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AN INFINITE-DIMENSIONAL SYMMETRY ALGEBRA IN STRING THEORY

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Abstract

Symmetry transformations of the space-time fields of string theory are generated by certain similarity transformations of the stress-tensor of the associated conformal field theories. This observation is complicated by the fact that, as we explain, many of the operators we habitually use in string theory (such as vertices and currents) have ill-defined commutators. However, we identify an infinite-dimensional subalgebra whose commutators are not singular, and explicitly calculate its structure constants. This constitutes a subalgebra of the gauge symmetry of string theory, although it may act on auxiliary as well as propagating fields. We term this object a *weighted tensor algebra*, and, while it appears to be a distant cousin of the W-algebras, it has not, to our knowledge, appeared in the literature before.

1. Introduction.

Closed string theory is the most promising candidate for a complete dynamics of elementary particles, yet its usual formulation as a first quantized theory is widely regarded as being rather unsatisfactory. First quantized string theory yields rules for the calculation of S-matrix elements in particular backgrounds, but obscures a more global (*i.e.* background independent) view of the theory, and shrouds in mystery its deeper principles. Symmetry, that most powerful tool of the theoretical physicist, must surely be counted among these principles.

In this paper we shall describe an infinite-dimensional subalgebra of the full symmetry algebra of a D-dimensional string theory. We shall also point out that many of the operators we habitually use in string theory are not well-defined—their commutators are singular.

There are several approaches to the study of the symmetries of string theory in the literature; let us briefly survey those of which we are aware. One natural approach is to attempt to formulate an action for a field theory and then study its invariances. Constructing a string field theory has proved a formidable task for reasons not obviously associated with the problem of symmetry. Nevertheless considerable progress has been made recently in this direction (see, for example, [1], and references therein to the earlier literature), although we think it fair to say that a manifestly background-independent formulation is still missing.

Other work has avoided the difficulties associated with going off-shell by working in the first-quantized formalism. The most physically direct approach is the *gedanken*-phenomenology of Gross, Mende and Manes [2], who derived expressions for the asymptotic form of scattering amplitudes at high energy. For our purposes, the important lesson from this work is that this asymptotic form is independent of the states on the external legs. This strongly suggests the existence of an enormous, spontaneously broken symmetry, but sheds little light on its actual form. Isberg, Lindstrom and Sundborg had the related idea of studying symmetries in the limit of zero tension [3], which should correspond to the same high energy regime.

A second approach [4], [5], treats the partition function of the non-linear sigma model (somewhat formally) as a generating functional for string amplitudes. This partition function is, of course, a functional integral, and, "changes of variables," in this functional integral generate invariances which, in turn, translate into Ward identities among amplitudes. This approach has the advantage of yielding directly the transformations on space-time fields, but (in our view) is hampered by the difficulties involved in defining the relevant functional measures.

The third type of approach focuses on algebraic aspects of string theory. It has long been recognized [6] that conserved currents ("current," meaning a (1,0) or (0,1) primary field) generate unbroken gauge symmetries. This notion was generalized to include currents that are conserved anywhere in the deformation class of a CFT [7]. However, this is still a special case of the method we shall employ in this paper; symmetries (broken and unbroken) are generated by (inner) automorphisms of the operator algebra of a deformation class of CFT's [8], [9]. Moore has recently used this method (in the Batalin-Vilkovisky formalism) to derive an infinite set of relations that were sufficient to determine all amplitudes with fewer than twenty-six external legs [10]. Physical consequences of the symmetries uncovered in this way have also been explored in [11].

This method will be reviewed in more detail in section 2. In section 3 we shall compute the symmetry subalgebra that is the main result of this paper, while in section 4 we shall explain why computing the *full* symmetry algebra is problematic. Indeed, section 4 contains a message of independent significance: operators that play important rôles in the theory (such as vertices and currents) *do not actually exist*, as presently defined, because their commutators are sick. We shall argue that this problem is not simply a technical detail to be corrected, but is probably insoluble. We find this problem troubling, but its full significance is obscure to us.

We end this introduction with a brief description of our infinite-dimensional symmetry subalgebra. As has been argued elsewhere [8], it is a *supersymmetry*, in that its generators do not commute with the generators of Lorentz transformations and mix excitations of different spin; it is spontaneously broken in flat space-time, because not all generators commute with the stress tensor of the free scalar CFT, and it transforms excitations of differing mass into one another.

The algebra may be described abstractly in terms of the following objects: consider the set of all covariant tensors of a D-dimensional vector space, paired with a k-tuple of positive-definite integer weights (k is the rank of the tensor, and each weight is associated with an index on the tensor). Introduce the operators Δ_r, where $r \leq k$ is a positive definite integer, defined by

$$\Delta_r \left\{\phi^{(k)}, w_i\right\} = \sum_{l=1}^{r} \left\{\phi^{(k)}, (w_i + \delta_{il})\right\}. \tag{1.1}$$

Then elements of the algebra consist of the pairs $\left\{\phi^{(k)}, w_i\right\}$, modulo the relations

$$\lambda \left\{\phi^{(k)}, w_i\right\} + \mu \left\{\psi^{(k)}, w_i\right\} - \left\{\lambda\phi^{(k)} + \mu\psi^{(k)}, w_i\right\} \cong 0 \tag{1.2}$$

$$\Delta_k \left\{\phi^{(k)}, w_i\right\} \cong 0 \tag{1.3}$$

$$\left\{\phi^{(k)}_{\ldots\mu\ldots\nu\ldots}, (\ldots, w_m, \ldots, w_n, \ldots)\right\} - \left\{\phi^{(k)}_{\ldots\nu\ldots\mu\ldots}, (\ldots, w_n, \ldots, w_m, \ldots)\right\} \cong 0 \tag{1.4}$$

Equation (1.2) says that, for fixed weights, tensor addition is vector space addition, but note that there is no such property for the weights. Equation (1.3) asserts that, given a basis element, incrementing each weight by one and summing yields an element that should be identified with zero. Equation (1.4) identifies two basis elements if their tensors are transposes on a pair of indices and the corresponding weights are interchanged, *e.g.* $\left\{\phi^{(2)}, (w_1, w_2)\right\} \cong \left\{\phi^{(2)T}, (w_2, w_1)\right\}$ where $\phi^{(2)T}_{\mu\nu} = \phi^{(2)}_{\nu\mu}$.

