

Classical Dynamics of Macroscopic Strings [†]

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In recent work, Dabholkar *et al.* constructed static “cosmic string” solutions of the low-energy supergravity equations of the heterotic string, and conjectured that these solitons are actually exterior solutions for infinitely long fundamental strings. In this paper we provide compelling dynamical evidence to support this conjecture by computing the dynamical force exerted by a solitonic string on an identical test-string limit, the Veneziano amplitude for the scattering of macroscopic winding states and the metric on moduli space for the scattering of two string solitons. All three methods yield trivial scattering in the low-energy limit.

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1. Introduction

Soliton solutions of string theory recently discovered by Dabholkar *et al.* [1,2] have the property, like BPS magnetic monopoles, that they exert zero static force on each other and can be superposed to form multi-soliton solutions with arbitrarily variable collective coordinates. In this paper we show that, in contradistinction to the BPS case, the velocity-dependent forces between these string solitons also vanish (i.e. we argue that the scattering is trivial). We also argue that this phenomenon provides further, dynamical evidence for the identification of the Dabholkar-Harvey soliton with the underlying fundamental string by comparing the scattering of these soliton solutions with expectations from a Veneziano amplitude computation for macroscopic fundamental strings.

In section 2, we first summarize the solution of Dabholkar *et al.*[1], who construct static “cosmic string” solutions of the low-energy supergravity equations of the heterotic string. These solutions have several remarkable properties. The most notable is the vanishing of the static force between parallel straight “cosmic” strings of like orientation. This feature is the result of a cancellation of the long-range forces due to exchange of axions, gravitons and dilatons, and is reminiscent of the cancellation of gauge and Higgs forces between BPS monopoles. Indeed, in perfect analogy with the BPS case, the no-force condition makes it possible to construct multi-soliton solutions with any number of parallel, like-oriented straight cosmic strings at arbitrary separations. Since these static properties are also possessed by fundamental strings winding around an infinitely large compactified dimension, Dabholkar *et al.* conjecture [2] that the soliton is actually the exterior solution for an infinitely long fundamental string.

We examine the scattering of these solitons in section 3 using the “test string” approximation. From the sigma model action describing the motion of a point-like test string in a general background of axion, graviton and dilaton fields, we derive an effective action for the motion of the center of mass coordinate of the test string in the special background provided by a string soliton. We of course find that the static force vanishes (this is how Dabholkar *et al.* constructed their ansatz for the string soliton). More remarkably, we find that the $O(v^2)$ velocity-dependent forces vanish as well. This result suggests that there is trivial scattering between these string solitons, at least in the test-string approximation.

We address the scattering problem in section 4 from the string theoretic point of view. In particular, we calculate the string four-point amplitude for the scattering of macroscopic winding state strings in the infinite winding radius limit. In this scenario,

we can best approximate the soliton scattering problem considered in section 3. We find that the Veneziano amplitude obtained also indicates trivial scattering in the large winding radius limit, thus providing evidence for the identification of the string soliton solutions with infinitely long macroscopic fundamental strings.

In section 5 we compute the metric on moduli space for the string soliton in $D = 4$ to lowest nontrivial order in the string tension. The geodesics of this metric represent the motion of quasi-static solutions in the static solution manifold and in the absence of a full time dependent solution provide a good approximation to the low-energy dynamics of the solitons. The metric is found to be flat, which again implies trivial scattering of the solitons, in agreement with the results of the previous two sections.

We conclude in section 6 with a discussion of our results. In particular, our findings provide compelling dynamical evidence for the identification of the solitonic string with the underlying fundamental string. We note that the role of these solitons in string theory parallels the role of soliton solutions in field theory in describing extended particle states.

The results of sections 3 and 4 have been previously summarized in [3]. The Manton scattering result of section 5 has recently been summarized in [4].

2. String Multi-Soliton Static Solution

In recent work[2,1], Dabholkar and others presented a low-energy analysis of macroscopic superstrings and discovered several interesting analogies between macroscopic superstrings and solitons in supersymmetric field theories. The main result of this work centers on the existence of exact multi-string solutions of the low-energy supergravity super-Yang-Mills equations of motion. In addition, Dabholkar *et al.* find a Bogomolnyi bound for the energy per unit length which is saturated by these solutions, just as the Bogomolnyi bound is saturated by magnetic monopole solutions in ordinary Yang-Mills field theory.

