Casimir energy with a Robin boundary: The multiple-reflection cylinder-kernel expansion

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Abstract. We compute the vacuum energy of a massless scalar field obeying a Robin boundary condition ($\frac{\partial}{\partial x}\varphi = \beta\varphi$) on one plate and the Dirichlet boundary condition ($\varphi = 0$) on a parallel plate. The Casimir energy density for general dimension is obtained as a function of a (the plate separation) and β by studying the cylinder kernel (alias an exponential ultraviolet cutoff); we construct an infinite-series solution as a sum over classical paths and observe that the method of construction has broader applications. The total Casimir energy is finite after subtraction of divergences associated with the individual plates, which do not affect the force between the plates. The series for the total energy is an alternative to the integral formula of Romeo and Saharian, with which it agrees numerically.

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1. Introduction

The scalar Casimir effect for parallel plates with Robin boundary conditions has been studied thoroughly by Romeo and Saharian [1]. (See also [2, 3, 4].) Our reexamination of that problem here has several (loosely related) points of novelty. (1) Instead of zeta-function or dimensional regularization, we employ an explicit exponential cutoff on the normal-mode frequencies, which is implemented by calculating with an elliptic Green function associated with the system — the cylinder kernel [5, 6, 7, 8, 9]. (2) Instead of dealing directly with the eigenvalues and eigenfunctions of the system, we take the dual approach of constructing the Green function as a sum over classical paths, which in this problem means following the repeated reflections of rays from the plates. The method for incorporating Robin boundary behavior into this scheme was presented in [10]. (3) We calculate a local energy density and study carefully its relation to the renormalized total energy, calculated in the same regularization scheme. Please note that the constant denoted by β here (see (2.5)) is called $-1/\beta$ in [1].

We start the discussion by comparing the cylinder kernel with the better-known heat kernel. Consider the Laplacian operator in a bounded region Ω of d-dimensional Euclidean space, with some self-adjoint boundary condition, and let $\phi_n(\mathbf{x})$ and ${\omega_n}^2$ be the corresponding eigenfunctions and eigenvalues. (In this section the spatial integrations are understood to be over Ω.) The local heat kernel is defined by

$$K(t,\mathbf{x},\mathbf{y}) = \sum_{n=1}^{\infty} \phi_n(\mathbf{x})\phi_n^*(\mathbf{y})e^{-t{\omega_n}^2}. \tag{1.1}$$

The global heat kernel is the trace over the local one,

$$K(t) = \mathrm{Tr}\ K = \int K(t,\mathbf{x},\mathbf{x})\, d^dx = \sum_{n=1}^{\infty} e^{-t{\omega_n}^2}. \tag{1.2}$$

The local cylinder kernel is defined by

$$T(t,\mathbf{x},\mathbf{y}) = \sum_{n=1}^{\infty} \phi_n(\mathbf{x})\phi_n^*(\mathbf{y})e^{-t\omega_n}. \tag{1.3}$$

Then the global cylinder kernel is

$$T(t) = \mathrm{Tr}\ T = \int T(t,\mathbf{x},\mathbf{x})\, d^dx = \sum_{n=1}^{\infty} e^{-t\omega_n}. \tag{1.4}$$

The local heat kernel and cylinder kernel can be viewed as the Green functions of certain differential-equation problems: The heat kernel solves the heat equation in the sense that

$$u(t,\mathbf{x}) = \int K(t,\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d^dy \tag{1.5}$$

is the unique solution of the initial-value problem

$$\frac{\partial u}{\partial t} - \nabla^2 u = 0, \quad u(0,\mathbf{x}) = f(\mathbf{x}). \tag{1.6}$$

It has a well known asymptotic expansion of the form

$$K(t) = \sum_{s=0}^{\infty} b_s t^{-\frac{d}{2}+\frac{s}{2}}. \tag{1.7}$$

The cylinder kernel can be characterized similarly:

$$u(t,\mathbf{x}) = \int T(t,\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d^d y \tag{1.8}$$

is the unique bounded solution of the problem

$$\frac{\partial^2 u}{\partial t^2} + \nabla^2 u = 0, \quad u(0,\mathbf{x}) = f(\mathbf{x}). \tag{1.9}$$

The counterpart of (1.7) for T is [8, 11, 12]

$$T(t) = \sum_{s=0}^{\infty} e_s t^{-d+s} + \sum_{s=d+1,\, s-d\,\mathrm{odd}}^{\infty} f_s t^{-d+s} \ln t. \tag{1.10}$$

Both (1.7) and (1.10), as well as (1.13) later, come in both local and global versions.

The heat-kernel expansion has been a powerful tool to investigate the divergence structure of the vacuum energy, but it doesn't contain the nonlocal geometrical information needed to compute the finite part. The cylinder-kernel coefficients in (1.10) do incorporate that information. Formally, we can relate the total Casimir energy to the global cylinder kernel by taking the t-derivative,

$$E = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\operatorname{Tr} T = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int T(t,\mathbf{x},\mathbf{x})\, d^d x, \tag{1.11}$$

and the simplest definition of the vacuum energy density is

$$T_{00}(\mathbf{x}) = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n \phi_n(\mathbf{x})\phi_n^*(\mathbf{x}) = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t} T(t,\mathbf{x},\mathbf{x}). \tag{1.12}$$

(Other definitions differ from (1.12) by total divergences, so that classically they yield the same total energy when surface energies are taken into account [2, 3, 8, 13]. Definition (1.12) corresponds to $\xi = \frac{1}{4}$ in a standard notation. The other definitions (other choices of ξ) will not be treated in this paper. Note that E is independent of ξ.)

In reality, the definitions of Casimir energy and vacuum energy density in (1.11) and (1.12) contain divergent terms. But the coefficients of the divergent terms are simple, local objects that can be absorbed by renormalization, or at least cancelled when calculating forces between rigid bodies. The finite Casimir energy is given by the term of order t in (1.10):

$$E = -\tfrac{1}{2} e_{d+1}\,. \tag{1.13}$$

We will discuss the structure of the divergent terms in detail in later sections, since it depends on the dimension.

The paper is organized as follows. In Section 2 we set up the notation and show how to construct the integral kernels for a slab with a Robin boundary. In Section 3 we calculate the energy density, and in Section 4 the total energy. Some implications and motivations are discussed in Section 5, and some fine points in appendices.