The commutator of two such generators is

$$\left[\{\psi_{\mu_\mathcal{S}}, w_\mathcal{S}\}, \{\chi_{\nu_\mathcal{T}}, v_\mathcal{T}\}\right] = \sum_{\substack{\mathcal{U}\subseteq\mathcal{S} \\ \mathcal{P}:\mathcal{U}\hookrightarrow\mathcal{T}}} \mathcal{C}_{\mathcal{U},\mathcal{P}} \Delta_{|\mathcal{S}-\mathcal{U}|}^{\left(-1+\sum_{i\in\mathcal{U}} w_i + v_{\mathcal{P}(i)}\right)} \left\{\psi_{\mu_\mathcal{S}} \chi_{\nu_\mathcal{T}} \prod_{i\in\mathcal{U}} \eta^{\mu_i \nu_{\mathcal{P}(i)}},\ w_{\mathcal{S}-\mathcal{U}} \oplus v_{\mathcal{T}-\mathcal{P}(\mathcal{U})}\right\}. \tag{1.5}$$

The indices on the tensors ψ and χ are themselves labeled by index sets $\mathcal{S}$ and $\mathcal{T}$, *e.g.* $1, \ldots, k$. The sum is over all pairs consisting of subsets $\mathcal{U} \subseteq \mathcal{S}$ and injections $\mathcal{P}: \mathcal{S} \hookrightarrow \mathcal{T}$ (or, equivalently, all pairs of subsets of $\mathcal{S}$ and $\mathcal{T}$ of the same size). $\mathcal{S} - \mathcal{U}$ denotes the complement of $\mathcal{U}$ in $\mathcal{S}$ and $|\mathcal{S}|$ denotes the number of elements in $\mathcal{S}$. η is the Minkowski metric of space-time. The weights, $w_{\mathcal{S}-\mathcal{U}} \oplus v_{\mathcal{T}-\mathcal{P}(\mathcal{U})}$, are the weights of ψ that do not correspond to indices in $\mathcal{U}$ together with the weights of χ that do not correspond to the indices in the image of $\mathcal{U}$ under $\mathcal{P}$. The superscript on Δ is a power. The coefficients are given by

$$\mathcal{C}_{\mathcal{U},\mathcal{P}} = \frac{\prod_{k\in\mathcal{U}} (-1)^{w_k} (w_k + v_{\mathcal{P}(k)} - 1)!}{\left(\left(\sum_{k\in\mathcal{U}} w_k + v_{\mathcal{P}(k)}\right) - 1\right)!} \tag{1.6}$$

The formal statement of this commutator is thus rather cumbersome and intimidating, but the basic idea is simply that of Wick's theorem. The sum over $\mathcal{U}$ and $\mathcal{P}$ is a sum over all possible contractions of the tensors ψ and χ. The weights are those of the uncontracted indices, except that the weights of ψ are augmented in a suitable way by the weights of the contracted indices through a power of the operator $\Delta_{|\mathcal{S}-\mathcal{U}|}$. The similarity to Wick's theorem is, of course, no accident.

To the best of our (limited) knowledge, algebras of this type have not appeared in the literature until now; we suggest that they be termed *weighted tensor algebras*.

2. Deformations of Conformal Field Theories and Symmetries of String Theory.

In this section we shall review earlier work [8], [9] on deformations of conformal field theories and symmetries of string theory. For more details the reader is referred to the original papers, or the review contained in [12].

We shall study the question of string symmetries by finding transformations of the space-time degrees of freedom that map one solution of the classical equations of motion to another that is physically equivalent. Since, "Solutions of the classical equations of motion," are, for the case of string theory [13], two-dimensional conformal field theories [14], we are thus interested in physically equivalent conformal field theories.

Any quantum mechanical theory (including a CFT) is defined by an algebra of observables, $\mathcal{A}$ (determined by the degrees of freedom of the theory and their equal-time commutation relations), a representation of that algebra and a distinguished element of $\mathcal{A}$ that generates temporal evolution (the Hamiltonian). (Note that for the same $\mathcal{A}$ we may have many choices of Hamiltonian, so that $\mathcal{A}$ may more properly be associated with a *deformation class* of theories than with one particular theory.) For a CFT, we further want $\mathcal{A}$ to be generated by local fields, $\Phi(\sigma)$ (operator valued distributions on a circle parameterized by σ), and we require not just a single distinguished operator, but two distinguished fields, $T(\sigma)$ and $\overline{T}(\sigma)$, in terms of which the Hamiltonian, H, and generator of translations, P, may be written

$$H = \int d\sigma (T(\sigma) + \overline{T}(\sigma)) \tag{2.1}$$

$$P = \int d\sigma (T(\sigma) - \overline{T}(\sigma)) \tag{2.2}$$

and that satisfy Virasoro×Virasoro:

$$[T(\sigma), T(\sigma')] = \frac{-ic}{24\pi}\delta'''(\sigma - \sigma') + 2iT(\sigma')\delta'(\sigma - \sigma') - iT'(\sigma')\delta(\sigma - \sigma') \tag{2.3a}$$

$$[\overline{T}(\sigma), \overline{T}(\sigma')] = \frac{ic}{24\pi}\delta'''(\sigma - \sigma') - 2i\overline{T}(\sigma')\delta'(\sigma - \sigma') + i\overline{T}'(\sigma')\delta(\sigma - \sigma') \tag{2.3b}$$

$$[T(\sigma), \overline{T}(\sigma')] = 0. \tag{2.3c}$$

Except on σ, a prime denotes differentiation. T and $\overline{T}$ are the non-vanishing components of the stress tensor, and must satisfy (2.3) if they are to generate conformal transformations. Also of interest are the so-called *primary fields* of dimension $(d, \overline{d})$, $\Phi(\sigma)$, defined by the conditions

$$\begin{aligned} [T(\sigma), \Phi_{(d,\overline{d})}(\sigma')] &= id\Phi_{(d,\overline{d})}(\sigma')\delta'(\sigma - \sigma') - (i/\sqrt{2})\partial\Phi_{(d,\overline{d})}(\sigma')\delta(\sigma - \sigma') \\ [\overline{T}(\sigma), \Phi_{(d,\overline{d})}(\sigma')] &= -i\overline{d}\Phi_{(d,\overline{d})}(\sigma')\delta'(\sigma - \sigma') - (i/\sqrt{2})\overline{\partial}\Phi_{(d,\overline{d})}(\sigma')\delta(\sigma - \sigma') \end{aligned} \tag{2.4}$$

Clearly, then, two CFTs will be physically equivalent if there is an isomorphism between the corresponding algebras of observables, $\mathcal{A}$, that maps stress tensor to stress tensor. (The mapping of primary to primary is then automatic). The simplest example of such an isomorphism is an inner automorphism, or similarity transformation:

$$\Phi(\sigma) \mapsto e^{ih}\Phi(\sigma)e^{-ih} \tag{2.5}$$

for any fixed operator h. Thus the physics will be unchanged if we change a CFT's stress tensor by just such a similarity transformation.

