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On the Spectrum and Scattering of W_3 Strings

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ABSTRACT

We present a detailed investigation of scattering processes in W_3 string theory. We discover further physical states with continuous momentum, which involve excitations of the ghosts as well as the matter, and use them to gain a better understanding of the interacting theory. The scattering amplitudes display factorisation properties, with states from the different sectors of the theory being exchanged in the various intermediate channels. We find strong evidence for the unitarity of the theory, despite the unusual ghost structure of some of the physical states. Finally, we show that by performing a transformation of the quantum fields that involves mixing the ghost fields with one of the matter fields, the structure of the physical states is dramatically simplified. The new formalism provides a concise framework within which to study the W_3 string.

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1. Introduction

Since the discovery of W_3 symmetry [1] in two-dimensional field theories much work has been carried out on the construction of W_3-string theories [2–12]. Most of these efforts have been concerned with the understanding of their physical spectra [2–10,12]. Until recently, attention had been focussed on physical states with standard ghost structure, which are direct analogues of the physical states in the usual 26-dimensional bosonic string. It has been recently realised that there are also physical states with non-standard ghost structure in the W_3 string, *i.e.* states that involve excitations of the ghost fields as well as the matter fields [13,9–12]. Unlike the two-dimensional string, where states of this kind occur too [14,15], they exist not only in the two-scalar W_3 string where the momenta are discrete, but also in the *multi*-scalar W_3 string where the momenta are in general continuous. The occurrence of states involving excitations of the ghost fields in the physical spectrum is rather unusual in a gauge theory, and normally one might expect that it would be associated with a violation of unitarity. However, as was discussed in [11], and will be explored further in this paper, it seems that these states *are* unitary, and furthermore, they are an essential ingredient of the W_3 string. In fact, as we recently showed [11], the existence of physical states with non-standard ghost structure resolves the long-standing puzzle of how to introduce interactions in the W_3 string.

As in ordinary string theory, interactions have to be introduced by hand. The guiding principle that determines acceptable interactions is that they should respect the symmetries of the theory. This means in the present context that they should lead to BRST-invariant scattering amplitudes. The stumbling-block to introducing interactions in the W_3 string was that for a long time only the physical states with standard ghost structure were known, and with these it appears to be impossible to write down any non-vanishing BRST-invariant amplitude. The reason for this is that these physical states, and indeed *all* physical states in the W_3 string, have specific values of momentum in a particular "frozen" direction, and the particular values that occur for the states with standard ghost structure imply that momentum conservation in this direction cannot be satisfied in interactions.

In [11], we presented a procedure for building gauge-invariant scattering amplitudes in the W_3 string. An essential aspect is that non-vanishing scattering amplitudes necessarily involve external physical states with non-standard ghost structure. The problem with momentum conservation in the frozen direction is avoided because these states have different values of frozen momentum from those of the states with standard ghost structure. In this paper we shall review and elaborate on this procedure, resolve some open problems in [11], and begin the unravelling of the general structure of W_3-string scattering.

The paper is organised as follows. Section 2 contains a summary of the main results in [11] and outlines several issues that we shall explore further in this paper.

We begin section 3 by finding new physical states of the W_3 string; they have higher levels, and lower ghost numbers, than the ones that were known previously. All the physical states we know fall into two categories: those with continuous on-shell spacetime momentum, and those where the on-shell spacetime momentum can only take a discrete value (in fact, zero). The new physical states enable us to gain a better understanding of the interactions of the W_3 string. In particular, we are now able to construct certain interactions that could not have been obtained with the physical states of [11] alone. We then explain how the physical states can be organised into multiplets and discuss their structure in detail. One of the main features that emerges from the examples of physical states with continuous momentum that we have constructed is that they all admit an effective spacetime interpretation. By this we mean that a physical state of this kind factorises into a direct product of a piece involving the ghost fields together with the frozen coordinate, and another piece involving the remaining coordinates which can then be viewed as the coordinates of the target spacetime. Moreover, we find that from the effective spacetime point of view, these states fall into three sectors, with effective conformal dimension $L_0^{\rm eff}=1$, $\frac{15}{16}$ and $\frac{1}{2}$.

In section 4 we use the known physical states to build new four-point and five-point scattering amplitudes. One of these is a four-point function for four physical states in the $L_0^{\rm eff}=1$ sector. It provides a useful non-trivial check of our procedure, since it should, and indeed does, agree with the corresponding result in ordinary bosonic string theory (at tree level). We also show that physical states in the $L_0^{\rm eff}=\frac{15}{16}$ sector play a special rôle in the theory in that there are only very limited possibilities for them to interact with other states. Specifically, we find that no more than two of the physical states in a non-vanishing N-point function can have $L_0^{\rm eff}=\frac{15}{16}$.

Section 5 is divided into two subsections. In the first we make some general observations about the physical spectrum of the W_3 string and derive necessary conditions for the level numbers and momenta in the frozen direction of physical states which admit an effective spacetime interpretation and have $L_0^{\rm eff}$ taking values in the set $\{1,\ \frac{15}{16},\ \frac{1}{2}\}$. In the second subsection we consider the unitarity of the theory. We show that if all the physical states with continuous momentum indeed admit an effective spacetime interpretation and have the above values of $L_0^{\rm eff}$, then the W_3 string is unitary. Further evidence for unitarity comes from looking at the residues of the poles of the scattering amplitudes.

Motivated by the fact that all the continuous-momentum physical states known so far admit an effective spacetime interpretation, in section 6 we introduce a formalism that exploits this. It consists of making a certain transformation of the quantum fields, which mixes the ghost fields with the frozen coordinate. It is a canonical transformation, in the sense that the new fields satisfy the same set of operator-product expansions as the original ones. In terms of these redefined fields, the known physical operators acquire a remarkably simple form. A striking example is provided by a level 8 physical operator, which reduces from 82 terms to 4 after transforming to the new fields. Moreover, this formalism will make more transparent some of the features of the W_3 string that we have encountered in this paper. The simplification achieved by this transformation is highly suggestive that the new formalism is the appropriate one for describing the W_3 string.

The paper ends with concluding remarks in section 7.

2. A review of the W_3 string

In this section we review the procedure and main results of [11]. In order to make the present paper as self-contained as possible, and in order to establish notation, a certain amount of repetition is unavoidable.

The key ingredient for determining the physical spectrum of the W_3 string is the construction of the BRST operator [16], which is given by

$$Q_B = \oint dz \Big[c\,(T + \tfrac{1}{2} T_{\rm gh}) + \gamma\,(W + \tfrac{1}{2} W_{\rm gh}) \Big], \tag{2.1}$$

and is nilpotent provided that the matter currents T and W generate the W_3 algebra with central charge $c = 100$, and that the ghost currents are chosen to be

$$\begin{aligned} T_{\rm gh} &= -2b\,\partial c - \partial b\, c - 3\beta\,\partial\gamma - 2\partial\beta\,\gamma\ , \qquad (2.2)\\ W_{\rm gh} &= -\partial\beta\, c - 3\beta\,\partial c - \tfrac{8}{261}\big[\partial(b\,\gamma\, T) + b\,\partial\gamma\, T\big] \\ &\quad + \tfrac{25}{1566}\Big(2\gamma\,\partial^3 b + 9\partial\gamma\,\partial^2 b + 15\partial^2\gamma\,\partial b + 10\partial^3\gamma\, b\Big), \qquad (2.3) \end{aligned}$$

where the ghost-antighost pairs $(c,\,b)$ and $(\gamma,\,\beta)$ correspond respectively to the T and W generators. A matter realisation of W_3 with central charge 100 can be given in terms of $n \geq 2$ scalar fields, as follows [17]:

$$\begin{aligned} T &= -\tfrac{1}{2}(\partial\varphi)^2 - Q\partial^2\varphi + T^{\rm eff}, \\ W &= -\frac{2i}{\sqrt{261}}\Big[\tfrac{1}{3}(\partial\varphi)^3 + Q\partial\varphi\partial^2\varphi + \tfrac{1}{3}Q^2\partial^3\varphi + 2\partial\varphi T^{\rm eff} + Q\partial T^{\rm eff}\Big], \end{aligned} \tag{2.4}$$

where $Q^2 = \frac{49}{8}$ and $T^{\rm eff}$ is an energy-momentum tensor with central charge $\frac{51}{2}$ that commutes with φ. Since $T^{\rm eff}$ has a fractional central charge, it cannot be realised simply by taking free scalar fields. We can however use d scalar fields X^μ with a background-charge vector a_μ:

$$T^{\rm eff} = -\tfrac{1}{2}\partial X_\mu \partial X^\mu - i a_\mu \partial^2 X^\mu, \tag{2.5}$$

with a_μ chosen so that $\frac{51}{2} = d - 12 a_\mu a^\mu$ [17].

Physical states are by definition states that are annihilated by the BRST operator (2.1) but that are not BRST trivial. Such states with standard ghost structure are of the form:

$$\big|\chi\big\rangle = \big|{-}{-}\big\rangle \otimes \big|\varphi\,, X\big\rangle. \tag{2.6}$$

Here $\big|{-}{-}\big\rangle$ is the standard ghost vacuum, given by

$$\big|{-}{-}\big\rangle = c_1\gamma_1\gamma_2\big|0\big\rangle. \tag{2.7}$$

The $SL(2,C)$ vacuum satisfies

$$\begin{aligned} c_n\big|0\big\rangle &= 0, \qquad n\ge 2; \qquad\qquad & b_n\big|0\big\rangle &= 0, \qquad n\ge -1, \qquad (2.8a)\\ \gamma_n\big|0\big\rangle &= 0, \qquad n\ge 3; \qquad\qquad & \beta_n\big|0\big\rangle &= 0, \qquad n\ge -2. \qquad (2.8b)\end{aligned}$$

The antighost fields b, β have ghost number $G=-1$, and the ghost fields c, γ have ghost number $G=1$.*

For standard states of the form (2.6), the condition of BRST invariance becomes [16]:

$$\begin{aligned} (L_0-4)\big|\varphi\,,X\big\rangle &= 0,\\ W_0\big|\varphi\,,X\big\rangle &= 0,\\ L_n\big|\varphi\,,X\big\rangle = W_n\big|\varphi\,,X\big\rangle &= 0, \qquad n\ge 1.\end{aligned} \tag{2.9}$$

The consequences of these physical-state conditions have been studied in detail in various papers [3–8,10]. The main features that emerge are the following. There are two kinds of excited states, namely those for which there are no excitations in the φ direction, and those where φ is excited too. The latter states are all BRST trivial, as has been discussed in [6,8,18]. For the former, we may write $\big|\varphi\,,X\big\rangle$ as

$$\big|\varphi\,,X\big\rangle = e^{\beta\varphi(0)}\big|\mathrm{phys}\big\rangle_{\mathrm{eff}}, \tag{2.10}$$

where $\big|\mathrm{phys}\big\rangle_{\mathrm{eff}}$ involves only the X^μ fields and not φ. (There will be no confusion between the frozen momentum β and the spin-3 antighost β.) The physical-state conditions (2.9) imply that

$$(\beta+Q)(\beta+\tfrac{6}{7}Q)(\beta+\tfrac{8}{7}Q)=0, \tag{2.11}$$

together with the effective physical-state conditions:

$$\begin{aligned} (L_0^{\mathrm{eff}}-\Delta)\big|\mathrm{phys}\big\rangle_{\mathrm{eff}} &= 0,\\ L_n^{\mathrm{eff}}\big|\mathrm{phys}\big\rangle_{\mathrm{eff}} &= 0, \qquad n\ge 1.\end{aligned} \tag{2.12}$$

The value of the effective intercept Δ is 1 when $\beta=-\frac{6}{7}Q$ or $-\frac{8}{7}Q$, and it equals $\frac{15}{16}$ when $\beta=-Q$. Thus these states of the W_3 string are described by two effective Virasoro-string spectra, for an effective energy-momentum tensor T^{eff} with central charge $c=\frac{51}{2}$ and intercepts $\Delta=1$ and $\Delta=\frac{15}{16}$. The first of these gives a mass spectrum similar to an ordinary

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* For states, we follow the convention in [11] that the ghost vacuum $\big|{-}{-}\big\rangle$ has ghost number $G=0$, which means that the $SL(2,C)$ vacuum $\big|0\big\rangle$ has ghost number $G=-3$. This implies that physical states of ghost number G are obtained by acting on $\big|0\big\rangle$ with operators of ghost number $(G+3)$.

string, with a massless vector at level 1, whilst the second gives a spectrum of purely massive states [5,6].

As we have indicated in the introduction, the W_3-string spectrum is much richer than simply that of the standard physical states we have discussed so far. Although the classification of physical states with non-standard ghost structure is as yet incomplete, some classes of such states have been found [9,11,12]. They contain excitations of the ghost and antighost fields as well as the matter fields. The level number ℓ of these states is defined with respect to the ghost vacuum $\big|--\big\rangle$ given in (2.7). Thus, for example, the $SL(2,C)$ vacuum $\big|0\big\rangle$ has level number $\ell=4$ and ghost number $G=-3$, since it can be written as $\beta_{-2}\beta_{-1}b_{-1}\big|--\big\rangle$. It is straightforward to see that at level ℓ, the allowed ghost numbers of *states* (not necessarily physical) lie in the interval

$$1-\big[\sqrt{4\ell+1}\big]\le G\le 1+\big[\sqrt{4\ell+1}\big], \tag{2.13}$$

where $\big[a\big]$ denotes the integer part of a.

All the physical states in the W_3 string occur in multiplets [9,11]. The members of a multiplet are obtained by (repeatedly) acting with the $G=1$ operators $a_\varphi\equiv[Q_B,\varphi]$ and $a_{X^\mu}\equiv[Q_B,X^\mu]$ on a physical state that we call a *prime state*. Until now the known examples of prime states occurred either at ghost number $G=0$ (in the case of states with standard ghost structure), or at $G=-1$ or $G=-3$ (in the case of states with non-standard ghost structure). The corresponding operators have ghost numbers $G=3$, $G=2$ or $G=0$. All physical states in the W_3 string occur with φ momentum frozen to specific values. Examples of physical states at level $\ell=0,1,2,3$ were given in [11] and used for calculating scattering amplitudes. All the prime operators of [11] have an effective spacetime interpretation, in the sense that they can be written as products of the form $V=U_{(\mathrm{gh},\varphi)}\,U^{\mathrm{eff}}$, where $U_{(\mathrm{gh},\varphi)}$ involves only the ghosts and the φ field, and U^{eff} involves only the effective spacetime coordinates X^μ. We have already seen from (2.6) and (2.10) that states with standard ghost structure give physical operators of this form, with U^{eff} having effective intercepts $L_0^{\mathrm{eff}}=1$ or $\frac{15}{16}$ as measured by T^{eff}. The prime operators with non-standard ghost structure discussed in [11] can have three possible values for L_0^{eff}, namely 1, $\frac{15}{16}$ or $\frac{1}{2}$. In section 3 we shall find new prime states at $\ell=4,\ G=-2$; $\ell=5,\ G=-2$; and $\ell=8,\ G=-3$.* They again all admit an effective spacetime interpretation, with the same three values of L_0^{eff}. By using these new states in scattering calculations, we shall be able to clarify some unresolved issues in [11]. We shall also describe the structure of the multiplets in the multi-scalar W_3 string. As we shall see, for the physical states described above, the ghost boosters a_φ and a_{X^μ} generate a quartet from a given prime state.

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* In [11], a physical state at $\ell=3$, $G=-1$ given by the operator ($A.12$) of [11] was inadvertently described as a prime state. In fact the prime state at this level occurs at $G=-2$, and corresponds to the operator given in ($A.7$) of this paper; the state used in [11] is obtained by acting with the ghost boosters a_φ and a_{X^μ} on this prime state.

Scattering amplitudes are obtained by integrating over the worldsheet coordinates of physical operators in correlation functions. For the W_3 string the correlation functions are, by definition, given by functional integrals over all the quantum fields of the theory, namely the ghosts b, c, β and γ, the φ field, and the effective spacetime coordinates X^μ. These functional integrals can, as usual, be conveniently calculated by using the techniques of conformal field theory.