2. Notation and main theorem

The cylinder kernel of the free massless scalar field in $\mathbf{R}^d$ is

$$T(t,\mathbf{x},\mathbf{y}) = C(d)t(t^2 + |\mathbf{x}-\mathbf{y}|^2)^{-(d+1)/2}, \tag{2.1}$$

where

$$C(d) = \pi^{-(d+1)/2}\,\Gamma\left(\frac{d+1}{2}\right). \tag{2.2}$$

The cylinder kernel (2.1) is the Green function of the equation (1.9) when Ω is the free space $\mathbf{R}^d$ with no boundary. Our initial considerations apply to Green functions for more general problems than (1.9). We know from the method of images that the Green function associated with a Dirichlet problem ($u(t,0)=0$) in a half-space $(0,\infty)\times\mathbf{R}^{d-1}$ is

$$G_D(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) - G(t,-x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) \tag{2.3}$$

and the Green function associated with a Neumann problem ($\frac{\partial}{\partial x}u(t,0) = 0$) in the half-space is

$$G_N(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) + G(t,-x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) \tag{2.4}$$

But when it comes to the Robin problem

$$\frac{\partial}{\partial x}u(t,0) = \beta u(t,0) \qquad (\beta > 0), \tag{2.5}$$

the elementary method of images doesn't apply any more. In [10, Theorem 2] it was shown that one can still construct the Green function for the Robin problem from the Green function for all of $\mathbf{R}^d$, G, by adding an integral correction to the Green function for the corresponding Neumann problem:

$$\begin{aligned} G_R(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) &= G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) + G(t,-x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) \\ &\quad - 2\beta\int_0^\infty e^{-\beta\varepsilon} G(t,-x-\varepsilon,\mathbf{x}_\perp,y,\mathbf{y}_\perp)\,d\varepsilon. \end{aligned} \tag{2.6}$$

All this leads us to the more general problem of how to construct the Green function for a slab, $\Omega = (0,a)\times\mathbf{R}^{d-1}$, or even $(a_1,a_2)\times\mathbf{R}^{d-1}$, with any kind of boundary conditions. It is helpful to define four families of operators D, N, R and $\tilde{R}$ this way:

$$D_a G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = -G(t,2a-x,\mathbf{x}_\perp,y,\mathbf{y}_\perp), \tag{2.7}$$

$$N_a G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = G(t,2a-x,\mathbf{x}_\perp,y,\mathbf{y}_\perp), \tag{2.8}$$

$$\begin{aligned} R_a G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) &= G(t,2a-x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) \\ &\quad - 2\beta\int_0^\infty e^{-\beta\varepsilon} G(t,2a-x-\varepsilon,\mathbf{x}_\perp,y,\mathbf{y}_\perp)\,d\varepsilon. \end{aligned} \tag{2.9}$$

$$\begin{aligned} \tilde{R}_a G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) &= G(t,2a-x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) \\ &\quad + 2\gamma\int_0^\infty e^{-\gamma\varepsilon} G(t,2a-x+\varepsilon,\mathbf{x}_\perp,y,\mathbf{y}_\perp)\,d\varepsilon. \end{aligned} \tag{2.10}$$

The functions

$$G_D(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = (1+D_a)G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp), \tag{2.11}$$

$$G_N(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = (1+N_a)G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp), \tag{2.12}$$

$$G_R(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = (1+R_a)G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) \tag{2.13}$$

respectively satisfy Dirichlet, Neumann, and Robin boundary conditions at $x = a$. Furthermore, G_D and G_N are Green functions (in particular, they have the correct Dirac-delta boundary behavior as $t \to 0$) both in the region to the left of a and in the region to the right of a, whereas G_R has that property to the right of a.

A Robin condition at a right-hand boundary, to be physically similar to (2.5), must be of the form

$$\frac{\partial}{\partial x}u(t,0) = -\gamma u(t,0) \qquad (\gamma > 0). \tag{2.14}$$

(The *inward* normal derivative must have the positive sign.) Then

$$G_{\tilde{R}}(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = (1+\tilde{R}_a)G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) \tag{2.15}$$

is the correct Green function for the region left of a.

Theorem 1 *Let $T(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp)$ be the cylinder kernel on all of $\mathbf{R}^d$; then the corresponding cylinder kernel of the slab $(0,a)\times\mathbf{R}^{d-1}$ with Robin boundary condition (2.5) at $x=0$ and Dirichlet boundary condition at $x=L$ is*

$$\begin{aligned} T_{RD}(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) &= \sum_{n=0}^{\infty}(D_aR_0)^nT + \sum_{n=1}^{\infty}(R_0D_a)^nT \\ &\quad + \sum_{n=0}^{\infty}(D_aR_0)^nD_aT + \sum_{n=1}^{\infty}(R_0D_a)^{n-1}R_0T. \end{aligned} \tag{2.16}$$

Here

$$\begin{aligned} (D_aR_0)^nT(x,y) &= (-1)^nT(x-2na,y) \\ &\quad + (-1)^{n+1}(2\beta)\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}T(x-\varepsilon-2na,y)\,d\varepsilon, \end{aligned} \tag{2.17}$$

$$\begin{aligned} (R_0D_a)^nT(x,y) &= (-1)^nT(x+2na,y) \\ &\quad + (-1)^{n+1}(2\beta)\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}T(x+\varepsilon+2na,y)\,d\varepsilon, \end{aligned} \tag{2.18}$$

$$\begin{aligned} (D_aR_0)^nD_aT(x,y) &= (-1)^{n+1}T(-x+2(n+1)a,y) \\ &\quad + (-1)^n(2\beta)\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}T(-x+\varepsilon+2(n+1)a,y)\,d\varepsilon, \end{aligned} \tag{2.19}$$

$$\begin{aligned} (R_0D_a)^{n-1}R_0T(x,y) &= (-1)^{n+1}T(-x-2(n-1)a,y) \\ &\quad + (-1)^n(2\beta)\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}T(-x-\varepsilon-2(n-1)a,y)\,d\varepsilon, \end{aligned} \tag{2.20}$$

where

$$L^1_{n-1}(x) = \sum_{j=1}^{n} \binom{n}{j} \frac{(-x)^{j-1}}{(j-1)!} \tag{2.21}$$

is a Laguerre polynomial [14, Chap. 22], [15, Secs. 7.4, 8.97]. Two notational abbreviations have been adopted: The variables $(t, \mathbf{x}_\perp, \mathbf{y}_\perp)$ *are suppressed because they undergo no alteration, and it is understood that the integral terms are to be omitted whenever* $n = 0$.

We provide the proof of Theorem 1 in Appendix A; it is parallel to the construction of the wave kernel in [10, Sec. 5]. We now comment on the structure of the formula, which is a sum over classical paths from $(x, \mathbf{x}_\perp)$ to $(y, \mathbf{y}_\perp)$, including integrations over time delays at the Robin boundary. The terms can be thought of as wave pulses in a generalized sense. Terms (2.17) experience an even number of reflections, starting at the left; terms (2.18) experience an even number of reflections, starting at the right. When $y = x$ these terms are constant and are equal in pairs (i.e., $D_a R_0 T(x, x)$ and $R_0 D_a T(x, x)$ are identical, etc., because (2.1) is a function of $|\mathbf{x} - \mathbf{y}|$); these classical paths are periodic orbits (when $\mathbf{y}_\perp = \mathbf{x}_\perp$ also) and will contribute the spatially uniform Casimir energy associated with the finiteness of L. Terms (2.19) experience an odd number of reflections, starting at the right; terms (2.20) experience an odd number of reflections, starting at the left. When $(y, \mathbf{y}_\perp) = (x, \mathbf{x}_\perp)$ these paths are bounce orbits (closed but not periodic) that contribute the localized vacuum energy of interaction of a quantum field with the boundaries.