The Dabholkar *et al.* solution may be outlined as follows. The action for the massless spacetime fields (graviton, axion and dilaton) in the presence of a source string can be written as[5,6,1]

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{g} \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\alpha\phi} H^2 \right) + S_\sigma, \quad (2.1)$$

with the source terms contained in the sigma model action S_σ given by

$$S_\sigma = -\frac{\mu}{2} \int d^2\sigma (\sqrt{\gamma} \gamma^{mn} \partial_m X^\mu \partial_n X^\nu g_{\mu\nu} e^{\alpha\phi} + \epsilon^{mn} \partial_m X^\mu \partial_n X^\nu B_{\mu\nu}), \quad (2.2)$$

with $\alpha = \sqrt{2/(D-2)}$ and γ_{mn} a worldsheet metric to be determined.

The sigma model action S_σ describes the coupling of the string to the metric, antisymmetric tensor field and dilaton. The first part of the action S above represents the effective action for the massless fields in the spacetime frame and whose equations of motion are equivalent to conformal invariance of the underlying sigma model. The combined action thus generates the equations of motion satisfied by the massless fields in the presence of a macroscopic string source[1].

The static solution to the equations of motion is given by [1]

$$\begin{aligned} ds^2 &= e^A[-dt^2 + (dx^1)^2] + e^B d\vec{x} \cdot d\vec{x} \\ A &= \frac{D-4}{D-2}E(r) & B &= -\frac{2}{D-2}E(r) \\ \phi &= \alpha E(r) & B_{01} &= -e^{E(r)}, \end{aligned} \quad (2.3)$$

where x^1 is the direction along the string, $r = \sqrt{\vec{x} \cdot \vec{x}}$ and

$$e^{-E(r)} = \begin{cases} 1 + \frac{M}{r^{D-4}} & D > 4 \\ 1 - 8G\mu \ln(r) & D = 4 \end{cases} \quad (2.4)$$

for a single static string source. The solution can be generalized to an arbitrary number of static string sources by linear superposition of solutions of the $(D-2)$ -dimensional Laplace's equation.

The force exerted on a test string moving in given background fields is obtained from the sigma model equation of motion[1]

$$\nabla_m(\gamma^{mn}\nabla_n X^\mu) = -\Gamma_{\nu\rho}^\mu \partial_m X^\nu \partial_n X^\rho \gamma^{mn} + \frac{1}{2}H^\mu{}_{\nu\rho} \partial_m X^\nu \partial_n X^\rho \epsilon^{mn}, \quad (2.5)$$

where $\Gamma_{\nu\rho}^\mu$ are the Christoffel symbols calculated from the sigma model metric $G_{\mu\nu} = g_{\mu\nu}e^{\alpha\phi}$. We make the usual distinction between the sigma model metric and the Einstein metric; spacetime indices are raised and lowered by contraction with $G_{\mu\nu}$; worldsheet indices are denoted by m and n .

We consider a stationary test string in the background of a source string located at the origin. Assume further that both strings run along the x_1 direction and have the same orientation. We use conformal gauge for the test string and get $X^0 = \tau$, $X^1 = \sigma$, $\gamma_{mn} = \text{diag}(-1, +1)$ and $\epsilon^{01} = +1$. The transverse force then vanishes[1]:

$$\frac{d^2}{d\tau^2}X^i = -2\Gamma_{00}^i + H^i{}_{10} = 0. \quad (2.6)$$

Note that if the test string and source string were oppositely oriented then the second term would appear with a negative sign and there would be a net attractive force. Also note that the no-force condition depends only on the general ansatz (2.3) and not on the precise form of the solution in (2.4).

The zero-force condition arises from the cancellation of long-range forces of exchange of the massless fields of the string (the graviton, axion and dilaton) and can be seen explicitly from (2.6). This is a perfect analog to the zero-force condition of Manton for magnetic monopoles, which requires that the attractive scalar exchange force precisely cancel the repulsive vector exchange force when the Bogomolnyi bound is attained. Dabholkar *et al.*[1] show that a similar Bogomolnyi bound is satisfied by their string soliton solutions, further strengthening the analogy with the monopoles. In the next section, we use the test-string approach to study the dynamics of these solitons.

3. Test String Approximation

We now turn to the dynamics of these string solitons. While the static force vanishes as a result of the cancellation of long-range forces of exchange, the force between two moving solitons is in general nonvanishing and depends on the velocities of the solitons. The most complete answer would be given by a full time-dependent solution of the equations of motion of the above action for the case of an arbitrary number of sources moving with arbitrary transverse velocities. These equations, however, are much more difficult to solve for moving sources than for a static configuration. Even a two-soliton solution is in general quite intractable for this class of actions. The next best step is the calculation of the Manton metric on moduli space[7], and this will be shown in section 5.