A physical operator, denoted generically by $V(z)$, is by definition a non-trivial BRST-invariant operator that gives rise to the physical state $V(0)\big|0\big\rangle$. In particular, it has conformal dimension 0, as measured by the total energy-momentum tensor $T^{\rm tot} = T + T_{\rm gh}$. (Note that $V(z)$ includes ghosts as well as the matter fields.) In an N-point scattering amplitude one can use the $SL(2,C)$ invariance of $\big|0\big\rangle$ to fix the worldsheet coordinates of three of the operators in the correlation function, leaving $(N-3)$ coordinates over which to integrate. For conformal covariance one must integrate over operators of dimension 1. As in string theory [19], this can be achieved by making the replacement

$$V(z) \to \frac{1}{2\pi i}\oint_z dw\, b(w) V(z), \tag{2.14}$$

where the subscript on the contour integral indicates that it should be evaluated around a path enclosing z. In fact this procedure not only preserves the projective structure but also gives scattering amplitudes that are invariant under the BRST transformations generated by (2.1) [11]. This is because the BRST variation of the right-hand side of (2.14) is a total derivative. Note that one cannot use the spin-3 antighost β in place of b in (2.14), since it would not give BRST-invariant amplitudes.

In the ordinary 26-dimensional bosonic string, the physical operators all have the form $V(z) = \big(c\,U\big)(z)$ (or its conjugate, $\big(\partial c\, c\, U\big)(z)$), where $U(z)$ involves only the coordinates X^μ but no ghosts. If the replacement (2.14) were not made in $(N \ge 4)$-point functions (and accordingly, the corresponding worldsheet coordinate were left unintegrated), the results would certainly be BRST invariant. However, they would trivially be zero, since the non-zero ghost inner product in string theory is $\big\langle 0\big|\partial^2 c\,\partial c\, c\big|0\big\rangle$ [19]. In the W_3 string, on the other hand, physical operators do not all have the standard ghost structure, so it is no longer *a priori* obvious that the replacement (2.14) is required in order to get non-zero results for BRST-invariant $(N \ge 4)$-point scattering amplitudes. Nevertheless, it seems that the corresponding correlation functions in the W_3 string vanish unless the replacement (2.14) is made, as we shall discuss in section 6. (The situation is different if physical states with discrete effective-spacetime momentum are involved; see section 4.)

In order to identify which correlation functions might be non-vanishing, it is useful first to consider two necessary conditions [11]. If these conditions are satisfied, then it becomes a matter of more detailed calculation to determine the result. The first of the necessary conditions is that the total ghost structure of the product of operators in the correlation function should be appropriate. Specifically, the non-vanishing ghost inner product in the W_3 string is given by

$$1 = \big\langle 0\big|c_{-1}c_0c_1\,\gamma_{-2}\gamma_{-1}\gamma_0\gamma_1\gamma_2\big|0\big\rangle = \tfrac{1}{576}\big\langle 0\big|\partial^2c\,\partial c\,c\,\partial^4\gamma\,\partial^3\gamma\,\partial^2\gamma\,\partial\gamma\,\gamma\big|0\big\rangle\ . \tag{2.15}$$

(See [11] for a more detailed discussion.) Note that in particular the total ghost number of the operators in a non-vanishing correlator must be $3+5=8$.

The second of the necessary conditions for obtaining a non-vanishing correlation function is that momentum conservation must be satisfied. Owing to the presence of the background charges, this implies that we must have $\sum_{i=1}^N p_i^\mu = -2a^\mu$ in the effective spacetime, together with

$$\sum_{i=1}^N \beta_i = -2Q \tag{2.16}$$

in the φ direction. For states of continuous spacetime momentum p_μ, as indeed we have in the multi-scalar W_3 string, momentum conservation in the X^μ directions can be straightforwardly satisfied. However, the momentum β in the φ direction can only take specific frozen values in physical states. (For physical states with standard ghost structure this follows from (2.11); it is true also for states with non-standard ghost structure, as we shall see again in the next section.) Thus it is in general non-trivial to satisfy momentum conservation in the φ direction. Note that (2.16) is an extremely restrictive condition; indeed it implies that ($N\ge 3$)-point correlation functions built exclusively from physical operators of standard ghost structure will vanish, as may be seen from (2.11). It is worth emphasising that even though φ does not admit the interpretation of being an observable spacetime coordinate, the functional integration over it, and in particular over its zero-mode, determines a very stringent W_3 selection rule [11].

Using the procedure outlined above, we calculated various three-point and four-point functions in [11], with the physical states given therein. A number of patterns emerged, which we shall be able to develop further with the results of the present paper. In particular, it was found that physical states seem to be characterised by their effective spacetime structure, in the sense that physical states with the same $L_0^{\rm eff}$ values but different ghost and φ dependence give identical results in correlation functions, modulo an overall constant factor. This factor depends on the normalisation of the states, and furthermore would be zero if, for example, a selection rule such as (2.16) forbade a particular interaction.

It was found in [11] that non-vanishing three-point functions can occur for physical operators with $L_0^{\rm eff}$ values of $\{1,\ 1,\ 1\}$, $\{\frac{15}{16},\ \frac{15}{16},\ 1\}$, $\{\frac12,\ \frac12,\ 1\}$ and $\{\frac{15}{16},\ \frac{15}{16},\ \frac12\}$. All other three-point functions vanish. These results for three-point functions are in one-to-one correspondence with the fusion rules of the two-dimensional Ising model, if one associates the $L_0^{\rm eff}=1$ sector with the identity operator $\mathbf{1}$, the $L_0^{\rm eff}=\frac{15}{16}$ sector with the spin operator σ, and the $L_0^{\rm eff}=\frac12$ sector with the energy operator ε. This makes more precise the numerological connection between W_3 strings and the Ising model which emerges from the analysis of the spectrum [3,6,10]. In particular, the central charge of the effective energy-momentum tensor is $\frac{51}{2} = 26 - \frac{1}{2}$, where 26 is the critical central charge of the usual bosonic string, and $\frac{1}{2}$ is the central charge of the Ising model; moreover, the set of $L_0^{\rm eff}$ values $\{1,\ \frac{15}{16},\ \frac{1}{2}\}$ can be written as $1 - \Delta_{\rm min}$, where 1 is the intercept of the usual bosonic string and $\Delta_{\rm min}$ takes the values of the dimensions of the primary fields $\{\mathbf{1},\ \sigma,\ \varepsilon\}$ of the Ising model.

One of the interesting results of [11] is that this connection between the W_3 string and the Ising model does not go both ways in higher-point functions. In particular, it was shown in [11] that the selection rule (2.16) forbids the existence of a four-point function for four physical operators with $L_0^{\rm eff} = \frac{15}{16}$, even though the four-point function $\big\langle \sigma\ \sigma\ \sigma\ \sigma \big\rangle$ is non-vanishing in the Ising model. We shall find further indications in section 4 from five-point and six-point functions that the $L_0^{\rm eff} = \frac{15}{16}$ sector plays a special rôle in the W_3 string. One of the general features that we observed in [11] is that four-point functions in the W_3 string exhibit a duality property, which means that they can be interpreted in terms of underlying three-point functions, with sets of intermediate states being exchanged in the s, t or u channels. In this paper we shall illustrate this factorisation property for a five-point function too.

3. New physical states and the multiplet structure

In this section, we shall construct some new physical states of the W_3 string with non-standard ghost structure. These are prime states at levels $\ell = 4$ and 5, with ghost number $G = -2$. We shall use these, and an $\ell = 8$, $G = -3$ prime state which was found in [12], to calculate more scattering amplitudes in the next section. In particular, the $\ell = 5$ prime state enables us to obtain a non-vanishing four-point function with four $L_0^{\rm eff} = 1$ operators, which does not exist for the states given in [11]. The above states will provide more evidence for the observation that physical states appear to be characterised by their effective spacetime structures. We shall also investigate the multiplet structures generated by the action of the ghost boosters a_φ and a_{X^μ}.

Until now, prime states have been discovered at ghost number $G = 0$ for states with standard ghost structure, and at $G = -1$ and $G = -3$ in the case of states with non-standard ghost structure. The two new prime states that we shall construct in this section occur at $G = -2$, and thus are of a different kind.

The general procedure that we use for finding physical states is as follows. At a given, fixed, level number ℓ and ghost number G we write down the most general state, as a sum of all n_G possible structures with arbitrary coefficients g_i. Requiring BRST invariance leads to n_{G+1} equations for the g_i's, giving m_G independent solutions. There are $n_{G-1} - m_{G-1}$ BRST-trivial states at ghost number G, where m_{G-1} denotes the number of BRST-invariant states at ghost number $G - 1$. The number of BRST-nontrivial physical states at ghost number G is therefore given by $m_G - (n_{G-1} - m_{G-1})$. Thus at a given level number ℓ we may start from the lowest ghost number given by (2.13), and systematically work up to the highest ghost number allowed by (2.13), finding all the non-trivial physical states at that level. This is quite a tedious process, which is best performed with the aid of a computer.

Applying this procedure at level 4, we have found a physical state at $G = -2$, with φ momentum $\beta = \frac{1}{7}Q$. The general solution has three parameters which means, since there are two BRST-trivial structures coming from Q_B acting on the two structures at $G = -3$, that there is one non-trivial solution at $G = -2$. The two trivial parameters may be used in order to give the physical state a convenient form. In fact, it turns out that they may be chosen so as to remove all terms involving excitations of the spacetime coordinates, X^μ, and thus the state acquires an effective spacetime interpretation, as a tachyon. The explicit form of the corresponding physical operator, which has ghost number $G = 1$, is given in the appendix in $(A.8)$. For the amplitudes that we shall be concerned with in this paper, only two of its thirteen terms contribute. They are

$$\mathbf{V}^1_{15/16}[\tfrac{1}{7}Q, p] = -\tfrac{3}{16}\sqrt{29}\, i\, \Big(3\sqrt{2}\, c\, \beta\, \gamma + 4\partial\varphi\, c + \cdots\Big) e^{\frac{1}{7}Q\varphi} e^{ip\cdot X} \ , \tag{3.1}$$

where the effective spacetime momentum p_μ satisfies the mass-shell condition

$$\tfrac{1}{2}p^\mu(p_\mu + 2a_\mu) = \tfrac{15}{16} \ . \tag{3.2}$$

In (3.1) we have used the notation of [11], where a physical operator V with ghost number G, $L_0^{\rm eff} = \Delta$ and momentum (β, p_μ) is denoted by $\mathbf{V}^G_\Delta[\beta, p]$. The operator (3.1) is a representative of a tachyon of the $L_0^{\rm eff} = \frac{15}{16}$ sector of the W_3 string. The operator $e^{ip\cdot X}$ may be replaced by an arbitrary excited effective physical operator $R(\partial X^\mu)e^{ip'\cdot X}$ with the same $L_0^{\rm eff}$ value. This allows the construction of new physical states at higher levels with X^μ excitations, that have an interpretation as excited effective spacetime states.

At level 5, we have found another new physical state at $G = -2$, with φ momentum $\beta = \frac{2}{7}Q$. Again we can use the freedom to add BRST-trivial terms so as to give the state an effective spacetime interpretation, as a tachyon. The corresponding physical operator is given in $(A.9)$. For the amplitudes that we shall be concerned with in this paper, only 5 of its 32 terms contribute, namely

$$\begin{aligned}\mathbf{V}^1_1[\tfrac{2}{7}Q, p] = \tfrac{1}{20}\sqrt{29}\, i\, \Big(&6\sqrt{2}\, c\, \partial\beta\, \gamma - 3\sqrt{2}\, c\, \beta\, \partial\gamma + 12\partial\varphi\, c\, \beta\, \gamma \\ &+ 4\sqrt{2}\, \partial\varphi\, \partial\varphi\, c + 2\partial^2\varphi\, c + \cdots\Big) e^{\frac{2}{7}Q\varphi} e^{ip\cdot X} \ ,\end{aligned} \tag{3.3}$$

where the effective spacetime momentum p_μ satisfies the mass-shell condition

$$\tfrac{1}{2}p^\mu(p_\mu + 2a_\mu) = 1 \ . \tag{3.4}$$

This operator is a representative of an effective tachyon of the $L_0^{\rm eff}=1$ sector. Again, one can replace $e^{ip\cdot X}$ by an arbitrary excited effective-spacetime physical operator with the same $L_0^{\rm eff}$ value.

The third physical state that we shall be using in this paper, and that was not given in [11], occurs at level 8 with ghost number $G=-3$ and φ momentum $\beta=\frac{4}{7}Q$; it was found in [12]. Once more, the freedom of adding BRST trivial parts can be used to remove all excitations in the X^μ directions, thus giving it the interpretation of an effective spacetime tachyon. As such it is a representative of the $L_0^{\rm eff}=\frac{1}{2}$ sector. It has 82 terms, and is given in $(A.10)$. In fact, for the scattering amplitudes that we shall calculate in the next section, only four of those terms contribute to the result, namely

$$\mathbf{V}^0_{1/2}[\tfrac{4}{7}Q,p]=\Big(\partial\varphi\, c\,\partial\beta+3\sqrt{2}\,c\,\partial\beta\,\beta\,\gamma-3\partial^2\varphi\, c\,\beta-\tfrac{3}{4}\sqrt{2}\,c\,\partial^2\beta+\cdots\Big)e^{\frac{4}{7}Q\varphi}e^{ip\cdot X}\ , \tag{3.5}$$

where

$$\tfrac{1}{2}p^\mu(p_\mu+2a_\mu)=\tfrac{1}{2}\ . \tag{3.6}$$

The fact that only the small subsets of terms (3.1), (3.3) and (3.5) contribute in scattering amplitudes involving the physical operators $(A.8)$, $(A.9)$ and $(A.10)$ respectively suggests that there is a great deal of redundancy in the formalism describing the W_3 string. In section 6, we shall introduce a new formalism that removes this redundancy.

The new physical operators that we have presented here have lower ghost numbers than any of those used in [11]. Specifically, $(A.8)$ and $(A.9)$ have $G=1$, and $(A.10)$ has $G=0$. An important consequence of this is that it enables us to obtain non-vanishing N-point functions for arbitrarily large N, whereas the physical operators of [11] only allowed their construction for $N\le 5$. This is because the total ghost number of operators in a non-vanishing N-point function must be equal to 8, and the insertion of many operators of the form (2.14) will rapidly cause this number to be exceeded unless the physical operators V have ghost numbers ≤ 1. We shall discuss this further at the end of section 4. This completes our discussion of new prime states of the W_3 string with continuous on-shell momenta.

Let us now turn our attention to the structure of the multiplets generated by the action of the ghost boosters a_φ and a_{X^μ} on prime states. The explicit form of the ghost boosters is given in [9] for the two-scalar W_3 string and the generalisation to the multi-scalar case is immediate. They are given in an alternative formulation in (6.21). Although we do not have a general proof, we have checked the structure of the multiplets in several examples. Acting on a prime state $\big|G\big\rangle$ at ghost number G with the $(d+1)$ ghost boosters, we obtain just two independent non-trivial physical states at ghost number $(G+1)$: $a_\varphi\big|G\big\rangle=\big|G+1\big\rangle_1$ and $a_{X^\mu}\big|G\big\rangle=(p^\mu+a^\mu)\big|G+1\big\rangle_2$. The fact that $a_{X^\mu}\big|G\big\rangle$ gives only one non-trivial state rather than d is quite surprising and results in a much simpler multiplet structure than one might have expected (we shall return to this point in section 6). The multiplet is completed by a single non-trivial state $\big|G+2\big\rangle$ obtained by acting either with a_φ on $\big|G+1\big\rangle_2$ or with

a_{X^μ} on $\big|G+1\big\rangle_1$. Thus it seems from our examples that each prime state $\big|G\big\rangle$ gives rise to a quartet of physical states $\{\big|G\big\rangle,\ \big|G+1\big\rangle_1,\ \big|G+1\big\rangle_2,\ \big|G+2\big\rangle\}$. The structure is reminiscent of an $N=2$ supermultiplet.