Because of the simplicity of the slab geometry, the series solution (2.16) is exact, in principle; no stationary-phase approximations, for instance, have been needed. In practice, it may become necessary to truncate the sum, considering only short paths.

Clearly the construction is more general than stated in Theorem 1. (1) It applies to other Green functions, such as the wave kernel [10] and heat kernel [16]. (2) Setting $\beta = 0$ in the theorem's formulas yields T_{ND}, the cylinder kernel for Neumann condition at $x = 0$ and Dirichlet condition at $x = a$. (3) For Neumann condition at $x = a$, T_{RN} is given by the same formulas with the factors (-1) omitted. (4) Formulas (2.7)–(2.15) have been formulated to allow one to get the analog of (2.16) for any standard boundary conditions by replacing R_0 and D_a by the corresponding operators. Except in the case $T_{R\tilde{R}}$, the analogs of (2.17)–(2.20) follow easily. (5) We believe that the same method can be applied in principle to nonflat boundaries and nonconstant β; of course, the result in such a case cannot be any more accurate than whatever classical-path or multiple-reflection approximation is used for the underlying Green functions G and G_D or G_N.

3. Vacuum energy densities of a Robin-Dirichlet slab and a single plate

In this section we calculate the vacuum energy density of a slab with Robin boundary at $x = 0$ and Dirichlet boundary at $x = a$ for general spatial dimension d. We also consider the density for a scalar field satisfying a Robin or Dirichlet boundary condition for an isolated plate. The single-plate effects completely account for the singular behavior of the density in the slab.

3.1. Vacuum energy density for two parallel plates

An infinite summation for the energy density follows from Theorem 1, the definition of vacuum energy density in (1.12), and the formula (2.1) for T. According to (2.16) the cylinder kernel is a sum of four types of terms,

$$\begin{aligned}(D_aR_0)^nT = &\frac{(-1)^n C(d)t}{(t^2+(2na)^2)^{\frac{d+1}{2}}}\\ &+\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\,\frac{(-1)^{n+1}(2\beta)C(d)t}{(t^2+(2na+\varepsilon)^2)^{\frac{d+1}{2}}}\,d\varepsilon \qquad (n\ge 0),\end{aligned} \tag{3.1}$$

$$\begin{aligned}(R_0D_a)^nT = &\frac{(-1)^n C(d)t}{(t^2+(2na)^2)^{\frac{d+1}{2}}}\\ &+\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\,\frac{(-1)^{n+1}(2\beta)C(d)t}{(t^2+(2na+\varepsilon)^2)^{\frac{d+1}{2}}}\,d\varepsilon \qquad (n\ge 1),\end{aligned} \tag{3.2}$$

$$\begin{aligned}(D_aR_0)^nD_aT = &\frac{(-1)^{n+1} C(d)t}{(t^2+(2na+2a-2x)^2)^{\frac{d+1}{2}}}\\ &+\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\,\frac{(-1)^{n}(2\beta)C(d)t}{(t^2+(2na+2a+\varepsilon-2x)^2)^{\frac{d+1}{2}}}\,d\varepsilon \qquad (n\ge 0),\end{aligned} \tag{3.3}$$

$$\begin{aligned}(R_0D_a)^{n-1}R_0T = &\frac{(-1)^{n+1} C(d)t}{(t^2+(2na-2a+2x)^2)^{\frac{d+1}{2}}}\\ &+\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\,\frac{(-1)^{n}(2\beta)C(d)t}{(t^2+(2na-2a+\varepsilon+2x)^2)^{\frac{d+1}{2}}}\,d\varepsilon \qquad (n\ge 1).\end{aligned} \tag{3.4}$$

On the other hand, in the classical-path interpretation the sum can be reorganized by number of reflections as

$$T_{RD}(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = T + (R_0T + D_aT) + (R_0D_aT + D_aR_0T) + \cdots. \tag{3.5}$$

We now implement the local version of (1.13). The first term in (3.5), T, corresponds to a path that experiences no reflection at all; the contribution of this term to the vacuum energy density is

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}T = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\frac{C(d)}{t^d} = \frac{C(d)}{2}\frac{d}{t^{d+1}}\bigg|_{t\to 0}. \tag{3.6}$$

This is the anticipated leading divergent term. It is the universal, x-independent formal vacuum energy of infinite empty flat space; speculations about "dark energy" aside, it is universally agreed that this term should be discarded.

The second and third terms, D_LT and R_0T, experience only one reflection on the boundary; they respectively contribute

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}D_aT = \frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}} \tag{3.7}$$

and

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}R_0T = -\frac{C(d)}{2}\frac{1}{(2x)^{d+1}} + \int_0^\infty e^{-\beta\varepsilon}\frac{\beta C(d)}{(\varepsilon+2x)^{d+1}}\,d\varepsilon. \tag{3.8}$$

These two terms are "dangerous" since we can see that the energy density contributed by D_aT goes (nonintegrably) to infinity at the Dirichlet boundary ($x \to a$) and the energy density contributed by R_0T diverges similarly at the Robin boundary ($x \to 0$).

Next we write down the contributions to vacuum energy density from paths that are reflected at least twice on the boundary, calculated through (1.12):

$$(D_aR_0)^nT: \qquad (-1)^{n+1}\frac{C(d)}{2(2na)^{d+1}} + (-1)^n\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\frac{\beta C(d)}{(2na+\varepsilon)^{d+1}}\,d\varepsilon \qquad (n\geq 1), \tag{3.9}$$

$$(R_0D_a)^nT: \qquad (-1)^{n+1}\frac{C(d)}{2(2na)^{d+1}} + (-1)^n\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\frac{\beta C(d)}{(2na+\varepsilon)^{d+1}}\,d\varepsilon \qquad (n\geq 1), \tag{3.10}$$

$$(D_aR_0)^nD_aT: \qquad (-1)^n\frac{C(d)}{2(2na+2a-2x)^{d+1}} + (-1)^{n+1}\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\frac{\beta C(d)}{(2na+2a+\varepsilon-2x)^{d+1}}\,d\varepsilon \qquad (n\geq 1), \tag{3.11}$$

$$(R_0D_a)^{n-1}R_0T: \qquad (-1)^n\frac{C(d)}{2(2na-2a+2x)^{d+1}} + (-1)^{n+1}\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\frac{\beta C(d)}{(2na-2a+\varepsilon+2x)^{d+1}}\,d\varepsilon \qquad (n\geq 2). \tag{3.12}$$

The total vacuum energy density is the sum of (3.7) through (3.12). The contributions from $(D_aR_0)^nT$ and $(R_0D_a)^nT$ are independent of x; they correspond to the periodic orbits. The contributions from $(D_aR_0)^nD_LT$ and $(R_0D_a)^{n-1}R_0T$ are x-dependent; they correspond to the bounce orbits. All these terms with at least two reflections are integrable. Thus, apart from the universal divergent term T, the dangerous single-reflection terms (3.7) and (3.8) are the only terms that could be called divergent (and at this local stage they are still pointwise finite). They can be related to the situation of a single plate, as we now verify.