Here we will take the preliminary “test-string” approach: we consider simply a test string moving in the background of a string soliton whose massless fields are given by (2.3). The advantage of this approach is that we obtain a first order approximation in a relatively simple manner. The effective Lagrangian for the motion of a test string in the background of the source string can be read off from the sigma model source action S_σ . We then solve the constraint equation for the worldsheet metric obtained by varying the worldsheet Lagrangian \mathcal{L} . The resultant solution for the worldsheet metric along with the static solution for the spacetime metric, antisymmetric tensor field and dilaton from the static ansatz for a single source string are then substituted into the Lagrangian, whose equations yield the dynamics of the test string in the source string background.

The constraint equation for the worldsheet metric is given by

$$F_{mn} - \frac{1}{2}\gamma_{mn} (\gamma^{ab}F_{ab}) = 0, \quad (3.1)$$

where $F_{mn} \equiv \partial_m X^\mu \partial_n X^\nu G_{\mu\nu}$. The solution to (3.1) is given by

$$\gamma_{mn} = \Omega(X^\rho) \partial_m X^\mu \partial_n X^\nu G_{\mu\nu}, \quad (3.2)$$

where $\Omega(X^\rho)$ is an arbitrary conformal factor. Note that the choice $\Omega = e^{-E}$ fixes the conformal gauge $\gamma^{mn} = \eta^{mn}$ in the static case. Substituting (3.2) and (2.3) in the worldsheet action gives

$$\mathcal{L} = -\mu \left[\sqrt{-\det(\eta_{mn}e^E + \partial_m X^\mu \partial_n X^\nu \delta_{\mu\nu})} - e^E \right]. \quad (3.3)$$

Naturally Ω drops out of \mathcal{L} . The relative sign of the two terms in (3.3) would have been “plus” for oppositely oriented source and test strings. Taking the transverse coordinates of the test string independent of x^1 , the action reduces to

$$\mathcal{L} = -\mu \left[e^E \left(1 - \frac{(\dot{X}^i)^2}{e^E} \right)^{\frac{1}{2}} - e^E \right]. \quad (3.4)$$

Expanding this in powers of velocity one easily obtains

$$\mathcal{L} = \frac{\mu}{2}(\dot{X}^i)^2 + O(\dot{X}^4). \quad (3.5)$$

So, the same ansatz which causes the static force on a test string to vanish, causes the lowest-order velocity-dependent force to vanish. In this approximation, then, we have trivial scattering (i.e. no deviation from initial trajectories) and the Manton metric on moduli space is flat. Note that nothing in this argument depended on the detailed form of E , so the same result holds for a string moving in a general multi-string background.

Needless to say, this is a rather surprising result and so we would like to obtain some kind of confirmation for this answer. We proceed to do so in the next two sections. In section 4 we consider the scattering problem from the purely string theoretic viewpoint, while in section 5 we explicitly compute the metric on moduli space in $D = 4$ to lowest nontrivial order in the string tension.

4. Veneziano Amplitude for Macroscopic Fundamental Strings

As mentioned earlier, it seems likely that the string solitons of Dabholkar *et al.* are to be identified with infinitely long fundamental strings of the underlying string theory. If that is so, the results we have just found should agree with the corresponding Veneziano amplitude calculation of string scattering. We now turn to this useful sanity check.

The scattering problem is set up in four dimensions, as the kinematics correspond essentially to a four dimensional scattering problem, and strings in higher dimensions generically miss each other anyway[8]. The precise compactification scheme is irrelevant to our purposes.

The winding state strings then reside in four spacetime dimensions (0123), with one of the dimensions, say x_3 , taken to be periodic with period R , called the winding radius. The winding number n describes the number of times the string wraps around the winding dimension:

$$x_3 \equiv x_3 + 2\pi Rn, \quad (4.1)$$

and the length of the string is given by $L = nR$. The integer m , called the momentum number of the winding configuration, labels the allowed momentum eigenvalues. The momentum in the winding direction is thus given by

$$p^3 = \frac{m}{R}. \quad (4.2)$$

The number m is restricted to be an integer so that the quantum wave function $e^{ip \cdot x}$ is single valued. The total momentum of each string can be written as the sum of a right momentum and a left momentum

$$p^\mu = p_R^\mu + p_L^\mu, \quad (4.3)$$

where $p_{R,L}^\mu = (E, E\vec{v}, \frac{m}{2R} \pm nR)$, \vec{v} is the transverse velocity and R is the winding radius. The mode expansion of the general configuration $X(\sigma, \tau)$ in the winding direction satisfying the two-dimensional wave equation and the closed string boundary conditions can be written as the sum of right moving pieces and left moving pieces, each with the mode expansion of an open string[9]

$$\begin{aligned} X(\sigma, \tau) &= X_R(\tau - \sigma) + X_L(\tau + \sigma) \\ X_R(\tau - \sigma) &= x_R + p_R(\tau - \sigma) + \frac{i}{2} \sum_{n=0}^{\infty} \frac{1}{n} \alpha_n e^{-2in(\tau - \sigma)} \\ X_L(\tau + \sigma) &= x_L + p_L(\tau + \sigma) + \frac{i}{2} \sum_{n=0}^{\infty} \frac{1}{n} \tilde{\alpha}_n e^{-2in(\tau + \sigma)}. \end{aligned} \quad (4.4)$$