Now, the stress tensor is parameterized by the space-time fields of the string. For example,

$$T(\sigma) = \tfrac{1}{2}G_{\mu\nu}(X)\partial X^\mu \partial X^\nu \tag{2.6}$$

corresponds to the space-time metric $G_{\mu\nu}$, with all other fields vanishing. Thus a similarity transformation (2.5) applied to T will, in general, produce a change in T which corresponds to a change in the space-time fields, *without changing the physics*. This change in the space-time fields is therefore a *symmetry* transformation. In this way, one may exhibit symmetries both familiar (general coordinate and two-form gauge transformations, regular non-abelian gauge transformations—including the Green-Schwarz modification—for the heterotic string [5]) and unfamiliar (an infinite class of spontaneously broken, level-mixing gauge symmetries).

We may clarify the way in which the change in the stress tensor may be interpreted as a change in the space-time fields by first considering the more general problem of deforming a conformal field theory (we now consider deformations which, while they preserve conformal invariance, *need not be symmetries*). It is straightforward to show that, to first order, the Virasoro algebras (2.3) are preserved by deforming the choice of stress tensor by a so-called *canonical deformation*,

$$\delta T(\sigma) = \delta\overline{T}(\sigma) = \Phi_{(1,1)}(\sigma) \tag{2.7}$$

where $\Phi_{(1,1)}(\sigma)$ is a primary field of dimension (1,1) with respect to the stress tensor. We reiterate: (2.7) does *not* in general correspond to a symmetry transformation, although it preserves conformal invariance. Since (1,1) primary fields are vertex operators for physical states, they are in natural correspondence with the space-time fields, and equation (2.7) makes the connection between changes of the stress tensor and changes of the space-time fields more transparent. Returning now to the problem of symmetries, if we take the generator h in equation (2.5) to be the zero mode of an infinitesimal (1,0) or (0,1) primary field (a current), then it is straightforward to see that its action on the stress tensor is necessarily a canonical deformation, as in equation (2.7), which may easily be translated into a change in the space-time fields. It is well known that *conserved* currents generate symmetries [6], but within the formalism described here, conservation is *not* necessary, a fact that does not seem to have been widely appreciated. Indeed, it may be shown that a non-conserved current generates a symmetry that is spontaneously broken by the particular background being considered [8].

Actually, we do not even need the generator, h, to be a current. By considering a few examples, it is easy to see that the canonical deformation of equation (2.7) corresponds to turning on space-time fields in a particular gauge (something like Landau or harmonic gauge), and so symmetries generated by zero-modes of currents preserve this gauge condition, since they generate canonical deformations. Furthermore, the commutator of zero modes of currents is not necessarily itself the zero-mode of a current. Thus restricting the generators h in this way we are dealing with a subset of the symmetry generators that do not even form a subalgebra.

On the other hand, equation (2.7) is *not* the most general infinitesimal deformation that preserves the Virasoro algebras (2.3). In [9] we showed that, for the massless degrees of freedom of the bosonic string in flat space, we could find a distinct deformation of the stress tensor for each solution of the linearized Brans-Dicke equations. This correspondence was found by considering the general translation invariant *ansatz* of naive dimension two for δT;

$$\delta T = H^{\nu\lambda}(X)\partial X_\nu \overline{\partial} X_\lambda + A^{\nu\lambda}(X)\partial X_\nu \partial X_\lambda + B^{\nu\lambda}(X)\overline{\partial} X_\nu \overline{\partial} X_\lambda + C^\nu(X)\partial^2 X_\nu + D^\lambda(X)\overline{\partial}^2 X_\lambda, \tag{2.8}$$

with a similar, totally independent *ansatz* for $\delta\overline{T}$. The fields $H^{\mu\nu}$ *etc.* turn out to be characterized in terms of solutions to the linearized Brans-Dicke equation when we demand that the deformation preserves (to first order) the Virasoro algebras (2.3). By considering this more general *ansatz*, we get more than just covariant equations of motion—we may also understand a larger set of symmetry generators, h. Indeed, any generator that preserves the form of the *ansatz* (2.8) must necessarily generate a change in the stress tensor that corresponds to a change in the space-time fields.

The condition that δT be of naive dimension two (with which we shall soon dispense) is preserved if h is of naive dimension zero. The condition of translation invariance is

$$[P, \delta T(\sigma)] = -i\delta T'(\sigma), \tag{2.9}$$

which may be preserved by demanding that h commute with P, the generator of translations, (2.2). (Equation (2.9) may also be thought of as a gauge condition, but not one that has any obvious interpretation in terms of the space-time fields). Taken together, these conditions restrict h to be the zero-mode of a field of naive dimension one.

The lesson to be drawn from this massless example is clear: the way to introduce space-time fields unconstrained by gauge conditions is to consider an *arbitrary translation invariant ansatz* for the deformation of the stress tensor, and to ask only that it preserve the Virasoro algebras. To move beyond the massless level, we simply drop the requirement that the naive dimension be two. We argued in [9] that this was likely to introduce auxiliary fields beyond the massless level, as with the superfield formulation of supersymmetric theories, but so be it. (Indeed, this whole formulation of string theory is rather akin to a superspace approach, with T and $\overline{T}$ as superfields and derivatives of the world-sheet scalars playing the rôle of the odd coordinates of superspace).

Having dropped any requirement on the naive dimension of δT, we know that *any* operator h that commutes with P will generate a symmetry transformation on our space-time fields (possibly including the auxiliaries). Furthermore, the centralizer of P is necessarily an algebra—the commutator of the zero modes of two fields is itself a zero-mode. We are therefore in a position to ask questions about the symmetry algebra that were beyond us as long as we restricted our generators to be zero-modes of currents.