3.2. Vacuum energy density for a single plate

Consider a single plate with Dirichlet boundary condition at $x = a$ and a single plate with Robin boundary condition at $x = 0$. The corresponding cylinder kernels can be constructed on the basis of (2.11) and (2.13) — in other words, by considering closed paths with at most one reflection. Namely,

$$T_D(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = T + D_aT \tag{3.13}$$

and

$$T_R(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = T + R_0T. \tag{3.14}$$

Each cylinder kernel again contains the trivial term T, which we discard. The corresponding vacuum energy densities are then

$$T_{00}(x,\mathbf{x}_\perp) = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}D_aT = \frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}}\,, \tag{3.15}$$

$$T_{00}(x,\mathbf{x}_\perp) = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}R_0T = -\frac{C(d)}{2}\frac{1}{(2x)^{d+1}} + \int_0^\infty e^{-\beta\varepsilon}\,\frac{\beta C(d)}{(\varepsilon+2x)^{d+1}}\,d\varepsilon. \tag{3.16}$$

For $\mathbf{x}$ inside the slab, these are precisely the "dangerous" terms of the slab case, (3.7) and (3.8).

4. Total vacuum energy of a slab with Robin and Dirichlet boundary conditions

In this section we consider the total vacuum energy (per unit $(d-1)$-dimensional area) of a slab with Robin boundary at $x=0$ and Dirichlet boundary at $x=a$ for general spatial dimension d. Within the cylinder-kernel analysis there are two approaches that could be taken. First, one can attempt to integrate the local energy density given by (3.7)–(3.12), arguing away part or all of the divergent integrals of the dangerous terms. It is clear from Section 3.2 that the singular behavior of those terms is identical to that of the separate single-plate systems. Therefore, one can formally subtract the two single-plate energies to obtain the finite (per unit area in $\mathbf{x}_\perp$) Casimir energy associated with the interaction of the two plates. Second, one can integrate $T(t,x,x)$ (before or after taking its t derivative) with t strictly positive to define a regularized total energy, then take t to 0 and appeal to "renormalization" to discard divergent terms (those proportional to negative powers of t or to $\ln t$). This is the global version of (1.13). The first method will be followed here, the second in Appendix B.

Integrating (3.7) and (3.8) over $(0,a)$, we formally obtain the contributions of the two dangerous terms, D_aT and R_0T, to the total energy as

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_0^a D_aT\,dx = \int_0^a \frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}}\,dx = \frac{C(d)}{4d}\frac{1}{(2a-2x)^d}\bigg|_0^a\,, \tag{4.1}$$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_0^a R_0T\,dx = \int_0^a\left[-\frac{C(d)}{2}\frac{1}{(2x)^{d+1}} + \int_0^\infty e^{-\beta\varepsilon}\frac{\beta C(d)}{(\varepsilon+2x)^{d+1}}\,d\varepsilon\right]dx. \tag{4.2}$$

The first is clearly divergent as $x\to a$, and the second is similarly divergent as $x\to 0$. But the divergence due to the Dirichlet plate at $x=a$ is cancelled by subtracting the vacuum energy of a single plate in the region $(-\infty,a)\times\mathbf{R}^{d-1}$:

$$\begin{aligned}-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\left(\int_0^a D_aT\,dx - \int_{-\infty}^a D_aT\,dx\right) &= \frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_{-\infty}^0 D_aT\,dx \\ &= -\frac{C(d)}{4d}\frac{1}{(2a)^d}\,.\end{aligned} \tag{4.3}$$

In the same way, for the Robin plate at $x = 0$ we subtract the vacuum energy of a single plate in the region $(0, \infty) \times \mathbf{R}^{d-1}$:

$$\begin{aligned} -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\left(\int_0^a R_0 T\,dx - \int_0^\infty R_0 T\,dx\right) &= \frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_a^\infty R_0 T\,dx \\ &= \frac{C(d)}{4d}\frac{1}{(2a)^d} - \int_0^\infty e^{-\beta\varepsilon}\,\frac{\beta C(d)}{2d(\varepsilon+2a)^d}\,d\varepsilon. \end{aligned} \tag{4.4}$$

For general $n \geq 1$, no divergent terms are involved. $(D_a R_0)^n T$ and $(R_0 D_a)^n T$ each contribute to the total energy

$$(-1)^{n+1}\frac{C(d)}{2^{d+2}n^{d+1}a^d} + (-1)^n\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\,\frac{\beta a C(d)}{(2na+\varepsilon)^{d+1}}\,d\varepsilon. \tag{4.5}$$

$(D_a R_0)^n D_a T$ contributes

$$\begin{aligned} &(-1)^n\frac{C(d)}{4d}\left[\frac{1}{(2na)^d} - \frac{1}{(2na+2a)^d}\right] \\ &\quad + (-1)^{n+1}\frac{\beta C(d)}{2d}\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\left[\frac{1}{(\varepsilon+2na)^d} - \frac{1}{(\varepsilon+2na+2a)^d}\right]d\varepsilon. \end{aligned} \tag{4.6}$$

For general $n \geq 2$, $(R_0 D_a)^{n-1} R_0 T$ contributes

$$\begin{aligned} &(-1)^n\frac{C(d)}{4d}\left[\frac{1}{(2na-2a)^d} - \frac{1}{(2na)^d}\right] \\ &\quad + (-1)^{n+1}\frac{\beta C(d)}{2d}\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\left[\frac{1}{(\varepsilon+2na-2a)^d} - \frac{1}{(\varepsilon+2na)^d}\right]d\varepsilon. \end{aligned} \tag{4.7}$$

Now we sum up all the terms to obtain the finite total energy

$$\begin{aligned} E = &\sum_{n=1}^\infty \frac{(-1)^{n+1}C(d)}{2^{d+1}n^{d+1}a^d} - \int_0^\infty e^{-\beta\varepsilon}\,\frac{2\beta a C(d)}{(\varepsilon+2a)^{d+1}}\,d\varepsilon - \int_0^\infty e^{-\beta\varepsilon}\,\frac{\beta C(d)}{2d(\varepsilon+4a)^d}\,d\varepsilon \\ &- \sum_{n=2}^\infty(-1)^n\int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\left[\frac{\beta C(d)}{2d(\varepsilon+2na-2a)^d} - \frac{\beta C(d)}{2d(\varepsilon+2na+2a)^d}\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left. - \frac{2\beta a C(d)}{(2na+\varepsilon)^{d+1}}\right]d\varepsilon. \end{aligned} \tag{4.8}$$

Because we take $\xi = \frac{1}{4}$, there is no additional surface contribution as in [1].

