_S(5[E_l, E_m, E_n, E_k]''' - 2E_i + 7E_j)^{1/30}$ $\mathcal{O}_S(5(E_l + E_m + E_n + E_k) - 7E_i + 2E_j)^{1/30}$
8	$(3, 6, 3)$	No Solution
9	$(4, 7, 3)$	No Solution

Table 8: Flux configurations for $G_S = SU(6)$ with $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ and $\alpha_3 = 2, 3$.

5.2.2 Spectrum from the Curves

With $G_S = SU(6)$, to obtain matter in $SU(5)$ GUT, it is required that $L_\Sigma \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$. In this case, there are three kinds of intersecting curves, $\Sigma_{SU(7)}$, $\Sigma_{SO(12)}$ and Σ_{E_6} with enhanced gauge groups $SU(7)$, $SO(12)$, and E_6, respectively. The breaking patterns are as shown in Eqs. (3.21)-(3.23). To achieve doublet-triplet splitting and make contact with the spectrum in the MSSM, we consider $U(1)^2$ flux configurations (L_1, L_2) already solved in the previous section. In this section we shall study the spectrum from the curves and show that the doublet-triplet splitting and non-minimal spectrum of the MSSM can be achieved. A detailed example can be found in section 5.2.3.

In local F-theory models, the gauge group on the curve along which S intersects with S' will be enhanced at least by one rank. In the present case of $G_S = SU(6)$, the possible enhanced gauge groups are $SU(7)$, $SO(12)$ and E_6. The matter fields transform as fundamental representation **6**, anti-symmetric tensor representation of rank two **15**, and anti-symmetric tensor representation of rank three **20** in $SU(6)$ can be engineered to localize on the curves with gauge groups $SU(7)$, $SO(12)$, and E_6, respectively. In order to split doublet and triplet states in Higgs and obtain the spectrum of the MSSM, $L_{1\Sigma}$, $L_{2\Sigma}$ and L'_Σ have to be non-trivial, which breaks G_Σ

into $G_{\rm std}\times U(1)^2$. The breaking patterns of $SU(7)$, $SO(12)$ and E_6 are as follows:

$$\begin{array}{ccccl}
SU(7) & \to & SU(6)\times U(1)' & \to & SU(3)\times SU(2)\times U(1)'\times U(1)_1\times U(1)_2 \\
\mathbf{48} & \to & \mathbf{35}_0+\mathbf{1}_0 & \to & (\mathbf{8},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{3})_{0,0,0}+(\mathbf{3},\mathbf{2})_{0,-5,0}+(\bar{\mathbf{3}},\mathbf{2})_{0,5,0} \\
 & & +\mathbf{6}_{-7}+\bar{\mathbf{6}}_7 & & +(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{2})_{0,3,6}+(\mathbf{3},\mathbf{1})_{0,-2,6} \\
 & & & & +(\mathbf{1},\bar{\mathbf{2}})_{0,-3,-6}+(\bar{\mathbf{3}},\mathbf{1})_{0,2,-6}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{2})_{-7,3,1} \\
 & & & & +(\mathbf{3},\mathbf{1})_{-7,-2,1}+(\mathbf{1},\mathbf{1})_{-7,0,-5}+(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}+(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1} \\
 & & & & +(\mathbf{1},\mathbf{1})_{7,0,5}
\end{array} \tag{5.58}$$

$$\begin{array}{ccccl}
SO(12) & \to & SU(6)\times U(1)' & \to & SU(3)\times SU(2)\times U(1)'\times U(1)_1\times U(1)_2 \\
\mathbf{66} & \to & \mathbf{35}_0+\mathbf{1}_0 & \to & (\mathbf{8},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{3})_{0,0,0}+(\mathbf{3},\mathbf{2})_{0,-5,0}+(\bar{\mathbf{3}},\mathbf{2})_{0,5,0} \\
 & & +\mathbf{15}_2+\overline{\mathbf{15}}_{-2} & & +(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{2})_{0,3,6}+(\mathbf{3},\mathbf{1})_{0,-2,6} \\
 & & & & +(\mathbf{1},\bar{\mathbf{2}})_{0,-3,-6}+(\bar{\mathbf{3}},\mathbf{1})_{0,2,-6}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{2})_{2,3,-4} \\
 & & & & +(\mathbf{3},\mathbf{1})_{2,-2,-4}+(\mathbf{1},\mathbf{1})_{2,6,2}+(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}+(\mathbf{3},\mathbf{2})_{2,1,2} \\
 & & & & +(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}+(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}+(\mathbf{1},\mathbf{1})_{-2,-6,-2}+(\mathbf{3},\mathbf{1})_{-2,4,-2} \\
 & & & & +(\bar{\mathbf{3}},\bar{\mathbf{2}})_{-2,-1,-2}
\end{array} \tag{5.59}$$

$$\begin{array}{ccccl}
E_6 & \to & SU(6)\times U(1)' & \to & SU(3)\times SU(2)\times U(1)'\times U(1)_1\times U(1)_2 \\
\mathbf{78} & \to & \mathbf{35}_0+\mathbf{1}_0+\mathbf{1}_{\pm 2} & \to & (\mathbf{8},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{3})_{0,0,0}+(\mathbf{3},\mathbf{2})_{0,-5,0}+(\bar{\mathbf{3}},\mathbf{2})_{0,5,0} \\
 & & +\mathbf{20}_1+\mathbf{20}_{-1} & & +(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{2})_{0,3,6}+(\mathbf{3},\mathbf{1})_{0,-2,6} \\
 & & & & +(\mathbf{1},\bar{\mathbf{2}})_{0,-3,-6}+(\bar{\mathbf{3}},\mathbf{1})_{0,2,-6}+(\mathbf{1},\mathbf{1})_{0,0,0}+(\mathbf{1},\mathbf{1})_{\pm 2,0,0} \\
 & & & & +[(\mathbf{1},\mathbf{1})_{1,6,-3}+(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}+(\mathbf{3},\mathbf{2})_{1,1,-3}+c.c] \\
 & & & & +[(\mathbf{1},\mathbf{1})_{-1,6,-3}+(\bar{\mathbf{3}},\mathbf{1})_{-1,-4,-3}+(\mathbf{3},\mathbf{2})_{-1,1,-3}+c.c].
\end{array} \tag{5.60}$$

Due to non-trivial $U(1)^2$ flux configurations on the bulk S, the last two $U(1)$ charges of the fields on the curves should be conserved in each Yukawa coupling. From the breaking patterns, we list possible Yukawa couplings of type $\Sigma\Sigma S$ and $\Sigma\Sigma\Sigma$ in Table 9. According to Table 9, the possible field content is shown in Table 10. In what follows, we shall focus on the case of $\Sigma\Sigma\Sigma$-type couplings and find all possible field configurations supported by the curves $\Sigma_{SU(7)}$, $\Sigma_{SO(12)}$, and Σ_{E_6} with given $U(1)^2$ flux configuration (L_1, L_2).

Let us start with the case of $\Sigma_{SU(7)}$ and consider $(\alpha_1,\alpha_2,\alpha_3)=(k,k,0)$ with $k=0,1,2,3$. When $(\alpha_1,\alpha_2,\alpha_3)=(0,0,0)$, which is the second case in Table 7, it is clear that we have $L_2=\mathcal{O}_S(5E_l-2E_i-3E_j)^{1/30}$ or $L_2=\mathcal{O}_S(-5E_l+3E_i+2E_j)^{1/30}$. We define $(n_1,n_2,n_3)=(N_{(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}},N_{(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}},N_{(\mathbf{1},\mathbf{1})_{7,0,5}})$. To avoid exotic fields, we require that $n_1\in\mathbb{Z}_{\geqslant 0}$. Given field configurations (n_1,n_2,n_3) on the curve $\Sigma_{SU(7)}$, the necessary conditions[14] for the homological class of the curve $\Sigma_{SU(7)}$ are

$$\begin{cases} (E_i-E_j)\cdot\Sigma_{SU(7)}=n_2-n_1 \\ (E_i-E_l)\cdot\Sigma_{SU(7)}=n_2-n_3. \end{cases} \tag{5.61}$$

[14] $L_{1\Sigma_{SU(7)}}=\mathcal{O}_{\Sigma_{SU(7)}}(\frac{1}{5}(n_1-n_2))$, $L_{2\Sigma_{SU(7)}}=\mathcal{O}_{\Sigma_{SU(7)}}(\frac{1}{30}(-3n_1-2n_2+5n_3))$, and $L'_{\Sigma_{SU(7)}}=\mathcal{O}_{\Sigma_{SU(7)}}(\frac{1}{42}(3n_1+2n_2+n_3))$.

