Spin Hall effect and irreversible thermodynamics:
Center-to-edge transverse current-induced voltage

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We examine the Dyakonov and Perel theory of the Spin Hall effect (SHE) from the viewpoint of irreversible thermodynamics, which is more constraining than the symmetry arguments of pure phenomenology. As thermodynamic driving forces we include the thermal gradient, the gradient of the electrochemical potential (rather than the potential gradient and density gradient separately), and the "internal" magnetic field that is thermodynamically conjugate to the magnetization. In turn, we obtain the form of bulk transport coefficients relating the fluxes to the thermodynamic forces. Relative to Dyakonov and Perel, in addition to the new terms due to thermal gradients, the Onsager relations require three new (nonlinear) terms in the current density, and minor revisions in the current density and spin current density. For a longitudinal current along a strip, the center-to-edge transverse voltage difference, due both to the $-\beta \vec{P} \times \vec{E}$ term of of the number current density $\vec{q}$ and to one of the new current density terms, is calculated. An ac capacitive probe likely would not significantly disturb this effect. An appendix explicitly relates the anomalous Hall Effect to the term in the (vector) number flux that is Onsager related to the SHE term in the (tensor) spin flux.

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I. INTRODUCTION

The spin Hall Effect (SHE), based on spin-orbit scattering off atoms (even for scattering off spinless atoms) [1,2] was first proposed by Dyakonov and Perel [3,4], both for semiconductors and metals. However, not until the SHE was independently recognized and named by Hirsch [5]—at a time when it was realized that spin manipulation had the prospect of being integrated with the practical world of electronic devices—did the SHE become a subject of intense interest [6]. In the SHE a longitudinal electric current in zero magnetic field causes a transverse spin current that is spin polarized in the second transverse direction. Unlike the Hall effect (HE), the SHE is not identical to its inverse; in the inverse spin Hall effect (ISHE) a longitudinal spin current causes a transverse electric current.

The ISHE was observed in the 1980s [7], and (with $\vec{S}$ the spin density) the Onsager-related $\vec{\nabla} \times \vec{S}$ term in the electric current [8], but the SHE has been observed only within the last decade [9–11]. The transported spin that cannot escape the transverse walls of the sample leads to an excess spin (beyond the equilibrium value) that is called the *spin accumulation* [3,4]; one of the predictions of the theory is that spin will accumulate along the sides of the sample in the presence of a longitudinal electric current.

For a system with a negative charge carrier, Dyakonov and Perel applied their theory to a sample in the shape of a wire, which at the time was the geometry most likely to be studied [3,4,12]. More recently Dyakonov rewrote his equations to use spin polarization density $\vec{P}$ rather than the spin (accumulation) density

$$\vec{S} = (\hbar/2)\vec{P}$$

as a variable. He then studied, for the now commonly available planar geometry, magnetic field dephasing (Hanle effect) on the spin accumulation, which leads to a magnetoresistance [13]. Lifshits and Dyakonov, motivated by a microscopic view of a scatterer yielding both ordinary and spin-orbit scattering, have have used the spin Boltzmann equation to study in detail two terms in the spin current [14], called spin-swapping terms because the spatial index (flow direction) and the spin index (spin direction) are interchanged. The spin-swapping terms were included phenomenologically in Ref. [4].

At least some aspects of this work by Dyakonov *et al.* [3,4,13,14] are consistent with the Onsager principle, but the work as a whole has not yet been approached from the viewpoint of irreversible thermodynamics, of which the Onsager principle is a part. It is the purpose of the present work to provide such an approach. We also include the effect of temperature gradients, thus touching on the area of "spin caloritronics" [15].

We find that, with a few additions, Refs. [3,4,13,14] are consistent with irreversible thermodynamics. Specifically, the Onsager principle predicts three new nonlinear terms in the electric current. One of these new terms reduces (see Sec. VIII) a predicted current-induced center-to-edge transverse voltage $\Delta V_{\perp}$ by a factor of 2. Table I gives a $\Delta V_{\perp}$ as large as 10^{-4} V for GaAs, but 10^{-8} V for Pt.

A. Dyakonov and Perel, and Hirsch

Both Dyakonov and Perel (DP), and Hirsch, have in mind the same physical mechanism of spin-orbit scattering. DP develop a theory for the electric current and spin current, with an underlying kinetic theory. Hirsch employs subtle physical arguments to estimate the SHE for a paramagnet, using an analogy to the anomalous Hall effect (AHE) for a ferromagnet. Such reasoning is valid because the ferromagnet and the paramagnet are related by a continuous transformation

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of a single order parameter. Therefore certain properties of the ferromagnet can be applied perturbatively to the paramagnet (in Hirsch's case, the AHE). DP remark, without proof, that their theory implies the AHE; Appendix A shows this explicitly. Hirsch also explicitly predicts the ISHE (implicit in DP), which has become a valuable tool.

B. Phenomenology vs irreversible thermodynamics

The present work is based on the methods of irreversible thermodynamics, rather than being a pure phenomenology, or being based on a Boltzmann equation. To illustrate the distinction between these three types of theory, we consider the bulk spin transfer torque (STT) and spin pumping (SP) in ferromagnets with nontrivial equilibrium configurations (e.g., a domain wall, so with $\vec{M}$ the magnetization it has $\partial_i \vec{M} \neq 0$). In STT a spin-polarized current drives the magnetic dynamics; in SP magnetic disequilibrium drives a spin-polarized current.

For both STT and SP, there are two such terms. Pure phenomenology gives four unconstrained coefficients. However, irreversible thermodynamics relates two pairs of these terms, leading to only two unconstrained terms. In addition, irreversible thermodynamics shows that the so-called adiabatic terms are dissipative (they are adiabatic in space but not in time), and contribute to the rate of entropy and heat production, whereas the so-called nonadiabatic terms are nondissipative, and do not contribute to the rate of entropy and heat production [16]. Transport theories can obtain these properties of the irreversible thermodynamics and also give specific values for the corresponding transport coefficients. Reference [17] develops a transport theory for the spin Hall effect in ferromagnets, but restricts itself to diagonal components for the spin, and thus does not consider certain transport coefficients of interest in the present work. Note also Ref. [18], which presents a sophisticated phenomenology for magnetic semiconductors, and applies it to the spin Hall effect. DP present certain relationships between their transport coefficients, obtained via kinetic theory. See Sec. V.

The above examples apply to the case of so-called off-diagonal dissipation coefficients. Irreversible thermodynamics has also been used to study diagonal dissipation coefficients. This is particularly relevant to ferromagnetic damping. There are two phenomenological theories of magnetic damping, one due to Landau and Lifshitz, and one due to Gilbert. However, three independent derivations of near-equilibrium magnetic damping each yields Landau-Lifshitz damping. This and related issues are discussed in Appendix B.

C. Outline

Section II gives the equations employed in Ref. [14] and points out that the spin polarization density $\vec{P}$ is employed, in the context of irreversible thermodynamics, in three ways—a clarification necessary for appreciating the complexity of the theory. Section III presents the thermodynamic variables and the thermodynamics of the system. Section IV presents the equations of motion for the system, and the constraint on the rate of entropy density production $R_s \geqslant 0$. Section V shows that, in the absence of temperature gradients (not considered

in Ref. [14]) the equations of Ref. [14] are not quite consistent with $R_s \geqslant 0$, and gives three additional terms that must be present in the number flux q_i. Section VI gives all of the fluxes, including the entropy flux (which has the same symmetry as the number flux), that are consistent with $R_s \geqslant 0$. Section VII applies two of the nonlinear terms in q_i—the $(\vec{P} \times \vec{E})$ term and the "new" $P_j \partial_i P_j$ term to the strip geometry, finding that there is a small transverse voltage difference between the center and the edges of the strip. Section VIII gives a summary and conclusions. Appendix A determines the AHE resistivity ρ_{AHE} using the theory of Dyakonov and Perel, and Appendix B discusses irreversible thermodynamics and magnetic damping.