The first term in E can be expressed by the Riemann η-function:

$$E_{ND} = \frac{C(d)}{2^{d+1}a^d}\eta(d+1)\,; \tag{4.9}$$

it is the known result for one Neumann and one Dirichlet plate. The integrals in (4.8) can be evaluated in terms of the incomplete gamma function [14, Sec. 6.5], [15, Sec. 8.35]. The resulting infinite summation presumably can't be converted to a closed form. However, the terms starting with $n = 4$ are relatively small and almost cancel each other, so the expression truncated to $n \leq 3$ is a good approximation for the total energy. (The

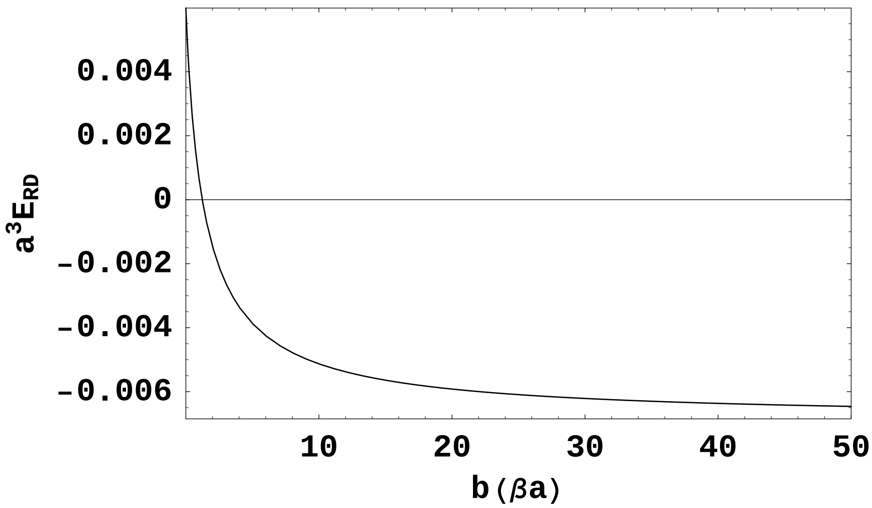

Figure 1. Total integrated Casimir energy per unit area multiplied by a^3, for $d = 3$ and $0 \le b = \beta a \le 5$. The graph of $E(a)$ itself for fixed $\beta \ne 0$ or $+\infty$ would have a minimum somewhere to the right of $a = 1.237/\beta$ and a singularity at the origin.

proof of this assertion is in Appendix C.) Explicitly, the total energy for $d = 3$ as a function of $b = \beta a$ (through order $n = 3$) is

$$\begin{aligned} E_{RD} = \frac{7\pi^2}{11520a^3} + \frac{1}{\pi^2 a^3}[&-b^3 e^{2b}\Gamma(-2,2b) + 3b^3 e^{4b}\Gamma(-2,4b) + b^3 e^{6b}\Gamma(-2,6b) \\ &- (19b^3/6)e^{8b}\Gamma(-2,8b) - 12b^4 e^{4b}\Gamma(-3,4b) - 72b^4 e^{6b}\Gamma(-3,6b) \\ &+ 56b^4 e^{8b}\Gamma(-3,8b) + 864b^5 e^{6b}\Gamma(-4,6b) - 256b^5 e^{8b}\Gamma(-4,8b) \\ &- 2880b^6 e^{6b}\Gamma(-5,6b)]. \end{aligned} \tag{4.10}$$

(Again, the b of Romeo and Saharian [1] is the negative reciprocal of our b.)

Note that at $\beta = 0$ the Robin boundary becomes a Neumann boundary and one recovers

$$a^3 E_{RD}\big|_{\beta\to 0} = a^3 E_{ND} = \frac{7\pi^2}{11520} = 0.00599. \tag{4.11}$$

When $\beta \to \infty$, the Robin boundary becomes a Dirichlet boundary, so we expect to recover the familiar result

$$a^3 E_{RD}\big|_{\beta\to\infty} = a^3 E_{DD} = -\frac{\pi^2}{1440} = -0.00685. \tag{4.12}$$

The graph of $a^3 E_{RD}$ as a function of $b = \beta a$ is given in Fig. 1, which (together with numerical calculations for larger b) confirms (4.11) and (4.12). (All computations and graphics were done with *Mathematica*.) The crossover from positive to negative energy occurs near $b = 1.237$, or $-\frac{1}{b} \approx -0.81$, in agreement with [1]. That reference states that this value marks a change from repulsive to attractive Casimir force, but that is incorrect: The zero of the force function $-\frac{\partial}{\partial a}E_{RD}$ occurs at some larger value of a.

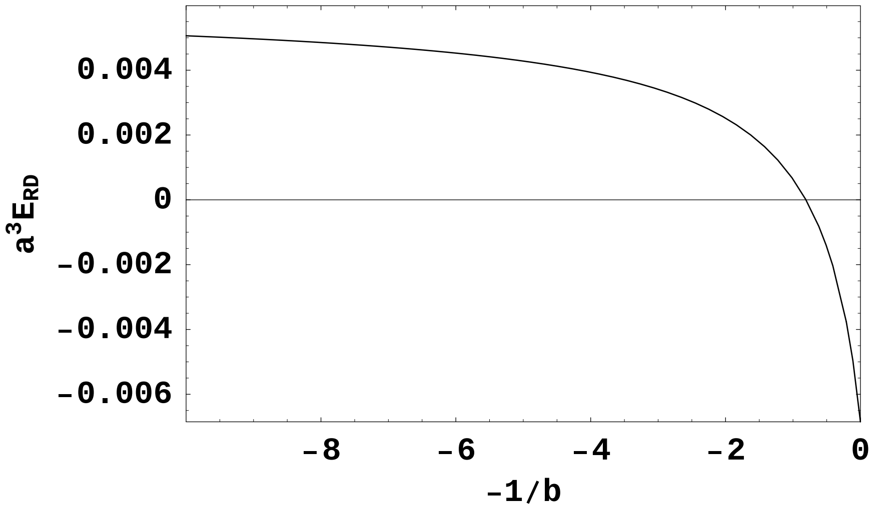

Figure 2. Total integrated Casimir energy per unit area multiplied by a^3, for $d = 3$ and $-10 \le -1/b = -1/\beta a \le 0$. This graph should be compared with Fig. 3 of [1], where $-1/b$ is called b_2. The zero of the total energy is at $-1/b \approx -0.81$.