Coupling	Representation	Configuration
$Q_L u_R H_u$	$(\mathbf{3},\mathbf{2})_{2,1,2}(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}(\mathbf{1},\mathbf{2})_{-7,3,1}$	$\Sigma_{SO(12)}\Sigma_{E_6}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{2,1,2}(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}(\mathbf{1},\mathbf{2})_{2,3,-4}$	$\Sigma_{SO(12)}\Sigma_{SO(12)}\Sigma_{SO(12)}$
	$(\mathbf{3},\mathbf{2})_{1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}(\mathbf{1},\mathbf{2})_{-7,3,1}$	$\Sigma_{E_6}\Sigma_{SO(12)}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{-1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}(\mathbf{1},\mathbf{2})_{-7,3,1}$	$\Sigma_{E_6}\Sigma_{SO(12)}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}(\mathbf{1},\mathbf{2})_{0,3,6}$	$\Sigma_{E_6}\Sigma_{E_6}S$
	$(\mathbf{3},\mathbf{2})_{-1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}(\mathbf{1},\mathbf{2})_{0,3,6}$	$\Sigma_{E_6}\Sigma_{E_6}S$
$Q_L d_R H_d$	$(\mathbf{3},\mathbf{2})_{2,1,2}(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{SO(12)}\Sigma_{SU(7)}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{2,1,2}(\bar{\mathbf{3}},\mathbf{1})_{0,2,-6}(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$\Sigma_{SO(12)}S\Sigma_{SO(12)}$
	$(\mathbf{3},\mathbf{2})_{1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{E_6}\Sigma_{SO(12)}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{-1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{E_6}\Sigma_{SO(12)}\Sigma_{SU(7)}$
	$(\mathbf{3},\mathbf{2})_{1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$\Sigma_{E_6}\Sigma_{SU(7)}\Sigma_{SO(12)}$
	$(\mathbf{3},\mathbf{2})_{-1,1,-3}(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$\Sigma_{E_6}\Sigma_{SU(7)}\Sigma_{SO(12)}$
	$(\mathbf{3},\mathbf{2})_{2,1,2}(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}(\mathbf{1},\bar{\mathbf{2}})_{0,-3,-6}$	$\Sigma_{SO(12)}\Sigma_{SO(12)}S$
$L_L e_R H_d$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{2,6,2}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{SU(7)}\Sigma_{SO(12)}\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}(\mathbf{1},\mathbf{1})_{1,6,-3}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{SO(12)}\Sigma_{E_6}\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}(\mathbf{1},\mathbf{1})_{-1,6,-3}(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$\Sigma_{SO(12)}\Sigma_{E_6}\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{1,6,-3}(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$\Sigma_{SU(7)}\Sigma_{E_6}\Sigma_{SO(12)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{-1,6,-3}(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$\Sigma_{SU(7)}\Sigma_{E_6}\Sigma_{SO(12)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}(\mathbf{1},\mathbf{1})_{2,6,2}(\mathbf{1},\bar{\mathbf{2}})_{0,-3,-6}$	$\Sigma_{SO(12)}\Sigma_{SO(12)}S$
$L_L N_R H_u$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{0,0,0}(\mathbf{1},\mathbf{2})_{-7,3,1}$	$\Sigma_{SU(7)}S\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}(\mathbf{1},\mathbf{1})_{-7,0,-5}(\mathbf{1},\mathbf{2})_{-7,3,1}$	$\Sigma_{SO(12)}\Sigma_{SU(7)}\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{7,0,5}(\mathbf{1},\mathbf{2})_{2,3,-4}$	$\Sigma_{SU(7)}\Sigma_{SU(7)}\Sigma_{SO(12)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}(\mathbf{1},\mathbf{1})_{0,0,0}(\mathbf{1},\mathbf{2})_{2,3,-4}$	$\Sigma_{SU(7)}S\Sigma_{SU(7)}$
	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}(\mathbf{1},\mathbf{1})_{-7,0,-5}(\mathbf{1},\mathbf{2})_{0,3,6}$	$\Sigma_{SU(7)}\Sigma_{SU(7)}S$

Table 9: The Yukawa couplings of the MSSM model from $G_S = SU(6)$.

if $L_2 = \mathcal{O}_S(5E_l - 2E_i - 3E_j)^{1/30}$. For the case of $L_2 = \mathcal{O}_S(-5E_l + 3E_i + 2E_j)^{1/30}$, the conditions are as follows:

$$\begin{cases} (E_i - E_j)\cdot\Sigma_{SU(7)} = n_2 - n_1 \\ (E_i - E_l)\cdot\Sigma_{SU(7)} = n_3 - n_1. \end{cases} \tag{5.62}$$

Note that the first condition of Eq. (5.61) and Eq. (5.62) is universal since it comes from the restriction of the universal supersymmetric line bundle $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ to the curve $\Sigma_{SU(7)}$. Note that there are no further constraints for $n_i,\ i = 1,2,3$ except $n_1 \in \mathbb{Z}_{\geqslant 0}$, $n_1 \neq n_2$, $3n_1 + 2n_2 \neq 5n_3$ and $3n_1 + 2n_2 + n_3 \neq 0$. The last three constraints follow from the conditions $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$, and $L'_\Sigma \neq \mathcal{O}_\Sigma$. Let us look at an example. Consider the case of $(n_1, n_2, n_3) = (0, 1, 0)$, Eq. (5.61) and Eq. (5.62) can be easily solved by $\Sigma = H - E_i - E_m$ and $\Sigma = H - E_i - E_l$, respectively.

	Q_L	u_R	d_R	e_R	L_L	H_u	H_d
M_0	$(\mathbf{3},\mathbf{2})_{2,1,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{0,2,-6}$	$(\mathbf{1},\mathbf{1})_{1,6,-3}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$(\mathbf{1},\mathbf{2})_{-2,3,-4}$	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$
M_1	$(\mathbf{3},\mathbf{2})_{2,1,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}$	$(\mathbf{1},\mathbf{1})_{2,6,2}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$
M_2	$(\mathbf{3},\mathbf{2})_{1,1,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$	$(\mathbf{1},\mathbf{1})_{2,6,2}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$
M_3	$(\mathbf{3},\mathbf{2})_{-1,1,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$	$(\mathbf{1},\mathbf{1})_{2,6,2}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$
M_4	$(\mathbf{3},\mathbf{2})_{2,1,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}$	$(\mathbf{1},\mathbf{1})_{-1,6,-3}$	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$
M_5	$(\mathbf{3},\mathbf{2})_{1,1,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$	$(\mathbf{1},\mathbf{1})_{-1,6,-3}$	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$
M_6	$(\mathbf{3},\mathbf{2})_{-1,1,-3}$	$(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$	$(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$	$(\mathbf{1},\mathbf{1})_{-1,6,-3}$	$(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$	$(\mathbf{1},\mathbf{2})_{-7,3,1}$	$(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$

Table 10: Field content in the MSSM from $G_S = SU(6)$

In this case, double and triplet states in the Higgs field $\bar{\mathbf{5}}_{7,-1}$ can be split without producing exotic fields. Let us look at one more case, $(\alpha_1, \alpha_2, \alpha_3) = (3, 3, 0)$. It follows from Table 7 that $L_2 = \mathcal{O}_S(E_j - E_i)^{7/30}$. The conditions for the homological class of the curve $\Sigma_{SU(7)}$ to support the field configurations (n_1, n_2, n_3) are

$$\begin{cases} (E_i - E_j) \cdot \Sigma_{SU(7)} = n_2 - n_1 \\ 2n_1 = n_2 + n_3. \end{cases} \tag{5.63}$$

This time we get one more constraint, $2n_1 = n_2 + n_3$. It follows that when $(\bar{\mathbf{3}}, \mathbf{1})_{7,2,-1}$ vanishes, the doublets always show up together with singlets. For the cases of $(\alpha_1, \alpha_2, \alpha_3) = (k, k, 0)$ with $k = 1, 2$, we summarize the results[15]in Table 11

$(\alpha_1, \alpha_2, \alpha_3)$	Conditions	L_2
$(0, 0, 0)$	$(E_i - E_l) \cdot \Sigma_{SU(7)} = n_2 - n_3$	$\mathcal{O}_S(5E_l - 2E_i - 3E_j)^{1/30}$
	$(E_i - E_l) \cdot \Sigma_{SU(7)} = n_3 - n_1$	$\mathcal{O}_S(-5E_l + 3E_i + 2E_j)^{1/30}$
$(1, 1, 0)^*$	$n_2 = n_3$	$\mathcal{O}_S(E_i - E_j)^{1/10}$
$(1, 1, 0)$	$([E_l, E_m]'') \cdot \Sigma_{SU(7)} = n_3 - n_1$	$\mathcal{O}_S(5[E_l, E_m]'' - 2E_i + 2E_j)^{1/30}$
$(2, 2, 0)$	$(-E_l + E_j) \cdot \Sigma_{SU(7)} = n_3 - n_1$	$\mathcal{O}_S(-5E_l - 2E_i + 7E_j)^{1/30}$
	$(E_l - E_i) \cdot \Sigma_{SU(7)} = n_3 - n_1$	$\mathcal{O}_S(5E_l - 7E_i + 2E_j)^{1/30}$
$(3, 3, 0)$	$2n_1 = n_2 + n_3$	$\mathcal{O}_S(E_j - E_i)^{7/30}$

Table 11: The conditions for $\Sigma_{SU(7)}$ supporting the field configurations (n_1, n_2, n_3) with $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$.