II. ON THREE USES OF THERMODYNAMIC VARIABLES

Theories of the SHE have employed any of the equivalent variables spin density $\vec{S}$, spin polarization density $\vec{P}$ (units of density), or magnetization $\vec{M}$ (units of magnetic moment density); they are all proportional with "universal" constants. Following Ref. [13] we will employ $\vec{P}$. We point out that $\vec{P}$ appears in the theory in three roles: (1) dynamical variable; (2) "structure constant" used to construct thermodynamic forces and sources with the correct number of vector indices and the correct spatial inversion properties; and (3) thermodynamic driving force.

To be specific, we consider the equations employed in Ref. [13]. The number density n is associated with the number flux $\vec{q}$, which is proportional to the electric current density $\vec{j}$, and $\vec{P}$ is associated with the polarization flux q_{ij}. With carrier charge $-e$ (electrons), electric field $\vec{E}$, mobility μ, and parameter (rather than variation) δ, Ref. [13] gives for $\vec{q}$ and q_{ij},

$$\vec{q} = -\frac{\vec{j}}{e} = -[\mu n \vec{E} + D\vec{\nabla}n] - \beta \vec{E} \times \vec{P} - \delta \vec{\nabla} \times \vec{P}, \quad (1)$$

$$q_{ij} = -\mu E_i P_j - D\partial_i P_j + \varepsilon_{ijk}[\beta n E_k + \delta \partial_k n]. \quad (2)$$

(In q_{ij} the first index is real space and the second index is spin space.) Observe that $\vec{q}$ and q_{ij} satisfy the equations of motion

$$\partial_t n + \vec{\nabla} \cdot \vec{q} = 0, \quad (3)$$

$$\partial_t P_j + \partial_i q_{ij} = -(\vec{\Omega} \times \vec{P})_j - \tau_s^{-1} P_j, \quad (4)$$

where the angular frequency $\vec{\Omega} = \gamma \vec{B}$ ($\vec{B} = \mu_0 \vec{H}$ is in units of Tesla, where μ_0 is the permeability of free space and the applied field $\vec{H}$ is in units of A/m). Note that $\vec{\Omega}$ and $\vec{P}$ have signatures under time reversal $\mathcal{T} : t \to -t$ that are both odd. In both (1) and (2), the terms in β and δ were introduced in Ref. [3], and are due to spin-orbit scattering. Their signature under time reversal is opposite the signature of the usual (dissipative terms), and therefore they are nondissipative.

In q_{ij}, which was the focus of Refs. [3,4], the $\epsilon_{ijk}(\beta n E_k + \delta \partial_k n)$ terms are responsible for the SHE. In $\vec{q}$ the $\vec{P} \times \vec{E}$ term, which is required by the Onsager principle from the SHE terms in q_{ij}, leads to an AHE in the presence of an out-of-plane magnetic field that is strong enough to overcome the demagnetization field acting on $\vec{P}$. This is shown in

Appendix A. (Dyakonov and Perel did not invoke the Onsager principle in obtaining their results.) The $\vec{\nabla} \times \vec{E}$ term in $\vec{q}$ is responsible for the ISHE.

With these equations at hand, we can now comment on the three types of usage of $\vec{P}$:

(1) $\vec{P}$ is a dynamical variable in the $\partial_t P_j$ term of (4).

(2) $\vec{P}$ is used twice as an order-parameter-related structure term: in $\vec{q}$ the term proportional to $\vec{E} \times \vec{P}$, and in q_{ij} the term proportional to $E_i P_j$. Systems for which $P_i = 0$ have no such order and no such structure term.

(3) $\vec{P}$ is used three times as a stand-in for the as-yet-undefined thermodynamic driving force $\vec{\psi}$, to which it is proportional: In $\vec{q}$ the term in $\vec{\nabla} \times \vec{P}$ and in q_{ij} the term in $\partial_i P_j$ both serve as thermodynamic forces that yield fluxes; and in $\partial_t P_j$ the P_j term serves as a thermodynamic force that leads to a source. We will show that $\vec{\psi}$ is the same as the (properly defined) spin accumulation potential $\vec{\mu}_s$:

$$\vec{\psi} \equiv \vec{\mu}_s.$$

Because $\vec{S} \sim \vec{P} \sim \vec{\mu}_s$, *spin accumulation* is often used for *spin accumulation potential*.

Note that the density operator matrix in spin space $\tilde{n}$ can be written as [13]

$$\tilde{n} = \tfrac{1}{2}(n\sigma_0 + \vec{n} \cdot \vec{\sigma}),$$

where σ_0 is the unit 2-by-2 matrix, $\vec{\sigma}$ is the set of Pauli spin matrices, and

$$\vec{n} \equiv \vec{P}.$$

Nevertheless we continue to use $\vec{P}$ of Ref. [13]. Other notations have also been employed for $\hat{n}$ and the variables it contains, sometimes without the factor of $\frac{1}{2}$.

Reference [14] adds four (nonlinear) terms Δq_{ij} to the q_{ij} given above:

$$\Delta q_{ij} = -\kappa_{so}\mu(P_i E_j - \delta_{ij}\vec{P} \cdot \vec{E}) - \kappa_{so}D(\partial_j P_i - \delta_{ij}\vec{\nabla} \cdot \vec{P}),$$

(5)

where κ_{so} is due to spin-orbit scattering (Ref. [14] uses κ, which we will reserve for the thermal conductivity, later). $q_{ij} + \Delta q_{ij}$ contains three terms bilinear in $\vec{P}$ and $\vec{E}$. They are all dissipative.

Our use of irreversible thermodynamics leads to a theory with the same structure as that of Dyakonov and Perel, with two exceptions:

(1) Wherever the electric field $\vec{E}$ appears by itself, it should be accompanied by a density gradient $\vec{\nabla} n$ with coefficient such that the combination may be condensed to a single effective electric $\vec{E}^*$ (see Sec. IV) that includes both [19]. This density gradient correction is significant for semiconductors but not for metals. For a Rashba potential where a true field $\vec{E}$ shifts the particle energy in a nearly two-dimensional situation, this comment does not hold because there $\vec{E}$ serves as a structure constant rather than as a thermodynamic force [20].

(2) By Onsager, the three terms in $q_{ij} + \Delta q_{ij}$ bilinear in $\vec{P}$ (as a structure term) and $\vec{E}$ (as a thermodynamic force) lead to three additional terms in q_i that are bilinear in $\vec{P}$ (as a structure term) and the gradient of its thermodynamic driving force $\vec{\psi} \equiv \vec{\mu}_s$. Because, as shown by Dyakonov and

Perel, $\vec{P}$ is produced by $\vec{E}$, these new terms are nonlinear in deviations from equilibrium. Section VIII presents the example of a transverse voltage whose value is reduced by a factor of 2 by one of the new terms.

In principle we treat the three-dimensional (3D) case, where there is bulk inversion asymmetry, normally associated with the Dresselhaus [21] interaction. However we also have in mind the two-dimensional case (2D), where there is surface inversion asymmetry, which can be due to either the Dresselhaus [21] or Rashba interaction [20].

III. METHOD OF IRREVERSIBLE THERMODYNAMICS

Although the method of irreversible thermodynamics is well established [22–27], often it is not applied, even when it is relevant. (It took some 30 years before irreversible thermodynamics was applied to distinguish between two proposed phenomenologies for magnetic damping—see Appendix B.) In the present case of the SHE, there has been a time interval of over 40 years between the initial, seminal, work [3,4] and the present application of irreversible thermodynamics to test its consistency. (A number of recent works [16,28,29] have applied irreversible thermodynamics to magnetic systems, but they have not considered the SHE). The method, briefly, is as follows:

(1) Define the appropriate variables for the system and give its thermodynamics, typically for the energy density in terms of intensive thermodynamic variables (such as temperature T) and the density of the conjugate extensive variable (such as entropy density s). This depends on the order, or symmetry, of the system.

(2) For each thermodynamic density X write down the conservation law or (when not appropriate, as for spin density, which is not conserved) the equation of motion in terms of an unknown flux j_i^X and (if X does not correspond to a conserved quantity) an unknown source R_X. It is the goal of irreversible thermodynamics to obtain these quantities:

$$\partial_t X + \partial_i j_i^X = R_X. \tag{6}$$

(3) Using the thermodynamics and the equations of motion, determine the rate of entropy density production R_s as a sum of products of fluxes (such as entropy flux j_i^s) with their appropriate thermodynamic forces (such as temperature gradient $\partial_i T$) and as a pure divergence involving fluxes.