The right moving and left moving components are then essentially independent parts with corresponding vertex operators, number operators and Virasoro conditions.

The winding configuration described by $X(\sigma, \tau)$ describes a soliton string state. It is therefore a natural choice for us to compare the dynamics of these states with the solitons solutions of the previous sections in order to determine whether we can identify the solutions of the supergravity field equations with infinitely long fundamental strings. Accordingly, we study the scattering of the winding states in the limit of large winding radius.

Our kinematic setup is as follows. We consider the scattering of two straight macroscopic strings in the CM frame with winding number n and momentum number $\pm m$ [9,8]. The incoming momenta in the CM frame are given by

$$\begin{aligned} p_{1R,L}^\mu &= (E, E\vec{v}, \frac{m}{2R} \pm nR) \\ p_{2R,L}^\mu &= (E, -E\vec{v}, -\frac{m}{2R} \pm nR). \end{aligned} \tag{4.5}$$

Let $\pm m'$ be the outgoing momentum number. For the case of $m = m'$, the outgoing momenta are given by

$$\begin{aligned} -p_{3R,L}^\mu &= (E, E\vec{w}, \frac{m}{2R} \pm nR) \\ -p_{4R,L}^\mu &= (E, -E\vec{w}, -\frac{m}{2R} \pm nR), \end{aligned} \tag{4.6}$$

where conservation of momentum and winding number have been used and where $\pm\vec{v}$ and $\pm\vec{w}$ are the incoming and outgoing velocities of the strings in the transverse $x - y$ plane. The outgoing momenta winding numbers are not *a priori* equal to the initial winding numbers, but must add up to $2n$. Conservation of energy for sufficiently large R then results in the above answer. This is also in keeping with the soliton scattering nature of the problem (i.e. the solitons do not change “shape” during a collision).

For now we have assumed no longitudinal excitation ($m = m'$). We will later relax this condition to allow for such excitation, but show that our answer for the scattering is unaffected by this possibility. It follows from this condition that $v^2 = w^2$. For simplicity we take $\vec{v} = v\hat{x}$ and $\vec{w} = v(\cos\theta\hat{x} + \sin\theta\hat{y})$, and thus reduce the problem to a two-dimensional scattering problem.

As usual, the Virasoro conditions $L_0 = \tilde{L}_0 = 1$ must hold, where

$$\begin{aligned} L_0 &= N + \frac{1}{2}(p_R^\mu)^2 \\ \tilde{L}_0 &= \tilde{N} + \frac{1}{2}(p_L^\mu)^2 \end{aligned} \tag{4.7}$$

are the Virasoro operators[9] and where N and \tilde{N} are the number operators for the right- and left-moving modes respectively:

$$\begin{aligned} N &= \sum \alpha_{-n}^\mu \alpha_{n\mu} \\ \tilde{N} &= \sum \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n\mu}, \end{aligned} \tag{4.8}$$

where we sum over all dimensions, including the compactified ones. It follows from the Virasoro conditions that

$$\begin{aligned} \tilde{N} - N &= mn \\ E^2(1 - v^2) &= 2N - 2 + \left(\frac{m}{2R} + nR\right)^2. \end{aligned} \tag{4.9}$$

In the following we set $n = 1$ and consider for simplicity the scattering of tachyonic winding states. For our purposes, the nature of the string winding states considered is irrelevant. A similar calculation for massless bosonic strings or heterotic strings, for example, will be slightly more complicated, but will nevertheless exhibit the same essential behaviour. For tachyonic winding states we have $N = \tilde{N} = m = 0$. Equation (4.9) reduces to

$$E^2(1 - v^2) = R^2 - 2. \tag{4.10}$$

The Mandelstam variables (s, t, u) are identical for right and left movers and are given by

$$\begin{aligned} s &= 4 \left[\frac{(R^2 - 2)v^2}{1 - v^2} - 2 \right] \\ t &= -2 \left[\frac{(R^2 - 2)v^2}{1 - v^2} \right] (1 + \cos \theta) \\ u &= -2 \left[\frac{(R^2 - 2)v^2}{1 - v^2} \right] (1 - \cos \theta). \end{aligned} \tag{4.11}$$