Similarly, we can extend the calculation to the curve $\Sigma_{SO(12)}$. Let us define $(s_1, s_2, s_3, s_4, s_5) = (N_{(\mathbf{3},\mathbf{2})_{2,1,2}}, N_{(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}}, N_{(\mathbf{3},\mathbf{1})_{2,-2,-4}}, N_{(\mathbf{1},\mathbf{2})_{2,3,-4}}, N_{(\mathbf{1},\mathbf{1})_{2,6,2}})$ and consider the case of $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)$, which is the third case in Table 7. It is

[15]For simplicity, we are not going to show the universal conditions $(E_i - E_j) \cdot \Sigma = w_2 - w_1,\ w \in \{n, s\}$ for $\Sigma_{SU(7)}$ and $\Sigma_{SO(12)}$, respectively and $(E_i - E_j) \cdot \Sigma = p_3 - p_1$ for Σ_{E_6} in Table 11, 12, and 13.

clear that we have $L_2 = \mathcal{O}_S(5[E_l, E_m]'' - 2E_i + 2E_j)^{1/30}$. The necessary conditions[16] for the homological class of the curve $\Sigma_{SO(12)}$ with field configurations $(s_1, s_2, s_3, s_4, s_5)$ are

$$\begin{cases} (E_i - E_j) \cdot \Sigma_{SO(12)} = s_2 - s_1 \\ ([E_l, E_m]'') \cdot \Sigma_{SO(12)} = s_2 - s_3, \end{cases} \tag{5.64}$$

and

$$\begin{cases} s_4 = s_3 + s_1 - s_2 \\ s_5 = 2s_1 - s_2. \end{cases} \tag{5.65}$$

Note that Eq. (5.65) impose severe restrictions on the configurations $(s_1, s_2, s_3, s_4, s_5)$. For example, one cannot simply set $(s_1, s_2, s_3, s_4, s_5) = (0, 0, 0, m, 0)$ to achieve the doublet-triplet splitting of Higgs $\mathbf{5}_{2,-4}$; it is easy to see that m is forced to be zero by the constraints in Eq. (5.65). This will cause trouble when we attempt to engineer the Higgs on the curve $\Sigma_{SO(12)}$ with doublet-triplet splitting. Consider the case of $s_4 > 0$ and set $s_1 = 0$. From the constraints in Eq. (5.65), we obtain $s_2 + (-s_3) < 0$. Note that to avoid exotic fields from $\Sigma_{SO(12)}$, it is required that $s_1,\ s_2 \in \mathbb{Z}_{\geqslant 0}$ and $s_3 \in \mathbb{Z}_{\leqslant 0}$. It follows that $0 \leqslant s_2 + (-s_3) < 0$, which leads to a contradiction. As a result, the appearance of $(\mathbf{3}, \mathbf{2})_{2,1,2}$ cannot be avoided on the curve $\Sigma_{SO(12)}$ as $N_{(\mathbf{1},\mathbf{2})_{2,3,-4}} = s_4 > 0$. If $s_4 > 0$, actually the most general non-trivial configurations are $(s_1, s_2, s_3, s_4, s_5) = (l, l+n-m, n, m, l+m-n)$, where $m,\ l \in \mathbb{Z}_{>0}$ and $m - l \leqslant n \leqslant 0$. Note that $(\mathbf{3}, \mathbf{2})_{2,1,2}$ is treated as matter in the MSSM, which requires that[17] $l \leqslant 3$. It follows that $1 \leqslant m \leqslant 3$ and $m \leqslant l \leqslant 3$. It turns out that there are finitely many non-trivial configurations. More precisely, the field configurations are as follows:

$$(s_1, s_2, s_3, s_4, s_5) = \left\{ \begin{array}{l} (1,0,0,1,2), (2,1,0,1,3), (2,0,-1,1,4), \\ (3,2,0,1,4), (3,1,-1,1,5), (3,0,-2,1,6), \\ (2,0,0,2,4), (3,1,0,2,5), (3,0,-1,2,6), \\ (3,0,0,3,6) \end{array} \right\}. \tag{5.66}$$

If $-3 \leqslant s_4 \leqslant 0$, with $0 \leqslant s_1,\ s_2 \leqslant 3$ and $-3 \leqslant s_3 \leqslant 0$, we have another branch of the configurations as follows:

$$(s_1, s_2, s_3, s_4, s_5) = \left\{ \begin{array}{l} (0,1,-1,-2,-1), (0,1,-2,-3,-1), (0,2,-1,-3,-2), \\ (1,0,-1,0,2), (1,0,-3,-2,2), (1,2,0,-1,0), \\ (1,2,-1,-2,0), (1,3,0,-2,-1), (1,3,-1,-3,-1), \\ (2,0,-2,0,4), (2,0,-3,-1,4), (2,1,-2,-1,3), \\ (2,3,0,-1,1), (2,1,-3,-2,3), (2,3,-1,-2,1), \\ (2,1,-1,0,3), (2,3,-2,-3,1), (3,0,-3,0,6), \\ (3,1,-2,0,5), (3,1,-3,-1,5), (3,2,-1,0,4), \\ (3,2,-2,-1,4), (3,2,-3,-2,4) \end{array} \right\}, \tag{5.67}$$

[16] $L_{1\Sigma_{SO(12)}} = \mathcal{O}_{\Sigma_{SO(12)}}(\frac{1}{5}(s_1 - s_2))$, $L_{2\Sigma_{SO(12)}} = \mathcal{O}_{\Sigma_{SO(12)}}(\frac{1}{30}(2s_1 + 3s_2 - 5s_3))$, and $L'_{\Sigma_{SO(12)}} = \mathcal{O}_{\Sigma_{SO(12)}}(\frac{1}{6}(2s_1 + s_3))$.

[17] We allow the cases in which three copies of matter fields can be distributed over different matter curves.

where all configurations[18] in (5.66) and (5.67) satisfy the conditions $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$, and $L'_\Sigma \neq \mathcal{O}_\Sigma$. With these configurations, one can solve the conditions for the intersection numbers, namely, the conditions in Eq. (5.64). Let us consider the case of $(s_1, s_2, s_3, s_4, s_5) = (1, 0, 0, 1, 2)$, it is clear that $\Sigma = 2H - E_l - E_m - E_j$ is a solution. For a more complicated case, for example $(s_1, s_2, s_3, s_4, s_5) = (3, 1, -1, 1, 5)$, the conditions can be solved by $\Sigma = 4H + E_p - 2E_j - 2E_l$ if $[E_l, E_m]'' = E_l - E_m$ and by $\Sigma = 4H + E_p - 2E_j - 2E_m$ if $[E_l, E_m]'' = E_m - E_l$.

Let us turn to another case. Consider the first case in Table 7, namely $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)^*$. The supersymmetric fractional line bundle L_2 is $\mathcal{O}_S(E_i - E_j)^{1/10}$. The necessary conditions are

$$\begin{cases} (E_i - E_j) \cdot \Sigma_{SO(12)} = s_2 - s_1 \\ s_1 = s_3, \end{cases} \tag{5.68}$$

and Eq. (5.65). Note that $(\bar{\mathbf{3}}, \mathbf{2})_{-2,-1,-2}$ and $(\mathbf{3}, \mathbf{1})_{2,-2,-4}$ are exotic fields in the MSSM. The constraint, $s_1 = s_3$ in Eq. (5.68) and Eq. (5.65) imply that $s_1 = s_3 = 0$. If $s_4 \geqslant 0$, by the constraints in Eq. (5.65), we obtain $(s_1, s_2, s_3, s_4, s_5) = (0, 0, 0, 0, 0)$. If $s_4 < 0$, we have general configurations $(s_1, s_2, s_3, s_4, s_5) = (0, n, 0, -n, -n)$, where $1 \leqslant n \leqslant 3$. However, these configurations violate the condition $L'_\Sigma \neq \mathcal{O}_\Sigma$. As a check, using the configurations in (5.66), (5.67), and taking the condition $s_1 = s_3$ into account, one can see that there are no solutions in this case.

Next we consider the fifth case in Table 7, namely $(\alpha_1, \alpha_2, \alpha_3) = (3, 3, 0)$. In this case, L_2 is $\mathcal{O}_S(E_j - E_i)^{7/30}$. The necessary conditions are

$$\begin{cases} (E_i - E_j) \cdot \Sigma_{SO(12)} = s_2 - s_1 \\ 2s_2 = s_1 + s_3, \end{cases} \tag{5.69}$$

and Eq. (5.65). It is easy to see that $s_2 = s_4$. If $s_2 = 0$, we obtain the non-trivial configurations $(s_1, s_2, s_3, s_4, s_5) = (k, 0, -k, 0, 2k)$, where $1 \leqslant k \leqslant 3$. Note that these configurations satisfy the conditions, $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$, and $L'_\Sigma \neq \mathcal{O}_\Sigma$. Let us turn to the case of $s_2 = m \in \mathbb{Z}_{>0}$. The general configurations are $(s_1, s_2, s_3, s_4, s_5) = (l, m, 2m - l, m, 2l - m)$ with $l \geqslant 2m > 0$. Note that $(\mathbf{3}, \mathbf{2})_{2,1,2}$ is treated as matter in the MSSM. As a result, we focus on the case of $l \leqslant 3$, which implies that $m = 1$ and $l = 2, 3$. It turns out that the allowed configurations are $(s_1, s_2, s_3, s_4, s_5) = \{(2, 1, 0, 1, 3), (3, 1, -1, 1, 5)\}$, where the configurations satisfy the conditions $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$, and $L'_\Sigma \neq \mathcal{O}_\Sigma$. Putting these two branches together, we obtain

$$(s_1, s_2, s_3, s_4, s_5) = \left\{ \begin{array}{l} (1, 0, -1, 0, 2), (2, 0 - 2, 0, 4), (3, 0, -3, 0, 6), \\ (2, 1, 0, 1, 3), (3, 1, -1, 1, 5) \end{array} \right\}. \tag{5.70}$$

As a check, from the field configurations in (5.66), (5.67) and the constraint $2s_2 = s_1 + s_3$, one can find that there are exactly five solutions as shown in (5.70).

[18] $s_3 < 0$ represents $N_{(\mathbf{3},\mathbf{1})_{2,-2,-4}} = 0$ and $N_{(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}} = -s_3$. The same rule can be applied to other s_i.

