

**Comment on: “The Casimir force on a piston in the spacetime  
with extra compactified dimensions” [Phys. Lett. B 668 (2008) 72]**

S. A. Fulling\*

*Departments of Mathematics and Physics,  
Texas A&M University, College Station, TX, 77843-3368 USA*

Klaus Kirsten†

*Department of Mathematics, Baylor University, Waco, TX 76798-7328 USA*

(Dated: June 1, 2018)

**Abstract**

We offer a clarification of the significance of the indicated paper of H. Cheng. Cheng’s conclusions about the attractive nature of Casimir forces between parallel plates are valid beyond the particular model in which he derived them; they are likely to be relevant to other recent literature on the effects of hidden dimensions on Casimir forces.

---

\*Electronic address: [fulling@math.tamu.edu](mailto:fulling@math.tamu.edu)

†Electronic address: [klaus.kirsten@baylor.edu](mailto:klaus.kirsten@baylor.edu)

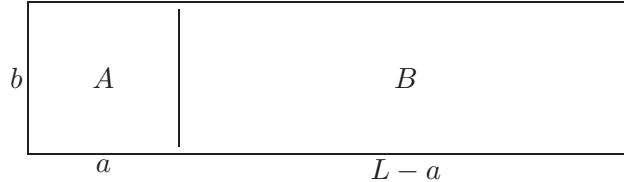


FIG. 1: A schematic Casimir piston. The interior plate is free to move.

The interesting Letter by Cheng [1] studies the effect of small Kaluza–Klein dimensions on the Casimir force between parallel plates in the macroscopic dimensions. It reaches conclusions about parallel plates different from those of a previous paper by the same author [2]. Here we offer some remarks that we believe make its significance clearer.

First, a historical observation. In early work on the Casimir force between parallel plates it was standard to enclose the movable plate in a large, finite box, so as to make all energies finite. This procedure is followed, for example, in [3, 4, 5]. That is, all these early calculations were done in what we would nowadays call a piston [6, 7]. At the end the transverse dimensions (such as  $b$  in Fig. 1) and the length  $L - a$  in Fig. 1 were taken to infinity; it was found that the results then were the same as those of a more naive calculation where the transverse boundaries and the distant plate (right end of Fig. 1) were never present. Emboldened by this consistency, many later authors did not bother to put in the spatial cutoffs.

Indeed, that looser procedure was followed by Cheng in [2] but not in [1]. The earlier paper concluded that the presence of compactified extra dimensions modified the Casimir force, making it repulsive under certain conditions. (In terms of Fig. 1, the narrow chamber  $A$  would tend to expand, not contract, in analogy with the well known (but heavily disputed) prediction for a conducting rectangular cavity [8].) However, the more recent paper [1] studies a fully finite configuration like that of Fig. 1 (but with extra dimensions, both small and large). One finds that the result of moving the rightmost plate to infinity is *not* the same as never having that plate at all. The force is always attractive.

This result is analogous to that in the original piston literature [6, 7], where careful calculation of the total vacuum energy in the rectangular cavities  $A$  and  $B$  yields an attractive force, notwithstanding [8]. The details of this analogy are worth dwelling upon. The point of [6, 7] (see also [9]) is that regarding chamber  $B$  as infinitely long (or as absent) obscures the fact that a change in the energy in  $B$  cancels a part of the change in the energy in

$A$  when the piston plate moves. On the other hand, the existence of the finite transverse dimension(s)  $b$  is essential for creating the repulsive force in  $A$  to begin with. That force arises from the change with  $a$  of the (negative) Casimir energy of interaction of the top and bottom plates in Fig. 1. In the calculations of Cheng the large transverse dimensions play no essential role in the considered limit  $b \rightarrow \infty$ . However, the small extra dimensions are now significant. The repulsive force found in Cheng’s first paper [2] is (minus) the  $a$ -derivative of the Casimir energy associated with the Kaluza–Klein dimensions, only cavity  $A$  being considered. In his second paper [1] the Kaluza-Klein vacuum energy is taken into account also for cavity  $B$ , so that the repulsive effect is cancelled.

What makes the model in [1] a “piston” is not just the presence of the third plate, but also the presence of compact transverse dimensions. Both elements are necessary, one to create the “paradox” of a repulsive force and the other to remove it. In the Kaluza-Klein context this situation was not obvious until Cheng pointed it out; see also [10]. As he stresses, experimental evidence is against any repulsion between large parallel plates; although originally offered [2] as evidence against the existence of Kaluza-Klein dimensions, this fact now [1] appears as additional confirmation that a careful “piston” analysis is the correct way to do the parallel-plate calculation if such dimensions do exist.

In closing we make a technical comment on Cheng’s calculations. (We note that more recent treatments [11, 12] of multidimensional pistons are more general than previous literature.) Both papers [1, 2] are based on a mode sum that corresponds, in the small compact dimensions, to the spectrum of a rectangular box with Neumann boundary conditions. It would be more natural to take periodic boundary conditions (i.e., to make the Kaluza–Klein space a torus) or to consider non-flat compact extra dimensions like a sphere. But in fact, the main conclusions of [1, 2] are independent of such details. Let  $N$  denote the manifold of the extra dimensions such that the space of our universe is given by  $\mathbb{R}^3 \times N$ , the piston living in  $\mathbb{R}^3$ . In the limit where  $b \rightarrow \infty$  in Fig. 1, the eigenfrequencies of the vacuum fluctuations are given by