(4) Determine the forms of the fluxes and sources, linear in the thermodynamic forces, and subject to the symmetries appropriate to the particular system being studied (for example, entropy flux proportional to temperature gradient and gradient of electrochemical potential). In the present case we will use the fluxes given above.

(5) Enforce the condition that the rate of entropy density production $R_s \geqslant 0$. At this point we apply the Onsager relations for the transport coefficients to ensure that: (1) the dissipative contributions to the rate of entropy production with the same form are equal (e.g., from entropy current driven by the gradient in electrochemical potential and from electric current driven by the gradient in temperature); and (2) nondissipative terms cancel. We now follow the specified procedure.

IV. THERMODYNAMICS

A. Definitions

For our magnetic system, with no spontaneous magnetization $\vec{M}$, the differential of the energy density ε will be written in terms of an entropy density s, a number density n, and (following Ref. [13]) a spin polarization density P_i directed along the direction $\hat{S}_i$ of the local spin density S_i. In terms of up and down spin number densities $n_\uparrow$ and $n_\downarrow$, we have

$$n = n_\uparrow + n_\downarrow, \quad P_i = \hat{S}_i(n_\uparrow - n_\downarrow). \quad (7)$$

Thermodynamically conjugate to s is the temperature T, and thermodynamically conjugate to n is the electrochemical potential

$$\tilde{\mu} = \mu_c - eV, \quad (8)$$

where μ_c is the chemical potential of electrons with charge $-e$ (with subscript c to distinguish μ_c from the mobility μ), e is the magnitude of the electronic charge, and V is the electrical potential. From this one can define the effective potential V^*, whose negative gradient gives the effective electric field $\vec{E}^*$:

$$V^* = V - \frac{1}{e}\mu_c = -\frac{\tilde{\mu}}{e}, \quad (9)$$

$$\vec{E}^* = \frac{1}{e}\vec{\nabla}\tilde{\mu} = -\vec{\nabla}V^* = -\vec{\nabla}V + \frac{1}{e}\vec{\nabla}\mu_c. \quad (10)$$

We employ $\vec{E}^*$ rather than $\vec{\tilde{E}}$ because the latter is awkward.

Considering μ_c to be a function only of n, we may take $\delta\mu_c = (\partial\mu_c/\partial n)\delta n$, and write

$$\vec{E}^* = \vec{E} + \frac{1}{e}\frac{\partial\mu_c}{\partial n}\vec{\nabla}n. \quad (11)$$

As observed earlier, when a field $\vec{E}$ appears in the equations as a thermodynamic force, it should be in the above combination with $\vec{\nabla}n$ [19]. Note that a temperature-dependent term leads to thermoelectric and magnetothermoelectric effects, which we obtain below.

With gyromagnetic ratio

$$\gamma \equiv \frac{|g|\mu_B}{\hbar}, \quad \mu_B \equiv \frac{e\hbar}{2m_e}, \quad (12)$$

where $g \approx -2$ is the g-factor and μ_B is the Bohr magneton (m_e is the bare electron mass), the magnetization $\vec{M}$ and spin density $\vec{S}$ are related by

$$\vec{M} = -\gamma\vec{S}. \quad (13)$$

Further, $\vec{S}$ and the polarization density $\vec{P}$ are related by

$$\vec{S} = \frac{\hbar}{2}\vec{P}, \quad (14)$$

and thus

$$\vec{M} = -\frac{|g|\mu_B}{2}\vec{P} = -a\vec{P}, \quad a \equiv \frac{|g|\mu_B}{2} = \frac{\gamma\hbar}{2}. \quad (15)$$

Note that for the common case where $|g| \approx 2$ we have $a \approx \mu_B$. In SI units $\vec{M}$ is given in A/m.

B. Thermodynamics

The thermodynamics of the system alone is given by

$$d\varepsilon_{\text{sys}} = Tds + \tilde{\mu}dn + \mu_0\vec{H}_{\text{int}} \cdot d\vec{M}, \quad (16)$$

where T is the temperature, s is the entropy density, μ_0 is the magnetic permeability of free space, and $\vec{H}_{\text{int}}$ is the (internal) field (in A/m) that is conjugate to $\vec{M}$.

Note that the energy density of interaction of $\vec{M}$ and an applied field $\vec{H}$ is

$$\varepsilon_I = -\mu_0\vec{M} \cdot \vec{H}, \quad (17)$$

and that in equilibrium $\vec{H}_{\text{int}} = \vec{H}$.

The magnetic induction is

$$\vec{B} = \mu_0(\vec{H} + \vec{M}), \quad (18)$$

and is in units of Tesla. Note that $\vec{M} \times \vec{B} = \mu_0\vec{M} \times \vec{H}$.

For paramagnetic systems (our present concern)

$$\vec{M} \approx \chi\vec{H}_{\text{int}}, \quad (19)$$

with χ dimensionless; $\vec{M}$, like $\vec{H}$, is in units of A/m. For fixed s and n, integration of (16) with (19) gives an energy of magnetization $\mu_0 M^2/2\chi$, as expected. The total energy, including the field interaction energy, satisfies

$$d\varepsilon = Tds + \tilde{\mu}dn - \mu_0\vec{H}^* \cdot d\vec{M}, \quad \vec{H}^* \equiv \vec{H} - \vec{H}_{\text{int}}. \quad (20)$$

In equilibrium $\vec{H}_{\text{int}} = H$, so $\vec{H}^* = \vec{0}$. We now shift thermodynamic variable from $\vec{M}$ to $\vec{P}$ via (15), which requires a shift from conjugate thermodynamic variable $\vec{H}^*$ to

$$\vec{\psi} = \frac{g\mu_B}{2}(\mu_0\vec{H}^*) = \frac{\gamma\hbar}{2}(\mu_0\vec{H}^*), \quad (21)$$

so that $\mu_0\vec{M} \cdot \vec{H}^* = \vec{P} \cdot \vec{\psi}$. Then the thermodynamics is

$$d\varepsilon = Tds + \tilde{\mu}dn + \vec{\psi} \cdot d\vec{P}. \quad (22)$$

For a paramagnetic system,

$$\vec{\psi} \approx \frac{\mu_0 a^2}{\chi}\vec{P} + \mu_0 a\vec{H} = \lambda\vec{P} + \mu_0 a\vec{H}, \quad \lambda \equiv \frac{\mu_0 a^2}{\chi}. \quad (23)$$

For dimensional checks note that $\mu_0 a^2$ has units of energy volume, $\vec{\psi}$ has units of energy, and $\vec{P}$ has units of density.

C. On notation

We now compare the present notation $\vec{H}^*$ for a conducting ferromagnet, introduced by Johnson and Silsbee [30] and earlier used by the author [16], with the (vector) *spin chemical potential* $\vec{\mu}_s$ employed by Refs. [31,32]. We will find that $\vec{H}^* \sim \vec{\psi} = \vec{\mu}_s$. Paramagnets and ferromagnets can be compared because they are related by a single continuous order parameter; when a paramagnet has a nonzero polarization (magnetization), that gives it the same symmetry as a ferromagnet. We will find that a term in the current that does not normally appear for paramagnets is what enables us to make the identification $\vec{H}^* \sim \vec{\psi} = \vec{\mu}_s$.

Although Refs. [31,32] do not employ thermodynamics, they do present the electric current, which can then be compared with the current of Ref. [16], which also does the thermodynamics. We take the majority spin current to be

down, and the down-spin conductivity $\sigma_\downarrow$ to exceed the up-spin conductivity $\sigma_\uparrow$. Let

$$\sigma \equiv \sigma^\uparrow + \sigma^\downarrow, \quad P \equiv (-\sigma^\uparrow + \sigma^\downarrow)/\sigma,$$

where P is the polarization of the spin current (*not* the magnitude of the spin polarization). Reference [32] gives (we restore a factor of e) the electric current density j_i as

$$ej_i = \sigma \partial_i \mu - P\sigma \partial_i (\hat{M} \cdot \vec{\mu}_s). \tag{24}$$

The second term in the current does not normally appear for a paramagnet. Reference [17], which uses the spin-diagonal Boltzmann equation to study the spin Hall effect in a ferromagnet, also obtains this second term, among others.