As we shall explain in section 4, there are severe obstacles to calculating the full centralizer of P, but in the next section we shall consider a tractable subalgebra. Any D-dimensional string theory has a formulation of its CFT in terms of at least D scalar fields, X, (among others). The algebra we shall consider is generated by zero modes of fields that do not contain functions of the X themselves, but rather only holomorphic derivatives thereof.

3. Symmetry Algebra.

In the last section, we explained why any zero-mode in the operator algebra of a deformation class of CFTs generates a symmetry transformation of string theory. The symmetry algebra is just the algebra of these zero-modes. In this section we shall explicitly calculate the structure constants of an infinite-dimensional subalgebra of this full symmetry algebra—that generated by arbitrary normal-ordered products of holomorphic derivatives of the free scalar field, X, that is, operators of the form:

$$h = \int d\sigma\, \phi_{\mu\nu\cdots\rho} \partial^{w_1} X^\mu \partial^{w_2} X^\nu \cdots \partial^{w_n} X^\rho. \tag{3.1}$$

As such, it differs from other infinite-dimensional algebras that have appeared in string theory (Virasoro, affine, W), all of which are infinite because of the infinite moding of a (usually) finite set of fields. By contrast, our algebra arises from the zero-modes of an infinite set of fields.

Here and throughout the paper, the light-cone "derivatives" ∂ and $\overline{\partial}$ are to be interpreted, "as if in free-field theory." That is,

$$\partial X^\mu(\sigma) = \left(\eta^{\mu\nu}\pi_\nu(\sigma) + X^{\mu\prime}(\sigma)\right)/\sqrt{2}, \tag{3.2}$$

$$\partial^2 X^\mu(\sigma) = \left(\eta^{\mu\nu}\pi_\nu{}'(\sigma) + X^{\mu\prime\prime}(\sigma)\right), \qquad \text{etc.}, \tag{3.3}$$

the symbol on the left being simply a shorthand notation for the operator on the right, which should make sense for *all* the CFTs in the deformation class, even though the object in equation (3.2), for example, is only the light-cone derivative of the scalar field X^μ for the particular case of free field theory.

Operators of the form of equation (3.1) are clearly specified by a tensor on R^D, ϕ, and a weight, w_i associated with each index μ_i. The operators are to be normal-ordered with respect to the free creation and annihilation operators, a fact which will be denoted, when necessary, by the usual normal ordering symbol, $: :$. In section 1, we gave an abstract specification of these elements in terms of tensors and their associated weights. The origin of the identifications in equations (1.2)–(1.4) should now be clear: the operators h are linear in the tensors for fixed weights, which explains equation (1.2); normal ordering means that the order of the weighted indices does not matter, equation (1.4); equation (1.3) is simply the statement that the integral of a total derivative vanishes (Δ_r differentiates the first r factors in h).

The operators of equation (3.1) are associated with a *deformation class* of CFTs, not with a particular field theory, and their equal-time commutators are similarly independent of any particular choice of CFT. We may therefore compute the structure constants in any convenient theory and yet know that that the result will hold for the entire deformation class. That is, the structure constants will apply to string *theory*, not just to some particular solution. It is clearly simplest to calculate in free field theory, where the integrands are holomorphic, and we may easily use the operator product expansion to compute commutators.

We want to calculate the equal-time commutators

$$[h_1, h_2] = [\int d\sigma\, \psi_{\mu\nu\cdots\rho}\partial^{w_1}X^\mu \cdots \partial^{w_n}X^\rho, \int d\sigma' \chi_{\kappa\lambda\cdots\sigma}\partial^{v_1}X^\kappa \cdots \partial^{v_m}X^\sigma] \tag{3.4}$$

Our starting point will be the operator product expansion on the complex plane

$$\psi_{\mu\nu\cdots\rho}\partial^{w_1}X^\mu\partial^{w_2}X^\nu \cdots \partial^{w_n}X^\rho(u)\, \chi_{\kappa\lambda\cdots\sigma}\partial^{v_1}X^\kappa\partial^{v_2}X^\lambda \cdots \partial^{v_m}X^\sigma(z) \tag{3.5}$$

In general this operator product expansion will include singular terms of the form $\frac{\Theta_m(w,\overline{w})}{(u-z)^m}$. Performing the contour integration and transforming back on the cylinder will produce terms $\delta^{(m-1)}(\sigma-\sigma')\Theta_m(\sigma')$. Since we are interested in the commutators of the zero modes of these operators, most of these terms will give zero upon integration over σ. The only non-zero result will arise from simple poles in the operator product expansion, and it is therefore the coefficient of this simple pole that we must calculate.

Taking into account that the two-point function for free bosons is $\langle X^\mu(z)X_\nu(w)\rangle = -\log(z-w)\delta^\mu_\nu$, we can prove the following formula by induction:

$$\partial^{w_i}X^\mu(u)\partial^{v_j}X_\nu(z) = \frac{(-1)^{w_i}(w_i+v_j-1)!}{(u-z)^{w_i+v_j}} + \ldots \tag{3.6}$$

The problem therefore reduces to a calculation of the simple pole in the operator product of the two integrands. This is not hard, the principle difficulty being figuring out how to write Wick's theorem in a way that sufficiently automates the calculation. This may be done as follows:

$$:\prod_{i\in\mathcal{S}}\partial^{w_i}X^{\mu_i}(u)::\prod_{j\in\mathcal{T}}\partial^{v_j}X^{\nu_j}(z): = \sum_{\substack{\mathcal{U}\subseteq\mathcal{S}\\ \mathcal{P}:\mathcal{U}\hookrightarrow\mathcal{T}}}\prod_{k\in\mathcal{U}}\langle\partial^{w_k}X^{\mu_k}(u)\partial^{v_{\mathcal{P}(k)}}X^{\nu_{\mathcal{P}(k)}}(z)\rangle$$
$$:\prod_{i\in\mathcal{S}-\mathcal{U}}\partial^{w_i}X^{\mu_i}(u)\prod_{j\in\mathcal{T}-\mathcal{P}(\mathcal{U})}\partial^{v_j}X^{\nu_j}(z):. \tag{3.7}$$