As discussed in [9,11], each physical state at ghost number G and momentum (β, p_μ) has a conjugate partner at ghost number $(2-G)$ and momentum $(-2Q-\beta, -2a_\mu - p_\mu)$. Thus associated with each prime state is a quartet and a conjugate quartet of physical states.

We have already remarked that for all the known examples, prime states either automatically have, or can be given, an effective spacetime interpretation. Remarkably, in all the examples we have checked, the freedom to add BRST-trivial parts is precisely sufficient to enable the boosted members of a multiplet to acquire an effective spacetime interpretation too. In one of our examples, the prime state involved an excited effective spacetime part, indicating that this effective spacetime interpretation is universal, and not just confined to effective tachyonic states.

These considerations lead us to conjecture that *all* the physical states (with continuous on-shell spacetime momentum p_μ — see below) in the W_3 string admit an effective spacetime interpretation, *i.e.* they can all be written in the form

$$\big|\text{ghost} + \varphi\big\rangle \otimes \big|\text{effective spacetime}\big\rangle\ . \tag{3.7}$$

We shall sharpen this conjecture in section 6. Moreover, we conjecture that the effective intercepts of all these physical states take values in the set $\{1,\ \frac{15}{16},\ \frac{1}{2}\}$.*

The above discussion of the effective-spacetime interpretation of physical states, and their multiplet structure, applies for all physical states with continuous on-shell spacetime momentum p_μ. There are, however, additional physical states in the spectrum of the multi-scalar W_3 string that occur only for fixed values of p_μ. In fact, in all the examples we know, these discrete states occur with $p_\mu = 0$. The simplest example is provided by the $SL(2,C)$ vacuum, $\big|0\big\rangle = \beta_{-2}\beta_{-1}b_{-1}\big|--\big\rangle$, which is a level $\ell=4$ state with ghost number $G=-3$. We shall adopt the notation $\mathbf{D}^G[\beta]$ to denote the physical operator with ghost number G and φ momentum β that gives a discrete state with ghost number $(G-3)$. Thus $\big|0\big\rangle$ corresponds to the operator $\mathbf{D}^0[0]=1$.

* In [13], physical states at $\ell=2$, $G=2$ in the two-scalar W_3 string were discussed, corresponding to an operator of the form $\partial^3 c\, c\, \partial^2\gamma\, \partial\gamma\, \gamma e^{\beta_1\varphi_1} e^{\beta_2\varphi_2}$. This is annihilated by Q_B provided that the exponentials have total conformal weight 2. However, contrary to what is asserted in [13], this state is in general BRST trivial, except for three values of (β_1, β_2). Two of these values, $(-\frac{12}{7}Q, -\frac{12}{7}a)$ and $(-\frac{12}{7}Q, -\frac{2}{7}a)$, correspond to states with $L_0^{\rm eff} = \frac{1}{2}$ which generalise to an $L_0^{\rm eff}=\frac{1}{2}$ state in the multi-scalar W_3 string. (In fact this is the conjugate of one of the quartet members from the prime state given by $\mathbf{V}^2_{1/2}[-\frac{2}{7}Q, p]$ in formula $(A.6)$.) The third momentum value, $(-Q, -\frac{17}{7}a)$, corresponds to a state with $L_0^{\rm eff} = -\frac{17}{16}$ that does not generalise to the multi-scalar W_3 string. Our conjecture that the effective intercept values are $\{1,\ \frac{15}{16},\ \frac{1}{2}\}$ relates to the physical states of the *multi*-scalar W_3 string, and thus the $L_0^{\rm eff} = -\frac{17}{16}$ state does not contradict it.

We shall first consider the prime discrete states, and discuss their multiplet structures later. There are two examples at level $\ell=1$, corresponding to the following operators [9]:

$$\mathbf{D}^2[-\tfrac{6}{7}Q] = \big(c\,\gamma - \frac{i}{3\sqrt{58}}\partial\gamma\,\gamma\big)e^{-\frac{6}{7}Q\varphi}\ , \tag{3.8}$$

$$\mathbf{D}^2[-\tfrac{8}{7}Q] = \big(c\,\gamma + \frac{i}{3\sqrt{58}}\partial\gamma\,\gamma\big)e^{-\frac{8}{7}Q\varphi}\ . \tag{3.9}$$

Apart from the $SL(2,C)$ vacuum, the next example of a prime discrete state that we know of occurs at level $\ell=6$, ghost number $G=-3$ [9], with the corresponding operator being denoted by $\mathbf{D}^0[\frac{2}{7}Q]$. (In the two-scalar W_3 string [9], it plays a similar rôle to the ground-ring generators of Witten [15].) From its detailed form, which is given explicitly in [9] for the two-scalar W_3 string, one can see that the freedom to add BRST-trivial parts is precisely sufficient to enable it to be given an effective spacetime interpretation; the result appears in $(A.14)$. This is presumably the generic situation for discrete prime states in the multi-scalar W_3 string. Higher-level discrete states can be generated by taking appropriate normal-ordered products of these examples. For instance, $\big(\mathbf{D}^0[\frac{2}{7}]\big)^{4m}$ gives the new discrete operator $\mathbf{D}^0[\frac{8}{7}mQ]$ at level $\ell=(4m^2+7m+4)$, and $\big(\mathbf{D}^0[\frac{2}{7}]\big)^{4m+1}$ gives the new discrete operator $\mathbf{D}^0[\frac{2}{7}(4m+1)Q]$ at level $\ell=(4m^2+11m+6)$ [9]. Thus we have discrete operators with $G=0$ and

$$\begin{aligned} \mathbf{D}^0[\tfrac{8}{7}mQ]: &\qquad \ell = 4m^2+7m+4\ ,\\ \mathbf{D}^0[\tfrac{2}{7}(4m+1)Q]: &\qquad \ell = 4m^2+11m+6\ . \end{aligned} \tag{3.10}$$

We now turn to the consideration of the multiplet structure associated with the prime discrete states. Here we find by checking several examples that the pattern is rather different from that for the continuous-momentum physical states; for d effective spacetime coordinates X^μ we get a multiplet with $4d$ members. It is convenient to decompose the ghost boosters a_{X^μ} into an operator $a^{//}$ which is parallel to the background charge vector a^μ and $(d-1)$ operators $(a_{X^\mu})^\perp$ which are perpendicular to the background charge vector. Acting on a prime discrete state $\big|G\big\rangle$ of ghost number G, we have

$$\begin{array}{lccc} G: & & \big|G\big\rangle & \\ G+1: & a_\varphi\big|G\big\rangle, & a^{//}\big|G\big\rangle, & (a_{X^\mu})^\perp\big|G\big\rangle \\ G+2: & a_\varphi\,a^{//}\big|G\big\rangle, & a_\varphi\,(a_{X^\mu})^\perp\big|G\big\rangle, & a^{//}\,(a_{X^\mu})^\perp\big|G\big\rangle \\ G+3: & & a_\varphi\,a^{//}\,(a_{X^\mu})^\perp\big|G\big\rangle\ . & \end{array} \tag{3.11}$$

Thus we see that the multiplicities of states at ghost numbers $\{G,\ G+1,\ G+2,\ G+3\}$ are $\{1,\ d+1,\ 2d-1,\ d-1\}$. The situation is very different from the case of prime states with continuous momentum, in that here each of the $(d+1)$ ghost boosters generates an independent non-trivial physical state at ghost number $(G+1)$. Note that the repeated application of the $(a_{X^\mu})^\perp$ operators does not give any non-trivial state, so (3.11) gives the complete multiplet. In some sense it is reminiscent of an $N=3$ supermultiplet.*

Another striking difference between the discrete states and the physical states with continuous momentum p_μ concerns their effective spacetime interpretation. First, we recall that *all* prime physical states, discrete or continuous, appear to admit an effective spacetime interpretation. For continuous-momentum physical states, this is true also for the entire multiplet (*i.e.* quartet) generated by the ghost boosters. For the discrete states, however, this is no longer the case; the multiplet members do not all admit an effective-spacetime interpretation, even though the prime state does. Specifically, it is states generated by the action of a_{X^μ} that lack an effective-spacetime interpretation. The reason for this is that the Lorentz index μ has to live either on the background-charge vector a^μ or on the effective-spacetime coordinates X^μ. If it lived solely on a^μ, then the a_{X^μ} operators would give only one independent physical state, whereas in fact they all give independent states in the multiplet. Thus in some (but not, as can easily be checked, all) of the terms in the boosted states the μ index must live on X^μ, implying that these states cannot all be given an effective-spacetime interpretation.

The discrete operators can be used to map physical states with continuous momentum p_μ into others by normal ordering, provided that the normal ordered product exists. This appears to be their most important rôle in the interacting W_3 string, since if one tries to view them as ordinary physical states in their own right, they give divergent results in $(N \ge 4)$-point amplitudes. We shall discuss this further in the next section.

For convenience, we summarise the prime operators that we shall be using in this paper in two tables. Their explicit forms are given in the appendix. (In Table 1 we have included a physical operator at level $\ell = 9$, which we shall find in section 6 in a new formalism.)

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* It seems that a natural generalisation of the decomposition of a_{X^μ} into $a^{//}$ and $(a_{X^\mu})^\perp$ for physical states with continuous momentum is to decompose a_{X^μ} into its components parallel and perpendicular to $(p_\mu + a_\mu)$. In our discussion of the quartet structure of the multiplets for continuous-momentum physical states, we saw that when a_{X^μ} acts on such physical states, it gives states of the form $(p_\mu + a_\mu)\big|\quad\big\rangle$. Thus $(a_{X^\mu})^\perp$ annihilates continuous-momentum physical states, and we see that the multiplet structure in (3.11) reduces to a quartet.

	G	$L_0^{\rm eff}$	β
	3	15/16	$-Q$
$\ell=0$	3	1	$-6Q/7, \quad -8Q/7$
	2	15/16	$-3Q/7$
$\ell=1$	2	1/2	$-4Q/7$
$\ell=2$	2	1/2	$-2Q/7$
$\ell=3$	1	1	0
$\ell=4$	1	15/16	$Q/7$
$\ell=5$	1	1	$2Q/7$
$\ell=8$	0	1/2	$4Q/7$
$\ell=9$	0	15/16	$5Q/7$

Table 1. Continuous-momentum physical operators

	G	β
$\ell=1$	2	$-6Q/7, \quad -8Q/7$
$\ell=4$	0	0
$\ell=6$	0	$2Q/7$

Table 2. Discrete physical operators

4. W_3-string scattering

In our previous paper [11] on the interacting W_3 string, we evaluated all the non-vanishing four-point functions for effective tachyons that could be built at tree level using the physical states given in that paper. As we have already remarked, they are characterised by their effective-spacetime structure, and in particular their $L_0^{\rm eff}$ values. For convenience, we list the structure of the results here. In an obvious notation, we have [11]:

$$\big\langle\tfrac12\ \tfrac12\ \tfrac12\ \tfrac12\big\rangle \sim \int_0^1 dx\ x^{-s/2-2}(1-x)^{-t/2-2}(1-x+x^2)\ , \tag{4.1a}$$

$$\big\langle\tfrac{15}{16}\ \tfrac{15}{16}\ \tfrac12\ \tfrac12\big\rangle \sim \int_0^1 dx\ x^{-s/2-2}(1-x)^{-t/2-31/16}(x-2)\ , \tag{4.1b}$$

$$\big\langle 1\ \tfrac{15}{16}\ \tfrac{15}{16}\ 1\big\rangle \sim \int_0^1 dx\ x^{-s/2-31/16}(1-x)^{-t/2-2}\ , \tag{4.1c}$$

$$\big\langle\tfrac{15}{16}\ \tfrac{15}{16}\ 1\ \tfrac12\big\rangle \sim \int_0^1 dx\ x^{-s/2-3/2}(1-x)^{-t/2-31/16}\ , \tag{4.1d}$$

$$\big\langle 1\ \tfrac12\ 1\ \tfrac12\big\rangle \sim \int_0^1 dx\ x^{-s/2-3/2}(1-x)^{-t/2-3/2}\ , \tag{4.1e}$$

where s, t (and u) are the Mandelstam variables:

$$\begin{aligned} s &\equiv -(p_1+p_2)^2 - 2a\cdot(p_1+p_2)\ , \\ t &\equiv -(p_2+p_3)^2 - 2a\cdot(p_2+p_3)\ , \\ u &\equiv -(p_1+p_3)^2 - 2a\cdot(p_1+p_3)\ . \end{aligned} \tag{4.2}$$

In [11] we observed that the above four-point functions have the correct crossing properties, and that they exhibit a W_3 duality behaviour in which they can be written in terms of the underlying three-point functions with the exchange of intermediate states in the s, t or u channels. From the underlying three-point functions, one might expect that there could be two more non-vanishing four-point functions, namely, $\big\langle 1\ 1\ 1\ 1\big\rangle$ and $\big\langle \frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\big\rangle$. For the latter, we proved in [11] that there cannot exist $L_0^{\rm eff}=\frac{15}{16}$ physical states with φ momenta that enable the selection rule (2.16) to be satisfied in this case. In other words

$$\big\langle \tfrac{15}{16}\ \tfrac{15}{16}\ \tfrac{15}{16}\ \tfrac{15}{16}\big\rangle = 0 \tag{4.3}$$

in the W_3 string. Later in this section we shall argue that in fact any N-point function with three or more $L_0^{\rm eff}=\frac{15}{16}$ physical states must vanish.

For the case of $\big\langle 1\ 1\ 1\ 1\big\rangle$ on the other hand, we argued in [11] that there was no reason to expect that it must vanish, and that our inability to construct it was simply due to an insufficient supply of examples of physical states with $L_0^{\rm eff}=1$. In fact with the new physical states given in this paper, we shall indeed be able to construct such a four-point function. The other new physical states given in section 3 enable us to construct further examples of four-point functions that provide additional evidence for the conjecture that they are characterised by the effective-spacetime structure of the physical states.

We begin the detailed discussion of our new results by considering the four-point function $\big\langle 1\ 1\ 1\ 1\big\rangle$ mentioned above. This can be constructed, for example, by taking the level $\ell=5$ operator $\mathbf{V}_1^1[\frac{2}{7}Q,p]$ given in $(A.9)$ together with physical operators given in [11]. One of them is the level $\ell=0$ operator $\mathbf{V}_1^3[-\frac{8}{7}Q,p]$ of standard ghost structure (of which we take two copies); the remaining operator, $\mathbf{W}_1^2[0,p]$, has level number $\ell=3$. It is obtained from $\mathbf{V}_1^1[0,p]$ given in $(A.7)$ by boosting; its explicit form is given in $(A.12)$ of [11], where it was inadvertently denoted as $\mathbf{V}_1^2[0,p]$. We choose to make the replacement (2.14) on the level $\ell=5$ operator. This replacement singles out all the terms involving an undifferentiated c ghost, and removes it, reducing the 32 terms in $(A.9)$ to 14. Of these 14 terms, only 5 can give rise to the ghost structure (2.15), namely those that do not involve a b antighost or any of its derivatives; they are given in (3.3). The $\mathbf{V}_1^3[-\frac{8}{7}Q,p_1]$ and $\mathbf{V}_1^3[-\frac{8}{7}Q,p_2]$ operators each have only one term, and 7 of the 11 terms of $\mathbf{W}_1^2[0,p_4]$ do not contribute. Thus there are $5\times 4=20$ different contributions to the final result. The techniques used for calculating these contributions were explained in [11]. Since the intermediate steps are rather involved, we shall only present the final result. Explicitly we find for this four-point function:

• $\underline{L_0^{\rm eff}=\{1,\ 1,\ 1,\ 1\}}$:

$$\begin{aligned}\int\oint_{z_3}&\big\langle 0\big|\mathbf{V}_1^3[-\tfrac{8}{7}Q,p_1](z_1)\ \mathbf{V}_1^3[-\tfrac{8}{7}Q,p_2](z_2)\,b(w)\,\mathbf{V}_1^1[\tfrac{2}{7}Q,p_3](z_3)\ \mathbf{W}_1^2[0,p_4](z_4)\big|0\big\rangle\\ &\qquad=-\frac{\sqrt{58}\,i}{10}\int_0^1 dx\,x^{-s/2-2}(1-x)^{-t/2-2}\ ,\end{aligned}\tag{4.4}$$

where the integrals at the front of the first line denote the integration over the worldsheet coordinate z_3, and the contour integral of (2.14), respectively. The expression (4.4) is identical (up to normalisation) to the four-point tachyon scattering amplitude in ordinary bosonic string theory. This is not surprising, in view of our observation that interactions in the W_3 string are characterised by the effective spacetime structure, and in particular the $L_0^{\rm eff}$ values, of the external states, and that the $L_0^{\rm eff}=1$ sector of the W_3 string has the closest resemblance to the usual bosonic string. Indeed the result provides a reassuring check that despite the complexity of the computation, the expected result emerges. In section 6, we shall present a new formalism within which it will become manifest that tree-level scattering of purely $L_0^{\rm eff}=1$ states in the W_3 string is the same as in the bosonic string, apart from the presence of the background charge.