On the other hand, Ref. [16], which does both the thermodynamics and irreversible thermodynamics of a ferromagnet, gives its Eqs. (A15) and (A16) for the number currents. The sum of the number currents yields, on neglecting all but the most essential terms,

$$j_{\uparrow i} + j_{\downarrow i} = -\frac{\sigma_\uparrow}{e^2} \partial_i \mu_\uparrow^* - \frac{\sigma_\downarrow}{e^2} \partial_i \mu_\downarrow^* + \cdots . \tag{25}$$

Here the magnetoelectrochemical potentials $\mu_\uparrow^*$ and $\mu_\downarrow^*$ are given in terms of the electrochemical potentials

$$\tilde{\mu}_\uparrow \equiv \mu_\uparrow - eV, \quad \tilde{\mu}_\downarrow \equiv \mu_\downarrow - eV,$$

where $\mu_\uparrow$ and $\mu_\downarrow$ are the chemical potentials and V is the voltage, via

$$\mu_\uparrow^* \equiv \tilde{\mu}_\uparrow + \mu_0 (\gamma \hbar/2) \hat{M} \cdot \vec{H}^*, \quad \mu_\downarrow^* \equiv \tilde{\mu}_\downarrow - \mu_0 (\gamma \hbar/2) \hat{M} \cdot \vec{H}^*. \tag{26}$$

Setting $\tilde{\mu}_\uparrow = \tilde{\mu}_\downarrow = \mu$ we then obtain for e times the electric current density $j_i = -e(j_{\uparrow i} + j_{\downarrow i})$ that

$$ej_i = \sigma \partial_i \mu - P\sigma \partial_i [\hat{M} \cdot (-\gamma \hbar/2)(\mu_0 \vec{H}^*)]. \tag{27}$$

Comparison with (21) yields that

$$\vec{\mu}_s = \frac{\gamma \hbar}{2} (\mu_0 \vec{H}^*). \tag{28}$$

This is identical with (21). Henceforth we will employ $\vec{\mu}_s$ of Refs. [31,32] rather than continue to use $\vec{\psi}$. By (23) we then have

$$\vec{\mu}_s \approx \frac{\mu_0 a^2}{\chi} \vec{P} + \mu_0 a \vec{H} = \lambda \vec{P} + \mu_0 a \vec{H}, \quad \lambda \equiv \frac{\mu_0 a^2}{\chi}. \tag{29}$$

V. IRREVERSIBLE THERMODYNAMICS DERIVATION

What follows refers only to the nonequilibrium parts of the spin currents, thus assuming that any equilibrium spin currents [33] are not subject to dissipation. Following Ref. [13] we employ $\vec{P}$ rather than $\vec{S}$ or $\vec{M}$. We employ j_i^ε, j_i^s, j_i^n, and q_{ij} for the energy flux, entropy flux, number flux, and polarization flux. For the source terms for the nonconserved entropy s and polarization P_i we employ $R_s \geqslant 0$ and R_j.

If, as we assume, the field source is constant in time, its energy does not change ($d\varepsilon_{\text{field}} = 0$), so with

$$\varepsilon_{\text{tot}} = \varepsilon + \varepsilon_{\text{field}}, \tag{30}$$

the thermodynamics is given by (20).

The equations of motion for ε, n, s, and P_j are

$$\partial_t \varepsilon + \partial_i j_i^\varepsilon = 0, \tag{31}$$

$$\partial_t s + \partial_i j_i^s = R_s, \tag{32}$$

$$\partial_t n + \partial_i q_i = 0, \tag{33}$$

$$\partial_t P_j + \partial_i q_{ij} = -\gamma \mu_0 (\vec{P} \times \vec{H})_j + R_j. \tag{34}$$

Use of (31)–(34) in the time derivative of Eq. (30) leads to an equation for the non-negative quantity R_s. We rewrite this equation, using partial integration as needed, so that it is a sum of two types of terms. First is a divergence; second is a sum over each flux times its corresponding thermodynamic force. Specifically, we have (on dropping the term involving $\vec{P} \times \vec{H}$ since, by (29), $\vec{\mu}_s \cdot (\vec{P} \times \vec{H}) = 0$)

$$0 \leqslant R_s = -\partial_i \big[j_i^\varepsilon - T j_i^s - \tilde{\mu} q_i - q_{ij} \mu_{s,j} \big] \\ - j_i^s \partial_i T - q_i \partial_i \tilde{\mu} - q_{ij} \partial_i \mu_{s,j} - R_j \mu_{s,j}. \tag{35}$$

The fluxes of Dyakonov and Perel in the language of irreversible thermodynamics

For later reference, we rewrite the fluxes of Dyakonov and Perel, $\vec{q}$ and q_{ij} (appended by Δq_{ij}) in the language of irreversible thermodynamics, where E_i^* is replaced by $(1/e)\partial_i \tilde{\mu}$ [34], and (where appropriate) $\vec{P}$ is replaced by $\vec{\mu}_s / \lambda$. Equations (1), (2), and (5) then become

$$\vec{q} = -\mu n \vec{E}^* - \beta \vec{E}^* \times \vec{P} - \delta \vec{\nabla} \times \vec{P} \\ = -\frac{\mu n}{e} \vec{\nabla} \tilde{\mu} - \frac{\beta}{e} (\vec{\nabla} \tilde{\mu} \times \vec{P}) - \frac{\delta}{\lambda} \vec{\nabla} \times \vec{\mu}_s, \tag{36}$$

$$q_{ij} = -\mu E_i^* P_j - D \partial_i P_j + \varepsilon_{ijk} \beta n E_k^* \\ = -\frac{\mu}{e} \partial_i \tilde{\mu} P_j - \frac{D}{\lambda} \partial_i \mu_{s,j} + \varepsilon_{ijk} \frac{\beta n}{e} \partial_i \tilde{\mu}, \tag{37}$$

$$\Delta q_{ij} = -\kappa_{so} \mu (P_i E_j^* - \delta_{ij} \vec{P} \cdot \vec{E}^*) \\ - \kappa_{so} D(\partial_j P_i - \delta_{ij} \vec{\nabla} \cdot \vec{P}) \\ = -\frac{\kappa_{so} \mu}{e} P_i (\partial_j \tilde{\mu}) + \frac{\kappa_{so} \mu}{e} \delta_{ij} \vec{P} \cdot \vec{\nabla} \tilde{\mu} \\ - \frac{\kappa_{so} D}{\lambda} \partial_j \mu_{s,i} + \frac{\kappa_{so} D}{\lambda} \delta_{ij} \vec{\nabla} \cdot \vec{\mu}_s. \tag{38}$$

Comparison of (4) and (34) yields

$$R_j = -\frac{1}{\tau_s} P_j = -\frac{1}{\tau_s \lambda} \mu_{s,j}. \tag{39}$$

The equations above distinguish between $\vec{P}$ used as a structure term and used to form the thermodynamic force source $\vec{\mu}_s$. (We will see that $\mu_{s,i}$ is a source associated with spin-flip and $\partial_j \mu_{s,i}$ is a force associated with diffusion.)

Note that β has units of mobility and δ has units of diffusivity. The spin-orbit-related parameter κ_{so} is dimensionless; [14] gives values of 0.3 (InSb) and 0.003 (GaAs). Reference [3], which uses $\vec{S}$ rather than $\vec{P}$, gives the relationship

$$\beta = \frac{e\delta}{k_B T}; \tag{40}$$

that relationship still holds, although both β and δ now have different units than in Ref. [3]. Further, Ref. [3] gives

$$\gamma_{so} = \frac{\beta}{\mu} = \frac{\delta}{D}, \tag{41}$$

where γ_{so} is a dimensionless measure of the effect of the spin-orbit interaction (Ref. [13] actually uses γ, rather than γ_{so}), and is about 10^{-2} for GaAs and 0.37×10^{-2} for Pt at room temperature. The fact that (40) and (41) were given by Dyakonov and Perel is an indication of the Boltzmann equation approach underlying their papers [3,4].