It is easy to see that $p_{iR} \cdot p_{jR} = p_{iL} \cdot p_{jL}$ holds for this configuration so that the tree level 4-point function reduces to the usual Veneziano amplitude for closed tachyonic strings[8]

$$\begin{aligned} A_4 &= \frac{\kappa^2}{4} B(-1 - s/2, -1 - t/2, -1 - u/2) \\ &= \left(\frac{\kappa^2}{4}\right) \frac{\Gamma(-1 - s/2)\Gamma(-1 - t/2)\Gamma(-1 - u/2)}{\Gamma(2 + s/2)\Gamma(2 + t/2)\Gamma(2 + u/2)}. \end{aligned} \tag{4.12}$$

This can be seen as follows. In the standard computation of the four point function for closed string tachyons, we rely on the independence of the right and left moving open

strings. For the tachyonic winding state, we also separate the right and left movers with vertex operators given by $V_R = e^{ip_R \cdot x_R}$ and $V_L = e^{ip_L \cdot x_L}$ respectively to arrive at the following expression for the amplitude

$$A_4 = \frac{\kappa^2}{4} \int d\mu_4(z) \prod_{i < j} |z_i - z_j|^{p_{iR} \cdot p_{jR}} |z_i - z_j|^{p_{iL} \cdot p_{jL}}. \quad (4.13)$$

From $p_{iR} \cdot p_{jR} = p_{iL} \cdot p_{jL}$, (4.13) reduces to the expression for the four-point amplitude of a nonwinding closed tachyonic string, from which the standard Veneziano amplitude in (4.12) results.

To compare the implications of A_4 with the classical scattering of section three we take $R \rightarrow \infty$. It is convenient to define $x \equiv \frac{(R^2 - 2)v^2}{1 - v^2} = s/4 + 2$, since the Mandelstam variables can be expressed solely in terms of x and θ . We now have $A_4 = A_4(x, \theta)$, which can be explicitly written as

$$A_4 = \left(\frac{\kappa^2}{4}\right) \frac{\Gamma(3 - 2x)\Gamma(-1 + x(1 + \cos \theta))\Gamma(-1 + x(1 - \cos \theta))}{\Gamma(-2 + 2x)\Gamma(2 - x(1 + \cos \theta))\Gamma(2 - x(1 - \cos \theta))}. \quad (4.14)$$

The problem reduces to studying A_4 in the limit $x \rightarrow \infty$. We now use the identity $\Gamma(1 - a)\Gamma(a) \sin \pi a = \pi$ to rewrite A_4 as

$$A_4 = \left(\frac{\kappa^2}{4\pi}\right) \left[\frac{\Gamma(-1 + x(1 + \cos \theta))\Gamma(-1 + x(1 - \cos \theta))}{\Gamma(-2 + 2x)} \right]^2 \times \left(\frac{\sin(\pi x(1 + \cos \theta)) \sin(\pi x(1 - \cos \theta))}{\sin 2\pi x} \right). \quad (4.15)$$

From the Stirling approximation $\Gamma(u) \sim \sqrt{2\pi} u^{u-1/2} e^{-u}$ for large u , we obtain in the limit $x \rightarrow \infty$

$$A_4 \sim \left[\frac{(x(1 + \cos \theta))^{x(1 + \cos \theta)} (x(1 - \cos \theta))^{x(1 - \cos \theta)}}{(2x)^{2x}} \right]^2 \times \left(\frac{\sin(\pi x(1 + \cos \theta)) \sin(\pi x(1 - \cos \theta))}{\sin 2\pi x} \right). \quad (4.16)$$

Note that the exponential terms cancel automatically. From (4.16) we notice that the powers of x in the first factor also cancel. A_4 then reduces in the limit $x \rightarrow \infty$ to

$$A_4 \sim \left(\frac{1 + \cos \theta}{2}\right)^{2x(1 + \cos \theta)} \left(\frac{1 - \cos \theta}{2}\right)^{2x(1 - \cos \theta)} \times \left(\frac{\sin(\pi x(1 + \cos \theta)) \sin(\pi x(1 - \cos \theta))}{\sin 2\pi x} \right). \quad (4.17)$$

The poles in the third factor in (4.17) are just the usual s -channel poles. It follows from (4.17) that for $\theta \neq 0, \pi$ $A_4 \rightarrow e^{-f(\theta)x}$ as $x \rightarrow \infty$, where f is some positive definite function of θ . Hence the 4-point function vanishes exponentially with the winding radius away from the poles.