The physical operators ($A.8$) and ($A.10$) provide further nice checks of the observation that what characterises an interaction is the effective spacetime structure of the external states. Using them, we have computed the following four-point functions:

• $\underline{L_0^{\rm eff}=\{1,\ \tfrac{15}{16},\ \tfrac{15}{16},\ 1\}}$:

$$\begin{aligned}\int\oint_{z_3}&\big\langle 0\big|\mathbf{V}_1^3[-\tfrac{6}{7}Q,p_1](z_1)\ \mathbf{V}_{15/16}^2[-\tfrac{3}{7}Q,p_2](z_2)\,b(w)\,\mathbf{V}_{15/16}^1[\tfrac{1}{7}Q,p_3](z_3)\ \mathbf{V}_1^3[-\tfrac{6}{7}Q,p_4](z_4)\big|0\big\rangle\\ &\qquad=0\ ,\end{aligned}\tag{4.5}$$

• $\underline{L_0^{\rm eff}=\{\tfrac{15}{16},\ 1,\ \tfrac{1}{2},\ \tfrac{15}{16}\}}$:

$$\begin{aligned}\int\oint_{z_3}&\big\langle 0\big|\mathbf{V}_{15/16}^3[-Q,p_1](z_1)\ \mathbf{V}_1^3[-\tfrac{6}{7}Q,p_2](z_2)\,b(w)\,\mathbf{V}_{1/2}^2[-\tfrac{2}{7}Q,p_3](z_3)\ \mathbf{V}_{15/16}^1[\tfrac{1}{7}Q,p_4](z_4)\big|0\big\rangle\\ &\qquad=0\ ,\end{aligned}\tag{4.6}$$

• $\underline{L_0^{\rm eff}=\{\tfrac{1}{2},\ \tfrac{15}{16},\ \tfrac{15}{16},\ \tfrac{1}{2}\}}$:

$$\begin{aligned}\int\oint_{z_3}&\big\langle 0\big|\mathbf{V}_{1/2}^2[-\tfrac{4}{7}Q,p_1](z_1)\ \mathbf{V}_{15/16}^3[-Q,p_2](z_2)\,b(w)\,\mathbf{V}_{15/16}^1[\tfrac{1}{7}Q,p_3](z_3)\ \mathbf{W}_{1/2}^3[-\tfrac{4}{7}Q,p_4](z_4)\big|0\big\rangle\\ &\qquad=-\frac{3\sqrt{58}\,i}{8}\int_0^1 dx\,x^{-s/2-31/16}(1-x)^{-t/2-2}(1+x)\ ,\end{aligned}\tag{4.7}$$

• $\underline{L_0^{\rm eff} = \{1,\ 1,\ \tfrac12,\ \tfrac12\}}$:

$$\int\oint_{z_3} \langle 0\big|\mathbf{W}_1^4[-\tfrac87 Q,p_1](z_1)\ \mathbf{V}_1^3[-\tfrac87 Q,p_2](z_2)\, b(w)\, \mathbf{V}_{1/2}^0[\tfrac47 Q,p_3](z_3)\ \mathbf{V}_{1/2}^2[-\tfrac27 Q,p_4](z_4)\big|0\rangle$$
$$= -\frac{\sqrt2}{4}\int_0^1 dx\, x^{-s/2-2}(1-x)^{-t/2-3/2}\ . \tag{4.8}$$

As in [11], we use the notation $\mathbf{W}_\Delta^G[\beta,p]$ for a physical operator which is obtained by acting once with the ghost boosters a_φ or a_{X^μ} on the prime operator $\mathbf{V}_\Delta^{G-1}[\beta,p]$.*

The four-point functions (4.5) and (4.6) provide an illustration of W_3 duality in the sense that they are zero because the underlying three-point functions vanish given the particular φ momenta of the external states. For example, in (4.5) we could expand in the s channel in which case the intermediate states are from the $L_0^{\rm eff}=\frac{15}{16}$ sector. Since no such operators exist with $\beta=-\frac57 Q$ (see section 5 for a discussion on this point), there does not exist a three-point function with $\mathbf{V}_1^3[-\frac67 Q,p_1]$ and $\mathbf{V}_{15/16}^2[-\frac37 Q,p_2]$ as external operators.

The four-point function (4.7) is equivalent to (4.1b), after interchanging the ordering of the operators. Since the operators in (4.7) and (4.1b) are different, this illustrates once more that interactions are characterised by the effective spacetime structure of the physical operators. In addition, the comparison of (4.7) with (4.1b) illustrates the crossing behaviour of W_3 scattering amplitudes, which in this particular case amounts to interchanging s with t, and x with $(1-x)$. Both these properties are also exhibited by (4.8), which is equivalent to (4.1e) after interchanging s with u, and x with $1/x$. The computation of (4.8) is surprisingly simple, bearing in mind that the operator $\mathbf{V}_{1/2}^0[\frac47 Q,p_3]$ given in (A.10) contains 82 terms. This is because making the replacement (2.14) selects just 27 terms, and of these all but the 4 arising from those given explicitly in (3.5) do not contribute.

Let us now construct a five-point function. This will provide further indications that the observations we have made for three-point and four-point functions are indeed general features of W_3-string theory. In particular, we shall exhibit the factorisation properties that are the generalisation of duality in four-point functions. Our example is built from five physical operators that were given in [11], namely $\mathbf{V}_{15/16}^2[-\frac37 Q,p_1]$, $\mathbf{V}_{15/16}^2[-\frac37 Q,p_2]$, $\mathbf{V}_{1/2}^2[-\frac47 Q,p_3]$, $\mathbf{V}_{1/2}^2[-\frac47 Q,p_4]$ and $\mathbf{W}_1^2[0,p_5]$ (which was inadvertently denoted by $\mathbf{V}_1^2[0,p_5]$

* Since, as we have discussed in section 3, there are two independent such boosted operators for a prime operator with continuous momentum p_μ, the choice of boosted operator $\mathbf{W}_\Delta^G[\beta,p]$ might seem ambiguous since an arbitrary linear combination could be taken. However, as we have seen, only the coefficient, but not the form, of the scattering amplitude should depend on which representative is taken. This implies that the scattering amplitude will have a universal structure with a combination-dependent coefficient; one degenerate combination will therefore give zero regardless of whether or not the particular amplitude is intrinsically non-vanishing. Any other choice of linear combination will reveal the true interaction.

in [11]). After some algebra, we find the result

$$\begin{aligned}\big\langle \tfrac{15}{16}\ \tfrac{15}{16}\ \tfrac{1}{2}\ \tfrac{1}{2}\ 1\big\rangle = \int_0^1 dx \int_0^1 dy\, & x^{|p_1+p_2|^2/2-2} y^{|p_4+p_5|^2/2-3/2}(1-x)^{|p_2+p_3|^2/2-31/16} \\ & (1-y)^{|p_3+p_4|^2/2-2}(1-xy)^{|p_2+p_4|^2/2-31/16}(xy+x-2)\ ,\end{aligned} \tag{4.9}$$

where we have introduced the Koba-Nielsen variables x and y, and $|p|^2 \equiv p\cdot(p+2a)$. This example is particularly well suited to showing both the similarities to, and the differences from, scattering amplitudes in the ordinary bosonic string. For five tachyons in the ordinary string, one would have

$$\begin{aligned}\int_0^1 dx \int_0^1 dy\, & x^{(p_1+p_2)^2/2-2} y^{(p_4+p_5)^2/2-2}(1-x)^{(p_2+p_3)^2/2-2} \\ & (1-y)^{(p_3+p_4)^2/2-2}(1-xy)^{(p_2+p_4)^2/2-2}\ .\end{aligned} \tag{4.10}$$

First, we note that (4.9) involves the background-charge vector a_μ in the effective spacetime, which is of course absent in the usual 26-dimensional bosonic string. both (4.9) and (4.10) display the property of factorisation. From (4.9) one can see for the W_3 string that states with $L_0^{\rm eff} = 1$, $\frac{15}{16}$ and $\frac{1}{2}$ are exchanged in the various intermediate channels, whereas the usual bosonic string amplitude (4.10) shows that all the exchanged states have, of course, intercept 1. Finally, we note the occurrence in (4.9) of the regular function $(xy+x-2)$. This function, like the $(1-x+x^2)$ in (4.1a), and the $(x-2)$ in (4.1b), comes from the functional integration over the spin-3 ghost system (β,γ) and the φ field. These functions are typical W_3 contributions to tree amplitudes.*

Not surprisingly, if we compute a five-point function $\big\langle 1\ 1\ 1\ 1\ 1\big\rangle$ in the W_3 string, we get the same result as (4.10), apart from the presence of the background charge which means that $(p_i+p_j)^2$ is replaced by $|p_i+p_j|^2$. An example is provided by taking the operators $\mathbf{W}_1^4[-\frac{8}{7}Q,p]$, $\mathbf{V}_1^3[-\frac{6}{7}Q,p]$, and three $\mathbf{V}_1^1[0,p]$ operators.

We shall now explore the rôle of the discrete physical states in more detail. If one attempts to treat them on the same footing as physical states with continuous momentum

* These same functions appear in the corresponding correlation functions of the Ising model, if one makes the association described at the end of section 2. However, as we discussed there, there is no one-to-one correspondence between non-zero correlators in the W_3 string and the Ising model. In particular, one gets the function $\cos\frac{\pi}{8}\sqrt{1+\sqrt{1+x}}+\sin\frac{\pi}{8}\sqrt{1-\sqrt{1-x}}$ from $\big\langle\sigma\ \sigma\ \sigma\ \sigma\big\rangle$ in the Ising model [20], whereas the function from $\big\langle\frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\big\rangle$ in the W_3 string is 0, as follows from (4.3). This is an indication that one cannot simply view the W_3 string as a product of effective Virasoro strings with the Ising model, since the latter gives a non-zero result (see [21]) for an amplitude that is zero in the W_3 string. Such a viewpoint does not capture the essence of the W_3 symmetry, since it does not involve functional integrations over all the quantum fields of the W_3 string. In particular, the selection rule (2.16) which comes from integration over the φ field, and is responsible for the vanishing of the four-point function $\big\langle\frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\big\rangle$ in the W_3 string, does not exist in such an approach. In [21] this model of effective Virasoro strings tensored with the Ising model emerged from applying the "group-theoretical method" to the effective Virasoro strings. Presumably if the method were applied instead to the W_3 string itself, it would describe W_3-string scattering.

p_μ, one finds that in $(N \geq 4)$-point functions they lead to divergent integrals. This can be easily understood; since the p_μ momentum of the discrete state is zero, it follows that one can always use the duality or factorisation properties of the amplitude to express it in terms of a sum over *on-shell* intermediate states at a three-point vertex with the discrete state and another physical state as external states, thereby giving a divergent result. A simple example is provided by looking at a four-point function in which one of the operators is the level 4 discrete operator $\mathbf{D}^0[0] = 1$, which corresponds to the physical state $\big|0\big\rangle$. One easily finds that the integral over x is logarithmically divergent. A less trivial example is provided by considering the four-point function built from the operators $\mathbf{W}^2_1[0, p_1]$, $\mathbf{V}^2_{15/16}[-\frac{3}{7}Q, p_2]$, $\mathbf{W}^3_{15/16}[-\frac{3}{7}Q, p_3]$ and the discrete operator $\mathbf{D}^2[-\frac{8}{7}Q]$. Following our procedure for calculating this four-point function, we obtain the result

$$\big\langle 1\ \tfrac{15}{16}\ \tfrac{15}{16}\ D\big\rangle = -120 \int_0^1 dx \left[3 + \frac{2}{x(1-x)}\right] . \tag{4.11}$$

This has the same logarithmic divergence as one finds in the previous example with the unit operator.

These divergences can obviously be avoided by not performing the offending integrations. More precisely, in an $(N \geq 4)$-point function in which m of the external physical states are discrete, no divergence will occur if only $(N-3-m)$ replacements (2.14), and corresponding integrations, are performed, rather than the usual $(N-3)$. Let us consider the following example, of a four-point function with the operators $\mathbf{V}^2_{1/2}[-\frac{4}{7}Q, p_1]$, $\mathbf{V}^2_{1/2}[-\frac{4}{7}Q, p_2]$, $\mathbf{W}^2_1[0, p_3]$ and the discrete operator $\mathbf{D}^2[-\frac{6}{7}Q]$. Following the above prescription, there will be no integration at all, and so *a priori* one might expect the result to depend on the cross-ratio x. Actually it does not, and the result is simply the constant $\frac{4}{87}\sqrt{58}\,i$. This is the result that one would expect for a three-point function. In fact this is precisely what it is; what has happened is that the discrete state $\mathbf{D}^2[-\frac{6}{7}Q]$ has normal ordered, by virtue of Wick's theorem, with another of the physical operators to produce a new physical operator. Thus the net result is that one is really just evaluating a three-point function of continuous-momentum physical operators.

This phenomenon generalises to any N-point function containing m discrete operators. Provided that they can normal order with some continuous-momentum physical operators so as to produce new ones, then Wick's theorem ensures that the N-point function reduces to an $(N-m)$-point function of continuous-momentum physical operators. If there is no such normal ordering possible, then the result will be zero. Indeed, a set of discrete operators with φ momenta β_i has a well-defined normal-ordered product with a continuous-momentum operator with φ momentum β if and only if $\beta \sum_i \beta_i + \sum_{i<j} \beta_i \beta_j$ is an integer. Provided that the m discrete operators can be distributed over the original continuous-momentum operators of the N-point function so that this condition is satisfied in each case, then a

well-defined $(N-m)$-point function exists; otherwise, it vanishes. We have checked this in several examples.

To end this section, we turn to a discussion of the special rôle of the $L_0^{\rm eff}=\frac{15}{16}$ state in the physical spectrum of the W_3 string. We have already seen that the selection rule (2.16) implies the vanishing of the four-point function $\langle\frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\rangle$ [11]. We shall now argue that all N-point functions with three or more external physical states from the $L_0^{\rm eff}=\frac{15}{16}$ sector are zero.

We shall first demonstrate this result in the case when there are an odd number of $L_0^{\rm eff}=\frac{15}{16}$ states in the N-point function. It follows from the fact that the φ momentum of *any* $L_0^{\rm eff}=\frac{15}{16}$ physical state is of the form $\beta=\frac{k}{7}Q$ where k is an odd integer, whereas for the $L_0^{\rm eff}=1$ and $L_0^{\rm eff}=\frac12$ sectors k must be an even integer. These properties can be proved from the mass-shell condition implied by the fact that $L_0^{\rm tot}$ annihilates physical states, namely

$$0=\ell-4-\tfrac12\beta^2-\beta Q+L_0^{\rm eff}\ , \tag{4.12}$$

where ℓ is the level number. Writing $\beta=\frac{k}{7}Q$, and recalling that $Q^2=\frac{49}{8}$, this gives

$$k^2+14k=16L_0^{\rm eff}+16\ell-64\ . \tag{4.13}$$

Now there is a general argument for *any* physical state that shows that k must certainly be an integer. This can be seen by considering the case when φ is taken to be a timelike field, in which case it is automatically periodic and hence its conjugate momentum is quantised [5,6] in units of $\frac17 Q$. (It can also be seen in all the physical states that have been found in the W_3 string.) Given that k is an integer, it is immediately clear that solutions of (4.13) when $L_0^{\rm eff}=\frac{15}{16}$ require k to be odd, whilst solutions when $L_0^{\rm eff}=1$ or $\frac12$ require k to be even. Thus it is impossible to satisfy the selection rule (2.16) if there is an odd number of $L_0^{\rm eff}=\frac{15}{16}$ external states.