$$\omega^2 = k_1^2 + k_2^2 + \left(\frac{n\pi}{D}\right)^2 + \lambda_i^2,$$

where  $n$  and  $i$  are positive integers,  $k_1^2 + k_2^2$  comes from the two free transversal dimensions in  $\mathbb{R}^3$ ,  $(n\pi/D)^2$  results from the Dirichlet plates ( $D = a$  for the left chamber and  $D = L - a$  for

the right chamber in Fig. 1), and  $\lambda_i^2$  are the eigenfrequencies in the additional dimensions,

$$-\Delta_N \varphi_i = \lambda_i^2 \varphi_i.$$

Under the assumption that  $\lambda_i^2 \geq 0$ , the Casimir force on the piston, as  $L \rightarrow \infty$ , can be shown to be

$$F = -\frac{\pi^2 g_0}{480a^4} + \frac{1}{8\pi^2} \sum_{n=1}^{\infty} \sum_i' \frac{\lambda_i^2}{n^2} \frac{\partial}{\partial a} \frac{1}{a} K_2(2an\lambda_i),$$

where  $g_0$  is the multiplicity of the states with  $\lambda_i = 0$  and the prime indicates that the summation over  $i$  omits these states. Arguing as in [6], using well known properties of the modified Bessel function  $K_\nu(z)$  [13],  $F$  can be seen to be negative (as in [1]) independently of any details of the topology or geometry of the extra dimensions (within the confines  $\lambda_i^2 \geq 0$ ). Note that this conclusion remains valid if we replace Dirichlet by Neumann boundary conditions on the plates, which implies we sum  $n$  from zero to infinity. This only introduces additional  $D$ -independent terms into the energy which are irrelevant for the force.

With only the left chamber in Fig. 1 considered, the answer in general will contain renormalization ambiguities. However, on the equilateral torus of radius  $R$  the force is finite and reads

$$F = -\frac{\pi^2}{480a^4} + \frac{\Gamma\left(\frac{d}{2} + 2\right)}{32\pi^{4+d/2}R^4} Z_d\left(\frac{d}{2} + 2\right) + \frac{1}{8\pi^2 R^2} \sum_{n=1}^{\infty} \sum_{n_1, \dots, n_d = -\infty}^{\infty}{}' \frac{n_1^2 + \dots + n_d^2}{n^2} \frac{\partial}{\partial a} \frac{1}{a} K_2\left(\frac{2an}{R} \sqrt{n_1^2 + \dots + n_d^2}\right),$$

where

$$Z_d(s) = \sum_{n_1, \dots, n_d = -\infty}^{\infty}{}' [n_1^2 + \dots + n_d^2]^{-s}.$$

This shows (as in [2]) that for  $a \gg R$  the force is positive and asymptotically constant when region  $B$  and its spatial cutoffs are neglected.

The only effect of replacing Neumann by periodic conditions in the extra dimensions is to multiply the contribution of each nonzero quantum number by 2, since it corresponds to both a left-moving and a right-moving mode. Of course, one does not know what the actual geometry of the Kaluza-Klein space is. The arguments we have just presented (whose details will be presented elsewhere [14]) show that the situation is qualitatively similar for any such geometry, with the caveat that the (presumably spurious) Lukosz-type repulsion

for a single chamber will not arise at all unless the (renormalized) Casimir energy of the Kaluza–Klein manifold is negative, as it is in a torus of any dimension. There are cases where that energy is positive — for example, a scalar field in a 2-dimensional rectangle with Dirichlet conditions or an electromagnetic field in a 3-dimensional rectangular box with perfect conductor conditions [15]. Note that this question is distinct from (although related to) the issue of whether a Lukosz repulsion exists within the Kaluza–Klein space itself: positive energy is not synonymous with positive force.

### Acknowledgments

Our work is supported by National Science Foundation Grants PHY-0554849 (TAMU) and PHY-0757791 (Baylor). S.A.F. enjoyed the hospitality of the Institute for Mathematics and Its Applications (U. Minnesota, Minneapolis) while the paper was written.

- 
- [1] H. Cheng, *Phys. Lett. B* **668** (2008) 72.
  - [2] H. Cheng, *Phys. Lett. B* **643** (2006) 311.
  - [3] M. Fierz, *Helv. Phys. Acta* **33** (1960) 855.
  - [4] E. A. Power, *Introductory Quantum Electrodynamics*, Elsevier, New York, 1964, Appendix.
  - [5] T. H. Boyer, *Ann. Phys.* **56** (1970) 474.
  - [6] R. M. Cavalcanti, *Phys. Rev. D* **69** (2004) 065015.
  - [7] M. P. Hertzberg, R. L. Jaffe, M. Kardar, and A. Scardicchio, *Phys. Rev. Lett.* **95** (2005) 250402.
  - [8] W. Lukosz, *Physica* **56** (1971) 109 .
  - [9] S. A. Fulling, L. Kaplan, K. Kirsten, Z. H. Liu, and K. A. Milton, [arXiv:0806.2468](https://arxiv.org/abs/0806.2468).
  - [10] A. Edery and V. Marachevsky, [arXiv:0810.3430](https://arxiv.org/abs/0810.3430).
  - [11] S. C. Lim and L. P. Teo, [arXiv:0808.0047](https://arxiv.org/abs/0808.0047).
  - [12] A. Edery and V. Marachevsky, *Phys. Rev. D* **78** (2008) 025021.
  - [13] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*. Academic Press, New York, 1965.
  - [14] S. A. Fulling and K. Kirsten, Kaluza-Klein models as pistons, in preparation.

[15] J. Ambjørn and S. Wolfram, *Ann. Phys.* **147** (1983) 1.