As discussed in section 1, $\mathcal{S}$ and $\mathcal{T}$ are index sets labeling the factors in the two normal-ordered operators whose product is to be taken. The sum is over all pairs of subsets $\mathcal{U} \in \mathcal{S}$ and injections $\mathcal{P}:\mathcal{U} \hookrightarrow \mathcal{T}$, which is to say, the sum over all contractions. We now substitute our formula for the contractions, equation (3.6), into equation (3.7) to obtain,

$$:\prod_{i\in\mathcal{S}}\partial^{w_i}X^{\mu_i}(u)::\prod_{j\in\mathcal{T}}\partial^{v_j}X^{\nu_j}(z): = \sum_{\substack{\mathcal{U}\subseteq\mathcal{S}\\ \mathcal{P}:\mathcal{U}\hookrightarrow\mathcal{T}}}\prod_{k\in\mathcal{U}}\frac{\eta^{\mu_k\nu_{\mathcal{P}(k)}}(-1)^{w_k}(w_k+v_{\mathcal{P}(k)}-1)!}{(u-z)^{w_k+v_{\mathcal{P}(k)}}}$$
$$:\prod_{i\in\mathcal{S}-\mathcal{U}}\partial^{w_i}X^{\mu_i}(u)\prod_{j\in\mathcal{T}-\mathcal{P}(\mathcal{U})}\partial^{v_j}X^{\nu_j}(z):. \tag{3.8}$$

The next step is to extract the simple pole from equation (3.8). To do this we must Taylor expand the u-dependent part of the normal-ordered operators about z:

$$f(u) = \sum_{n=0}^{\infty}\frac{1}{n!}\partial^n f(z)(u-z)^n, \tag{3.9}$$

and observe that, for a given pair $\mathcal{U}$ and $\mathcal{P}$, the leading pole in equation (3.8) is of order $\sum_{k\in\mathcal{U}} w_k + v_{\mathcal{P}(k)}$. To pick out the simple pole, we must therefore take the term with $n = \sum_{k\in\mathcal{U}}(w_k+v_{\mathcal{P}(k)})-1$ from the Taylor expansion, equation (3.9). This yields

$$:\prod_{i\in\mathcal{S}}\partial^{w_i}X^{\mu_i}(u)::\prod_{j\in\mathcal{T}}\partial^{v_j}X^{\nu_j}(z): = \sum_{\substack{\mathcal{U}\subseteq\mathcal{S}\\ \mathcal{P}:\mathcal{U}\hookrightarrow\mathcal{T}}}\frac{\prod_{k\in\mathcal{U}}\eta^{\mu_k\nu_{\mathcal{P}(k)}}(-1)^{w_k}(w_k+v_{\mathcal{P}(k)}-1)!}{(u-z)(\sum_{k\in\mathcal{U}}(w_k+v_{\mathcal{P}(k)})-1)!}$$
$$:\partial^{\left(\sum_{k\in\mathcal{U}}(w_k+v_{\mathcal{P}(k)})-1\right)}\Big(\prod_{i\in\mathcal{S}-\mathcal{U}}\partial^{w_i}X^{\mu_i}\Big)\prod_{j\in\mathcal{T}-\mathcal{P}(\mathcal{U})}\partial^{v_j}X^{\nu_j}(z): + \ldots \tag{3.10}$$

where the ellipsis refers to other powers of $(u-z)$ which are of no interest to us.

Upon contracting both sides of equation (3.10) with the coefficient tensors $\psi_{\mu_\mathcal{S}}$ and $\chi_{\nu_\mathcal{T}}$ (here, the subscript $\mu_\mathcal{S}$ indicates the full set of indices indexed by $\mathcal{S}$), we read off the commutator,

$$[h_1,h_2] = \sum_{\substack{\mathcal{U}\subseteq\mathcal{S}\\ \mathcal{P}:\mathcal{U}\hookrightarrow\mathcal{T}}}\int d\sigma\,\psi_{\mu_\mathcal{S}}\chi_{\nu_\mathcal{T}}\frac{\prod_{k\in\mathcal{U}}\eta^{\mu_k\nu_{\mathcal{P}(k)}}(-1)^{w_k}(w_k+v_{\mathcal{P}(k)}-1)!}{(\sum_{k\in\mathcal{U}}(w_k+v_{\mathcal{P}(k)})-1)!}$$
$$:\partial^{\left(\sum_{k\in\mathcal{U}}(w_k+v_{\mathcal{P}(k)})-1\right)}\Big(\prod_{i\in\mathcal{S}-\mathcal{U}}\partial^{w_i}X^{\mu_i}\Big)\prod_{j\in\mathcal{T}-\mathcal{P}(\mathcal{U})}\partial^{v_j}X^{\nu_j}(\sigma):. \tag{3.11}$$

This is precisely the commutator given, in more abstract form, in section 1, so our calculation is completed.

4. Problems with More General Commutators.

In the last section, we computed explicitly the commutators of zero-modes of *holomorphic* operators constructed out of derivatives of scalar fields. We wish to emphasize that we see no problem with this calculation, but we limited ourselves to this subalgebra precisely because there are problems associated with more general commutators. This section is devoted to a description of these difficulties, and as such its only logical connection to its predecessor is that it explains why we restricted the scope of our work as we did.

We were able to study the subalgebra of the previous section because the fields in question are holomorphic and their operator products contained only poles. Unfortunately, this is not always, or even usually, the case in string theory. When fields contain functions of a scalar field (not just derivatives) the short distance singularities may be more complicated since

$$e^{ip\cdot X_L(z)}e^{iq\cdot X_L(w)} \sim \frac{e^{i(p+q)\cdot X_L(w)}}{(z-w)^{-p\cdot q}}, \tag{4.1}$$

and $p\cdot q$ need not be an integer.

Moreover, we are *not* just interested in the commutators of (anti-)holomorphic fields; any operator containing both $\partial^w X$ and $\overline{\partial^v} X$ as factors will not be holomorphic, and the same is true for any non-constant function of $X = X_L(z) + X_R(\overline{z})$. This is important because the relationship between the operator product expansion and equal-time commutators holds only if one of the fields is (anti-)holomorphic. (Recall that demonstrating this relationship involves deforming contours and invoking Cauchy's theorem.)