For an even number of $L_0^{\rm eff}=\frac{15}{16}$ external states (or indeed any number ≥ 4), we can make use of the factorisation property of the N-point function. This enables us to view the N-point function as a "four-point function" where three legs are external physical $L_0^{\rm eff}=\frac{15}{16}$ states and the fourth is an intermediate state connecting to the rest of the diagram. Two distinct cases then arise. If the intermediate states in the fourth leg are in the $L_0^{\rm eff}=1$ or $\frac12$ sectors, then from the duality of the four-point function itself, it can be seen to vanish because of the vanishing of its underlying three-point functions. If on the other hand the intermediate states of the fourth leg are in the $L_0^{\rm eff}=\frac{15}{16}$ sector, then the four-point function vanishes by virtue of (4.3).

We have checked these properties in a number of examples. More specifically, we have verified explicitly that N-point functions such as $\langle\frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\ \frac12\rangle$, $\langle\frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\ \frac12\ \frac12\rangle$, and $\langle\frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\ \frac{15}{16}\rangle$ all vanish. This provides a non-trivial check, since there are candidate amplitudes in which the selection rules (2.15) and (2.16) are satisfied.

The conclusion of the argument above is that any N-point function in the W_3 string with three or more external $L_0^{\rm eff}=\frac{15}{16}$ states vanishes, *i.e.*

$$\big\langle \tfrac{15}{16}\ \tfrac{15}{16}\ \tfrac{15}{16}\ \cdots\big\rangle = 0\ , \tag{4.14}$$

where $\cdots$ represents *any* set of physical states. This shows that indeed the $L_0^{\rm eff}=\frac{15}{16}$ sector plays a special rôle in the W_3 string.

In the $L_0^{\rm eff}=1$ and $\frac12$ sectors, on the other hand, no such feature as (4.14) occurs. Indeed it is possible to construct non-vanishing scattering amplitudes with an arbitrarily large numbers of physical operators from either of these sectors. For $L_0^{\rm eff}=1$, the $\ell=3$ prime operator $\mathbf{V}_1^1[0,p]$ at ghost number $G=1$ can, with the replacement (2.14), be inserted arbitrarily-many times in amplitudes without disturbing the total φ momentum or ghost number. For $L_0^{\rm eff}=\frac12$, one can in the same vein insert arbitrary numbers of pairs of the physical operators $\mathbf{V}^2_{1/2}[-\frac47 Q,p]$ and $\mathbf{V}^0_{1/2}[\frac47 Q,p]$, at levels $\ell=1$ and 8, and ghost numbers $G=2$ and 0.

An interesting consequence of (4.14) is that at tree level $L_0^{\rm eff}=\frac12$ states will not appear in intermediate channels if all the external physical states lie in the $L_0^{\rm eff}=1$ and $\frac{15}{16}$ sectors. In loop amplitudes, on the other hand, states from all three sectors will run around the loops. This is illustrated by the fact that the one-loop partition function of states with standard ghost structure (which all have $L_0^{\rm eff}=1$ or $\frac{15}{16}$) is not modular invariant [10]. This is to be expected, since we know that the physical spectrum consists not only of the states with standard ghost structure, but also of states with non-standard ghost structure, which include in particular states in the $L_0^{\rm eff}=\frac12$ sector.

5. More on the spectrum, and the no-ghost theorem

5.1 The physical spectrum revisited

All the physical states with continuous momenta p_μ that we know of display the remarkable property that they can be written as direct products of a factor involving the ghost pairs (b,c) and (β,γ) together with the frozen coordinate φ, and a factor involving only the X^μ coordinates, as in (3.7). This means that all the known physical states with continuous momentum admit an interpretation as effective spacetime physical states. In other words, they are highest-weight states of $T^{\rm eff}$ given by (2.5). It is worth noting that this is a highly non-trivial feature. For example if we consider the $\ell=8$ prime state in, for simplicity, the two-scalar case, there are 22 BRST-trivial structures at $G=-3$ which have to, and indeed do, conspire to permit the removal of 134 terms that involve excitations in the effective spacetime, thereby leaving 82 terms. (Since, by construction, these terms do not involve excitations in the effective spacetime, their form, and number, is independent of the dimension of the effective spacetime. They are in fact precisely the terms given in (A.10).) The occurrence of this phenomenon in numerous examples leads us to conjecture that *all* physical states with continuous momentum p_μ (prime states *and* quartet members) admit an effective spacetime interpretation.

The effective intercepts $L_0^{\rm eff}$ of the known continuous-momentum physical states all take values in the set $\{1,\ \frac{15}{16},\ \frac{1}{2}\}$. The abundance of the known examples leads us to believe that this too is a general feature of all the continuous-momentum physical states of the W_3 string.

Given that the φ momentum of physical states is quantised in the form $\frac{k}{7}Q$, where k is an integer, we can enumerate all the possible level numbers and φ momenta of all physical states which admit an effective spacetime interpretation with $L_0^{\rm eff}$ lying in the set $\{1,\ \frac{15}{16},\ \frac{1}{2}\}$. We shall present the argument for the case where the effective spacetime state is a tachyon. From the mass-shell condition (4.13), it follows that

$$k+7=\pm\sqrt{16\ell+16L_0^{\rm eff}-15}\ . \qquad (5.1)$$

If we first consider the sector with $L_0^{\rm eff}=1$, we see that for k to be an integer it follows that we must have $\ell=4p^2+p$, with p an arbitrary integer. Similar arguments can be applied to the other two sectors, giving

$$\begin{aligned} L_0^{\rm eff}&=1: &\qquad \ell&=4p^2+p, &\qquad k&=-7\pm(8p+1)\ , &\qquad (5.2a)\\ L_0^{\rm eff}&=\tfrac{15}{16}: & \ell&=p^2, & k&=-7\pm 4p\ , & (5.2b)\\ L_0^{\rm eff}&=\tfrac{1}{2}: & \ell&=4p^2+3p+1, & k&=-7\pm(8p+3)\ , & (5.2c)\end{aligned}$$

where p is an arbitrary integer in each case. It is instructive to tabulate these results for the first few level numbers:

$L_0^{\rm eff}=1$	
ℓ	$k_-,\quad k_+$
0	$-8,\quad -6$
3	$-14,\quad 0$
5	$-16,\quad 2$
14	$-22,\quad 8$
18	$-24,\quad 10$
33	$-30,\quad 16$
39	$-32,\quad 18$

$L_0^{\rm eff}=15/16$	
ℓ	$k_-,\quad k_+$
0	$-7,\quad -7$
1	$-11,\quad -3$
4	$-15,\quad 1$
9	$-19,\quad 5$
16	$-23,\quad 9$
25	$-27,\quad 13$
36	$-31,\quad 17$

$L_0^{\rm eff}=1/2$	
ℓ	$k_-,\quad k_+$
1	$-10,\quad -4$
2	$-12,\quad -2$
8	$-18,\quad 4$
11	$-20,\quad 6$
23	$-26,\quad 12$
28	$-28,\quad 14$
46	$-34,\quad 20$

Table 3. Allowed levels and φ momenta for effective tachyons

The results presented in (5.2a–c) and the above table give the level numbers ℓ of physical states in the W_3 string that are tachyons from the effective spacetime point of view. As we have discussed earlier, one can always replace the tachyon by an excited effective spacetime physical state with the same $L_0^{\rm eff}$ value. If this effective state has effective level number $\ell_{\rm eff}$, then the ℓ values given above are shifted according to $\ell \to \ell + \ell_{\rm eff}$.

The conditions given in (5.2a–c), and presented in Table 3, represent *necessary* conditions, derived from the mass-shell constraint, for the existence of continuous-momentum physical operators of the kinds we are discussing. There is no *a priori* reason to expect that they must exist. However comparison with the known physical operators (see Table 1), shows that up to and including level 9, all the operators in Table 3 do actually occur, at the indicated $L_0^{\rm eff}$ values. The φ momenta of all the physical operators in Table 1 correspond to the larger of the allowed k values, namely $\beta = \frac{k_+}{7} Q$; the k_- value corresponds to the conjugate physical operator in each case. The operators with standard ghost structure, at $\ell = 0$, are an exception in that both k_+ and k_- occur for both the operators and their conjugates.

A similar discussion can be given for discrete states. Although, as we have explained in section 3, they do not in general admit an effective spacetime interpretation (although the prime discrete states do) we may still still use the mass-shell condition (4.13), with $L_0^{\rm eff} = 0$ since they have $p_\mu = 0$. Thus we find that they can occur at level numbers and φ momenta given by

$$\ell = 4p^2 + p + 1, \qquad k = -7 \pm (8p+1) \ , \tag{5.3}$$

where p is an arbitrary integer. The first few examples are:

ℓ	k_-, k_+
1	-8, -6
4	-14, 0
6	-16, 2
15	-22, 8
19	-24, 10
34	-30, 16
40	-32, 18

Table 4. Allowed levels and φ momenta for discrete operators

The level $\ell = 1$, 4 and 6 discrete operators in the table are the ones we discussed in section 3. In fact, as we explained there, we can take normal ordered products of the discrete operators to create new ones. From equation (3.10) we see that discrete operators in Table 4 at $\ell \ge 15$ with φ momentum $\frac{k_+}{7} Q$ can be obtained from normal ordered powers of the $\ell = 6$ operator. Similarly by normal ordering discrete operators with continuous-momentum physical operators, new such operators are created. In a certain sense, a discrete operator can thus be viewed as a "screening charge" in the φ direction times the identity operator in the effective spacetime directions. The discrete operator $\mathbf{D}^0[\frac{2}{7}Q]$ at $\ell = 6$ and $G = 0$ given in $(A.14)$ provides a nice illustration. Since it has φ momentum $\beta = \frac{2}{7}Q$, it could in principle enable one to step up through the k_+ values of the three sectors in Table 3. Since physical operators with non-negative φ momentum will certainly have a non-vanishing normal-ordered product with appropriate powers of $\mathbf{D}^0[\frac{2}{7}Q]$, it may be that one should regard the first level number in each sector at which k_+ is non-negative as a foundation from which higher levels in that sector can be obtained. As an example, we have explicitly checked that normal ordering $\mathbf{D}^0[\frac{2}{7}Q]$ given in $(A.14)$ with the $\ell = 3$ prime operator $(A.7)$ of the $L_0^{\rm eff} = 1$ sector gives the $\ell = 5$ prime operator $(A.9)$.

5.2 Unitarity of the W_3 string

Let us now consider the question of the unitarity of the W_3 string. For physical states of standard ghost structure, unitarity was first proven in [6]. The argument consists of observing that since such states all admit an effective spacetime interpretation (see (2.6) and (2.10)), proving unitarity for these states amounts to proving unitarity for an effective Virasoro string theory with central charge $c = \frac{51}{2}$ and intercepts $a = 1$ and $a = \frac{15}{16}$. A standard result from ordinary string theory is that the unitarity bounds derived from level 1 and level 2 states are sufficient to ensure unitarity at all excited levels. As discussed, for example, in [6], the unitarity of level states requires $a \le 1$, whilst level 2 unitarity requires

$$a \le \frac{37 - c - \sqrt{(c-1)(c-25)}}{16} \qquad \text{or} \qquad a \ge \frac{37 - c + \sqrt{(c-1)(c-25)}}{16}\ . \tag{5.4}$$

For the case of $c = \frac{51}{2}$, these bounds imply that the effective spacetime intercept must satisfy

$$a \le \tfrac{1}{2} \qquad \text{or} \qquad \tfrac{15}{16} \le a \le 1\ . \tag{5.5}$$

Thus the physical states in the W_3 string with standard ghost structure precisely saturate the limits of the second interval. This means that they are unitary. (In fact because they saturate the limits, it means that there are null states, characteristic of spacetime gauge symmetries, as well as positive-norm states.)

This reasoning extends to *all* physical states in the W_3 string that admit an effective spacetime interpretation. All the known examples of continuous-momentum physical states do indeed admit such an interpretation, and furthermore, their effective intercept values lie in the set $\{1,\ \frac{15}{16},\ \frac{1}{2}\}$. Thus all the known continuous-momentum physical states in the W_3 string satisfy the unitarity conditions (5.5). In fact they all saturate one or another of the limits in (5.5).

If, as we have conjectured in this paper, all continuous-momentum physical states of the W_3 string admit an effective spacetime interpretation, with $L_0^{\rm eff}$ values lying in the set $\{1,\ \frac{15}{16},\ \frac12\}$, it follows that the argument above constitutes a proof of unitarity of the W_3 string.

Discrete states do not upset this conclusion. This is because, unlike the continuous-momentum physical states whose unitarity we have already addressed, a discrete state is a scalar under the effective-spacetime Lorentz group rather than a tensor, and so the issue of negative-norm states does not arise.

Another indication of the unitarity of the W_3 string can be seen by looking at the residues of the poles in intermediate channels of ($N \ge 4$)-point functions. Let us consider first the example of the four-point function $\big\langle 1\ 1\ 1\ 1\big\rangle$ given in (4.4). (This is of the same form as the four-point tachyon amplitude in the bosonic string.) The integral can be written as

$$\big\langle 1\ 1\ 1\ 1\big\rangle \sim \frac{\Gamma(-s/2-1)\Gamma(-t/2-1)}{\Gamma(-s/2-t/2-2)}\ . \tag{5.6}$$

Using the standard expression for the behaviour of the Γ function near a pole, $\Gamma(-n+\epsilon) = (-1)^n/n!\,\epsilon^{-1}$, and expanding in the t-channel poles, $t/2+1 = n-\epsilon$, we find for the leading-order in s:

$$\big\langle 1\ 1\ 1\ 1\big\rangle \sim \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{(s/2)^n}{n-1-t/2}\ . \tag{5.7}$$

The fact that residues at all the poles have the same sign implies that the propagators of all the intermediate states in the sum have the same sign, and correspondingly they do not have negative norm. Similar arguments can be applied to the other channels, and to all the other N-point functions of the W_3 string. For example, expanding the $\big\langle \frac12\ \frac12\ \frac12\ \frac12\big\rangle$ result given in (4.1a) in the t channel gives

$$\big\langle \tfrac12\ \tfrac12\ \tfrac12\ \tfrac12\big\rangle \sim \sum_{n=1}^{\infty} \frac{1}{n!}\,\frac{(s/2)^n}{n-1-t/2} \tag{5.8}$$

to leading order in s.