VI. SYMMETRY-ALLOWED STRUCTURE OF FLUXES

We now derive the irreversible thermodynamics. For clarity, we will present the fluxes (in this usage we consider the source R_i to be a generalized type of flux) in three stages. First, we give the usual fluxes when there is no spin-orbit interaction. Second, we include terms when there are spin-orbit interactions, but we do not use P_j in constructing the fluxes. Third, we construct the fluxes with spin-orbit interactions and with P_j. For q_i^n, j_i^s, and q_{ij} each term must be odd under spatial inversion, but for R_i each term must be even under spatial inversion. After the fluxes are constructed we will compare with the forms given above, due to DP.

Using Onsager coefficients L (superscripts are associated with the "flux" to the left, subscripts are associated with the "force" to the right), the first set of linearized thermodynamic fluxes is given by

$$q_i^{(1)} = -L_T^n \partial_i T - L_{\tilde{\mu}}^n \partial_i \tilde{\mu}, \tag{42}$$

$$j_i^{s(1)} = -L_T^s \partial_i T - L_{\tilde{\mu}}^s \partial_i \tilde{\mu}, \tag{43}$$

$$q_{ij}^{(1)} = -L_{\mu_s}^{P1} \partial_i \mu_{s,j}, \tag{44}$$

$$R_i^{(1)} = -L_{\mu_s}^P \mu_{s,i}. \tag{45}$$

Comparison with (36), (37), (38), and (39) yields

$$L_{\tilde{\mu}}^n = \frac{\mu n}{e} = \frac{\sigma}{e^2}, \quad \sigma \equiv ne\mu, \tag{46}$$

$$L_{\mu_s}^{P1} = \frac{D}{\lambda}, \quad L_{\mu_s}^P = \frac{1}{\tau_s \lambda}. \tag{47}$$

With κ the thermal conductivity, $\mathcal{S}$ the Seebeck coefficient, and Π the Peltier coefficient, comparison of (42) and (43) with conventional definitions yields

$$L_T^s = \frac{\kappa}{T}, \quad L_T^n = -\frac{\sigma \mathcal{S}}{e}, \quad L_{\tilde{\mu}}^s = -\frac{\Pi \sigma}{eT}. \tag{48}$$

The second set of linearized thermodynamic fluxes is

$$q_i^{(2)} = -L_{\mu_s}^n \varepsilon_{ijk} \partial_j \mu_{s,k} = -L_{\mu_s}^n (\vec{\nabla} \times \vec{\mu})_{s,i}, \tag{49}$$

$$j_i^{s(2)} = -L_{\mu_s}^s \varepsilon_{ijk} \partial_j \mu_{s,k} = -L_{\mu_s}^s (\vec{\nabla} \times \vec{\mu})_{s,i}, \tag{50}$$

$$q_{ij}^{(2)} = -L_T^P \varepsilon_{ijk} \partial_k T - L_{\tilde{\mu}}^P \varepsilon_{ijk} \partial_k \tilde{\mu}$$
$$- L_{\mu_s}^{P2} \partial_j \mu_{s,i} - L_{\mu_s}^{P3} \delta_{ij} \partial_k \mu_{s,k}, \tag{51}$$

$$R_i^{(2)} = 0. \tag{52}$$

Except for those associated with the entropy flux, these all have corresponding terms in Ref. [13]. Comparison with (36), (37), (38), and (39) yields

$$L_{\mu_s}^n = \frac{\delta}{\lambda}, \quad L_{\tilde{\mu}}^P = -\frac{\beta n}{e}, \quad L_{\mu_s}^{P2} = \frac{\kappa_{so}D}{\lambda}, \quad L_{\mu_s}^{P3} = -\frac{\kappa_{so}D}{\lambda}. \tag{53}$$

Two (thermal) terms were not discussed in Ref. [13]:

$$L_{\mu_s}^s, \quad L_T^P.$$

The third set of linearized thermodynamic fluxes is

$$q_i^{(3)} = -L_{\mu_s P}^{n1} P_j \partial_i \mu_{s,j} - L_{\mu_s P}^{n2} P_j \partial_j \mu_{s,i} - L_{\mu_s P}^{n3} P_i \partial_j \mu_{s,j}$$
$$- L_{TP}^n (\vec{P} \times \vec{\nabla}T)_i - L_{\tilde{\mu}P}^n (\vec{P} \times \vec{\nabla}\tilde{\mu})_i, \tag{54}$$

$$j_i^{s(3)} = -L_{\mu_s P}^{s1} P_j \partial_i \mu_{s,j} - L_{\mu_s P}^{s2} P_j \partial_j \mu_{s,i} - L_{\mu_s P}^{s3} P_i \partial_j \mu_{s,j}$$
$$- L_{TP}^s (\vec{P} \times \vec{\nabla}T)_i - L_{\tilde{\mu}P}^s (\vec{P} \times \vec{\nabla}\tilde{\mu})_i, \tag{55}$$

$$q_{ij}^{(3)} = -L_{\tilde{\mu}P}^{P1} P_j \partial_i \tilde{\mu} - L_{\tilde{\mu}P}^{P2} P_i \partial_j \tilde{\mu} - L_{\tilde{\mu}P}^{P3} \delta_{ij} P_k \partial_k \tilde{\mu}$$
$$- L_{TP}^{P1} P_j \partial_i T - L_{TP}^{P2} P_i \partial_j T - L_{TP}^{P3} \delta_{ij} P_k \partial_k T, \tag{56}$$

$$R_i^{(3)} = 0. \tag{57}$$

Only the terms in $q_{ij}^{(3)}$ associated with gradients of $\tilde{\mu}$ are already contained in Ref. [14]. In addition to the sets of terms involving the entropy flux and involving temperature gradients, the terms associated with the number flux are new. Of these sixteen new terms, the theory of Dyakonov and Perel makes statements about four of them:

$$L_{\tilde{\mu}P}^n = -\frac{\beta}{e}, \quad L_{\tilde{\mu}P}^{P2} = \frac{\mu}{e}, \quad L_{\tilde{\mu}P}^{P3} = \frac{\kappa_{so}\mu}{e}, \quad L_{\tilde{\mu}P}^{P1} = -\frac{\kappa_{so}\mu}{e}. \tag{58}$$

The others remain undetermined:

$$L_{\mu_s P}^{n1}, L_{\mu_s P}^{n2}, L_{\mu_s P}^{n3}, L_{\mu_s P}^{s1}, L_{\mu_s P}^{s2}, L_{\mu_s P}^{s3},$$
$$\times L_{TP}^n, L_{TP}^s, L_{\tilde{\mu}P}^s, L_{TP}^{P1}, L_{TP}^{P1}, L_{TP}^{P1}.$$

Nevertheless, because of various Onsager relations—soon to be derived—the three new terms associated with $\vec{q}$—$L_{\mu_s P}^{n1}$, $L_{\mu_s P}^{n2}$, $L_{\mu_s P}^{n3}$—can be related to various coefficients determined by Ref. [14].

VII. RATE OF BULK ENTROPY PRODUCTION

We now turn to the rate of bulk entropy (density) production R_s of (35). A term in $q_{ij}\partial_i \mu_{s,j}$ involving $(\partial_i \mu_{s,j})(\partial_j \mu_{s,i})$ will have to be rewritten to enable us to "complete the square." To this purpose we employ the identity

$$(\partial_j A_i)(\partial_i A_j) = \partial_i (A_j \partial_j A_i - A_i \partial_j A_j) + (\vec{\nabla} \cdot \vec{A})^2 \tag{59}$$

for $A_i = \mu_{s,i}$. When used later, this enables us to write the rate of entropy production in terms of $(\vec{\nabla} \cdot \vec{A})^2$ and $(\partial_i \mu_{s,j})^2$ without any $(\partial_i \mu_{s,j})(\partial_j \mu_{s,i})$ terms. This is relevant to ensuring that, after the fluxes have been expressed in terms of forces, none of the off-diagonal terms in R_s are too large—otherwise they could dominate, thus permitting R_s to take on either sign, contrary to $R_s \geqslant 0$. In what follows we will assume that $\vec{B}_a$

is uniform, so its gradient is zero. A uniform but nonzero $\vec{B}_a$ will affect the equilibrium value of $\vec{\mu}_s$ and $\vec{P}$, which can be handled as appropriate.

Now note that the divergence term, involving unknown fluxes, must be zero, since divergences can be of either sign. Therefore, no matter what the coefficients in the fluxes, setting the divergence to zero gives j_i^ε in terms of the other fluxes (and an unphysical curl term, whose divergence is zero).