Nor are these difficulties easily circumvented. For example, one might try a, "conformal block," representation of a non-holomorphic operator as a sum of products of holomorphic and anti-holomorphic fields, and then try to construct the full commutator out of the commutators for the (anti-)holomorphic components. Unfortunately, a simple example shows that this approach is inadequate. Consider the commutator $\left[A\overline{A}(\sigma), B\overline{B}(\sigma')\right]$, where A and B are holomorphic and $\overline{A}$ and $\overline{B}$ are anti-holomorphic. Then,

$$\left[A\overline{A}(\sigma), B\overline{B}(\sigma')\right] = \left[A(\sigma), B(\sigma')\right]\overline{B}(\sigma')\overline{A}(\sigma) + A(\sigma)B(\sigma')\left[\overline{A}(\sigma), \overline{B}(\sigma')\right]. \tag{4.2}$$

Here we have assumed that, as is usually the case, the holomorphic and anti-holomorphic operators are constructed from disjoint, mutually commuting sets of creation and annihilation operators.

The problem with equation (4.2) is that each commutator will typically be very local, being constructed out of $\delta(\sigma-\sigma')$ and its derivatives. Unfortunately this means that we must take the product of the fields outside the commutator (*e.g.* $\overline{B}(\sigma')\overline{A}(\sigma)$ in the first term) at the same point—a clear signal of possible trouble. Note that the product of fields outside the commutators is *not* normal-ordered. At the very least, then, there is a problem to be addressed over and above knowing the commutators of the (anti-)holomorphic blocks.

Evidently, we need to calculate commutators with greater care. We shall therefore proceed by defining composite fields through a point-splitting regularization and renormalization, calculating the commutators at finite splitting and finally taking the splitting to zero. This rather long-winded method yields the correct result when applied to holomorphic operators, and so we may be reasonably confident of its validity. However, we shall see that in many cases of interest, taking the point-splitting to zero does not yield a finite limit.

It will be instructive to sketch first a calculation that *does* work, and the obvious choice is the Virasoro algebra of the stress tensor of a single free scalar, which we define by point splitting as follows:

$$\begin{aligned} T(\sigma) &= \tfrac{1}{2}{:}\partial X\partial X(\sigma){:} \\ &= \lim_{\epsilon\to 0} \tfrac{1}{2}\partial X(\sigma)\partial X(\sigma+\epsilon) + \frac{1}{4\pi\epsilon^2} \end{aligned} \tag{4.3}$$

Hence we may calculate the commutator:

$$\begin{aligned} 4\left[T(\sigma), T(\sigma')\right] &= \lim_{\epsilon,\epsilon'\to 0}\left[\partial X(\sigma)\partial X(\sigma+\epsilon), \partial X(\sigma')\partial X(\sigma'+\epsilon')\right] \\ &= \lim_{\epsilon,\epsilon'\to 0} i\left(\partial X(\sigma+\epsilon)\partial X(\sigma'+\epsilon')\delta'(\sigma-\sigma') + \partial X(\sigma)\partial X(\sigma'+\epsilon')\delta'(\sigma+\epsilon-\sigma')\right. \\ &\qquad \left. +\partial X(\sigma')\partial X(\sigma)\delta'(\sigma+\epsilon-\sigma'-\epsilon') + \partial X(\sigma')\partial X(\sigma+\epsilon)\delta'(\sigma-\sigma'-\epsilon')\right) \\ &= 4i{:}\partial X(\sigma)\partial X(\sigma'){:}\delta'(\sigma-\sigma') - \lim_{\epsilon,\epsilon'\to 0}\frac{i}{2\pi}\left\{\frac{\delta'(\sigma-\sigma')}{(\sigma+\epsilon-\sigma'-\epsilon')^2}\right. \\ &\qquad \left. +\frac{\delta'(\sigma+\epsilon-\sigma')}{(\sigma-\sigma'-\epsilon')^2} + \frac{\delta'(\sigma+\epsilon-\sigma'-\epsilon')}{(\sigma-\sigma')^2} + \frac{\delta'(\sigma-\sigma'-\epsilon')}{(\sigma+\epsilon-\sigma')^2}\right\} \end{aligned} \tag{4.4}$$

Now using $f(\sigma)\delta'(\sigma-\sigma') = f(\sigma')\delta'(\sigma-\sigma') - f'(\sigma')\delta(\sigma-\sigma')$, equation (4.4) becomes

$$
\begin{aligned}
[T(\sigma), T(\sigma')] = 2iT(\sigma')\delta'(\sigma-\sigma') - iT'(\sigma')\delta(\sigma-\sigma') - \lim_{\epsilon,\epsilon'\to 0} \frac{i}{8\pi} \Bigg\{ &\frac{\delta'(\sigma-\sigma') + \delta'(\sigma+\epsilon-\sigma'-\epsilon')}{(\epsilon-\epsilon')^2} \\
+ \frac{\delta'(\sigma+\epsilon-\sigma') + \delta'(\sigma-\sigma'-\epsilon')}{(\epsilon+\epsilon')^2} &+ 2\frac{\delta(\sigma-\sigma') - \delta(\sigma+\epsilon-\sigma'-\epsilon')}{(\epsilon-\epsilon')^3} \\
+2\frac{\delta(\sigma-\sigma'-\epsilon') - \delta(\sigma+\epsilon-\sigma')}{(\epsilon+\epsilon')^3} \Bigg\} &\qquad (4.5)
\end{aligned}
$$

It is a very non-trivial fact that the right-hand side of equation (4.5) does have a sensible limit, the non-trivial term being $\frac{-i}{24\pi}\delta'''(\sigma-\sigma')$, in agreement with equation (2.3). (Of course, to show this properly we should convolve these distributions with suitable test functions, but our slightly heuristic arguments wherein we, "differentiate," delta-functions carry over directly to a more pedantic proof.) It is precisely the fact that these singular terms do not always group themselves into nice derivatives of delta-functions that is the problem we wish to draw attention to in this section.

