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Observer-based stabilization of one-sided Lipschitz systems with application to flexible link manipulator

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Abstract

This article is concerned with the observer-based output feedback stabilization problem for a class of nonlinear systems that satisfies the one-sided Lipschitz and the quadratically inner-bounded conditions. The system model under consideration encompasses the classical Lipschitz nonlinear system as a special case. For such a system, we design the output feedback controller via constructing a full-order Luenberger-type state observer. Sufficient conditions that guarantee the existence of observer-based output feedback are established in the form of linear matrix inequalities, which are readily solved by the available numerical software. Moreover, the proposed observer-based output feedback designs are applied to a flexible link manipulator system. Finally, simulation study on the manipulator system is given to demonstrate the effectiveness of the developed control design.

Keywords

Observer-based stabilization, one-sided Lipschitz systems, output feedback, flexible link manipulator

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Introduction

In the design of feedback control systems, the knowledge of system state plays a key role. However, in engineering practice it may be quite difficult, sometimes even impossible, to directly measure all the system state variables through sensors.[1] In those situations, a state observer is usually needed, and then the so-called observer-based control can be carried out using the estimated state.[2–4] For linear systems, the observer-based control is readily achieved due to the *separation principle*. However, for nonlinear systems, the observer-based control problem becomes quite difficult. In fact, for a general nonlinear system, the state estimation by itself is still an open problem. Therefore, the observer design associated with the observer-based control problems of nonlinear systems has received considerable attention in the past two decades; see, for example, Maurice et al.,[5] Rajamani,[6] Zhu and Han,[7] Talole et al.,[8] Shaker and Tahavori,[9] Shaker and How,[10] Raghavan and Hedrick[11] and Song and Hedrick,[12] and the references therein.

It is known that for a general nonlinear system, the so-called separation principle may not be valid. As a consequence, the observer-based stabilization for general nonlinear plants becomes quite challenging. Therefore, in recent years, many research efforts are mainly focused on some special kinds of

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nonlinearities.[6–12] For instance, the classical Lipschitz nonlinear system was popularly studied. Indeed, in practice many physical systems globally or locally are Lipschitz continuous. Till now, various effective methods have been provided to study the design of Lipschitz state observers and the associated observer-based output feedback control issue, for example, the Riccati equation–based approach developed in Rajamani,[6] Zhu and Han,[7] Talole et al.,[8] Shaker and Tahavori,[9] Shaker and How,[10] and Raghavan and Hedrick[11] and the linear matrix inequality (LMI)-based output feedback design proposed in Kheloufi et al.,[2] Lien,[4] and Song and Hedrick.[12]

More recently, the one-sided Lipschitz condition has been introduced in the design of nonlinear state observer.[13–16] It is shown that the one-sided Lipschitz encompasses a large class of nonlinearities and can reduce conservatism in the existing designs. Abbaszadeh and Marquez[17] further studied the state estimation problem of one-sided Lipschitz systems by introducing an additional restrict called quadratically inner-bounded condition. For such systems, less conservative designs on both full-order and reduced-order observers have been considered by Zhang et al.[18,19] where the Riccati equation and the LMI approaches were, respectively, introduced. In the discrete-time case, the state estimation issue was investigated in Zhang et al.[20] Moreover, the observer design for such systems with unknown inputs was addressed in Zhang et al.[21] However, it should be noted that most of the above-mentioned references are focused on the observer deign issue. A more challenging problem is the observer-based stabilization or output feedback control for one-sided Lipschitz systems. This also motivates this study.

In this article, based on Lyapunov stability theory,[22] we consider the observer-based stabilization for a general class of nonlinear systems satisfying the one-sided Lipschitz and the quadratically inner-bounded conditions. In the existing literature, to the best of our knowledge, only Fu et al.[23] address the similar problem. However, Fu et al.[23] actually used the one-sided Lipschitz property defined in Hu,[13] which is scaled by a positive definite matrix P. How to verify that property together with observer synthesis conditions is still an open problem.[13] In our study, we assume that the system nonlinear function satisfies the standard one-sided Lipschitz condition and is quadratically inner-bounded.[17] On the other hand, to design the observer-based controller, the challenge is to analyze the Lyapunov stability of closed-loop systems.[9] Generally, one needs to solve a kind of bilinear matrix inequalities (BMIs).[12] Many previous studies on observer-based stabilization were designed by applying the obtained BMI conditions. As we know, solving the BMIs is nondeterministic polynomial-time hard (NP-hardness),[24] and up to now there is no effective algorithm to numerically solve a BMI problem (see, for example, Kheloufi et al.[2] and Boyd et al.[25]). Recently, in order to overcome this limitation, Kheloufi et al.[2] have developed a less conservative LMI approach for a class of linear uncertain systems. Indeed, to solve observer-based output feedback design of such systems, they employed the famous *Young's relation*[25] to develop the BMI and then transformed it into an LMI form.[2]

In this study, we also employ Young's relation to deduce the observer-based output feedback controller synthesis conditions for nonlinear systems that satisfy the one-sided Lipschitz condition and quadratically inner-bounded condition. The systems under consideration include the classical Lipschitz nonlinear systems as special cases. Motivated by the methods proposed in Kheloufi et al.[2] and Zhang et al.,[18] we formulate the observer-based controller design issue for such systems into solving a set of LMIs, which are readily tractable via numerical software. Moreover, as the application of observer-based control design, we use the developed stabilization approach to a flexible link manipulator system. The simulation results on the manipulator system are also provided to demonstrate the effectiveness of the proposed observer-based stabilization approach.