6. Mixing the φ and ghost fields

The fact that all the physical states with continuous momentum admit an effective spacetime interpretation suggests that there may be a more appropriate description of the W_3 string, where this feature is more manifest. Motivated by this, we start by noting that we can write the BRST operator Q_B of (2.1) as

$$Q_B = \oint dz \Big[\widetilde{c}\,T^{\rm eff} + \text{more}\Big]\ , \tag{6.1}$$

where all the dependence on the spacetime coordinates X^μ is contained in the first term, and

$$\widetilde{c} \equiv c + \tfrac{7}{174}\sqrt{58}\, i\, \partial\gamma - \tfrac{8}{261} b\, \partial\gamma\, \gamma - \tfrac{4}{87}\sqrt{29}\, i\, \partial\varphi\, \gamma\ . \tag{6.2}$$

This suggests that simplifications might occur if we were to treat $\widetilde{c}$ rather than c as the ghost for the spin-2 symmetry. We thus look for a corresponding set of redefinitions for b, γ, β and φ such that the new fields have the same set of operator-product expansions as the original ones. This leads uniquely to

$$\begin{aligned}
&\widetilde{c} \equiv c + \tfrac{7}{174}\sqrt{58}\, i\, \partial\gamma - \tfrac{8}{261} b\, \partial\gamma\, \gamma - \tfrac{4}{87}\sqrt{29}\, i\, \partial\varphi\, \gamma \\
&\widetilde{b} \equiv b\ , \\
&\widetilde{\gamma} \equiv \gamma\ , \\
&\widetilde{\beta} \equiv \beta + \tfrac{7}{174}\sqrt{58}\, i\, \partial b - \tfrac{8}{261} \partial b\, b\, \gamma + \tfrac{4}{87}\sqrt{29}\, i\, \partial\varphi\, b\ , \\
&\widetilde{\varphi} \equiv \varphi - \tfrac{4}{87}\sqrt{29}\, i\, b\, \gamma\ .
\end{aligned} \tag{6.3}$$

These redefinitions may be inverted, to give

$$\begin{aligned}
&c \equiv \widetilde{c} - \tfrac{7}{174}\sqrt{58}\, i\, \partial\widetilde{\gamma} - \tfrac{8}{261} \widetilde{b}\, \partial\widetilde{\gamma}\, \widetilde{\gamma} + \tfrac{4}{87}\sqrt{29}\, i\, \partial\widetilde{\varphi}\, \widetilde{\gamma}\ , \\
&b \equiv \widetilde{b}\ , \\
&\gamma \equiv \widetilde{\gamma}\ , \\
&\beta \equiv \widetilde{\beta} - \tfrac{7}{174}\sqrt{58}\, i\, \partial\widetilde{b} - \tfrac{8}{261} \partial\widetilde{b}\, \widetilde{b}\, \widetilde{\gamma} - \tfrac{4}{87}\sqrt{29}\, i\, \partial\widetilde{\varphi}\, \widetilde{b}\ , \\
&\varphi \equiv \widetilde{\varphi} + \tfrac{4}{87}\sqrt{29}\, i\, \widetilde{b}\, \widetilde{\gamma}\ .
\end{aligned} \tag{6.4}$$

Since they preserve the operator products, the transformations (6.3) are in some sense canonical.

It is now a straightforward matter to use (6.4) to rewrite the BRST operator, the ghost boosters a_φ and a_{X^μ}, and the physical operators in terms of the new fields. We shall first consider the physical operators. Remarkably, the higher-level physical operators become greatly simplified. The results for the physical operators with continuous momentum presented in the appendix become:

- $\underline{\ell = 0,\ G = 3}$:

$$\mathbf{V}^3_1[-\tfrac{8}{7}Q, p] = \widetilde{c}\, \partial\widetilde{\gamma}\, \widetilde{\gamma}\, e^{-\frac{8}{7}Q\tilde{\varphi}} e^{ip\cdot X}\ , \tag{6.5}$$

$$\mathbf{V}^3_1[-\tfrac{6}{7}Q, p] = \widetilde{c}\, \partial\widetilde{\gamma}\, \widetilde{\gamma}\, e^{-\frac{6}{7}Q\tilde{\varphi}} e^{ip\cdot X}\ , \tag{6.6}$$

$$\mathbf{V}^3_{15/16}[-Q, p] = \widetilde{c}\, \partial\widetilde{\gamma}\, \widetilde{\gamma}\, e^{-Q\tilde{\varphi}} e^{ip\cdot X}\ . \tag{6.7}$$

- $\underline{\ell = 1,\ G = 2}$:

$$\mathbf{V}^2_{15/16}[-\tfrac{3}{7}Q, p] = \widetilde{c}\, \widetilde{\gamma} e^{-\frac{3}{7}Q\tilde{\varphi}} e^{ip\cdot X}\ , \tag{6.8}$$

$$\mathbf{V}^2_{1/2}[-\tfrac{4}{7}Q, p] = \widetilde{c}\, \widetilde{\gamma} e^{-\frac{4}{7}Q\tilde{\varphi}} e^{ip\cdot X}\ . \tag{6.9}$$

• $\underline{\ell=2,\ G=2}$:

$$\mathbf{V}^2_{1/2}[-\tfrac{2}{7}Q,p]=\widetilde{c}\Big(\sqrt{2}\,\partial\widetilde{\varphi}\,\widetilde{\gamma}-\tfrac{3}{2}\partial\widetilde{\gamma}\Big)e^{-\frac{2}{7}Q\tilde{\varphi}}e^{ip\cdot X}\ . \tag{6.10}$$

• $\underline{\ell=3,\ G=1}$:

$$\mathbf{V}^1_1[0,p]=\widetilde{c}\,e^{ip\cdot X}\ . \tag{6.11}$$

• $\underline{\ell=4,\ G=1}$:

$$\mathbf{V}^1_{15/16}[\tfrac{1}{7}Q,p]=-\tfrac{3}{16}\sqrt{29}\,i\,\widetilde{c}\,\Big(3\sqrt{2}\,\widetilde{\beta}\,\widetilde{\gamma}+4\,\partial\widetilde{\varphi}\Big)e^{\frac{1}{7}Q\tilde{\varphi}}e^{ip\cdot X}\ . \tag{6.12}$$

• $\underline{\ell=5,\ G=1}$:

$$\mathbf{V}^1_1[\tfrac{2}{7}Q,p]=\tfrac{1}{20}\sqrt{29}\,i\,\widetilde{c}\,\Big(6\sqrt{2}\,\partial\widetilde{\beta}\,\widetilde{\gamma}-3\sqrt{2}\,\widetilde{\beta}\,\partial\widetilde{\gamma}+12\partial\widetilde{\varphi}\,\widetilde{\beta}\,\widetilde{\gamma}+4\sqrt{2}\,\partial\widetilde{\varphi}\,\partial\widetilde{\varphi}\,+2\partial^2\widetilde{\varphi}\Big)e^{\frac{2}{7}Q\tilde{\varphi}}e^{ip\cdot X}\ . \tag{6.13}$$

• $\underline{\ell=8,\ G=0}$:

$$\mathbf{V}^0_{1/2}[\tfrac{4}{7}Q,p]=\widetilde{c}\,\Big(\partial\widetilde{\varphi}\,\partial\widetilde{\beta}+3\sqrt{2}\,\partial\widetilde{\beta}\,\widetilde{\beta}\,\widetilde{\gamma}-3\,\partial^2\widetilde{\varphi}\,\widetilde{\beta}-\tfrac{3}{4}\sqrt{2}\,\partial^2\widetilde{\beta}\Big)e^{\frac{4}{7}Q\tilde{\varphi}}e^{ip\cdot X}\ . \tag{6.14}$$

• $\underline{\ell=9,\ G=0}$:

$$\begin{aligned}\mathbf{V}^0_{15/16}[\tfrac{5}{7}Q,p]=\widetilde{c}\,\Big(&\partial\widetilde{\varphi}\,\partial\widetilde{\varphi}\,\partial\widetilde{\beta}-\tfrac{3}{8}\sqrt{2}\,\partial\widetilde{\varphi}\,\partial^2\widetilde{\beta}-2\sqrt{2}\,\partial^2\widetilde{\varphi}\,\partial\widetilde{\beta}-3\,\partial^2\widetilde{\varphi}\,\partial\widetilde{\varphi}\,\widetilde{\beta}-\tfrac{3}{8}\sqrt{2}\,\partial^3\widetilde{\varphi}\,\widetilde{\beta}\\&+\tfrac{15}{4}\sqrt{2}\,\partial\widetilde{\varphi}\,\partial\widetilde{\beta}\,\widetilde{\beta}\,\widetilde{\gamma}-3\,\partial\widetilde{\beta}\,\widetilde{\beta}\,\partial\widetilde{\gamma}+\tfrac{27}{16}\partial^2\widetilde{\beta}\,\widetilde{\beta}\,\widetilde{\gamma}-\tfrac{1}{2}\partial^3\widetilde{\beta}\Big)e^{\frac{5}{7}Q\tilde{\varphi}}e^{ip\cdot X}\ .\end{aligned} \tag{6.15}$$

The $\ell=1$ and $\ell=6$ discrete operators given in the appendix become

• $\underline{\ell=1,\ G=2}$:

$$\mathbf{D}^2[-\tfrac{6}{7}Q]=\Big(\widetilde{c}\,\widetilde{\gamma}+\tfrac{2}{87}\sqrt{58}\,i\,\partial\widetilde{\gamma}\,\widetilde{\gamma}\Big)e^{-\frac{6}{7}Q\tilde{\varphi}}\ , \tag{6.16}$$

$$\mathbf{D}^2[-\tfrac{8}{7}Q]=\Big(\widetilde{c}\,\widetilde{\gamma}+\tfrac{5}{87}\sqrt{58}\,i\,\partial\widetilde{\gamma}\,\widetilde{\gamma}\Big)e^{-\frac{8}{7}Q\tilde{\varphi}}\ . \tag{6.17}$$

• $\underline{\ell=6,\ G=0}$:

$$\mathbf{D}^0[\tfrac{2}{7}Q]=\Big[\tfrac{261}{10}\widetilde{c}\,\widetilde{\beta}-\tfrac{1}{5}\sqrt{29}\,i\,\Big(6\sqrt{2}\,\partial\widetilde{\beta}\,\widetilde{\gamma}-3\sqrt{2}\,\widetilde{\beta}\,\partial\widetilde{\gamma}+12\partial\widetilde{\varphi}\,\widetilde{\beta}\,\widetilde{\gamma}+4\sqrt{2}\,\partial\widetilde{\varphi}\,\partial\widetilde{\varphi}\,+2\partial^2\widetilde{\varphi}\Big)\Big]e^{\frac{2}{7}Q\tilde{\varphi}}\ . \tag{6.18}$$

Several comments are now in order. In all the above expressions, the implicit normal ordering is with respect to the tilded fields. Since the $(b\,\gamma)$ term in the redefinition of φ is nilpotent, the exponential terms of the form $e^{\lambda\tilde{\varphi}}$ reduce to

$$\Big(1-\tfrac{4}{87}\sqrt{29}\,i\,\lambda\,b\,\gamma\Big)e^{\lambda\varphi} \tag{6.19}$$

in terms of the original fields.

It is instructive to compare the physical operators as given above with their expressions in terms of the original fields given in the appendix. In particular, the comparisons between the expressions (6.13) and (A.9) for the $\ell = 5$ operator, where 32 terms reduce to 5, and between the expressions (6.14) and (A.10) for the $\ell = 8$ operator, where 82 terms reduce to 4, are quite striking. The level $\ell = 9$ operator given in (6.15) does not appear in the appendix. In fact we have obtained it only using the new formalism of this section. In the original formalism it involves 414 terms, thus providing another example of the power of the new formalism.

The calculation of scattering amplitudes in the W_3 string can be performed equivalently in terms of the new quantum fields given by (6.3). Since the transformed fields satisfy the same operator-product expansions as the original ones, the Jacobian of the transformation is simply the identity, and one can calculate with the tilded fields in exactly the same way as we did with the untilded ones. For example, (2.15) is identical in its tilded form. Interestingly, we see that the terms in (6.12), (6.13) and (6.14) are of precisely the same form as those in (3.1), (3.3) and (3.5), which were the only terms in (A.8), (A.9) and (A.10) that contributed in scattering amplitudes. This explains the high degree of redundancy in the representation of physical states in terms of the original fields, where many of the terms in a physical state seemed to be irrelevant in interactions. The new fields $\widetilde{c}$, $\widetilde{b}$, $\widetilde{\gamma}$, $\widetilde{\beta}$ and $\widetilde{\varphi}$ thus provide what one might call a "minimal" formalism in which this redundancy is removed.

All the above physical operators with continuous momentum p_μ have the form

$$U(\widetilde{\beta},\widetilde{\gamma},\widetilde{\varphi})\,\widetilde{c}\,e^{ip\cdot X}\ . \tag{6.20}$$

The $\widetilde{c}\,e^{ip\cdot X}$ factor is reminiscent of an effective string theory. We conjecture that *all* prime physical operators with continuous momentum p_μ have the form (6.20) (or with $\partial\widetilde{c}\widetilde{c}$ in the case of the conjugate physical operator). The discrete operators, on the other hand, do not factorise in this way.

The ghost boosters a_φ and a_{X^μ} take on simpler forms in terms of the new fields:

$$\begin{aligned} a_{\tilde\varphi} &\equiv [Q_B,\widetilde{\varphi}] = \widetilde{c}\,\partial\widetilde{\varphi} - Q\,\partial\widetilde{c} \\ &\qquad + \tfrac{1}{261}\sqrt{29}\,i\,\Big(19\,\partial^2\widetilde{\gamma} + 36\,\widetilde{\beta}\,\partial\widetilde{\gamma}\,\widetilde{\gamma} + 24\,\partial\widetilde{\varphi}\,\partial\widetilde{\varphi}\,\widetilde{\gamma} - 24Q\,\partial\widetilde{\varphi}\,\partial\widetilde{\gamma}\Big)\ , \\ a_{X^\mu} &\equiv [Q_B, X^\mu] = \widetilde{c}\,\partial X^\mu - i\,a^\mu\partial\widetilde{c}\ . \end{aligned} \tag{6.21}$$

Note that $a_{\tilde\varphi}$ is equivalent to a_φ modulo a BRST-trivial term, since $[Q_B, b\,\gamma]$ is BRST trivial. It is clear from (6.21) why the a_{X^μ} ghost boosters generate just one physical operator at ghost number $(G+1)$ by acting on a prime operator at ghost number G, as we described in section 3: for an operator of the form (6.20), we get

$$a_{X^\mu}\,U(\widetilde{\beta},\widetilde{\gamma},\widetilde{\varphi})\,\widetilde{c}\,e^{ip\cdot X} = i\,(-1)^G(p^\mu + a^\mu)\,U(\widetilde{\beta},\widetilde{\gamma},\widetilde{\varphi})\,\partial\widetilde{c}\,\widetilde{c}\,e^{ip\cdot X}\ . \tag{6.22}$$

From the form of the ghost boosters (6.21), it becomes clear that the entire quartet generated from the physical operator (6.20) will admit an effective spacetime interpretation. Since, on the other hand, the discrete operators cannot be written in the form (6.20), a_{X^μ} generates d independent discrete operators rather than just one. Correspondingly, these boosted operators do not all admit an effective spacetime interpretation.

The introduction of the tilded fields makes manifest an observation that we made previously in section 2. There, we observed that in ordinary string theory one would get zero in ($N \geq 4$)-point scattering amplitudes unless the replacement (2.14) is made for $(N-3)$ of the operators, since each operator involves a c ghost, but that it was *a priori* not so clear that the same would be true in the W_3 string. However, as we have now seen by using the tilded fields, all known physical operators with continuous momentum involve a factor of $\widetilde{c}$, and thus the replacement (2.14) is necessary for the same reason as in ordinary string theory. Since discrete operators do not have this structure, the same reasoning does not apply, which is consistent with our findings in section 4 where the replacement (2.14) had to be omitted in order to get finite and non-zero results in amplitudes involving them.

Another feature of the W_3 string that becomes manifest in the new formalism is that tree-level N-point amplitudes where all the external states have $L_0^{\rm eff} = 1$ are identical to those of the ordinary bosonic string. As an example, for effective tachyons we can take two $\ell = 0$ operators $\mathbf{W}_1^4[-\frac{8}{7}Q, p]$ and $\mathbf{V}_1^3[-\frac{6}{7}Q, p]$, and $(N-2)$ operators $\mathbf{V}_1^1[0, p]$ given in (6.11). As usual, one could replace the tachyons by excited effective spacetime states if desired.