Let us take a look at some solutions for the curve satisfying Eq. (5.69). For the the case of $(s_1, s_2, s_3, s_4, s_5) = (2, 1, 0, 1, 3)$, it is easy to see that $\Sigma = H - E_j - E_s$ solves the first equation in Eq. (5.69). For the case of $(s_1, s_2, s_3, s_4, s_5) = (2, 0, -2, 0, 4)$, $\Sigma = 3H - 2E_j - E_p$ can be a solution. From these examples, we expect that if we choose $\Sigma_{SO(12)}$ to house Higgs fields, it will be difficult to achieve doublet-triple splitting without introducing extra chiral fields. For other $U(1)^2$ flux configurations corresponding to the case of $(\alpha_1, \alpha_2, \alpha_3) = (k, k, 0)$ with $k = 0, 2$, the analysis is similar to the case of $k = 1$. We summarize the results in Table 12.

$(\alpha_1, \alpha_2, \alpha_3)$	Conditions	L_2
$(0, 0, 0)$	$(E_i - E_l) \cdot \Sigma_{SO(12)} = s_3 - s_1$	$\mathcal{O}_S(5E_l - 2E_i - 3E_j)^{1/30}$
	$(E_i - E_l) \cdot \Sigma_{SO(12)} = s_2 - s_3$	$\mathcal{O}_S(-5E_l + 3E_i + 2E_j)^{1/30}$
$(1, 1, 0)^*$	$s_1 = s_3$	$\mathcal{O}_S(E_i - E_j)^{1/10}$
$(1, 1, 0)$	$([E_l, E_m]'') \cdot \Sigma_{SO(12)} = s_2 - s_3$	$\mathcal{O}_S(5[E_l, E_m]'' - 2E_i + 2E_j)^{1/30}$
$(2, 2, 0)$	$(-E_l + E_j) \cdot \Sigma_{SO(12)} = s_2 - s_3$	$\mathcal{O}_S(-5E_l - 2E_i + 7E_j)^{1/30}$
	$(E_l - E_i) \cdot \Sigma_{SO(12)} = s_2 - s_3$	$\mathcal{O}_S(5E_l - 7E_i + 2E_j)^{1/30}$
$(3, 3, 0)$	$2s_2 = s_1 + s_3$	$\mathcal{O}_S(E_j - E_i)^{7/30}$

Table 12: The conditions for $\Sigma_{SO(12)}$ supporting the field configurations $(s_1, s_2, s_3, s_4, s_5)$ with $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ and constraints $2s_1 = s_2 + s_5$, $s_4 = s_3 + s_1 - s_2$.

In addition to doublet-triplet splitting problem, we also would like to study the matter spectrum. According to Table 10, the matter fields can come from the curves $\Sigma_{SU(7)}$, $\Sigma_{SO(12)}$, and Σ_{E_6}. The configurations of the fields and the conditions of the intersection numbers on the curves $\Sigma_{SU(7)}$ and $\Sigma_{SO(12)}$ have been studied earlier in this section. Next we are going to analyze the case of Σ_{E_6}. Note that for the case of M_0 in Table 10, to engineer $3 \times d_R$ on the bulk, it is required to set $\alpha_3 = 3$. However, it gives rise to exotic fields $(\mathbf{1}, \mathbf{2})_{3,6}$ and $(\mathbf{1}, \bar{\mathbf{2}})_{-3,-6}$ on the bulk. In what follows, we are going to focus on the case of $(\alpha_1, \alpha_2, \alpha_3) = (k, k, 0)$ on the bulk.

Let us start with the case of $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$. It is clear that $L_2 = \mathcal{O}_S(5E_l - 2E_i - 3E_j)^{1/30}$ or $L_2 = \mathcal{O}_S(-5E_l + 3E_i + 2E_j)^{1/30}$. We define $(p_1, p_2, p_3, p_4, p_5, p_6) = (N_{(\mathbf{3},\mathbf{2})_{1,1,-3}}, N_{(\mathbf{3},\mathbf{2})_{-1,1,-3}}, N_{(\bar{\mathbf{3}},\mathbf{1})_{1,-4,-3}}, N_{(\bar{\mathbf{3}},\mathbf{1})_{-1,-4,-3}}, N_{(\mathbf{1},\mathbf{1})_{1,6,-3}}, N_{(\mathbf{1},\mathbf{1})_{-1,6,-3}})$. The necessary conditions[19] for the curve Σ_{E_6} are as follows:

$$\begin{cases} (E_i - E_j) \cdot \Sigma_{E_6} = p_3 - p_1 \\ (E_i - E_l) \cdot \Sigma_{E_6} = p_2 + p_3, \end{cases} \tag{5.71}$$

and

$$\begin{cases} p_4 = p_2 + p_3 - p_1 \\ p_5 = 2p_1 - p_3 \\ p_6 = p_1 + p_2 - p_3, \end{cases} \tag{5.72}$$

[19] $L_{1\Sigma_{E_6}} = \mathcal{O}_{\Sigma_{E_6}}(\frac{1}{5}(p_1 - p_3))$, $L_{2\Sigma_{E_6}} = \mathcal{O}_{\Sigma_{E_6}}(-\frac{1}{30}(3p_1 + 5p_2 + 2p_3))$, and $L'_{\Sigma_{E_6}} = \mathcal{O}_{\Sigma_{E_6}}(\frac{1}{2}(p_1 - p_2))$.

if $L_2 = \mathcal{O}_S(5E_l - 2E_i - 3E_j)^{1/30}$. For the case of $L_2 = \mathcal{O}_S(-5E_l + 3E_i + 2E_j)^{1/30}$, the conditions are

$$\begin{cases} (E_i - E_j) \cdot \Sigma_{E_6} = p_3 - p_1 \\ (E_i - E_l) \cdot \Sigma_{E_6} = -p_1 - p_2, \end{cases} \tag{5.73}$$

and Eq. (5.72), where $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ has been used. Note that the first condition in Eq. (5.71) and Eq. (5.72) are universal since they come from the restriction of the universal supersymmetric line bundle $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ to the curve Σ_{E_6} and from the consistency of the definition of $(p_1, p_2, p_3, p_4, p_5, p_6)$, respectively and that Eq. (5.72) impose severe restrictions on the configurations $(p_1, p_2, p_3, p_4, p_5, p_6)$. For example, one can simply set $(p_1, p_2, p_3, p_4, p_5, p_6) = (n, 0, 0, 0, 0, 0)$ to engineer n copies of $(\mathbf{3}, \mathbf{2})_{1,1,-3}$ on the curve Σ_{E_6}. Then by constraints in Eq. (5.72), n is forced to be vanishing in order to avoid the exotic fields. Let us look at some examples of the non-trivial configurations. It is easy to see that if $p_1 = p_3 = 0$, we obtain non-trivial configurations $(p_1, p_2, p_3, p_4, p_5, p_6) = (0, l, 0, l, 0, l)$, where $l \in \mathbb{Z}_{>0}$. When $p_2 = p_4 = 0$, the non-trivial configurations are $(p_1, p_2, p_3, p_4, p_5, p_6) = (m, 0, m, 0, m, 0)$ with $m \in \mathbb{Z}_{>0}$. If $p_3 = p_4 = 0$, it follows that $(p_1, p_2, p_3, p_4, p_5, p_6) = (n, n, 0, 0, 2n, 2n)$, where $n \in \mathbb{Z}_{>0}$. However, these configurations violate the conditions $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$. Therefore, we need to find more general non-trivial configurations. For the matter fields in the MSSM, we require that the number of the matter field is equal to or less than three. As a result, we impose the conditions $1 \leqslant p_i \leqslant 3,\ i = 1, 2, 3, 4$ in this case. By the constraints in Eq. (5.72), we obtain the following configurations

$$(p_1, p_2, p_3, p_4, p_5, p_6) = \left\{ \begin{array}{l} (0, r, 1-r, 1, r-1, 2r-1), (1, r, 1-r, 0, r+1, 2r), \\ (0, q, 2-q, 2, q-2, 2q-2), (1, q, 2-q, 1, q, 2q-1), \\ (2, q, 2-q, 0, q+2, 2q), (0, v, 3-v, 3, v-3, 2v-3), \\ (1, v, 3-v, 2, v-1, 2v-2), (2, v, 3-v, 1, v+1, 2v-1), \\ (3, v, 3-v, 0, v+3, 2v), (1, t, 4-t, 3, t-2, 2t-3), \\ (2, t, 4-t, 2, t, 2t-2), (3, t, 4-t, 1, t+2, 2t-1), \\ (2, u, 5-u, 3, u-1, 2u-3), (3, u, 5-u, 2, u+1, 2u-2), \\ (3, 3, 3, 3, 3, 3) \end{array} \right\}, \tag{5.74}$$

where $r = 0, 1$, $q = 0, 1, 2$, $v = 0, 1, 2, 3$, $t = 1, 2, 3$, and $u = 2, 3$. Taking the conditions of $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$ into account, the resulting configurations are as follows:

$$(p_1, p_2, p_3, p_4, p_5, p_6) = \left\{ \begin{array}{l} (0, 1, 1, 2, -1, 0), (1, 0, 2, 1, 0, -1), (1, 2, 0, 1, 2, 3), \\ (2, 1, 1, 0, 3, 2), (0, 1, 2, 3, -2, -1), (0, 2, 1, 3, -1, 1), \\ (1, 3, 0, 2, 2, 4), (1, 0, 3, 2, -1, -2), (2, 0, 3, 1, 1, -1), \\ (2, 3, 0, 1, 4, 5), (3, 1, 2, 0, 4, 2), (3, 2, 1, 0, 5, 4), \\ (1, 2, 2, 3, 0, 1), (2, 1, 3, 2, 1, 0), (2, 3, 1, 2, 3, 4), \\ (3, 2, 2, 1, 4, 3) \end{array} \right\}. \tag{5.75}$$