Let us therefore now exhibit an example where the above method fails; consider the commutator $\mathcal{C} = \left[e^{ip\cdot X}\partial X(\sigma),\, e^{iq\cdot X}\partial X(\sigma')\right]$. As in equation (4.3), we must first define what we mean by the normal-ordered operators in this commutator:

$$
:e^{ip\cdot X}\partial X(\sigma): = \lim_{\epsilon\to 0}\left\{ :e^{ip\cdot X(\sigma)}:\partial X(\sigma+\epsilon) - \frac{i\sqrt{2}p\,:e^{ip\cdot X(\sigma)}:}{4\pi\epsilon}\right\}. \qquad (4.6)
$$

Then the commutator we wish to calculate is

$$
\begin{aligned}
\mathcal{C} = \left[e^{ip\cdot X}\partial X(\sigma)\,,\, e^{iq\cdot X}\partial X(\sigma')\right] = \lim_{\epsilon,\epsilon'\to 0}\Bigg[&:e^{ip\cdot X(\sigma)}:\partial X(\sigma+\epsilon) - \frac{i\sqrt{2}p:e^{ip\cdot X(\sigma)}:}{4\pi\epsilon}\,, \\
&:e^{iq\cdot X(\sigma')}:\partial X(\sigma'+\epsilon') - \frac{i\sqrt{2}q\,:e^{iq\cdot X(\sigma')}:}{4\pi\epsilon'}\Bigg]. \quad (4.7)
\end{aligned}
$$

With a little algebra this expands to

$$
\begin{aligned}
\mathcal{C} = \lim_{\epsilon,\epsilon'\to 0}\big\{&(q/\sqrt{2})\delta(\sigma-\sigma'+\epsilon)\,:e^{ip\cdot X(\sigma'-\epsilon)}::e^{iq\cdot X(\sigma')}:\partial X(\sigma'+\epsilon') \\
&- (p/\sqrt{2})\delta(\sigma-\sigma'-\epsilon')\,:e^{iq\cdot X(\sigma')}::e^{ip\cdot X(\sigma'+\epsilon')}:\partial X(\sigma'+\epsilon+\epsilon') \\
&+ i\delta'(\sigma-\sigma'+\epsilon-\epsilon')\,:e^{ip\cdot X(\sigma'+\epsilon'-\epsilon)}::e^{iq\cdot X(\sigma')}: + p\delta(\sigma-\sigma'+\epsilon-\epsilon')\,:e^{ip\cdot X(\sigma'+\epsilon'-\epsilon)}::e^{iq\cdot X(\sigma')}: \\
&+ (ip^2/4\pi\epsilon)\delta(\sigma-\sigma'-\epsilon')\,:e^{iq\cdot X(\sigma')}::e^{ip\cdot X(\sigma'+\epsilon')}: \\
&- (iq^2/4\pi\epsilon')\delta(\sigma-\sigma'+\epsilon)\,:e^{ip\cdot X(\sigma'-\epsilon)}::e^{iq\cdot X(\sigma')}:\big\} \qquad (4.8)
\end{aligned}
$$

As in the case of the Virasoro algebra (the last step in equation (4.4)), we must now rewrite the right hand side of equation (4.8) in terms of normal-ordered operators, and then attempt to take the limits. In order to do this we need one more piece of information—how to normal-order products of exponentials at nearby points. The answer is,

$$
:e^{ip\cdot X(\sigma)}::e^{iq\cdot X(\sigma+\epsilon)}: = :e^{ip\cdot X(\sigma)+iq\cdot X(\sigma+\epsilon)}:\epsilon^{p\cdot q/2\pi}, \qquad (4.9)
$$

where we assume that $\epsilon > 0$. Of course $p\cdot q$ can take any real value, so that in normal ordering these exponentials (and so in normal ordering any product of functions of X) we may get arbitrarily disgusting singularities. It is the appearance of these non-integer powers of ϵ that, more than anything else, dooms our attempt to make sense of this operator algebra, since, in contrast to the Virasoro case above, they are clearly *not* going to group themselves into derivatives of delta-functions. Nevertheless, let us carry the calculation through to the bitter end. Expressing equation (4.8) in terms of normal-ordered operators yields

$$\begin{aligned}\mathcal{C} = \lim_{\epsilon,\epsilon'\to 0}\big\{&(-p/\sqrt{2})\epsilon'^{p\cdot q/2\pi}\delta(\sigma-\sigma'-\epsilon')\,{:}e^{iq\cdot X(\sigma')+ip\cdot X(\sigma'+\epsilon')}\partial X(\sigma'+\epsilon+\epsilon'){:}\\ &+(q/\sqrt{2})\epsilon^{p\cdot q/2\pi}\delta(\sigma-\sigma'+\epsilon)\,{:}e^{iq\cdot X(\sigma')+ip\cdot X(\sigma'-\epsilon)}\partial X(\sigma'+\epsilon'){:}\\ &+|\epsilon-\epsilon'|^{p\cdot q/2\pi}\left\{i\delta'(\sigma-\sigma'+\epsilon-\epsilon')+p\delta(\sigma-\sigma'+\epsilon-\epsilon')\right\}{:}e^{iq\cdot X(\sigma')+ip\cdot X(\sigma'+\epsilon'-\epsilon)}{:}\\ &-\frac{ip\cdot q}{4\pi(\epsilon+\epsilon')}\big(\epsilon'^{p\cdot q/2\pi}\delta(\sigma-\sigma'-\epsilon')\,{:}e^{iq\cdot X(\sigma')+ip\cdot X(\sigma'+\epsilon')}{:}\\ &-\epsilon^{p\cdot q/2\pi}\delta(\sigma-\sigma'+\epsilon)\,{:}e^{iq\cdot X(\sigma')+ip\cdot X(\sigma'-\epsilon)}{:}\big)\big\}.\end{aligned} \tag{4.10}$$

Clearly, equation (4.10) has no sensible limit as $\epsilon, \epsilon' \to 0$ for generic $p \cdot q < 0$, and we have failed in our attempt to define the commutator of equation (4.7).

We have shown, then, that not all *normal-ordered* fields have well-defined commutators. Perhaps this is surprising at first sight, since we might as well have been dealing with free-field theory, where we normally think of normal-ordering as being sufficient. However, a little thought convinces us that this need not be the case. Any calculation in an interacting field theory may be reduced to a calculation in free-field theory with composite operators. The reason is simply that, in calculating some Greens function (for example), operators at different times may be represented as operators at some fixed time acted on by exponentials of the Hamiltonian (a composite operator). However, we know full well that there are divergences in such objects, and that normal ordering is not a sufficient palliative—there are additional, logarithmic divergences which need counterterms at each order in perturbation theory. Indeed, the divergences that gave rise to the difficulties in equation (4.10) are precisely these logarithmic divergences. $\epsilon^{p\cdot q/2\pi}$ may not look like $\ln \epsilon$, but that's only because powers of logarithms have been summed; if we expand $e^{ip\cdot X(\sigma)}$ we will have to deal with contractions of powers of $X(\sigma)$, which yield logarithms.