5. Conclusion

In Casimir theory — and in the general study of partial differential equations and the spectral theory of differential operators — the Robin boundary condition is of theoretical interest as the simplest step beyond the standard Dirichlet and Neumann problems for any particular geometrical configuration. The Robin condition also has physical applications: it arises naturally in place of the Neumann condition for half of the modes of the electromagnetic field in the presence of a curved boundary, it mocks up in a simple way the effect of a boundary between two media, and it may have cosmological significance in the brane-world scenario [17, 18, and references therein].

Our numerical results agree with those of Romeo and Saharian [1] to the extent that they have been compared. Because we use different notations to express the Robin boundary condition, the counterpart of b_2 in their notation is our $-1/b$. For a more direct comparison, we plot in Fig. 2 the total energy E_{RD} with respect to $-1/b$. The result matches [1, Fig. 3] very well, including the location of the zero.

The formula in [1] for the total energy is a rather complicated integral. Ours is an infinite sum whose terms fall off fairly rapidly, so reasonable accuracy can be attained by truncating the series. At least in the case where only one of the boundaries is Robin, the individual terms in the series can be evaluated in terms of known special functions, the Laguerre polynomials. The scope of this paper has not allowed us to tackle the case of two Robin boundaries in such detail, nor to study in much depth the questions of how the signs of the energy and the force depend on the parameters. Finally, we have restricted attention to positive Robin constants; the negative case is of more dubious physical significance, and the construction of the cylinder kernel in that case requires different mathematics [10].

We have taken pains to calculate the local energy density (albeit for only the easiest choice of the conformal coupling parameter, $\xi = \frac{1}{4}$) and to conduct the calculation of the total energy in the same framework. It has been known for many years [19] that vacuum energy densities in flat space are pointwise finite (apart from the ubiquitous zero-point energy of every quantized field) but (in general) nonintegrable near boundaries. The Robin condition introduces a new (less singular) divergent term in addition to those familiar from the more elementary conditions. Direct calculations of total energy lead immediately to formal divergences. When an ultraviolet cutoff (in particular, the cylinder-kernel approach) is used, the divergent terms depend on the cutoff parameter t polynomially or logarithmically, and these terms have a close relation to the divergent integrals of the energy density [8]. In odd dimensions the divergences associated with the Robin constant include one of the logarithmic class. "Analytic" regularization schemes (dimensional and zeta functions) automatically remove the polynomial terms. However, it is not clear that this nonchalance is physically justified. The energy density serves as a source in the gravitational field equation, so its singular behavior at boundaries cannot just be ignored [19, 9]. Also, the traditional approach to Casimir forces, while plausible for predicting attractions between rigid bodies, has been strongly criticized when applied to deformations of bodies [20, 21, 22, 23]. It may be that the divergent terms in the vacuum energy (or the related divergent integrals of the energy density) can be absorbed into terms in the equations of motion representing the mechanical response of the materials in the bodies, but there is generally no justification for simply setting those terms to zero. In the end a successful physical analysis of a particular system of experimental relevance must be based on a more realistic and complete model, but in the meantime a clear understanding of the (relatively tractable) vacuum-energy calculations is needed in order to diagnose the problems and to determine the limits of validity of the theory.

In the parallel-plate problem we have shown that the only divergent terms are directly associated with the individual plates. Therefore, they are not functions of the plate separation and do not contribute to the force between the plates. (This was, of course, known already, but our treatment of the total energy in the same framework as the energy density removes a certain mysticism from the renormalization and promises to elucidate the physics in more complicated situations in the future.) Within the remaining finite energy (4.8), the Neumann term (4.9) is a Casimir energy in the strictest sense: it is associated with the discretization of modes and with periodic orbits of the underlying classical system. The β-dependent terms, on the other hand, include contributions from (4.6) and (4.7) associated with closed but nonperiodic orbits (those with an odd number of reflections). Unless there is a nonobvious cancellation, these terms do contribute to the force. (We thank an anonymous referee for pointing out this fact.)

The construction of the cylinder kernel as a multiple-scattering expansion is a powerful method for calculating local spectral and vacuum effects, which demands further development.

Acknowledgments

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Appendix A. Proof of Theorem 1

Appendix A.1. Proof of (2.16)

After rearrangement the proposed series is

$$\begin{aligned}T_{RD}(x,y) &= T + \sum_{n=0}^{\infty}(R_0D_a)^n R_0 T + \sum_{n=0}^{\infty}(D_aR_0)^n D_a T + \sum_{n=0}^{\infty}(D_aR_0)^n D_a R_0 T \\ &\qquad + \sum_{n=1}^{\infty}(R_0D_a)^{n-1} R_0 D_a T \\ &= (1+R_0)\sum_{n=0}^{\infty}(D_aR_0)^n T + (1+R_0)\sum_{n=0}^{\infty}(D_aR_0)^n D_a T \\ &= (1+D_a)\sum_{n=0}^{\infty}(R_0D_a)^n T + (1+D_a)\sum_{n=0}^{\infty}(R_0D_a)^n R_0 T.\end{aligned} \tag{A.1}$$

Because of the falloff of T as a function of x (see (2.1)) the series converges (absolutely). Therefore, it is easy to see that it satisfies the cylinder equation (1.9) inside the slab and the proper boundary condition at $t = 0$. Finally, by virtue of (2.11) and (2.13), it satisfies both the Dirichlet condition at $x = L$ and the Robin condition at $x = 0$.

Appendix A.2. Proof of (2.17), etc.

When $n = 1$,

$$D_aR_0T(x,y) = -T(x-2a,y) + (2\beta)\int_0^{\infty} e^{-\beta\varepsilon}T(x-\varepsilon-2a,y)\,d\varepsilon, \tag{A.2}$$

so (2.17) is satisfied when $n = 1$. Suppose that when $n = m$ (2.17) is satisfied:

$$\begin{aligned}(D_aR_0)^m T(x,y) =& (-1)^m T(x-2ma,y) \\ &+ (-1)^{m+1}(2\beta)\int_0^{\infty} L^1_{m-1}(2\beta\varepsilon)e^{-\beta\varepsilon}T(x-\varepsilon-2ma,y)\,d\varepsilon.\end{aligned} \tag{A.3}$$

Then when $n = m+1$,

$$\begin{aligned}(D_aR_0)^{m+1}T(x,y) &= (-1)^{m+1}T(x-2ma-2a,y) \\ &+ (-1)^{m+2}(2\beta)\left[\int_0^{\infty}(1+L^1_{m-1}(2\beta\varepsilon))e^{-\beta\varepsilon}T(x-\varepsilon-2ma-2a,y)\,d\varepsilon\right. \\ &\left. - \int_0^{\infty} L^1_{m-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\,d\varepsilon\int_0^{\infty} e^{-\beta\eta}T(x-\varepsilon-\eta-2ma-2a,y)\,d\eta\right].\end{aligned} \tag{A.4}$$