Once we get allowed configurations, it is not difficult to calculate the homological classes of the curves, which satisfy Eq. (5.71) or Eq. (5.73). For example, consider the case of $(p_1, p_2, p_3, p_4, p_5, p_6) = (0, 1, 1, 2, -1, 0)$, one can check that

$\Sigma = 3H - E_i + E_l$ solves Eq. (5.71). Let us look at one more complicated example, $(p_1, p_2, p_3, p_4, p_5, p_6) = (3, 2, 2, 1, 4, 3)$. In this case, $\Sigma = 6H + 3E_i + 2E_j - 2E_l$ is a solution of Eq. (5.73). Next we consider the case of $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)$. It is clear that we have $L_2 = \mathcal{O}_S(5[E_l, E_m]'' - 2E_i + 2E_j)^{1/30}$. The necessary conditions are

$$\begin{cases} (E_i - E_j) \cdot \Sigma_{E_6} = p_3 - p_1 \\ ([E_l, E_m]'') \cdot \Sigma_{E_6} = -p_1 - p_2, \end{cases} \tag{5.76}$$

and Eq. (5.72). Note that the constraints are the same as the previous case, $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$. As a result, the allowed configurations are the same as (5.75). Let us take a look at the classes of the curves, which solve Eq. (5.76). For simplicity, we focus on the case of $[E_l, E_m]'' = E_l - E_m$ and consider $(p_1, p_2, p_3, p_4, p_5, p_6) = (1, 0, 2, 1, 0, -1)$, it is not difficult to see that $\Sigma = H - E_i - E_m$ is a solution. For the case of $(p_1, p_2, p_3, p_4, p_5, p_6) = (2, 1, 1, 0, 3, 2)$, $\Sigma = 4H + 2E_l - E_j - E_m$ can solve Eq. (5.76).

Let us turn to the first case in Table 7, namely $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 0)^*$. In this case, L_2 is $\mathcal{O}_S(E_i - E_j)^{1/10}$ and the necessary conditions for the homological class of Σ_{E_6} with given configurations $(p_1, p_2, p_3, p_4, p_5, p_6)$ are

$$\begin{cases} (E_i - E_j) \cdot \Sigma_{E_6} = p_3 - p_1 \\ p_2 + p_3 = 0, \end{cases} \tag{5.77}$$

and Eq. (5.72). Note that to avoid exotic fields, we require that $p_1,\ p_2,\ p_3,\ p_4 \in \mathbb{Z}_{\geqslant 0}$. The constraint, $p_2 + p_3 = 0$ in Eq. (5.77) implies that $p_2 = p_3 = 0$. By the constraints in Eq. (5.72), we obtain $(p_1, p_2, p_3, p_4, p_5, p_6) = (0, 0, 0, 0, 0, 0)$, which means that there are no non-trivial configurations in this case. As a check, by the configurations in (5.75) and the constraint $p_2 + p_3 = 0$, it is easy to see that there is indeed no solution, namely all configurations in (5.75) are completely ruled out by the constraint $p_2 + p_3 = 0$.

For the case of $(\alpha_1, \alpha_2, \alpha_3) = (3, 3, 0)$, we have $L_2 = \mathcal{O}_S(E_j - E_i)^{7/30}$. Given the configuration $(p_1, p_2, p_3, p_4, p_5, p_6)$, the necessary conditions are

$$\begin{cases} (E_i - E_j) \cdot \Sigma_{E_6} = p_3 - p_1 \\ p_3 = 2p_1 + p_2, \end{cases} \tag{5.78}$$

and Eq. (5.72). Since $(\mathbf{3}, \mathbf{2})_{1,1,-3}$, $(\mathbf{3}, \mathbf{2})_{-1,1,-3}$, $(\bar{\mathbf{3}}, \mathbf{1})_{1,-4,-3}$, and $(\bar{\mathbf{3}}, \mathbf{1})_{1,-4,-3}$ are all matter in the MSSM, we require that $p_i \leqslant 3,\ i = 1, 2, 3, 4$. By the second condition in Eq. (5.78), we have $(p_1, p_2) = (1, 0),\ (0, 1),\ (0, 2),\ (0, 3)$, or $(1, 1)$. Since $p_4 \leqslant 3$, it follows that the allowed configurations are $(p_1, p_2, p_3, p_4, p_5, p_6) = (0, 1, 1, 2, -1, 0)$, $(1, 0, 2, 1, 0, -1)$, and $(1, 1, 3, 3, -1, -1)$. Recall that in order to obtain matter in the MSSM, it is required that $L_{1\Sigma} \neq \mathcal{O}_\Sigma$, $L_{2\Sigma} \neq \mathcal{O}_\Sigma$ and $L'_\Sigma \neq \mathcal{O}_\Sigma$. As a result, the resulting configurations are

$$(p_1, p_2, p_3, p_4, p_5, p_6) = \left\{\ (0, 1, 1, 2, -1, 0),\ (1, 0, 2, 1, 0, -1)\ \right\}. \tag{5.79}$$

As a check, using the configurations in (5.75) and the constraint $p_3 = 2p_1 + p_2$, one can see that the resulting configurations are the same as that in (5.79). Now let us solve the classes of the curves satisfying Eq. (5.78). For these two configurations, the first condition in Eq. (5.78) can be solved by $\Sigma = H - E_i - E_l$. For other $U(1)^2$ flux configurations corresponding to the case of $(\alpha_1, \alpha_2, \alpha_3) = (k, k, 0)$ with $k = 2$, the analysis is similar to the case of $k = 0, 1$. We summarize the results in Table 13.

$(\alpha_1, \alpha_2, \alpha_3)$	Conditions	L_2
$(0,0,0)$	$(E_i - E_l)\cdot\Sigma = p_2 + p_3$	$\mathcal{O}_S(5E_l - 2E_i - 3E_j)^{1/30}$
	$(E_i - E_l)\cdot\Sigma = -p_1 - p_2$	$\mathcal{O}_S(-5E_l + 3E_i + 2E_j)^{1/30}$
$(1,1,0)^*$	$p_2 + p_3 = 0$	$\mathcal{O}_S(E_i - E_j)^{1/10}$
$(1,1,0)$	$([E_l, E_m]'')\cdot\Sigma = -p_1 - p_2$	$\mathcal{O}_S(5[E_l, E_m]'' - 2E_i + 2E_j)^{1/30}$
$(2,2,0)$	$(-E_l + E_j)\cdot\Sigma = -p_1 - p_2$	$\mathcal{O}_S(-5E_l - 2E_i + 7E_j)^{1/30}$
	$(E_l - E_i)\cdot\Sigma = -p_1 - p_2$	$\mathcal{O}_S(5E_l - 7E_i + 2E_j)^{1/30}$
$(3,3,0)$	$p_3 = 2p_1 + p_2$	$\mathcal{O}_S(E_j - E_i)^{7/30}$

Table 13: The conditions for Σ_{E_6} supporting the field configurations $(p_1, p_2, p_3, p_4, p_5, p_6)$ with $L_1 = \mathcal{O}_S(E_j - E_i)^{1/5}$ and constraints $p_4 = p_2 + p_3 - p_1$, $p_5 = 2p_1 - p_3$, and $p_6 = p_1 + p_2 - p_3$.

After analyzing the spectrum from the curves, it is clear that we are unable to obtain a minimal spectrum of the MSSM, but non-minimal spectra with doublet-triplet splitting can be obtained. In the next section we will give examples of non-minimal spectra for the MSSM.

5.3 Non-minimal Spectrum for the MSSM: Examples

In the previous section we already analyzed the spectrum from the curves $\Sigma_{SU(7)}$, $\Sigma_{SO(12)}$, and Σ_{E_6}. With some physical requirements, we obtain all field configurations supported by the curves. In what follows, we shall give examples of the non-minimal MSSM spectra using the results shown in section 5.2.2.

In what follows, we shall focus on the case M_1 in Table 13. In this case, Q_L and e_R are localized on the curves with $G_\Sigma = SO(12)$. u_R comes from Σ_{E_6} and d_R, L_L, H_u and H_d live on $\Sigma_{SU(7)}$. It is not difficult to see that in the examples considered, we are unable to get a minimal spectrum of the MSSM without exotic fields. However, it is possible to construct non-minimal spectra of the MSSM. One possible way is that we can make the exotic fields form trilinear couplings with conserved $U(1)$ charges so that they can decouple from the low-energy spectrum. According to the results in Table 7, let us consider the $U(1)^2$ flux configuration $L_1 = \mathcal{O}_S(E_1 - E_2)^{1/5}$ and $L_2 = \mathcal{O}_S(5E_3 - 2E_2 - 3E_1)^{1/30}$, which corresponds to the case of $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$