In general, for finite R and fixed v the strings may scatter into longitudinally excited final states, *i.e.* states not satisfying the above assumption that $m' = m$. The 4-point amplitude for each transition still vanishes exponentially with R . A simple counting argument shows that the total number of possible final states for a given R is bounded by a polynomial function of R . This counting argument proceeds as follows:

Without loss of generality, we may assume that our incoming states have $N = \tilde{N} = m = 0$ with fixed R and v . We relax the assumption of no longitudinal excitation to obtain outgoing states with nonzero m . We still consider $n = 1$ winding states for simplicity. Our scattering configuration can now be described by the incoming momenta

$$\begin{aligned} p_{1R,L}^\mu &= (E, E\vec{v}, \pm R) \\ p_{2R,L}^\mu &= (E, -E\vec{v}, \pm R). \end{aligned} \tag{4.18}$$

and the outgoing momenta

$$\begin{aligned} -p_{3R,L}^\mu &= (E_1, E_1\vec{w}_1, \frac{m}{2R} \pm R) \\ -p_{4R,L}^\mu &= (E_2, -E_2\vec{w}_2, -\frac{m}{2R} \pm R). \end{aligned} \tag{4.19}$$

Note that in general E_1 and E_2 are not equal to E . Without loss of generality, we take m to be positive. From conservation of momentum, however, we have

$$\begin{aligned} E_1 + E_2 &= 2E \\ E_1\vec{w}_1 &= E_2\vec{w}_2. \end{aligned} \tag{4.20}$$

It follows from the energy momentum relations for the ingoing and outgoing momenta that

$$\begin{aligned} E^2(1 - v^2) &= R^2 - 2 \\ E_1^2(1 - w_1^2) &= 2N_1 - 2 + \left(\frac{m}{2R} + R\right)^2 \\ E_2^2(1 - w_2^2) &= 2N_2 - 2 + \left(-\frac{m}{2R} + R\right)^2, \end{aligned} \tag{4.21}$$

where N_1 and N_2 are the number operators for the the right movers of the outgoing states.

Subtracting the third equation in (4.21) from the second equation and using (4.20) we obtain the relation

$$N_1 - N_2 + m = (E_1 - E_2)E. \quad (4.22)$$

From the first equation in (4.21) it follows that E is bounded by some multiple of R for fixed v . It then follows from the first equation in (4.20) that both E_1 and E_2 are bounded by a multiple of R . So from (4.22) we see that $N_1 - N_2 + m$ is bounded by some quadratic polynomial in R . We now add the last two equations in (4.21) to obtain

$$E_1^2(1 - w_1^2) + E_2^2(1 - w_2^2) = 2N_1 + 2N_2 + 2R^2 + \frac{m^2}{2R^2} - 4. \quad (4.23)$$

The left hand side of (4.23) is clearly bounded by a quadratic polynomial in R . It follows that $N_1 + N_2$ is also bounded by a quadratic polynomial, and that so are N_1 and N_2 and also, then, $N_1 - N_2$. From the boundedness of $N_1 - N_2 + m$ it therefore follows that m is bounded by a polynomial in R . Therefore the total number of possible distinct excited states (numbered by m) is bounded by a polynomial in R . The above argument also goes through for the case of a nonzero initial momentum number. For each transition, however, one can show that the Veneziano amplitude is dominated by an exponentially vanishing function of R , from a calculation entirely analogous to the zero-longitudinal excitation case worked out above. Hence the total square amplitude of the scattering (obtained by summing the square amplitudes of all possible transitions) is still dominated by a factor which vanishes exponentially with the radius, except at the poles at $\theta = 0, \pi$ corresponding to forward and backward scattering, which are physically equivalent for identical bosonic strings. This is in agreement with the trivial scattering found in section 3 and provides further evidence for the identification of the solitonic string solution found in [1] with the fundamental string.

The above argument can be repeated for any other type of string, including the heterotic string[10]. The kinematics differ slightly from the tachyonic case but the 4-point function is still dominated by an exponentially vanishing factor in the large radius limit. Hence the scattering is trivial, again in agreement with the result found in section 3.

5. Metric on Moduli Space in $D = 4$

In the low-velocity limit, multi-soliton solutions trace out geodesics in the static solution manifold, with distance defined by the Manton metric on moduli space manifold [7]. In the absence of a full time-dependent solution to the equations of motion, these geodesics represent a good approximation to the low-energy dynamics of the solitons. For BPS monopoles, the Manton procedure was implemented by Atiyah and Hitchin [11,12].