It might be thought that restricting our attention to $(1,1)$ primary fields will save us, since they are associated with conformal deformations of the stress tensor, and conformal field theories are ultra-violet finite. Alas, this is not so; in general, $(1,1)$ primaries preserve conformal invariance only to *first* order in the deformation, and deforming by a *finite* (1,1) primary will, in general, yield a non-conformal theory which needs further renormalization. (Deformations where this does not occur are sometimes called *strictly marginal*.)

This concludes our arguments in support of the proposition that the *normal ordered* operators encountered in string theory are frequently sick. However, even the astute reader who accepts these arguments may nevertheless argue that this problem is easily fixed. Surely it is sufficient to add the appropriate logarithmic counterterms to the definitions of our operators. That is, we should add terms involving $\ln \epsilon$ to the right hand sides of equations (4.9), (4.6) and, perhaps, (4.3). Alas! we shall argue that our experience with renormalizable field theories indicates that such a strategy need not succeed.

The reason is that the task we have set ourselves is more demanding than the one we usually face when renormalizing a field theory. In that case we need only make sense of *one* composite operator, namely the Hamiltonian. We, on the other hand, must make sense of an *infinite* number of composite operators, assigning fixed counterterms to each in such a way that the commutator of any pair is well defined. Consider equation (4.10), the result of calculating the commutator in equation (4.7). To the first operator in (4.7) we must add counterterms *depending only on* p, while the corresponding counterterms with $p \to q$ should be added to the second operator. However, it is very hard to imagine how such counterterms could cancel the $\epsilon^{p\cdot q/2\pi}$ terms in equation (4.10).

There is a simpler and more compelling example of this difficulty in renormalizable field theory.* In that case, the renormalization procedure makes sense of an interacting Hamiltonian, H, that is formally written as the sum of a free part and an interacting part,

$$H = H_0 + gH_I, \tag{4.11}$$

where g is the coupling. However, we shall argue that H_0 and gH_I are *not* in fact *separately* well defined. The reason is that g depends on the regulator, ϵ; typically, $g(\epsilon) = 1/(1+\beta\ln(\mu\epsilon))$, where μ is some arbitrary renormalization scale and β is some calculable number. The *only* freedom we have in defining H is in the variation of μ. On the other hand, if H_0 and gH_I were separately well defined, we would be able to add them with an arbitrary relative weight: $H = H_0 + \lambda gH_I$, with λ some finite (ϵ-independent) number. This, however, does *not* correspond to a finite (ϵ-independent) rescaling of μ (it requires an unacceptable change in β also, to put it back in canonical form). We therefore conclude, *reductio ad absurdum*, that H_0 and gH_I are *not* simultaneously well-defined (although their sum is). Thus the problem of simultaneously defining several composite operators is not, in general, solved by the conventional renormalization procedure.

As we said earlier, we are not sure what to make of this observation. It would be premature to conclude that there is something wrong with string theory, since we seem to be able to construct perfectly satisfactory scattering amplitudes. However, it does suggest that arguments based on the existence of an algebra of composite operators need to be treated with caution. Perhaps the way round this difficulty is to use the operator formalism [15], where, despite its name, the operator algebra is superfluous. Indeed, there are proposals for how to make finite deformations of conformal field theories within that formalism [16], a problem that would seem to require solving or evading the difficulty pointed out here.

5. Conclusions.

Let us review the results presented in this paper. We gave a rather sketchy outline of previous work on deformations of conformal field theories and its relation to the symmetries of string theory. For the purposes of this paper, the important idea to be drawn from this work is that symmetries are generated by a subalgebra of the full operator algebra of (the deformation class of) a conformal field theory. The subalgebra in question is the centralizer of the generators of world-sheet translations, which is a fancy way of saying all zero modes of quantum fields. It should also be noted that this statement may only be true after the introduction of auxiliary space-time fields, as in supersymmetric field theories.

However, most of the operators of interest are composite; they involve products of elementary fields at the same point, and so are potentially sick. We showed that such fears are justified; many of the normal-ordered composite operators that we habitually use in string theory (including, for example, vertices and currents) do not have well defined commutators.

Nevertheless, we did identify an infinite set of fields that do have well-defined commutators between themselves, and we computed the commutators of an arbitrary pair of zero modes. This then constitutes an infinite-dimensional subalgebra of the full symmetry of string theory.

This algebra is unfamiliar to us, and differs from most of the infinite algebras that have appeared in physics in that the infinite-dimensionality does not arise through the moding of a finite number of fields, but rather through having an infinite number of fields in the first place. The closest relative of our algebra would appear to be W_∞ [17], which may be realized in terms of the fields of equation (3.1) with weights $w_i = 1$ [18]. However, to obtain W_∞ it is necessary to retain *all* modes of these fields, while the zero-modes form only the Cartan subalgebra. The apparent kinship is therefore rather distant.

* This argument was developed in conversation with Alex Kovner.

Our algebra naturally possesses the general properties of string symmetries remarked in earlier work [8], [9]: it is a supersymmetry in that its generators do not commute with the generators of Lorentz transformations and mix excitations of different spin; it is spontaneously broken in flat space-time because not all generators commute with the stress tensor of the free scalar CFT, and it transforms excitations of differing mass into one another.

Naively, the symmetries we are discussing should be local symmetries. If the constant tensors ψ and χ in the generators were to depend upon the scalar field, $X^\mu(\sigma)$, the locality would be apparent, but it is precisely the appearance of X-dependence that wrought havoc with commutators, as we saw in section 4. In some sense, then, our sub-algebra would appear to be the *global* part of a gauge symmetry. This is rather encouraging, since global parts are usually the repository of most physics information (relations between couplings, conserved charges, *etc.*). However, the full global algebra should presumably include both holomorphic and anti-holomorphic derivatives, and prior work suggests that transformations involving *propagating* (as opposed to auxiliary) degrees of freedom would be generated by operators more evenly balanced between the two types of derivatives. Thus the physical importance of our sub-algebra remains to be determined. Commutators between such mixed generators are not calculable by the methods of section 3, but may yield those of section 4. This question is currently under investigation.

As we have remarked, the pathologies associated with the various composite operators of string theory are something of an enigma. Despite this problem, they seem to permit us to calculate perfectly fine scattering amplitudes. We can only hope that further work will shed some light on this puzzle.

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