on the bulk. To obtain three copies of Q_L and e_R, we engineer two curves $\Sigma^1_{SO(12)}$ and $\Sigma^2_{SO(12)}$ with field content $(2,0,-2,0,4)$ and $(1,0,-1,0,2)$, respectively. The exotic fields are $2\times(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$ and one singlet on $\Sigma^1_{SO(12)}$. For the curve $\Sigma^2_{SO(12)}$, we get exotic fields $1\times(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$ and two singlets. To get three copies of u_R, we arrange two curves, $\Sigma^1_{E_6}$ and $\Sigma^2_{E_6}$ with field content $(3,1,2,0,4,2)$ and $(2,1,1,0,3,2)$, respectively. We have exotic fields $3\times(\mathbf{3},\mathbf{2})_{1,1,-3}$, $1\times(\mathbf{3},\mathbf{2})_{-1,1,-3}$ and six singlets on $\Sigma^1_{E_6}$. On $\Sigma^2_{E_6}$, the exotic fields are $2\times(\mathbf{3},\mathbf{2})_{1,1,-3}$, $1\times(\mathbf{3},\mathbf{2})_{-1,1,-3}$ and five singlets. Since the rest of the fields in the case of M_1 come from the curves with $G_\Sigma = SU(7)$, we can easily engineer $3\times d_R$, $3\times L_L$, $1\times H_u$ and $1\times H_d$ on individual curves, denoted respectively by $\Sigma^1_{SU(7)}$, $\Sigma^2_{SU(7)}$, $\Sigma^u_{SU(7)}$, and $\Sigma^d_{SU(7)}$. Note that $(\mathbf{3},\mathbf{2})_{\pm1,1,-3}$, $(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$, and $(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$ can form trilinear couplings. To make the exotic fields form the couplings, we introduce one extra curve $\Sigma^\Phi_{SU(7)}$ with $\Phi=(\mathbf{1},\bar{\mathbf{2}})_{7,-3,-1}$. Now we arrange $\Sigma^1_{SO(12)}$ intersects $\Sigma^1_{E_6}$ and $\Sigma^2_{E_6}$, so does $\Sigma^2_{SO(12)}$. The curve $\Sigma^u_{SU(7)}$ passes through the intersection point of $\Sigma^1_{SO(12)}$ and $\Sigma^1_{E_6}$ and that of $\Sigma^2_{SO(12)}$ and $\Sigma^2_{E_6}$. The vertices of the triple intersections $(\Sigma^1_{SO(12)},\Sigma^1_{E_6},\Sigma^u_{SU(7)})$ and $(\Sigma^2_{SO(12)},\Sigma^2_{E_6},\Sigma^u_{SU(7)})$ represent the coupling $Q_L u_R H_u$. Another two vertices are formed by triple intersections $(\Sigma^1_{SO(12)},\Sigma^2_{E_6},\Sigma^\Phi_{SU(7)})$ and $(\Sigma^2_{SO(12)},\Sigma^1_{E_6},\Sigma^\Phi_{SU(7)})$, which represent the coupling $\Theta\Psi\Phi$ and $\widetilde{\Theta}\Psi\Phi$, where $\Theta=(\mathbf{3},\mathbf{2})_{1,1,-3}$, $\widetilde{\Theta}=(\mathbf{3},\mathbf{2})_{-1,1,-3}$, and $\Psi=(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$. When Φ gets a vev, the exotic fields are decoupled through the coupling, which means that at low energy, those fields will not show up in the spectrum. To obtain the coupling $Q_L d_R H_d$, one can arrange two curves $\Sigma^1_{SU(7)}$, and $\Sigma^d_{SU(7)}$ intersect $\Sigma^1_{SO(12)}$ at one point. For the coupling $L_L e_R H_d$, one can let the curve $\Sigma^2_{SU(7)}$ intersect $\Sigma^d_{SU(7)}$ at another point on $\Sigma^1_{SO(12)}$. The intersection point of $\Sigma^u_{SU(7)}$ and $\Sigma^2_{SU(7)}$ represents the coupling $L_L N_R H_u$. To sum up, the superpotential is as follows:

$$\mathcal{W} \supset \mathcal{W}_{\rm MSSM} + \Theta\Psi\Phi + \widetilde{\Theta}\Psi\Phi + \cdots . \tag{5.80}$$

As mentioned earlier, through the last two couplings in (5.80), we obtain a non-minimal MSSM spectrum at low energy. Note that in this case, H_u and H_d come from the curves $\Sigma^u_{SU(7)}$ and $\Sigma^d_{SU(7)}$, respectively. As shown in section 5.2.2, doublet-triplet splitting can be achieved by $U(1)^2$ gauge fluxes. Therefore, a non-minimal spectrum of the MSSM with doublet-triple splitting can be achieved in a local F-theory model where $G_S = SU(6)$ and with $U(1)^2$ gauge fluxes. As shown in section 5.2.2, given the field configurations, one can calculate the homological classes of the curves supporting the configurations. For the present example, we simply summarize the field content and the classes of the curves in Table 14. Note that in the previous example there are some exotic singlets. Following similar procedure, these singlets can be lifted via trilinear couplings. Let us consider the following example. To obtain three copies of Q_L and e_R, we engineer two curves $\widetilde{\Sigma}^1_{SO(12)}$ and $\widetilde{\Sigma}^2_{SO(12)}$ with field

‡With one additional singlet.

§With two additional singlets.

¶With six additional singlets.

‖With five additional singlets.

Multiplet	Curve	Σ	g_Σ	$L_{1\Sigma}$	$L_{2\Sigma}$	L'_Σ
$2\times Q_L$ $+3\times e_R{}^{\ddagger}$	$\Sigma^1_{SO(12)}$	$5H+2E_2-2E_3$ $-2E_4-2E_5$	0	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)^{2/5}$	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)^{7/15}$	$\mathcal{O}_{\Sigma^1_{SO(12)}}(1)^{1/3}$
$1\times Q_L{}^{\S}$	$\Sigma^2_{SO(12)}$	$4H+E_2-E_3$ $-2E_4-2E_5$	0	$\mathcal{O}_{\Sigma^2_{SO(12)}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^2_{SO(12)}}(1)^{7/30}$	$\mathcal{O}_{\Sigma^2_{SO(12)}}(1)^{1/6}$
$2\times u_R{}^{\P}$	$\Sigma^1_{E_6}$	$5H+3E_3-E_1$	0	$\mathcal{O}_{\Sigma^1_{E_6}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^1_{E_6}}(-1)^{3/5}$	$\mathcal{O}_{\Sigma^1_{E_6}}(1)$
$1\times u_R{}^{\|}$	$\Sigma^2_{E_6}$	$4H+2E_3-E_1$	0	$\mathcal{O}_{\Sigma^2_{E_6}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^2_{E_6}}(-1)^{13/30}$	$\mathcal{O}_{\Sigma^2_{E_6}}(1)^{1/2}$
$3\times d_R$	$\Sigma^1_{SU(7)}$	$4H+E_2+E_3$ $-2E_1$	0	$\mathcal{O}_{\Sigma^1_{SU(7)}}(1)^{3/5}$	$\mathcal{O}_{\Sigma^1_{SU(7)}}(-1)^{3/10}$	$\mathcal{O}_{\Sigma^1_{SU(7)}}(1)^{3/14}$
$3\times L_L$	$\Sigma^2_{SU(7)}$	$4H+E_3+E_1$ $-2E_2$	0	$\mathcal{O}_{\Sigma^2_{SU(7)}}(-1)^{3/5}$	$\mathcal{O}_{\Sigma^2_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\Sigma^2_{SU(7)}}(1)^{1/7}$
$1\times H_u$	$\Sigma^u_{SU(7)}$	$H-E_1-E_3$	0	$\mathcal{O}_{\Sigma^u_{SU(7)}}(1)^{1/5}$	$\mathcal{O}_{\Sigma^u_{SU(7)}}(1)^{1/15}$	$\mathcal{O}_{\Sigma^u_{SU(7)}}(-1)^{1/21}$
$1\times H_d$	$\Sigma^d_{SU(7)}$	$H-E_2-E_4$	0	$\mathcal{O}_{\Sigma^d_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\Sigma^d_{SU(7)}}(-1)^{1/15}$	$\mathcal{O}_{\Sigma^d_{SU(7)}}(1)^{1/21}$
$1\times\Phi$	$\Sigma^\Phi_{SU(7)}$	$3H-E_1-E_3$ $-2E_2$	0	$\mathcal{O}_{\Sigma^\Phi_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\Sigma^\Phi_{SU(7)}}(-1)^{1/15}$	$\mathcal{O}_{\Sigma^\Phi_{SU(7)}}(1)^{1/21}$

Table 14: An example for a non-minimal MSSM spectrum from $G_S = SU(6)$ with the $U(1)^2$ gauge flux configuration $L_1 = \mathcal{O}_S(E_1-E_2)^{1/5}$ and $L_2 = \mathcal{O}_S(5E_3-2E_2-3E_1)^{1/30}$.