In this section we compute the Manton metric on moduli space for the scattering of the soliton string solutions in $D = 4$ although we expect that the same result will hold for arbitrary $D \geq 4$. We find that the metric is flat to lowest nontrivial order in the string tension. This result implies trivial scattering of the string solitons and is consistent with the results of the previous two sections, and thus provides even more compelling evidence for the identification of the string soliton with the underlying fundamental string.

We first return to the solution of [1]. For $D = 4$, $\phi = E$ and the metric simplifies to

$$ds^2 = -dt^2 + (dx^1)^2 + e^{-E} d\vec{x} \cdot d\vec{x}. \quad (5.1)$$

Manton's procedure may be summarized as follows. We first invert the constraint equations of the system (Gauss' law for the case of BPS monopoles). The corresponding time dependent field configuration does not in general satisfy the time dependent field equations, but provides an initial data point for the fields and their time derivatives. Another way of saying this is that the initial motion is tangent to the set of exact static solutions. The resultant kinetic action obtained by replacing the solution to the constraints in the action defines a metric on the parameter space of static solutions. This metric defines geodesic motion on the moduli space[7].

We now assume that each string source possesses velocity $\vec{\beta}_n, n = 1, \dots, N$ in the two-dimensional transverse space (23). This will appear in the contribution of the sigma-model source action to the equations of motion in the form of "moving" $\delta^{(2)}(\vec{x} - \vec{a}_n(t))$, where $\vec{a}_n(t) \equiv \vec{A}_n + \vec{\beta}_n t$ (here \vec{A}_n is the initial position of the n th string source).

The equations of motion following from S are complicated and nonlinear (see [1]), and it is remarkable that such a simple ansatz as section 2 could provide a solution in the static limit. In the time dependent case, we are even less likely to be so fortunate. In order to make headway in solving even the $O(\beta)$ time dependent constraints, we make the simplifying assumption that $8G\mu \ll 1$ (this is equivalent to assuming that each cosmic

string produces a small deficit angle). It turns out that to linear order in μ an $O(\beta)$ solution to the constraint equations is given by

$$\begin{aligned}
e^{-E(\vec{x},t)} &= 1 - 8G\mu \sum_{n=1}^N \ln(\vec{x} - \vec{a}_n(t)) \\
g_{00} &= -g_{11} = -1, \quad g^{00} = -g^{11} = -1 \\
g_{ij} &= e^{-E}\delta_{ij}, \quad g^{ij} = e^E\delta_{ij} \\
g_{0i} &= 8G\mu \sum_{n=1}^N \vec{\beta}_n \cdot \hat{x}_i \ln(\vec{x} - \vec{a}_n(t)), \quad g^{0i} = e^E g_{0i} \\
H_{10j} &= \partial_j e^E \\
H_{1ij} &= \partial_i g_{0j} - \partial_j g_{0i},
\end{aligned} \tag{5.2}$$

where $i, j = 2, 3$.

The kinetic Lagrangian is obtained by replacing the expressions for the fields in (5.2) in S . Since (5.2) is a solution to order β , the leading order terms in the action (after the quasi-static part) is of order β^2 . The source part of the action (S_2) now represents moving string sources, and its only contribution to the kinetic Lagrangian density is of the form $(\mu/2)\beta_n^2$ for each source. The nontrivial elements of the metric on moduli space must therefore be read off from the $O(\beta^2)$ part of the massless fields effective action. In principle one must add a Gibbons-Hawking surface term (GHST) in order to cancel the double derivative terms in S (see [13–17]). In this case, however, the GHST vanishes to $O(\beta^2)$. To lowest nontrivial order in μ , the kinetic Lagrangian density is computed to be

$$\mathcal{L}_{kin} = \frac{1}{2\kappa^2} \left(2\dot{E}^2 - (\partial_m g_{0k})^2 \right). \tag{5.3}$$

Henceforth we simplify to the case of two strings with velocities $\vec{\beta}_1$ and $\vec{\beta}_2$ and positions \vec{a}_1 and \vec{a}_2 . Let $\vec{X}_n \equiv \vec{x} - \vec{a}_n, n = 1, 2$. Our moduli space consists of the configuration space of the relative separation vector $\vec{a} \equiv \vec{a}_2 - \vec{a}_1$. We now compute the metric on moduli space by integrating (5.3) over the (23) space. It turns out that the self-terms vanish on integration over the two-space. We are then left with the interaction terms, which may be written as

$$\mathcal{L}_{int} = \frac{64G^2\mu^2}{\kappa^2} \left[\frac{2(\vec{\beta}_1 \cdot \vec{X}_1)(\vec{\beta}_2 \cdot \vec{X}_2)}{X_1^2 X_2^2} - \frac{(\vec{\beta}_1 \cdot \vec{\beta}_2)(\vec{X}_1 \cdot \vec{X}_2)}{X_1^2 X_2^2} \right]. \tag{5.4}$$