After using the identity (59) to modify the divergence term in (35) we find that the energy flux in (35) takes the form

$$j_i^\varepsilon = T j_i^s + \tilde{\mu} q_i + q_{ij}\mu_{s,j} - (\mu_{s,j}\partial_j\mu_{s,i} - \mu_{s,i}\partial_j\mu_{s,j}). \quad (60)$$

Substitution of the fluxes of the previous section into (35) yields 28 terms. Specifically, q_i contributes eight terms, j_i^s eight terms, q_{ij} eleven terms, and R_i one term. Of these, six terms directly involve the squares of the thermodynamic forces, two are identically zero because they involve self-cross products, and the remaining 20 have the form of products of different thermodynamic forces, with ten repeats. It is to these repeats that the Onsager principle applies, reducing the number of off-diagonal terms to ten, for a total of 16 independent transport coefficients.

For the *on-diagonal* terms in R_s we find

$$0 \leqslant R_s = L_T^s (\vec{\nabla} T)^2 + L_{\tilde{\mu}}^n (\vec{\nabla}\tilde{\mu})^2 + L_{\mu_s}^{P1}(\partial_i\mu_{s,j})^2$$
$$+ \left(L_{\mu_s}^{P2} + L_{\mu_s}^{P3}\right)(\vec{\nabla}\cdot\vec{\mu}_s)^2 + L_{\mu_s}^P(\vec{\mu}_s)^2 + \cdots. \quad (61)$$

All of these diagonal terms must have non-negative coefficients, so

$$L_T^s \geqslant 0, \quad L_{\tilde{\mu}}^n \geqslant 0, \quad L_{\mu_s}^{P1} \geqslant 0,$$
$$L_{\mu_s}^{P2} + L_{\mu_s}^{P3} \geqslant 0, \quad L_{\mu_s}^P \geqslant 0. \quad (62)$$

For the *off-diagonal* terms in R_s we find

$$0 \leqslant R_s = (\partial_i T)(\partial_i\tilde{\mu})\left(L_{\tilde{\mu}}^s + L_T^n\right) + (\vec{\nabla}T\cdot\vec{\nabla}\times\vec{\mu}_s)\left(L_{\mu_s}^s + L_T^P\right) + (\vec{\nabla}T\cdot\vec{P}\times\vec{\nabla}\tilde{\mu})\left(L_{\tilde{\mu}P}^s - L_{TP}^n\right)$$

$$+ (\partial_i T)(P_j\partial_i\mu_{s,j})\left(L_{\mu_s P}^{s1} + L_{TP}^{P1}\right) + (\partial_i T)(P_j\partial_j\mu_{s,i})\left(L_{\mu_s P}^{s2} + L_{TP}^{P2}\right) + (\partial_i T)(P_i\partial_j\mu_{s_j})\left(L_{\mu_s P}^{s3} + L_{TP}^{P3}\right)$$

$$+ (\vec{\nabla}\tilde{\mu}\cdot\vec{\nabla}\times\vec{\mu}_s)\left(L_{\mu_s}^n + L_{\tilde{\mu}}^P\right) + (\partial_i\tilde{\mu})(P_j\partial_i\mu_{s,j})\left(L_{\mu_s P}^{n1} + L_{\tilde{\mu}P}^{P1}\right) + (\partial_i\tilde{\mu})(P_j\partial_j\mu_{s,i})\left(L_{\mu_s P}^{n2} + L_{\tilde{\mu}P}^{P2}\right)$$

$$+ (\partial_i\tilde{\mu})(P_i\partial_j\mu_{s_j})\left(L_{\mu_s P}^{n3} + L_{\tilde{\mu}P}^{P3}\right) + \cdots. \quad (63)$$

The terms even (odd) under time reversal are dissipative (nondissipative). In effect, the Onsager principle states that the two contributions to R_s from a product of two thermodynamic forces are equal (or equal and opposite) if the combined term (including structure terms, like P_i) is even (or odd) under time reversal. The Onsager principle thus ensures that the nondissipative terms (which can change sign under time reversal) do not contribute to the rate of entropy production, and that the "force" A acting on the flux b conjugate to the force B causes the same rate of dissipation as the force B acting on the flux a conjugate to the force A.

Application of the Onsager principle then leads to

$$L_{\tilde{\mu}}^s = L_T^n, \quad L_{\mu_s}^s = -L_T^P, \quad L_{\tilde{\mu}P}^s = L_{TP}^n, \quad L_{\mu_s}^n = -L_{\tilde{\mu}}^P,$$

$$L_{\mu_s P}^{s1} = L_{TP}^{P1}, \quad L_{\mu_s P}^{s2} = L_{TP}^{P2}, \quad L_{\mu_s P}^{s3} = L_{TP}^{P3},$$

$$L_{\mu_s P}^{n1} = L_{\tilde{\mu}P}^{P1}, \quad L_{\mu_s P}^{n2} = L_{\tilde{\mu}P}^{P2}, \quad L_{\mu_s P}^{n3} = L_{\tilde{\mu}P}^{P3}. \quad (64)$$

The first of these leads to

$$L_T^n = -\frac{\sigma \mathcal{S}}{e} = L_{\tilde{\mu}}^s = -\frac{\Pi\sigma}{eT}, \quad (65)$$

so

$$\Pi = T\mathcal{S}, \quad (66)$$

a thermoelectric relation due to Kelvin.

In terms of coefficients that have been defined in Ref. [14] we have

$$L_{\mu_s P}^{n1} = L_{\tilde{\mu}P}^{P1} = \frac{\mu}{e}, \quad (67)$$

$$L_{\mu_s P}^{n2} = L_{\tilde{\mu}P}^{P2} = \frac{\kappa_{so}\mu}{e}, \quad (68)$$

$$L_{\mu_s P}^{n3} = L_{\tilde{\mu}P}^{P3} = -\frac{\kappa_{so}\mu}{e}, \quad (69)$$

$$L_{\mu_s}^n = -L_{\tilde{\mu}}^P \quad \Rightarrow \quad \frac{\delta}{\lambda} = \frac{\beta n}{e}. \quad (70)$$

The last of these is consistent with the relationship (40), on using λ of (29) and the appropriate χ.

If we neglect the terms involving temperature gradients, there are four diagonal coefficients and four off-diagonal coefficients. We do not write down the nine inequalities, obtained by completing the squares, necessary to ensure that the products of the off-diagonal terms (which can be of either sign, according to the direction of a given thermodynamic force) do not overwhelm the corresponding diagonal terms. Some of them require transport coefficients that are higher order in $\vec{P}$, which we do not consider.

Neglecting the effects of temperature gradients in $q_i^{(3)}$, the new terms augment (36) by

$$\Delta q_i = -L_{\mu_s P}^{n1} P_j\partial_i\mu_{s,j} - L_{\mu_s P}^{n2} P_j\partial_j\mu_{s,i} - L_{\mu_s P}^{n3} P_i\partial_j\mu_{s,j}$$

$$= -\frac{\mu}{e}\lambda P_j\partial_i P_j - \frac{\kappa_{so}\mu}{e}\lambda P_j\partial_j P_i - \frac{\kappa_{so}\mu}{e}\lambda P_i\partial_j P_j, \quad (71)$$

where the second equality arises on using (58) and (29). In what follows we will need only the first of these terms.

VIII. CENTER-TO-EDGE VOLTAGE: A SECOND-ORDER EFFECT

Consider a long strip of width L with normal along z, in a uniform field $\vec{E}_0$ along x, and $-L/2 \leqslant y \leqslant L/2$. We now show that this leads to a center-to-edge voltage, whose value is affected by the first term in (71).

Because the spin-orbit term $\beta \varepsilon_{ijk} E_k^*$ in q_{yz} is nonzero, to satisfy the boundary conditions

$$q_{yz}|_{y=\pm L/2} = 0 \qquad (72)$$

requires a polarization $P_z(y)$. The $\delta \vec{\nabla} \times \vec{P}$ term in $\vec{q}$ then leads to a small correction $\Delta q_x(y)$ to the number current density q_x, which in turn leads to a small correction ΔR to the resistance R measured along x. For this geometry, including an applied field H_a normal to the plane, Ref. [13] calculated $\Delta q_x(y)$ and then the magnetoresistance $R(H_a)$ that arises from the accumulation of P_z at the edge.