Finally, we give the form of the BRST operator in terms of the new fields:

$$
\begin{aligned}
Q_B = Q_B^{\rm eff} + \oint dz \Big[& \widetilde{c}\, T_{\widetilde{\varphi}} - 3\, \widetilde{c}\, \widetilde{\beta}\, \partial\widetilde{\gamma} - 2\, \widetilde{c}\, \partial\widetilde{\beta}\, \widetilde{\gamma} \\
& - \tfrac{4}{261}\sqrt{29}\, i\, \widetilde{\gamma} \Big(2(\partial\widetilde{\varphi})^3 + 6Q\, \partial^2\widetilde{\varphi}\, \partial\widetilde{\varphi} + \tfrac{19}{4}\, \partial^3\widetilde{\varphi} + 9\, \partial\widetilde{\varphi}\, \widetilde{\beta}\, \partial\widetilde{\gamma} + 3Q\, \partial\widetilde{\beta}\, \partial\widetilde{\gamma} \Big) \Big] \ ,
\end{aligned}
\tag{6.23}
$$

where

$$
Q_B^{\rm eff} \equiv \oint dz\, \widetilde{c} \Big(T^{\rm eff} + \tfrac{1}{2} T_{\tilde{c},\tilde{b}} \Big) \ , \tag{6.24}
$$

and

$$
T_{\widetilde{\varphi}} \equiv -\tfrac{1}{2}(\partial\widetilde{\varphi})^2 - Q\, \partial^2\widetilde{\varphi} \ , \tag{6.25}
$$

$$
T_{\tilde{c},\tilde{b}} \equiv -2\, \widetilde{b}\, \partial\widetilde{c} - \partial\widetilde{b}\, \widetilde{c} \ . \tag{6.26}
$$

Note that although $Q_B^{\rm eff}$ is not nilpotent, since $T^{\rm eff}$ has central charge $\frac{51}{2}$ and $T_{\tilde{c},\tilde{b}}$ has central charge -26, it does have the property that it is nilpotent on the subspace spanned by $\widetilde{c}$, $\widetilde{\gamma}$, $\widetilde{\beta}$, $\widetilde{\varphi}$ and X^μ, but not on $\widetilde{b}$, in the sense that

$$
[Q_B^{\rm eff}, \{Q_B^{\rm eff}, \widetilde{b}\,\}] = -\tfrac{1}{24}\partial^3\widetilde{c} \ . \tag{6.27}
$$

Although $Q_B^{\rm eff}$ is not itself nilpotent, by adding all the other terms involving $\widetilde{c}$ in (6.23) we obtain a nilpotent operator Q_0. In fact writing $Q_B = Q_0 + Q_1$, with

$$\begin{aligned} Q_0 &= Q_B^{\rm eff} + \oint dz\, \Big(\widetilde{c}\, T_{\tilde\varphi} - 3\,\widetilde{c}\,\widetilde{\beta}\,\partial\widetilde{\gamma} - 2\,\widetilde{c}\,\partial\widetilde{\beta}\,\widetilde{\gamma}\Big) \\ &= \oint dz\,\widetilde{c}\Big(T^{\rm eff} + T_{\tilde\varphi} + T_{\tilde\gamma,\tilde\beta} + \tfrac{1}{2}\,T_{\tilde c,\tilde b}\Big)\ , \\ Q_1 &= -\tfrac{4}{261}\sqrt{29}\, i\, \oint dz\, \widetilde{\gamma}\, \Big(2(\partial\widetilde{\varphi})^3 + 6Q\,\partial^2\widetilde{\varphi}\,\partial\widetilde{\varphi} + \tfrac{19}{4}\,\partial^3\widetilde{\varphi} + 9\,\partial\widetilde{\varphi}\,\widetilde{\beta}\,\partial\widetilde{\gamma} + 3Q\,\partial\widetilde{\beta}\,\partial\widetilde{\gamma}\Big)\ , \end{aligned} \tag{6.28}$$

where $T_{\tilde\gamma,\tilde\beta} = -3\,\widetilde{\beta}\,\partial\widetilde{\gamma} - 2\,\partial\widetilde{\beta}\,\widetilde{\gamma}$, the BRST charge separates into two independently nilpotent parts. The nilpotency of Q_0 is easy to see because the total central charge of $T^{\rm eff} + T_{\tilde\varphi} + T_{\tilde\gamma,\tilde\beta}$ is 26, and the nilpotency of Q_1 follows from that of Q_B and Q_0. The splitting is very natural if we view it from another angle: Q_0 and Q_1 have different $(\widetilde{c},\widetilde{b})$ and $(\widetilde{\gamma},\widetilde{\beta})$ ghost numbers. Q_0 has $(\widetilde{c},\widetilde{b})$ ghost number 1 and $(\widetilde{\gamma},\widetilde{\beta})$ ghost number 0, while Q_1 has $(\widetilde{c},\widetilde{b})$ ghost number 0 and $(\widetilde{\gamma},\widetilde{\beta})$ ghost number 1. Thus we have

$$\begin{aligned} Q_B &= Q_0 + Q_1\ , \\ Q_0^2 &= Q_1^2 = \big\{Q_0,\ Q_1\big\} = 0\ . \end{aligned} \tag{6.29}$$

This grading of the BRST operator provides an elegant starting point for studying its cohomology.

7. Conclusions

In this paper we have investigated in detail the structure of the spectrum and the interactions of the open W_3 string. We have seen that the physical spectrum displays a surprising variety of states. As well as the physical states of standard ghost structure, at ghost number $G = 0$, which have been known for a long time, it has been found that it also contains prime states at ghost numbers $G = -1$, $G = -2$ and $G = -3$, and discrete prime states at $G = -1$ and $G = -3$. The prime states at $G = -2$ were found for the first time in this paper. We have reason to suspect that this is not the end of the story; it seems likely that physical states at more negative ghost numbers can exist too, both with continuous and with discrete momenta. We have also clarified the structure of the multiplets generated by the action of the ghost boosters a_φ and a_{X^μ} on the prime states. For continuous-momentum prime states they generate a quartet of physical states, at ghost numbers $\{G,\ G+1,\ G+2\}$ with multiplicities $\{1,\ 2,\ 1\}$. For discrete prime states they generate a $(4d)$-plet of discrete states at ghost numbers $\{G,\ G+1,\ G+2,\ G+3\}$ with multiplicities $\{1,\ d+1,\ 2d-1,\ d-1\}$.

All of the N-point functions that we have calculated in [11] and in this paper have involved physical states that are tachyons from the effective-spacetime point of view. As we have pointed out already, these effective tachyons can be replaced by excited effective physical states with the same $L_0^{\rm eff}$ values. (It is for states such as these that our unitarity discussion in section 5 applied.) The scattering amplitudes that we have presented can easily be modified to cover the case where such physical states with excitations in the effective spacetime are involved. This is because the excited physical states still have the form (3.7), and so the contribution from the φ and ghost fields will be the same as in the purely tachyonic case. Thus the standard techniques developed for calculating tree amplitudes for excited states in the bosonic string can be carried over to the W_3 string.

All the known physical states with continuous momentum p_μ admit an effective spacetime interpretation as given in (3.7), and have effective intercepts $L_0^{\rm eff} = 1$, $\frac{15}{16}$ or $\frac{1}{2}$. We have conjectured that these features will hold for all physical states in the W_3 string with continuous p_μ momentum.

An important consequence of the fact that physical states with continuous momentum p_μ admit such an effective spacetime interpretation is that scattering amplitudes factorise into two parts. The first part involves only the effective spacetime states and the second involves only the ghost fields and the φ coordinate. The result of the second part, and therefore the entire amplitude, is universal in the sense that for a given set of $L_0^{\rm eff}$ values for the external states, the result from the ghost and φ fields is always the same, up to a constant coefficient. This coefficient is zero if any selection rule is violated. We have encountered several selection rules in this paper. One is that the total ghost structure must be of the form (2.15); another is the φ-momentum conservation law (2.16); and another is that an $(N \geq 4)$-point function will vanish if any of its underlying three-point functions vanishes.

It is very striking that the frozen coordinate φ is more akin to the ghost fields than to the effective coordinates X^μ. In fact the momentum conservation law (2.16) is rather similar to the condition that the total ghost structure must be as in (2.15) to have a non-vanishing inner product. This similarity becomes particularly evident if one bosonises the ghosts. In fact in section 6, we have shown that by performing a redefinition of the quantum fields in which the ghosts and the φ field are mixed, great simplifications occur. In our previous paper on the interacting W_3 string, we argued that the existence of physical states involving excitations of the ghost fields was suggestive of the fact that it may be inappropriate to look for a classical correspondence principle for the W_3 string [11]. The results of this paper, and particularly of section 6, reinforce this suggestion. They also strengthen our belief that a complete understanding of W-geometry can only be achieved at the quantum level, when the ghosts are included.

In this paper we have been concerned only with tree-level amplitudes. The evaluation of loop amplitudes remains an interesting open question. We expect that some new features should arise. For example, one of the results in this paper is that the tree level amplitudes involving only $L_0^{\rm eff} = 1$ states are the same as in the ordinary bosonic string. At loop level, however, states with $L_0^{\rm eff} = \frac{15}{16}$ and $\frac{1}{2}$ will presumably also run around the loops, giving results that may differ from those of string theory. Since the $L_0^{\rm eff}=1$ sector of the W_3 string includes massless states, we may thus expect that the low energy limit of the W_3 string might differ from that of the usual string.

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NOTE ADDED

After this preprint was circulated, a paper by Freeman and West on related topics appeared [23]. They assert that the procedure we introduced in [11] for calculating W_3 string scattering amplitudes is incorrect, and that it gives results that violate unitarity and the assumptions of S-matrix theory. This is not the case, and in fact the authors of [23] explicitly use the procedure of [11] for computing W_3 scattering amplitudes, including our use of ghost boosters and the replacement (2.14) in order to achieve the correct ghost structures for non-vanishing correlation functions.

The authors of [23] introduce a screening charge $S=\oint dz\,U(z)$ in order to obtain further scattering amplitudes beyond those that we have computed. It is easy to see that this screening charge is related by a descent equation to our $\ell=6$ discrete state $D^0[\frac{2}{7}Q]$ given in (6.18). In fact, in the new formalism introduced in section 6 of this paper, their screening current U, after removing a total derivative, is simply given by $U=\widetilde{\beta}\,e^{\frac{2}{7}Q\tilde{\varphi}}$, and satisfies $\{Q_B,U\}=\frac{10}{261}\partial D^0[\frac{2}{7}Q]$. Conversely, we see that U can be obtained from $D^0[\frac{2}{7}Q]$ by making the replacment (2.14). In fact the screening charge S given in [23] is nothing but a special case of our standard replacement of a spin 0 physical state by a spin 1 current that can be integrated over the worldsheet to give BRST-invariant amplitudes. Not surprisingly, therefore, the new amplitudes obtained in [23] by including screening charges are in on-to-one correspondence with amplitudes that we can compute by including discrete states. In particular, we have checked that the scattering amplitude $\big\langle\frac{15}{16}\,\frac{15}{16}\,\frac{15}{16}\,D\,\frac{15}{16}\big\rangle$ for four $L^{\rm eff}=\frac{15}{16}$ states (one from ghost-boosting (6.7) and three from (6.8), as in [23]) and the discrete state $D=D^0[\frac{2}{7}Q]$ agrees with the result for the four-point function $\big\langle\frac{15}{16}\,\frac{15}{16}\,\frac{15}{16}\,\frac{15}{16}\,S\big\rangle$ obtained in [23].

In all the examples of scattering amplitudes involving discrete states that we discussed in section 4, the discrete states could normal order with external physical states to give new physical states. The new result of [23] amounts to the observation that there are further amplitudes that can be obtained, using the formalism that we presented in [11] and this paper, by including discrete states that cannot normal order with the external physical states. By this means, one can indeed obtain scattering amplitudes involving more than two $L^{\rm eff} = \frac{15}{16}$ external states. Our statements on this point in section 4 apply only to the case where no discrete states are involved.

Our formalism for describing W_3 string scattering does not suffer from any of the deficiencies mentioned in [23], and indeed, since the authors of [23] perform all their calculations by using the procedures that we have developed, (or, in the case of amplitudes involving screening charges, a procedure that is directly equivalent to ours), their results necessarily agree with ours.

We are grateful to the authors of [23] for encouraging us to look further at scattering amplitudes involving discrete states. S.S. is grateful to Jose M. Figueroa-O'Farrill for discussions about the screening charge.

APPENDIX

In this appendix we present the explicit forms of the various prime physical operators that we use in this paper. In cases where there is a freedom to add BRST-trivial pieces, this is always precisely sufficient to enable us to write the operators so that they admit an effective spacetime interpretation. It is in this form that we present the operators. We first list the physical operators with continuous momentum p_μ, and then those with discrete momentum.

Prime operators with continuous momentum

- $\underline{\ell = 0,\ G = 3}$:

$$\mathbf{V}^3_1[-\tfrac{8}{7}Q, p] = c\,\partial\gamma\,\gamma\, e^{-\frac{8}{7}Q\varphi} e^{ip\cdot X}\ , \tag{A.1}$$

$$\mathbf{V}^3_1[-\tfrac{6}{7}Q, p] = c\,\partial\gamma\,\gamma\, e^{-\frac{6}{7}Q\varphi} e^{ip\cdot X}\ , \tag{A.2}$$

$$\mathbf{V}^3_{15/16}[-Q, p] = c\,\partial\gamma\,\gamma\, e^{-Q\varphi} e^{ip\cdot X}\ . \tag{A.3}$$

- $\underline{\ell = 1,\ G = 2}$:

$$\mathbf{V}^2_{15/16}[-\tfrac{3}{7}Q, p] = \Big(c\,\gamma + \frac{i}{3\sqrt{58}}\partial\gamma\,\gamma\Big)\, e^{-\frac{3}{7}Q\varphi} e^{ip\cdot X}\ , \tag{A.4}$$

$$\mathbf{V}^2_{1/2}[-\tfrac{4}{7}Q, p] = \Big(c\,\gamma - \frac{i}{3\sqrt{58}}\partial\gamma\,\gamma\Big)\, e^{-\frac{4}{7}Q\varphi} e^{ip\cdot X}\ . \tag{A.5}$$

- $\underline{\ell = 2,\ G = 2}$:

$$\begin{aligned}\mathbf{V}^2_{1/2}[-\tfrac{2}{7}Q, p] =& \frac{-i}{\sqrt{29}}\Big(\partial\varphi\,\partial\gamma\,\gamma + \sqrt{58}\, i\,\partial\varphi\, c\,\gamma - \tfrac{3}{2}\sqrt{29}\, i\, c\,\partial\gamma \\ &- \tfrac{2}{3}\sqrt{2}\,\partial^2\gamma\,\gamma - \tfrac{1}{3}\sqrt{2}\ b\, c\,\partial\gamma\,\gamma\Big) e^{-\frac{2}{7}Q\varphi} e^{ip\cdot X}\ .\end{aligned} \tag{A.6}$$

- $\underline{\ell = 3,\ G = 1}$:

$$\mathbf{V}^1_1[0, p] = \Big(c - \tfrac{8}{261} b\,\partial\gamma\,\gamma - \tfrac{4}{87}\sqrt{29}\, i\,\partial\varphi\,\gamma + \tfrac{7}{174}\sqrt{58}\, i\,\partial\gamma\Big) e^{ip\cdot X}\ . \tag{A.7}$$

- $\underline{\ell = 4,\ G = 1}$:

$$\begin{aligned}\mathbf{V}^1_{15/16}[\tfrac{1}{7}Q, p] = \Big(& b\, c\,\partial\gamma + \tfrac{5}{348}\sqrt{58}\, i\, b\,\partial^2\gamma\,\gamma - \tfrac{27}{16}\beta\,\partial\gamma\,\gamma - \tfrac{9}{16}\sqrt{58}\, i\, c\,\beta\,\gamma - \tfrac{1}{2}\sqrt{2}\,\partial\varphi\, b\, c\,\gamma \\ &+ \tfrac{1}{58}\sqrt{29}\, i\,\partial\varphi\, b\,\partial\gamma\,\gamma - \tfrac{3}{4}\sqrt{29}\, i\,\partial\varphi\, c - \partial\varphi\,\partial\varphi\,\gamma + \tfrac{9}{8}\sqrt{2}\,\partial\varphi\,\partial\gamma + \tfrac{1}{16}\partial b\, c\,\gamma \\ &+ \tfrac{35}{928}\sqrt{58}\, i\,\partial b\,\partial\gamma\,\gamma - \tfrac{3}{4}\sqrt{2}\,\partial^2\varphi\,\gamma + \tfrac{1}{2}\partial^2\gamma\Big) e^{\frac{1}{7}Q\varphi} e^{ip\cdot X}\ .\end{aligned} \tag{A.8}$$