Let $\theta = \varepsilon + \eta$; then

$$\begin{aligned}
\int_0^\infty L^1_{m-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\,d\varepsilon \int_0^\infty e^{-\beta\eta} T(x-\varepsilon-\eta-2ma-2a,y)\,d\eta \\
&= \int_0^\infty L^1_{m-1}(2\beta\varepsilon)\,d\varepsilon \int_0^\infty e^{-\beta\theta} T(x-\theta-2ma-2a,y)\,d\theta \\
&= -\int_0^\infty \sum_{j=1}^{m} \binom{m}{j} \frac{(-2\beta\theta)^{j-1}}{j!}\, e^{-\beta\theta} T(x-\theta-2ma-2a,y)\,d\theta.
\end{aligned} \tag{A.5}$$

Thus

$$\begin{aligned}
(D_aR_0)^{m+1}T(x,y) &= (-1)^{m+1}T(x-2ma-2a,y) \\
&\quad + (-1)^{m+2}(2\beta)\int_0^\infty \left[1 + L^1_{m-1}(2\beta\varepsilon) + \sum_{j=1}^{m}\binom{m}{j}\frac{(-2\beta\theta)^{j-1}}{j!}\right] \\
&\qquad\qquad \times e^{-\beta\varepsilon}T(x-\varepsilon-2ma-2a,y)\,d\varepsilon \\
&= (-1)^{m+1}T(x-2ma-2a,y) \\
&\quad + (-1)^{m+2}(2\beta)\int_0^\infty L^1_{m+1-1}(2\beta\varepsilon)e^{-\beta\varepsilon}T(x-\varepsilon-2ma-2a,y)\,d\varepsilon.
\end{aligned} \tag{A.6}$$

That means that (2.17) is satisfied also when $n = m+1$. The formulas (2.18)–(2.20) can be proved by induction in the same way.

Appendix B. Boundary divergences in the total energy

Here we analyze the total energy by the global approach. That is, we integrate $T_{RD}(t,x,x)$ to get the global cylinder kernel $T_{RD}(t)$ before taking its t derivative and examining the limit $t \to 0$. We concentrate on the case $d = 3$ (hence $C(d) = \pi^{-2}$), and we discard from the outset the universal divergent term T of (3.6).

Appendix B.1. Regularized energy for a single plate

For the infinite space to the right of a Robin plate at $x = 0$ the integrated cylinder kernel is, from (3.12),

$$\begin{aligned}
R_0T(t) &= \int_0^\infty R_0T(t,x,x)\,dx \\
&= \int_0^\infty \frac{1}{\pi^2}\frac{t}{(t^2+(2x)^2)^2}\,dx - \int_0^\infty e^{-\beta\varepsilon}d\varepsilon \int_0^\infty \frac{2\beta}{\pi^2}\frac{t}{(t^2+(2x+\varepsilon)^2)^2}\,dx \\
&= \frac{1}{4\pi^2t^2}\left[\frac{2tx}{t^2+4x^2} + \arctan\frac{2x}{t}\right]_0^\infty \\
&\quad - \frac{\beta}{2\pi^2t^2}\int_0^\infty e^{-\beta\varepsilon}d\varepsilon\left[\frac{t(2x+\varepsilon)}{t^2+(2x+\varepsilon)^2} + \arctan\frac{2x+\varepsilon}{t}\right]_0^\infty .
\end{aligned} \tag{B.1}$$

For later comparison with the case of two plates, it is convenient to keep the lower-limit and upper-limit contributions separate.

From the upper limit at ∞ one gets (for $\beta \neq 0$)

$$\frac{1}{4\pi^2 t^2}\,\frac{\pi}{2} - \frac{\beta}{2\pi^2 t^2}\int_0^\infty e^{-\beta\varepsilon}\frac{\pi}{2}\,d\varepsilon = +\frac{1}{8\pi t^2} - \frac{1}{4\pi t^2} = -\frac{1}{8\pi t^2}\,. \tag{B.2}$$

The discontinuity at $\beta = 0$ is only apparent, because we shall now see that the contribution from the ε integral is cancelled by a like term from the lower limit.

From the lower limit 0 one gets

$$\begin{aligned}&\frac{\beta}{2\pi^2 t^2}\int_0^\infty e^{-\beta\varepsilon} d\varepsilon\left[\frac{t\varepsilon}{t^2+\varepsilon^2} + \arctan\frac{\varepsilon}{t}\right]\\ &= \frac{\beta}{2\pi^2 t}[\sin\beta t\,(\tfrac{\pi}{2} - \operatorname{Si}\beta t) - \cos\beta t\,\operatorname{Ci}\beta t] + \frac{1}{2\pi^2 t^2}[\cos\beta t\,(\tfrac{\pi}{2} - \operatorname{Si}\beta t) + \sin\beta t\,\operatorname{Ci}\beta t].\end{aligned} \tag{B.3}$$

The sine integral function Si and cosine integral function Ci have Taylor expansions

$$\operatorname{Si}(z) = \sum_{k=0}^\infty \frac{(-1)^k z^{2k+1}}{(2k+1)!(2k+1)}\,, \tag{B.4}$$

$$\operatorname{Ci}(z) = \gamma + \ln z + \sum_{k=1}^\infty \frac{(-1)^k z^{2k}}{(2k)!(2k)}\,, \tag{B.5}$$

where γ is Euler's constant. Therefore, the expansion of (B.3) at small t is

$$\frac{1}{4\pi t^2} - \frac{\beta}{2\pi^2 t} + \frac{\beta^2}{8\pi^2} + \frac{\beta^3}{6\pi^2}\,t\ln(\beta t) + \frac{(3\gamma-4)\beta^3}{18\pi^2}\,t + O(t^2). \tag{B.6}$$

The total regularized energy from (B.2) and (B.6) is

$$E_R(t) = +\frac{1}{8\pi t^3} - \frac{\beta}{4\pi^2 t^2} - \frac{\beta^3}{12\pi^2}\ln(\beta t) - \frac{(3\gamma-1)\beta^3}{36\pi^2} + O(t^1). \tag{B.7}$$

Similarly, the integrated cylinder kernel to the left of an isolated Dirichlet plate at $x = a$ is

$$\begin{aligned} D_aT(t) &= \int_{-\infty}^a D_aT(t,x,x)\,dx\\ &= -\frac{t}{\pi^2}\int_{-\infty}^a \frac{dx}{(t^2+(2a-2x)^2)^2} = -\frac{t}{\pi^2}\int_0^\infty \frac{du}{(t^2+4u^2)^2}\\ &= -\frac{1}{4\pi^2 t^2}\left[\frac{2tu}{(t^2+4u^2)} + \arctan\frac{2u}{t}\right]_0^\infty\\ &= -\frac{1}{8\pi t^2}\,,\end{aligned} \tag{B.8}$$

which corresponds to a regularized energy

$$E_D(t) = -\frac{1}{8\pi t^3}\,. \tag{B.9}$$

Appendix B.2. Regularized energy for the slab

We must integrate the terms (3.1)–(3.4) from 0 to a. Recall that only the terms D_aT and R_0T contain divergences. In all the other terms the denominator of the integrand remains nonzero even when both t and ε are zero, and therefore one can differentiate and pass to the limit $t \to 0$ before integrating; that is, their contributions are precisely those already presented in (4.5)–(4.7).