content $(2,1,-2,-1,3)$ and $(1,2,-1,-2,0)$, respectively. Clearly the exotic fields are $1\times(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$, $2\times(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$, and $1\times(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$ on $\widetilde{\Sigma}^1_{SO(12)}$. For the curve $\widetilde{\Sigma}^2_{SO(12)}$, we get exotic fields $2\times(\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$, $1\times(\bar{\mathbf{3}},\mathbf{1})_{-2,2,4}$, and $2\times(\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$. To get three copies of u_R, we arrange two curves, $\widetilde{\Sigma}^1_{E_6}$ and $\widetilde{\Sigma}^2_{E_6}$ with field content $(2,1,1,0,3,2)$ and $(3,1,2,0,4,2)$, respectively. We have exotic fields $2\times(\mathbf{3},\mathbf{2})_{1,1,-3}$, $1\times(\mathbf{3},\mathbf{2})_{-1,1,-3}$, and five singlets on $\widetilde{\Sigma}^1_{E_6}$. On $\widetilde{\Sigma}^2_{E_6}$, the exotic fields are $3\times(\mathbf{3},\mathbf{2})_{1,1,-3}$, $1\times(\mathbf{3},\mathbf{2})_{-1,1,-3}$, and six singlets. Since the rest of the fields in the case of M_1 come from the curves with $G_\Sigma = SU(7)$, we can easily engineer $3\times d_R$, $3\times L_L$, $1\times H_u$ and $1\times H_d$ on individual curves, denoted respectively by $\widetilde{\Sigma}^1_{SU(7)}$, $\widetilde{\Sigma}^2_{SU(7)}$, $\widetilde{\Sigma}^u_{SU(7)}$, and $\widetilde{\Sigma}^d_{SU(7)}$. Note that these exotic fields can form trilinear couplings with triplets on $\Sigma_{SU(7)}$. To make the exotic fields form the couplings, we introduce three extra curves $\Sigma^{\Upsilon_1}_{SU(7)}$, $\Sigma^{\bar{\Upsilon}_2}_{SU(7)}$, and $\Sigma^{\Upsilon'_3}_{SU(7)}$ with $\Upsilon_1 = (\mathbf{3},\mathbf{1})_{-7,-2,1}$, $\bar{\Upsilon}_2 = (\bar{\mathbf{3}},\mathbf{1})_{7,2,-1}$, and $\Upsilon_3+\Lambda$, respectively, where $\Upsilon_3 = (\mathbf{3},\mathbf{1})_{-7,-2,1}$ and $\Lambda = (\mathbf{1},\mathbf{1})_{-7,0,-5}$. The superpotential is as follows:

$$\mathcal{W} \supset \mathcal{W}_{\rm MSSM} + \Xi\Delta\Upsilon_1 + \Xi\widetilde{\Delta}\Upsilon_1 + \Theta\Pi\bar{\Upsilon}_2 + \widetilde{\Theta}\Pi\bar{\Upsilon}_2 + \Psi\Lambda\Upsilon_3 + \cdots, \quad (5.81)$$

where $\Xi = (\bar{\mathbf{3}},\mathbf{1})_{2,-4,2}$, $\Delta = (\mathbf{1},\mathbf{1})_{1,6,-3}$, $\widetilde{\Delta} = (\mathbf{1},\mathbf{1})_{-1,6,-3}$, and $\Pi = (\mathbf{1},\bar{\mathbf{2}})_{-2,-3,4}$. When Υ_1, $\bar{\Upsilon}_2$, and Υ_3 get vevs, the exotic fields are decoupled via the couplings, which means that at low energy, those fields will not show up in the spectrum. For

the couplings in $\mathcal{W}_{\rm MSSM}$, the arrangement of the curves is similar to the previous example. We are not going to repeat that. In this example, we obtain a non-minimal MSSM spectrum at low energy. The field content and the classes of the curves are summarized in Table 15.**

Multiplet	Curve	Σ	g_Σ	$L_{1\Sigma}$	$L_{2\Sigma}$	L'_Σ
$2\times Q_L$ $+3\times e_R$	$\widetilde{\Sigma}^1_{SO(12)}$	$5H-E_1-4E_3$ $-E_5$	0	$\mathcal{O}_{\widetilde{\Sigma}^1_{SO(12)}}(1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{SO(12)}}(1)^{17/30}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{SO(12)}}(1)^{1/3}$
$1\times Q_L$	$\widetilde{\Sigma}^2_{SO(12)}$	$4H+E_1-2E_3$ $+E_6$	0	$\mathcal{O}_{\widetilde{\Sigma}^2_{SO(12)}}(-1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{SO(12)}}(1)^{13/30}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{SO(12)}}(1)^{1/6}$
$1\times u_R$	$\widetilde{\Sigma}^1_{E_6}$	$4H+2E_3-E_1$	0	$\mathcal{O}_{\widetilde{\Sigma}^1_{E_6}}(1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{E_6}}(-1)^{13/30}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{E_6}}(1)^{1/2}$
$2\times u_R$	$\widetilde{\Sigma}^2_{E_6}$	$5H+3E_3-E_1$	0	$\mathcal{O}_{\widetilde{\Sigma}^2_{E_6}}(1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{E_6}}(-1)^{3/5}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{E_6}}(1)$
$3\times d_R$	$\widetilde{\Sigma}^1_{SU(7)}$	$4H+E_2+E_3$ $-2E_1$	0	$\mathcal{O}_{\widetilde{\Sigma}^1_{SU(7)}}(1)^{3/5}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{SU(7)}}(-1)^{3/10}$	$\mathcal{O}_{\widetilde{\Sigma}^1_{SU(7)}}(1)^{3/14}$
$3\times L_L$	$\widetilde{\Sigma}^2_{SU(7)}$	$4H+E_3+E_1$ $-2E_2$	0	$\mathcal{O}_{\widetilde{\Sigma}^2_{SU(7)}}(-1)^{3/5}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^2_{SU(7)}}(1)^{1/7}$
$1\times H_u$	$\widetilde{\Sigma}^u_{SU(7)}$	$3H+E_2-E_4$	0	$\mathcal{O}_{\widetilde{\Sigma}^u_{SU(7)}}(1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^u_{SU(7)}}(1)^{1/15}$	$\mathcal{O}_{\widetilde{\Sigma}^u_{SU(7)}}(-1)^{1/21}$
$1\times H_d$	$\widetilde{\Sigma}^d_{SU(7)}$	$H-E_2-E_4$	0	$\mathcal{O}_{\widetilde{\Sigma}^d_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^d_{SU(7)}}(-1)^{1/15}$	$\mathcal{O}_{\widetilde{\Sigma}^d_{SU(7)}}(1)^{1/21}$
$1\times \Upsilon_1$	$\widetilde{\Sigma}^{\Upsilon_1}_{SU(7)}$	$H-E_2-E_3$	0	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon_1}_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon_1}_{SU(7)}}(1)^{1/10}$	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon_1}_{SU(7)}}(-1)^{1/14}$
$1\times \bar{\Upsilon}_2$	$\widetilde{\Sigma}^{\bar{\Upsilon}_2}_{SU(7)}$	$2H-E_1-E_4$ $-E_5$	0	$\mathcal{O}_{\widetilde{\Sigma}^{\bar{\Upsilon}_2}_{SU(7)}}(1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^{\bar{\Upsilon}_2}_{SU(7)}}(-1)^{1/10}$	$\mathcal{O}_{\widetilde{\Sigma}^{\bar{\Upsilon}_2}_{SU(7)}}(1)^{1/14}$
$1\times \Upsilon_3$ $+1\times \bar{\Lambda}$	$\widetilde{\Sigma}^{\Upsilon'_3}_{SU(7)}$	$H-E_2-E_4$	0	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon'_3}_{SU(7)}}(-1)^{1/5}$	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon'_3}_{SU(7)}}(-1)^{1/15}$	$\mathcal{O}_{\widetilde{\Sigma}^{\Upsilon'_3}_{SU(7)}}(-1)^{2/21}$

Table 15: An example for a non-minimal MSSM spectrum from $G_S=SU(6)$ with the $U(1)^2$ gauge flux configuration $L_1=\mathcal{O}_S(E_1-E_2)^{1/5}$ and $L_2=\mathcal{O}_S(5E_3-2E_2-3E_1)^{1/30}$.

6 Conclusions

In this paper we demonstrate how to obtain $U(1)^2$ gauge flux configurations (L_1,L_2) with an exotic-free bulk spectrum of the local F-theory model with $G_S=SU(6)$. In this case each configuration is constructed by two fractional line bundles, which are

In this example Q_L and u_R are localized on different curves. The Yukawa coupling $Q_L u_R H_u$ descended from **10105 can be expressed as $[\widetilde{\Sigma}^1_{SO(12)}(1,2)+\widetilde{\Sigma}^2_{SO(12)}(3)][\widetilde{\Sigma}^1_{E_6}(1)+\widetilde{\Sigma}^2_{E_6}(2,3)][\widetilde{\Sigma}^u_{SU(7)}]$ generating nonvanishing diagonal elements in the Yukawa mass matrix, where the indices in the parenthesis represent the generations.

well-defined in the sense that up to a linear transformation of the $U(1)$ charges, an $U(1)^2$ flux configuration can be associated with a polystable bundle of rank two with structure group $U(1)^2$. Under physical assumptions, we obtain all flux configurations as shown in Table 7 and Table 8. For the case of $G_S = SO(10)$, as shown in [12] there is a no-go theorem which states that for an exotic-free spectrum, there are no solutions for $U(1)^2$ gauge fluxes.

To build a model of the MSSM, we study the field configurations localized on the curves with non-trivial gauge fluxes induced from the restriction of the flux configurations on the bulk S. With the non-trivial induced fluxes, the enhanced gauge group G_Σ will be broken into $G_{\rm std} \times U(1)$. Under physical assumptions, we obtain all field configurations localized on the curves with $G_\Sigma = SU(7)$, $G_\Sigma = SO(12)$ and $G_\Sigma = E_6$. Form the breaking patterns, we know that Higgs fields are localized on the curves $\Sigma_{SU(7)}$ and $\Sigma_{SO(12)}$. On the curve $\Sigma_{SU(7)}$, we found that doublet-triplet splitting can be achieved. However, it is impossible to get the splitting on the curve $\Sigma_{SO(12)}$ without raising exotic fields, which means that when building models, we should engineer the Higgs fields on the curve $\Sigma_{SU(7)}$ instead of $\Sigma_{SO(12)}$. Unlike Higgs fields, matter fields in the MSSM are distributed over the curves $G_\Sigma = SU(7)$, $G_\Sigma = SO(12)$ and $G_\Sigma = E_6$. With the solved field configurations, it is clear that it is extremely difficult to get the minimal spectrum of the MSSM without exotic fields. However, if those exotic fields can form trilinear couplings with the doublets or triplets on the curves with $G_\Sigma = SU(7)$, the exotic fields can be lifted from the massless spectrum when these doublets or triplets get vevs. In order to achieve this, we introduce extra curves to support these doublets or triplets coupled to exotic fields. With this procedure, we can construct a non-minimal spectrum of the MSSM with doublet-triple splitting. It would be interesting to study mechanisms breaking non-minimal gauge group G_S down to $G_{\rm std}$ other than $U(1)^2$ gauge fluxes.