A. Nonzero E_y and $\Delta V_\perp = V(L/2) - V(0)$

For $P_z \neq 0$, the $\beta \vec{P} \times \vec{E}^*$ term in $\vec{q}$ of (36) leads to a nonzero q_y. Thus (recall that Dyakonov's number flux is $\vec{q}$), to satisfy

$$q_y|_{y=\pm L/2} = 0 \qquad (73)$$

at the edges requires an E_y^*. This leads to a small but not insignificant nonzero transverse voltage difference

$$\Delta V_\perp = V(\pm L/2) - V(0) = -\int_0^{L/2} E_y \, dy, \qquad (74)$$

which might be observable by ac capacitative techniques (these do not change the boundary conditions on either the current q_y or the spin current q_{yz}).

The new term $-(\mu/e)P_z \partial_y P_z$ from (71) is of the same order of magnitude as the $\beta \vec{P} \times \vec{E}^*$ term in (36), so its effect must be included. Since Ref. [13] does not present $P_z(y)$, we derive it now. We employ a perturbation theory approach in powers of E_0.

B. Evaluating $P_z(y)$

To lowest order we take

$$q_y \approx 0, \quad q_x \approx -\mu n E_0. \qquad (75)$$

In the bulk, but not necessarily near the leads along x, $\partial_x n = 0$.

Assuming that P_z depends only on y, we consider only q_{yz} to be nonzero. To lowest order it is, by (37),

$$q_{yz} = -D \partial_y P_z + \beta n E_0. \qquad (76)$$

From (4) with $\partial_t P_z = 0$ we have

$$\partial_i q_{iz} = \partial_y q_{yz} = -\frac{P_z}{\tau_s}. \qquad (77)$$

Combining (76) and (77) yields

$$-D \partial_y^2 P_z = -\frac{P_z}{\tau_s}. \qquad (78)$$

With L_s the spin-flip-diffusion time, the solution of this is given by

$$P_z = A \sinh\left(\frac{y}{L_s}\right) + B \cosh\left(\frac{y}{L_s}\right), \quad L_s^2 \equiv D\tau_s, \qquad (79)$$

for arbitrary A and B. Application of the boundary conditions (72) at $y = \pm L/2$ leads to A and B, and thence

$$P_z = \frac{\beta n E_0 L_s}{D} \frac{\sinh\left(\frac{y}{L_s}\right)}{\cosh\left(\frac{L}{2L_s}\right)}. \qquad (80)$$

C. Evaluating E_y and $\Delta V_\perp$

By (36) and the first term of (54) we have, with (67) and $\mu_{s,z} = \lambda P_z$,

$$q_y = -\mu n E_y^* + \beta E_0 P_z - \frac{\mu}{e} P_z \partial_y \mu_{s,z}. \qquad (81)$$

The last term is the correction due to the new term in (71).

Because q_x is nearly constant in space, the $\vec{\nabla} \cdot \vec{q} = 0$ condition also leads to q_y being constant in space. Since $q_y = 0$ at $y = \pm L/2$, we have $q_y = 0$ for all y, a condition that determines E_y and $\Delta V_\perp$. Specifically, from (81) with $q_y = 0$, and using $\partial_y P_z$ explicitly (but not yet P_z) we find that

$$E_y^* = \frac{\beta E_0}{\mu n} P_z \left(1 - \frac{\mu \lambda n}{eD} \frac{\cosh\left(\frac{y}{L_s}\right)}{\cosh\left(\frac{L}{2L_s}\right)}\right). \qquad (82)$$

By (70) and (41) we have

$$\frac{\mu \lambda n}{eD} = 1.$$

Then, substituting P_z yields

$$E_y^* = \bar{E} \sinh\left(\frac{y}{L_s}\right)\left(1 - \frac{\cosh\left(\frac{y}{L_s}\right)}{\cosh\left(\frac{L}{2L_s}\right)}\right),$$

$$\bar{E} \equiv \frac{\beta^2 E_0^2 L_s}{\mu D \cosh(L/2L_s)}. \qquad (83)$$

Use of E_y^* above in Gauss's law in the form

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon}(-e \delta n), \qquad (84)$$

where ϵ is the dielectric constant, with (11) leads to an equation for δn:

$$\partial_y E_y^* = -\frac{e}{\epsilon}\left(\delta n - l_D^2 \partial_y^2 \delta n\right), \quad l_D^2 \equiv \frac{\epsilon}{e^2} \frac{\partial \mu_c}{\partial n}. \qquad (85)$$

Here l_D is the electrical screening length.

With $c_{I1} \equiv (1 - l_D^2/L_s^2)^{-1}$ and $c_{I2} \equiv (1 - 4l_D^2/L_s^2)^{-1}$, the inhomogeneous solution δn_I is given by

$$\delta n_I = -\frac{\bar{E}\epsilon}{eL_s} c_{I1} \cosh\frac{y}{L_s} + \frac{\bar{E}\epsilon}{eL_s} c_{I2} \frac{\cosh\frac{2y}{L_s}}{\cosh\frac{L}{2L_s}}. \qquad (86)$$

Requiring that the homogeneous solution be symmetric about $y = 0$, it must have the form

$$\delta n_H = K \cosh\left(\frac{y}{l_D}\right), \qquad (87)$$

TABLE I. Table with estimated experimental values. ρ_{GaAs} estimated via $\rho = (ne\mu)^{-1}$, with $n = 10^{22}/\text{m}^3$ from Ref. [35]. $j_{\max}^{\text{GaAs}}$ estimated by scaling $j_{\max}^{\text{Pt}}$ by $\rho_{\text{Pt}}/\rho_{\text{GaAs}}$. $E_0^{\max} \equiv \rho j_{\max}$.

Material	γ_{so}	μ (m²/V s)	τ_s (s)	ρ (ohm m)	$j_{\max}$ (A/m²)	$E_0^{\max}$ (V/m)	$\Delta V_\perp^{\max}$ (V)
GaAs	0.01	0.54 (Ref. [35])	10^{-7} (Ref. [36])	0.00117	0.86×10^6	0.54×10^4	7.0×10^{-5}
Pt	0.0037	0.01 (Ref. [37])	10^{-12} (Ref. [38])	9.6×10^{-8} (Ref. [39])	1.0×10^{11} (Ref. [40])	1.0×10^4	6.8×10^{-9}

where the overall charge neutrality requirement

$$\int_{-L/2}^{L/2} dy(\delta n_I + \delta n_H) = 0 \tag{88}$$

gives

$$K = \frac{\epsilon \bar{E}}{e l_D}(c_{I1} - c_{I2})\frac{\sinh\left(\frac{L}{2L_s}\right)}{\sinh\left(\frac{L}{2l_D}\right)}. \tag{89}$$

Note that the dimensionless coefficient $c_{I1} - c_{I2}$ is of order l_D^2/L_s^2, which is very small. Comparison of $\bar{E}$ and K shows that δn_H is smaller than δn_I by a factor of l_D/L_s.

With $\delta n = \delta n_I + \delta n_H$ we solve Gauss's law for E_y:

$$E_y = \bar{E} \sinh\left(\frac{y}{L_s}\right)\left(c_{I1} - c_{I2}\frac{\cosh\left(\frac{y}{L_s}\right)}{\cosh\left(\frac{L}{2L_s}\right)}\right)$$
$$- K\frac{e l_D}{\epsilon}\sinh\left(\frac{y}{l_D}\right). \tag{90}$$

As just noted, typically $l_D \ll l_s$, so we now can take $c_{I1} \approx c_{I2} \approx 1$ for the term in $\bar{E}$.

For the term in K, although the integral over $K \sinh(y/l_D)$ grows exponentially in L/l_D, this is compensated by a similarly exponentially growing denominator in K. The integral over y multiplies the $\bar{E}$ term by a factor on the order of L_s, whereas it multiplies the K term by a factor of only l_D. The net effect is that the transverse voltage produced by $\delta n_H \sim K$ is smaller by l_D^2/L_s^2 than the transverse voltage produced by $\delta n_I \sim \bar{E}$. We thus neglect the effect of δn_H on the transverse field and voltage.