• $\underline{\ell=5,\ G=1}$:

$$\begin{aligned}\mathbf{V}^1_1[\tfrac{2}{7}Q,p] = \Big(& b\,c\,\beta\,\partial\gamma\,\gamma - \tfrac{2}{15}b\,c\,\partial^2\gamma - \tfrac{2}{1305}\sqrt{58}\,i\,b\,\partial^2\gamma\,\partial\gamma - \tfrac{2}{783}\sqrt{58}\,i\,b\,\partial^3\gamma\,\gamma + \tfrac{2}{5}\beta\,\partial^2\gamma\,\gamma \\
& - \tfrac{3}{20}\sqrt{58}\,i\,c\,\beta\,\partial\gamma + \tfrac{3}{10}\sqrt{58}\,i\,c\,\partial\beta\,\gamma - \tfrac{11}{15}\sqrt{2}\,\partial\varphi\,b\,c\,\partial\gamma - \tfrac{4}{435}\sqrt{29}\,i\,\partial\varphi\,b\,\partial^2\gamma\,\gamma \\
& + \tfrac{9}{10}\sqrt{2}\,\partial\varphi\,\beta\,\partial\gamma\,\gamma + \tfrac{3}{5}\sqrt{29}\,i\,\partial\varphi\,c\,\beta\,\gamma + \tfrac{8}{15}\partial\varphi\,\partial\varphi\,b\,c\,\gamma - \tfrac{8}{1305}\sqrt{58}\,i\,\partial\varphi\,\partial\varphi\,b\,\partial\gamma\,\gamma \\
& + \tfrac{1}{5}\sqrt{58}\,i\,\partial\varphi\,\partial\varphi\,c + \tfrac{4}{15}\sqrt{2}\,\partial\varphi\,\partial\varphi\,\partial\varphi\,\gamma - \tfrac{11}{15}\partial\varphi\,\partial\varphi\,\partial\gamma + \tfrac{1}{6}\sqrt{2}\,\partial\varphi\,\partial b\,c\,\gamma \\
& - \tfrac{91}{2610}\sqrt{29}\,i\,\partial\varphi\,\partial b\,\partial\gamma\,\gamma - \tfrac{4}{15}\sqrt{2}\,\partial\varphi\,\partial^2\gamma - \tfrac{13}{870}\sqrt{58}\,i\,\partial b\,b\,c\,\partial\gamma\,\gamma - \tfrac{5}{12}\partial b\,c\,\partial\gamma \\
& - \tfrac{19}{1305}\sqrt{58}\,i\,\partial b\,\partial^2\gamma\,\gamma + \tfrac{11}{10}\partial\beta\,\partial\gamma\,\gamma + \tfrac{1}{3}\sqrt{2}\,\partial^2\varphi\,b\,c\,\gamma - \tfrac{1}{145}\sqrt{29}\,i\,\partial^2\varphi\,b\,\partial\gamma\,\gamma \\
& + \tfrac{1}{10}\sqrt{29}\,i\,\partial^2\varphi\,c + \tfrac{14}{15}\partial^2\varphi\,\partial\varphi\,\gamma - \tfrac{23}{60}\sqrt{2}\,\partial^2\varphi\,\partial\gamma - \tfrac{1}{30}\partial^2 b\,c\,\gamma \\
& - \tfrac{19}{1044}\sqrt{58}\,i\,\partial^2 b\,\partial\gamma\,\gamma + \tfrac{1}{5}\sqrt{2}\,\partial^3\varphi\,\gamma - \tfrac{2}{45}\partial^3\gamma\Big)e^{\frac{2}{7}Q\varphi}e^{ip\cdot X}\ . \qquad (A.9)\end{aligned}$$

• $\underline{\ell=8,\ G=0}$:

$$\begin{aligned}&\mathbf{V}^0_{1/2}[\tfrac{4}{7}Q,p] = \\
&\Big(\tfrac{20}{261}\sqrt{2}\,b\,\beta\,\partial^3\gamma\,\gamma - \tfrac{10}{29}\sqrt{2}\,b\,\beta\,\partial^2\gamma\,\partial\gamma + \tfrac{4}{29}\sqrt{29}\,i\,b\,c\,\beta\,\partial^2\gamma - \tfrac{4}{87}\sqrt{29}\,i\,b\,c\,\partial\beta\,\partial\gamma \\
&+ \tfrac{2}{29}\sqrt{29}\,i\,b\,c\,\partial^2\beta\,\gamma - \tfrac{20}{261}\sqrt{2}\,b\,\partial\beta\,\partial^2\gamma\,\gamma + \tfrac{17}{87}\sqrt{2}\,b\,\partial^2\beta\,\partial\gamma\,\gamma + \tfrac{1}{522}\sqrt{2}\,b\,\partial^4\gamma \\
&+ \tfrac{4}{87}\sqrt{29}\,i\,\beta\,\partial^3\gamma + 3\sqrt{2}\,c\,\partial\beta\,\beta\,\gamma - \tfrac{3}{4}\sqrt{2}\,c\,\partial^2\beta + \tfrac{16}{87}\partial\varphi\,b\,\beta\,\partial^2\gamma\,\gamma + \tfrac{4}{87}\sqrt{29}\,i\,\partial\varphi\,b\,c\,\partial\beta\,\gamma \\
&+ \tfrac{32}{87}\partial\varphi\,b\,\partial\beta\,\partial\gamma\,\gamma - \tfrac{52}{783}\partial\varphi\,b\,\partial^3\gamma + \partial\varphi\,c\,\partial\beta - \tfrac{16}{261}\partial\varphi\,\partial\varphi\,\partial\varphi\,\partial b\,\gamma - \tfrac{32}{261}\sqrt{2}\,\partial\varphi\,\partial\varphi\,\partial b\,b\,c\,\gamma \\
&- \tfrac{32}{22707}\sqrt{29}\,i\,\partial\varphi\,\partial\varphi\,\partial b\,b\,\partial\gamma\,\gamma - \tfrac{4}{87}\sqrt{29}\,i\,\partial\varphi\,\partial\varphi\,\partial b\,c + \tfrac{10}{87}\sqrt{2}\,\partial\varphi\,\partial\varphi\,\partial b\,\partial\gamma \\
&+ \tfrac{4}{87}\sqrt{29}\,i\,\partial\varphi\,\partial\varphi\,\partial\beta\,\gamma + \tfrac{2}{87}\sqrt{2}\,\partial\varphi\,\partial\varphi\,\partial^2 b\,\gamma + \tfrac{104}{261}\partial\varphi\,\partial b\,b\,c\,\partial\gamma + \tfrac{64}{7569}\sqrt{58}\,i\,\partial\varphi\,\partial b\,b\,\partial^2\gamma\,\gamma \\
&- \tfrac{28}{87}\partial\varphi\,\partial b\,\beta\,\partial\gamma\,\gamma - \tfrac{4}{29}\sqrt{58}\,i\,\partial\varphi\,\partial b\,c\,\beta\,\gamma - \tfrac{5}{58}\sqrt{58}\,i\,\partial\varphi\,\partial\beta\,\partial\gamma - \tfrac{8}{261}\partial\varphi\,\partial^2 b\,b\,c\,\gamma \\
&+ \tfrac{32}{7569}\sqrt{58}\,i\,\partial\varphi\,\partial^2 b\,b\,\partial\gamma\,\gamma + \tfrac{1}{58}\sqrt{58}\,i\,\partial\varphi\,\partial^2 b\,c - \tfrac{23}{174}\partial\varphi\,\partial^2 b\,\partial\gamma - \tfrac{1}{29}\sqrt{58}\,i\,\partial\varphi\,\partial^2\beta\,\gamma \\
&- \tfrac{1}{87}\partial\varphi\,\partial^3 b\,\gamma - \tfrac{8}{29}\sqrt{2}\,\partial b\,b\,c\,\beta\,\partial\gamma\,\gamma + \tfrac{4}{87}\sqrt{2}\,\partial b\,b\,c\,\partial^2\gamma - \tfrac{172}{7569}\sqrt{29}\,i\,\partial b\,b\,\partial^2\gamma\,\partial\gamma \\
&+ \tfrac{268}{68121}\sqrt{29}\,i\,\partial b\,b\,\partial^3\gamma\,\gamma + \tfrac{8}{87}\sqrt{2}\,\partial b\,\beta\,\partial^2\gamma\,\gamma + \tfrac{8}{29}\sqrt{29}\,i\,\partial b\,c\,\beta\,\partial\gamma - \tfrac{7}{87}\sqrt{29}\,i\,\partial b\,c\,\partial\beta\,\gamma \\
&- \tfrac{67}{174}\sqrt{2}\,\partial b\,\partial\beta\,\partial\gamma\,\gamma + \tfrac{2}{87}\sqrt{2}\,\partial b\,\partial^3\gamma + \tfrac{15}{29}\sqrt{29}\,i\,\partial\beta\,\beta\,\partial\gamma\,\gamma - \tfrac{2}{87}\sqrt{29}\,i\,\partial\beta\,\partial^2\gamma \\
&+ \tfrac{8}{87}\partial^2\varphi\,b\,\beta\,\partial\gamma\,\gamma + \tfrac{68}{261}\partial^2\varphi\,b\,\partial^2\gamma + \tfrac{15}{58}\sqrt{58}\,i\,\partial^2\varphi\,\beta\,\partial\gamma - 3\partial^2\varphi\,c\,\beta + \tfrac{8}{87}\sqrt{29}\,i\,\partial^2\varphi\,\partial\varphi\,b\,c \\
&- \tfrac{20}{87}\sqrt{2}\,\partial^2\varphi\,\partial\varphi\,b\,\partial\gamma - \tfrac{4}{29}\sqrt{29}\,i\,\partial^2\varphi\,\partial\varphi\,\beta\,\gamma + \tfrac{32}{261}\partial^2\varphi\,\partial\varphi\,\partial\varphi\,b\,\gamma - \tfrac{10}{87}\sqrt{2}\,\partial^2\varphi\,\partial\varphi\,\partial b\,\gamma \\
&- \tfrac{40}{261}\partial^2\varphi\,\partial b\,b\,c\,\gamma - \tfrac{8}{7569}\sqrt{58}\,i\,\partial^2\varphi\,\partial b\,b\,\partial\gamma\,\gamma + \tfrac{3}{58}\sqrt{58}\,i\,\partial^2\varphi\,\partial b\,c + \tfrac{17}{58}\partial^2\varphi\,\partial b\,\partial\gamma \\
&+ \tfrac{4}{29}\sqrt{58}\,i\,\partial^2\varphi\,\partial\beta\,\gamma + \tfrac{4}{29}\sqrt{29}\,i\,\partial^2\varphi\,\partial^2\varphi + \tfrac{8}{87}\partial^2\varphi\,\partial^2 b\,\gamma + \tfrac{2}{29}\sqrt{2}\,\partial^2 b\,b\,c\,\partial\gamma \\
&+ \tfrac{20}{22707}\sqrt{29}\,i\,\partial^2 b\,b\,\partial^2\gamma\,\gamma + \tfrac{13}{58}\sqrt{2}\,\partial^2 b\,\beta\,\partial\gamma\,\gamma + \tfrac{1}{29}\sqrt{29}\,i\,\partial^2 b\,c\,\beta\,\gamma - \tfrac{1}{174}\sqrt{2}\,\partial^2 b\,\partial b\,c\,\gamma \\
&- \tfrac{167}{15138}\sqrt{29}\,i\,\partial^2 b\,\partial b\,\partial\gamma\,\gamma + \tfrac{4}{87}\sqrt{2}\,\partial^2 b\,\partial^2\gamma + \tfrac{15}{116}\sqrt{29}\,i\,\partial^2\beta\,\partial\gamma + \tfrac{1}{29}\sqrt{58}\,i\,\partial^3\varphi\,b\,c \\
&- \tfrac{53}{261}\partial^3\varphi\,b\,\partial\gamma - \tfrac{2}{29}\sqrt{58}\,i\partial^3\varphi\,\beta\,\gamma + \tfrac{28}{261}\sqrt{2}\,\partial^3\varphi\,\partial\varphi\,b\,\gamma - \tfrac{2}{87}\sqrt{29}\,i\,\partial^3\varphi\,\partial\varphi\end{aligned}$$

$$- \tfrac{14}{261}\partial^3\varphi\,\partial b\,\gamma + \tfrac{7}{783}\sqrt{2}\,\partial^3 b\,b\,c\,\gamma + \tfrac{331}{68121}\sqrt{29}\,i\,\partial^3 b\,b\,\partial\gamma\,\gamma - \tfrac{1}{1044}\sqrt{29}\,i\,\partial^3 b\,c$$
$$+ \tfrac{23}{696}\sqrt{2}\,\partial^3 b\,\partial\gamma + \tfrac{8}{261}\partial^4\varphi\,b\,\gamma + \tfrac{1}{87}\sqrt{58}\,i\,\partial^4\varphi \,+ \tfrac{1}{522}\sqrt{2}\,\partial^4 b\,\gamma\Big)e^{\frac{4}{7}Q}e^{ip\cdot X}\ . \qquad (A.10)$$

Prime operators with discrete momentum

- $\underline{\ell = 1,\ G = 2}$:

$$\mathbf{D}^2[-\tfrac{6}{7}Q] = \Big(c\,\gamma - \frac{i}{3\sqrt{58}}\partial\gamma\,\gamma\Big)e^{-\frac{6}{7}Q\varphi}\ , \qquad (A.11)$$

$$\mathbf{D}^2[-\tfrac{8}{7}Q] = \Big(c\,\gamma + \frac{i}{3\sqrt{58}}\partial\gamma\,\gamma\Big)e^{-\frac{8}{7}Q\varphi}\ . \qquad (A.12)$$

- $\underline{\ell = 4,\ G = 0}$:

$$\mathbf{D}^0[0] = 1\ . \qquad (A.13)$$

- $\underline{\ell = 6,\ G = 0}$:

$$\mathbf{D}^0[\tfrac{2}{7}Q] = \Big(b\,\beta\,\partial\gamma\,\gamma - \tfrac{3}{5}\sqrt{58}\,i\,b\,c\,\beta\,\gamma - \tfrac{14}{15}b\,\partial^2\gamma - \tfrac{21}{20}\sqrt{58}\,i\,\beta\,\partial\gamma + \tfrac{261}{10}c\,\beta$$
$$- \tfrac{6}{5}\sqrt{29}\,i\,\partial\varphi\,b\,c - \tfrac{11}{15}\sqrt{2}\,\partial\varphi\,b\,\partial\gamma - \tfrac{6}{5}\sqrt{29}\,i\,\partial\varphi\,\beta\,\gamma - \tfrac{4}{5}\sqrt{58}\,i\,\partial\varphi\,\partial\varphi + \tfrac{8}{15}\partial\varphi\,\partial\varphi\,b\,\gamma$$
$$+ \tfrac{1}{15}\sqrt{2}\,\partial\varphi\,\partial b\,\gamma - \tfrac{1}{5}\partial b\,b\,c\,\gamma + \tfrac{7}{290}\sqrt{58}\,i\,\partial b\,b\,\partial\gamma\,\gamma - \tfrac{9}{20}\sqrt{58}\,i\,\partial b\,c - \tfrac{97}{60}\partial b\,\partial\gamma$$
$$- \tfrac{6}{5}\sqrt{58}\,i\,\partial\beta\,\gamma - \tfrac{8}{5}\sqrt{29}\,i\,\partial^2\varphi + \tfrac{8}{15}\sqrt{2}\,\partial^2\varphi\,b\,\gamma - \tfrac{8}{15}\partial^2 b\,\gamma\Big)e^{\frac{2}{7}Q\varphi}\ . \qquad (A.14)$$

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