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References

[1] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory" Cambridge University Press 1987.

[2] G. Aldazabal, L. E. Ibanez, F. Quevedo and A. M. Uranga, JHEP **0008**, 002 (2000) [arXiv:hep-th/0005067].

[3] H. Verlinde and M. Wijnholt, JHEP **0701**, 106 (2007) [arXiv:hep-th/0508089]; D. Malyshev and H. Verlinde, Nucl. Phys. Proc. Suppl. **171**, 139 (2007) [arXiv:0711.2451 [hep-th]], and references therein.

[4] R. Blumenhagen, M. Cvetic, P. Langacker and G. Shiu, Ann. Rev. Nucl. Part. Sci. **55**, 71 (2005) [arXiv:hep-th/0502005], and references therein.

[5] R. Blumenhagen, M. Cvetic and T. Weigand, Nucl. Phys. B **771**, 113 (2007) [arXiv:hep-th/0609191]; L. E. Ibanez and A. M. Uranga, JHEP **0703**, 052 (2007) [arXiv:hep-th/0609213]; B. Florea, S. Kachru, J. McGreevy and N. Saulina, JHEP **0705**, 024 (2007) [arXiv:hep-th/0610003]; R. Blumenhagen, M. Cvetic, D. Lust, R. Richter and T. Weigand, Phys. Rev. Lett. **100**, 061602 (2008) [arXiv:0707.1871 [hep-th]].

[6] R. Blumenhagen, V. Braun, T. W. Grimm and T. Weigand, arXiv:0811.2936 [hep-th].

[7] C. Vafa, Nucl. Phys. B **469**, 403 (1996) [arXiv:hep-th/9602022]; D. R. Morrison and C. Vafa, Nucl. Phys. B **473**, 74 (1996) [arXiv:hep-th/9602114]; D. R. Morrison and C. Vafa, Nucl. Phys. B **476**, 437 (1996) [arXiv:hep-th/9603161].

[8] F. Denef, arXiv:0803.1194 [hep-th].

[9] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, Nucl. Phys. B **481**, 215 (1996) [arXiv:hep-th/9605200].

[10] A. Sen, Nucl. Phys. B **475**, 562 (1996) [arXiv:hep-th/9605150].

[11] C. Beasley, J. J. Heckman and C. Vafa, JHEP **0901**, 058 (2009) [arXiv:0802.3391 [hep-th]].

[12] C. Beasley, J. J. Heckman and C. Vafa, arXiv:0806.0102 [hep-th].

[13] R. Donagi and M. Wijnholt, arXiv:0802.2969 [hep-th]; R. Donagi and M. Wijnholt, arXiv:0808.2223 [hep-th]; M. Wijnholt, arXiv:0809.3878 [hep-th].

[14] J. J. Heckman and C. Vafa, arXiv:0809.1098 [hep-th]; J. J. Heckman and C. Vafa, arXiv:0809.3452 [hep-ph]; J. J. Heckman, G. L. Kane, J. Shao and C. Vafa, arXiv:0903.3609 [hep-ph]; J. J. Heckman, A. Tavanfar and C. Vafa, arXiv:0906.0581 [hep-th].

[15] J. J. Heckman and C. Vafa, arXiv:0811.2417 [hep-th].

[16] V. Bouchard, J. J. Heckman, J. Seo and C. Vafa, arXiv:0904.1419 [hep-ph].

[17] J. J. Heckman and C. Vafa, arXiv:0904.3101 [hep-th].

[18] A. Font and L. E. Ibanez, arXiv:0811.2157 [hep-th].

[19] A. Font and L. E. Ibanez, JHEP **0909**, 036 (2009) [arXiv:0907.4895 [hep-th]].

[20] L. Randall and D. Simmons-Duffin, arXiv:0904.1584 [hep-ph].

[21] J. Jiang, T. Li, D. V. Nanopoulos and D. Xie, arXiv:0811.2807 [hep-th]; J. Jiang, T. Li, D. V. Nanopoulos and D. Xie, arXiv:0905.3394 [hep-th]; T. Li, arXiv:0905.4563 [hep-th].

[22] R. Blumenhagen, arXiv:0812.0248 [hep-th].

[23] J. L. Bourjaily, arXiv:0901.3785 [hep-th]; J. L. Bourjaily, arXiv:0905.0142 [hep-th].

[24] C.-M. Chen and Y.-C. Chung, Nucl. Phys. B **824**, 273 (2010) [arXiv:0903.3009 [hep-th]].

[25] J. P. Conlon and E. Palti, arXiv:0907.1362 [hep-th].

[26] J. P. Conlon and E. Palti, arXiv:0910.2413 [hep-th].

[27] S. Cecotti, M. C. N. Cheng, J. J. Heckman and C. Vafa, arXiv:0910.0477 [hep-th].

[28] J. Marsano, N. Saulina and S. Schafer-Nameki, JHEP **0908**, 030 (2009) [arXiv:0904.3932 [hep-th]]; J. Marsano, N. Saulina and S. Schafer-Nameki, JHEP **0908**, 046 (2009) [arXiv:0906.4672 [hep-th]].

[29] R. Donagi and M. Wijnholt, arXiv:0904.1218 [hep-th].

[30] C. Cordova, arXiv:0910.2955 [hep-th].

[31] H. Hayashi, R. Tatar, Y. Toda, T. Watari and M. Yamazaki, Nucl. Phys. B **806**, 224 (2009) [arXiv:0805.1057 [hep-th]]; A. Collinucci, F. Denef and M. Esole, JHEP **0902**, 005 (2009) [arXiv:0805.1573 [hep-th]]; A. P. Braun, A. Hebecker, C. Ludeling and R. Valandro, arXiv:0811.2416 [hep-th]; G. Aldazabal, P. G. Camara and J. A. Rosabal, arXiv:0811.2900 [hep-th]; A. Collinucci, arXiv:0812.0175 [hep-th]; B. Andreas and G. Curio, arXiv:0902.4143 [hep-th]; R. Tatar, Y. Tsuchiya and T. Watari, Nucl. Phys. B **823**, 1 (2009) [arXiv:0905.2289 [hep-th]]; A. Collinucci, arXiv:0906.0003 [hep-th]; R. Blumenhagen, T. W. Grimm, B. Jurke and T. Weigand, JHEP **0909**, 053 (2009) [arXiv:0906.0013 [hep-th]]; P. Aluffi and M. Esole, arXiv:0908.1572 [hep-th]; R. Blumenhagen, T. W. Grimm, B. Jurke and T. Weigand, arXiv:0908.1784 [hep-th]; T. W. Grimm, T. W. Ha, A. Klemm and D. Klevers, arXiv:0909.2025 [hep-th]; K. S. Choi, arXiv:0910.2571 [hep-th].

[32] H. Hayashi, T. Kawano, R. Tatar and T. Watari, arXiv:0901.4941 [hep-th].

[33] H. Hayashi, T. Kawano, Y. Tsuchiya and T. Watari, arXiv:0910.2762 [hep-th].

[34] E. I. Buchbinder, JHEP **0809**, 134 (2008) [arXiv:0805.3157 [hep-th]].

[35] J. J. Heckman, J. Marsano, N. Saulina, S. Schafer-Nameki and C. Vafa, arXiv:0808.1286 [hep-th]; J. Marsano, N. Saulina and S. Schafer-Nameki, arXiv:0808.1571 [hep-th]; J. Marsano, N. Saulina and S. Schafer-Nameki, arXiv:0808.2450 [hep-th].

[36] R. Blumenhagen, J. P. Conlon, S. Krippendorf, S. Moster and F. Quevedo, JHEP **0909**, 007 (2009) [arXiv:0906.3297 [hep-th]].

[37] J. J. Heckman, A. Tavanfar and C. Vafa, arXiv:0812.3155 [hep-th].

[38] S. H. Katz and C. Vafa, Nucl. Phys. B **497**, 146 (1997) [arXiv:hep-th/9606086].

[39] S. Donaldson,"Anti Self-Dual Yang-Mills Connections over Complex Algebraic Surfaces and Stable Vector Bundles," Proc. London Math. Soc. 50 1 (1985).

[40] K. Uhlenbeck and S.-T. Yau,"On the existence of Hermitian Yang-Mills connections in stable bundles," Comm. Pure App. Math. 39 257 (1986), 42 703 (1986).

[41] R. Slansky, Phys. Rept. **79**, 1 (1981).

[42] Demazure, Pinkham et Teissier, "Seminaire sur les Singularities des Surfaces," Ecole Polytechnique, 1976-1977.

[43] Y. I. Manin, Cubic forms: Algebra, geometry, arithmetic. North-Holland Publishing Co., Amsterdam, second ed., 1986. Translated from the Russian by M. Hazewinkel.

[44] R. Hartshorne, "Algebraic geometry," New York : Springer-Verlag, 1977.

[45] P. Griffith and J. Harris, "Principles of Algebraic Geometry," Wiley NY 1994.

[46] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, "Self-Duality in Four-Dimensional Riemannian Geometry," Proc. R. Soc. Lond. A 362 (1978) 425-461.