Performing the integral of (74) using only the $\bar{E}$ term in (90) yields

$$\Delta V_\perp \approx -\frac{(\beta E_0 L_s)^2}{\mu D}\left[1 - \frac{1}{\cosh\left(\frac{L}{2L_s}\right)} - \frac{1}{4}\frac{\cosh\left(\frac{L}{L_s}\right) - 1}{\left[\cosh\left(\frac{L}{2L_s}\right)\right]^2}\right]. \tag{91}$$

The last of the three terms in the brackets is due to the new term—the correction due to the application of irreversible thermodynamics. In the limit where $L \gg L_s$, this becomes

$$\Delta V_\perp \approx -\frac{(\beta E_0 L_s)^2}{2\mu D} = -\gamma_{so}^2 \frac{\mu \tau_s E_0^2}{2}. \tag{92}$$

Without the new term, for $L \gg L_s$, $\Delta V_\perp$ would have been twice as large as in (92).

Table I gives estimated experimental values, where $j_{\max}$ is the maximum current before the sample burns up, and we take

$$E_0^{\max} = \rho j_{\max}. $$

For GaAs the estimated maximum center-to-edge transverse voltage $\Delta V_\perp$ is on the order of 10^{-4} V, but for Pt it is on the order of 10^{-8} V. The former holds much more promise of measurement.

IX. SUMMARY AND CONCLUSIONS

We have applied the methods of irreversible thermodynamics to the theory of nonmagnetic conductors with a spin-1/2 degree of freedom and spin-orbit scattering. In addition to the terms of Refs. [13,14], we find three additional terms that enter the electric current. When applied to a transverse voltage difference that is second order in the applied electric field, we find that the additional term cancels half of the voltage computed in its absence, and that together the two terms yield a measurable voltage. Such measurement, however, cannot be done with conventional probes because they might draw current and/or spin accumulation, but a capacitive measurement—perhaps one that is done with a longitudinal ac field—might be effective as a detection method.

APPENDIX A: THE ANOMALOUS HALL EFFECT RESISTIVITY FROM THE SPIN HALL EFFECT COEFFICIENT

For our geometry, with $\vec{B} = B\hat{z}$ strong enough to align $\vec{M}$ along $\vec{B}$, application of Pugh's empirical result for the HE resistivity of ferromagnetic materials [41,42] takes the form

$$\rho_H \equiv \frac{E_y}{j_x} = R_0 B + R_1 M. \tag{A1}$$

Here Pugh employed emu, so B and M have the same units; and R_0 and R_1 are material-dependent constants. The second term is what is known as the AHE. Dyakonov and Perel remark that this form follows from the $-\beta \vec{P} \times \vec{E}$ term in their $\vec{q}$. Because this result is not well known, we present it here.

As a start, we rewrite (A1) as

$$\rho_H = \rho_{\text{H,O}} + \rho_{\text{AHE}}. \tag{A2}$$

To derive each term, we first add the Lorentz force $\vec{v} \times \vec{B}$ to $\vec{E}^*$ in $\vec{q}$ of (1). If $\vec{B}$ is strong enough to pull $\vec{M}$ out of the plane, so $\vec{M}$ is directed along $\vec{B}$, then $\vec{P}$ is opposite to $\vec{B}$. Moreover, the $\vec{\nabla} \times \vec{P}$ term is now negligible.

We now focus on q_y, which in steady state will go to zero:

$$q_y \approx -\mu n(E_y^* + v_x B) - \beta E^* P_z. \tag{A3}$$

Using $\sigma = ne\mu$, $\vec{P} = -(2/\gamma\hbar)\vec{M}$, $\vec{j} = -e\vec{q}$, and

$$j_x = \sigma E_x^* = -nev_x,$$

(A3) leads to

$$\vec{j}_y \approx \sigma\left[E_y^* - \frac{\sigma}{ne}E_x^*\left(B + \frac{2\beta ne^2}{\sigma^2\gamma\hbar}M\right)\right]. \tag{A4}$$

Setting $j_y = 0$ gives

$$E_y^* = \frac{\sigma}{ne}E_x^*\left(B + \frac{2\beta ne^2}{\sigma^2\gamma\hbar}M\right). \tag{A5}$$

Now note that the transverse (Hall) voltage V_H is actually determined by the electron chemical potential, and therefore $V_H = E_y^* d$ yields E_y^*. From this we find that

$$\rho_H \equiv \frac{E_y^*}{j_x} = \frac{B}{ne} + \frac{2}{\sigma}\left(\frac{\beta}{\mu}\right)\left(\frac{M}{\gamma\hbar n}\right). \tag{A6}$$

In the above equation the second two ratios are dimensionless, so this obviously is a resistivity. Comparison with (A2) then yields the ordinary and anomalous Hall resistivities.

$$\rho_{H,O} = \frac{B}{ne}, \quad \rho_{AHE} = \frac{2}{\sigma}\left(\frac{\beta}{\mu}\right)\left(\frac{M}{\gamma\hbar n}\right). \tag{A7}$$

We have thus seen how the theory of Dyakonov and Perel predicts an AHE, and gives a specific form for the anomalous Hall resistivity.

APPENDIX B: MAGNETIC DAMPING AND IRREVERSIBLE THERMODYNAMICS

The original proposal for the magnetic damping part $\partial_t\vec{M}_d$ of $\partial_t\vec{M}$, by Landau and Lifshitz (LL) [43], was a phenomenology of the form

$$\partial_t\vec{M}_{LL,d} = -\lambda\hat{M}\times(\vec{M}\times\vec{B}_a), \tag{B1}$$

where $\vec{B}_a$ includes the applied and anisotropy fields, and (if present) a nonuniform exchange field.

However, Kelly's rotational hysteresis measurements on permalloy with in-plane anisotropy K found unusually large damping at low frequencies, which Gilbert could not fit with

the Landau-Lifshitz form [44]. Gilbert then proposed the magnetic damping form

$$\partial_t\vec{M}_{G,d} = \alpha\hat{M}\times\partial_t\vec{M}, \tag{B2}$$

which he could fit to the data with a frequency-dependent α (as large as 9 for a frequency of 150 kHz in a field "above saturation" of about 10 Oe, or 10^{-3} T) and a fixed gyromagnetic ratio.

A later study by Mayfield [45] concluded that the excess damping at low frequency was, "as first pointed out by J. C. Slonczewski," due to "abrupt reorientations of M, which must occur when $K/M \leqslant H \leqslant 2K/M$." Here K is the in-plane uniaxial anisotropy constant that develops on cooling the system through the Curie temperature; this leads to two local minima in the indicated field range. (One may consider that $H = B_a/\mu_0$.) In other words, the dissipation that Kelly observed, and which prompted Gilbert to develop an alternate phenomenology that, like Ref. [43], assumed small variations in magnetization space, likely was due to the system undergoing large variations in magnetization space. The possibility of explaining such damping quantitatively awaited a Fokker-Planck theory of statistical fluctuations, which was not provided until a few years later by Brown [46,47].

Besides the theory of Gilbert, a number of alternate theories of magnetic damping were developed, including one by Callen [48] that assumed the LL form for transverse damping but added a longitudinal damping term that he studied using magnons. However, Landau-Lifshitz damping is supported by many independent derivations using irreversible thermodynamics, where $\hat{M}\times\vec{B}_a$ is the thermodynamic driving force [49–51]. Thus irreversible thermodynamics has distinguished unambiguously between the two distinct phenomenologies of Landau and Lifshitz and of Gilbert.

From the viewpoint of irreversible thermodynamics the primary objection to Gilbert damping is that it is not driven by a thermodynamic force (which would have a unique signature under time reversal)—here $\hat{M}\times\vec{B}_a$. It is also somewhat peculiar in that the damping term itself ($\hat{M}\times\partial_t\vec{M}$) is proportional to the quantity whose time derivative ($\partial_t\vec{M}$) is to be determined.

Most systems are not uniform. Nonuniformity can introduce two-magnon scattering (inhomogeneous) linewidths [52]; further, surface scattering is not included in (bulk) Landau-Lifshitz damping [53]. Modern samples have small values of $\lambda/\gamma = \alpha$, making the LL and Gilbert forms of damping nearly the same; theories based on α have been rather successfully applied to real materials [54,55].

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