For the divergent terms we could recycle the calculations (B.1) and (B.8), replacing the upper limit ∞ with a. However, the difference would be the negatives of the integrals from a to ∞, and to them the same argument as above applies: these are perfectly finite contributions to the energy, even when $t = 0$, and they have already been computed in (4.3)–(4.4).

All that remains to be considered is the sum of the regularized energies (B.7) and (B.9). (Recall that we have already discarded the ubiquitous t^{-4} term.) The terms of order t^{-3} cancel, but this is an artifact of our model, since Dirichlet and Neumann plates have divergent surface energies that are equal and opposite. According to the prescription (1.13) we should discard *all* the terms in the series that diverge as $t \to 0$. In the present case, because there is a logarithmic term in (B.7), we encounter the well known scale ambiguity: because the numerical factor inside the argument of the logarithm is arbitrary, the "finite part" of E_R, hence that of E_{RD}, is defined only up to an arbitrary numerical multiple of β^3. Ignoring E_R entirely in calculating E_{RD} yields the prescription of Sec. 4. The ambiguous β^3 term does not depend on a and hence does not affect the force between the plates. It does, of course, depend on β; one must feel some trepidation in ignoring it (or even the power-law divergent terms) in situations where β is allowed to vary.

Appendix C. Why we can discard terms with $n \geq 4$

The expression of the total energy in (4.8) is an infinite summation, but we shall prove for case $d = 3$ that all terms after $n = 3$ are quite small, so it's reasonable to discard them. Note that when $d = 3$, $C(d) = 1/\pi^2$ and hence the β-dependent part of the remainder is

$$R_3 = \sum_{n=4}^{\infty} \frac{(-1)^n}{\pi^2} \int_0^{\infty} L^1_{n-1}(2\beta\varepsilon) e^{-\beta\varepsilon} f_n(2\beta\varepsilon)\, d(\beta\varepsilon), \tag{C.1}$$

where

$$f_n(2\beta\varepsilon) = \frac{(2\beta)^3}{6(2\beta\varepsilon + 4(n-1)\beta a)^3} - \frac{(2\beta)^3}{6(2\beta\varepsilon + 4(n+1)\beta a)^3} - \frac{2a(2\beta)^4}{(2\beta\varepsilon + 4n\beta a)^4}\,. \tag{C.2}$$

Let $2\beta\varepsilon = x$; then these equations can be written as

$$R_3 = \sum_{n=4}^{\infty} \frac{(-1)^n}{2\pi^2} \int_0^{\infty} L^1_{n-1}(x) e^{-\frac{x}{2}} f_n(x)\, dx \tag{C.3}$$

and

$$f_n(x) = \frac{(2\beta)^3}{6(x+4(n-1)\beta a)^3} - \frac{(2\beta)^3}{6(x+4(n+1)\beta a)^3} - \frac{2a(2\beta)^4}{(x+4n\beta a)^4}. \quad (C.4)$$

It is straightforward to show that $f_n(x)$ is a decreasing function and $f_n(x) \geq 0$ for any $x \geq 0$, so

$$f_n(x) \leq f_n(0) = \frac{1}{a^3}\left(\frac{1}{6(2n-2)^3} - \frac{1}{6(2n+2)^3} - \frac{2}{(2n)^4}\right). \quad (C.5)$$

From a mean value theorem for integrals one has

$$\int_0^\infty L_{n-1}^1(x)e^{-\frac{x}{2}} f_n(x)\, dx = f_n(0)\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}\, dx, \quad \text{where } 0 < \eta < \infty. \quad (C.6)$$

It follows that

$$|R_3| \leq \frac{1}{2\pi^2}\sum_{n=4}^{\eta} f_n(0)\left|\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}\, dx\right|. \quad (C.7)$$

From (8.971.2) and (8.971.5) in [15] we get the integral

$$\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}\, dx + \int_0^\eta L_{n-2}^1(x)e^{-\frac{x}{2}}\, dx = -2\, e^{-\frac{x}{2}} L_{n-1}^0(x)\Big|_0^\eta. \quad (C.8)$$

Telescoping this recursion, we get

$$\begin{aligned} &\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}\, dx + (-1)^{n-2}\int_0^\eta L_0^1(x)e^{-\frac{x}{2}}\, dx \\ &\quad = \sum_{m=2}^{n}(-1)^{n-m}\left(\int_0^\eta L_{m-1}^1(x)e^{-\frac{x}{2}}\, dx + \int_0^\eta L_{m-2}^1(x)e^{-\frac{x}{2}}\, dx\right) \\ &\quad = \sum_{m=2}^{n}(-1)^{n-m}\left(-2\, e^{-\frac{x}{2}} L_{m-1}^0(x)\Big|_0^\eta\right). \end{aligned} \quad (C.9)$$

But $L_0^1(x) = 1$, so

$$\begin{aligned} \left|\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}\, dx\right| &\leq 2\sum_{m=1}^{n}\left|e^{-\frac{x}{2}} L_{m-1}^0(x)\Big|_0^\eta\right| \\ &\leq 2\sum_{m=1}^{n}\left(\left|e^{-\frac{\eta}{2}} L_{m-1}^0(\eta)\right| + \left|L_{m-1}^0(0)\right|\right). \end{aligned} \quad (C.10)$$

From [14, (22.14.13)],

$$\left|e^{-\frac{\eta}{2}} L_{m-1}^0(\eta)\right| \leq 1, \quad (C.11)$$

hence

$$\left|\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}\, dx\right| \leq 4n. \quad (C.12)$$

Now we continue (C.7):

$$
\begin{aligned}
|R_3| &\leq \frac{1}{2\pi^2}\sum_{n=4}^{\infty} 4n f_n(0) \\
&= \frac{2}{\pi^2 a^3}\sum_{n=4}^{\infty} n\left(\frac{1}{6(2n-2)^3}-\frac{1}{6(2n+2)^3}-\frac{2}{(2n)^4}\right) \\
&= \frac{1}{4\pi^2 a^3}\sum_{n=4}^{\infty} n\,\frac{10n^4-9n^2+3}{3n^4(n-1)^3(n+1)^3} \\
&\leq \frac{1}{4\pi^2 a^3}\sum_{n=4}^{\infty} \frac{10n}{3(n-1)^3(n+1)^3} = 1.5\times 10^{-4}\,\frac{1}{a^3}\,,
\end{aligned}
\tag{C.13}
$$

which is roughly 2 percent of $|E_{DD}| = \pi^2/1440a^3$. (The actual error in our numerical calculations is at most 0.1%.)

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