The most general answer obtained by integrating (5.4) over the transverse two-space is of the form

$$L_{int}(\vec{a}) = 2f(a)\vec{\beta}_1 \cdot \vec{\beta}_2 + 2g(a)(\vec{\beta}_1 \cdot \hat{a})(\vec{\beta}_2 \cdot \hat{a}). \tag{5.5}$$

We compute f and g by integrating (5.4) for only two configurations. In both cases, $\vec{\beta}_1$ is parallel to $\vec{\beta}_2$. The first case has the velocities parallel to \vec{a} and yields

$$L_{int}(a) = (2f + 2g)\beta_1\beta_2 \quad (5.6)$$

while the second case has the velocities perpendicular to \vec{a} and yields

$$L_{int}(a) = 2f\beta_1\beta_2. \quad (5.7)$$

In this way we can compute both f and g . A slightly tedious but straightforward computation yields

$$g = -2f = -\frac{64g^2\mu^2\pi}{\kappa^2} \left(1 - \frac{\ln 2}{2}\right), \quad (5.8)$$

and thus all the metric elements are constant. In two-dimensions, this implies that the metric on moduli space is flat (being of the form $dr^2 + Ar^2d\theta^2$, where A is a constant), and therefore has straight-line geodesics in the static solution manifold. To this approximation, then, the low-energy scattering is trivial, i.e. the strings do not deviate from their initial trajectories.

6. Discussion

In this section we first summarize the results obtained in this paper and then discuss their physical implications. In section 2 we outlined Dabholkar *et al.*'s multi-string soliton solutions of the $D = 10$ supergravity super Yang-Mills field equations. These solitons resemble multi-BPS monopole solutions in that their existence derives from a zero-force condition. Other similarities include the saturation of a Bogomolnyi bound. The zero force condition for parallel string solitons with the same orientation arises as a result of the cancellation of long-range forces of exchange of the massless modes of the string (graviton, axion and dilaton), just as the zero force condition for equally charged monopoles results from the cancellation of the attractive coulomb force resulting from scalar (Higgs) exchange with the repulsive coulomb force resulting from vector (gauge) exchange. The force cancellation was seen from the test string approximation, which examines the force on a static test soliton in the background of the fields produced by a source soliton.

We used the test-string approach in the next section to study the dynamics of the string solitons. In particular, we considered the motion of a test string in the background of a source string. Surprisingly, we found that the velocity dependent forces vanish as well.

Were we to extrapolate this result to soliton-soliton scattering, we would expect trivial scattering (i.e. no deviation from initial scattering trajectories).

In section 4 we approached the string soliton scattering from a string-theoretic (vertex operator) point of view. If the string soliton solutions of Dabholkar *et al.* are to be identified with infinitely long fundamental strings, the result of section 3 should agree with the corresponding Veneziano amplitude calculation of fundamental string scattering. We performed this computation for the scattering of two macroscopic winding state strings in the large R limit and once again found trivial scattering.

We returned in section 5 to the study of the dynamics of string solitons with a computation of the Manton metric on moduli space for these solutions. This metric describes the geodesics traced out by the multi-soliton solutions in the static solution manifold in the low velocity limit. A calculation of this metric in this section for the case $D = 4$ to lowest nontrivial order in the string tension yielded a flat metric, which also implies trivial scattering.

The agreement between the results of sections 3 and 5 on the one hand and that of section 4 provides compelling dynamical evidence for the identification of the string soliton with the underlying fundamental string. It is therefore likely that these solitons can be used to describe extended string states in semi-classical string theory, in much the same way that solitons in ordinary field theory are used to describe extended particle states. We are especially interested in discovering inherently “string-like” solutions, whose behaviour differs from already known configurations in field theory and which will give us a better understanding of string-theoretic effects in spacetime at the Planck scale. The solutions studied here seem to exhibit rather surprising behaviour (trivial scattering). This dynamic behaviour differs markedly from that of the magnetic monopole, with which the string solitons share several static features, such as a zero static force condition and the saturation of a Bogomolnyi bound. In this scenario, it seems that in the low-energy limit the soliton strings also obey a zero dynamical force.

It is therefore likely that a further study of these and related solutions (such as the fivebrane solutions in [18–20] and their exact extensions in [21–23]) in string theory will lead us eventually to a better understanding of nonperturbative string theory.

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