We shall use the following notation in this article: $\mathbb{R}^n$ represents the n-dimensional vector space, $\mathbb{R}^{m\times n}$ denotes the space of all m by n real matrices, and A^T indicates the transpose of a matrix A. The inner product of vectors $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y\rangle$, that is, $\langle x, y\rangle = x^T y$. $\|x\|$ stands for the Euclidean norm of vector x. A positive definite matrix P is denoted by $P>0$. The symbol $*$ stands for the symmetric term in a matrix. I_n represents the identity matrix of dimension n.

Problem statement and preliminaries

Consider a class of continuous-time nonlinear systems described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + f(x(t)) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ are the state vector, the control input, and the measured output of the system, respectively. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are the known constant matrices. The vector-valued function $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ represents the nonlinearity of the system. Throughout this article, without loss of generality, we assume that $f(0) = 0$, which implies that the unforced system (i.e. $u(t)\equiv 0$) has the origin as an equilibrium point.

In what follows, we shall recall the Lipschitz property and the one-sided Lipschitz property of the vector-valued function $f(x)$, respectively. The quadratic inner-boundedness condition[17] is also introduced, which is very helpful for obtaining LMI-type synthesis conditions.

Definition 1. The vector-valued function $f(x)$ is said to be *Lipschitz* if there exists a constant $\alpha > 0$ such that[17]

$$\|f(x_1) - f(x_2)\| \leq \alpha \|x_1 - x_2\| \tag{2}$$

where α is called the *Lipschitz constant*.

Definition 2. The vector-valued function $f(x)$ is said to be *one-sided Lipschitz* if there exists a constant $\rho \in \mathbb{R}$ such that[17]

$$\langle f(x_1) - f(x_2), x_1 - x_2 \rangle \leq \rho \|x_1 - x_2\|^2 \tag{3}$$

where $\rho \in \mathbb{R}$ is called the *one-sided Lipschitz constant*.

Definition 3. The vector-valued function $f(x)$ is said to be *quadratic inner-boundedness* if there exist two constants $\beta, \gamma \in \mathbb{R}$ such that[17]

$$\|f(x_1) - f(x_2)\|^2 \leq \beta \|x_1 - x_2\|^2 + \gamma \langle x_1 - x_2, f(x_1) - f(x_2) \rangle \tag{4}$$

It is worthwhile to point out that the one-sided Lipschitz constant ρ can be positive, zero, or even negative, while the Lipschitz constant α must be positive. Additionally, the one-sided Lipschitz constants can be chosen much smaller than the Lipschitz constants.[13–17] Moreover, if a function is Lipschitz, it is both one-sided Lipschitz and quadratically inner-bounded; however, the converse is not ture.[17] Therefore, the nonlinear plant considered in our work encompasses the Lipschitz nonlinear system as a special case (see Figure 1).

In this article, we focus our attention on the observer-based output feedback control for system (1) as well as its application to a flexible link manipulator system. In the next section, we will consider the observer-based stabilization design. We end this section by introducing the following two useful Lemmas.

Lemma 1. For two matrices X and Y with appropriate dimensions and a scalar $\varepsilon > 0$, we have the following inequality[27]

$$XY + Y^T X^T \leq \varepsilon X X^T + \varepsilon^{-1} Y^T Y \tag{5}$$

Lemma 2 (the Schur complement Lemma; see, for example, Boyd et al.[25]). For a real symmetric matrix Q, the following statements are equivalent:

1. $Q := \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} < 0.$
2. $Q_{11} < 0$, and $Q_{22} - Q_{12}^T Q_{11}^{-1} Q_{12} < 0.$
3. $Q_{22} < 0$, and $Q_{11} - Q_{12} Q_{22}^{-1} Q_{12}^T < 0.$

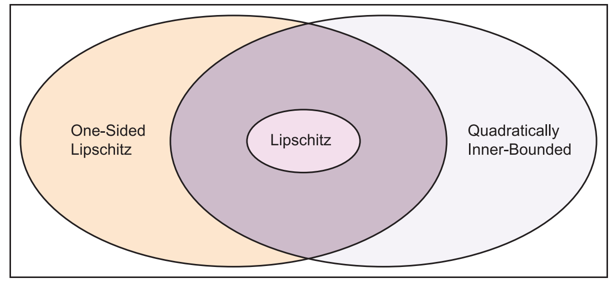

Figure 1. The Lipschitz, one-sided Lipschitz, and quadratically inner-bounded function sets.[26]

Observer-based stabilization design

In this section, we address the observer-based stabilization problem for system (1) under conditions (3) and (4). As usual, we first employ the known Luenberger-like observer to estimate the state and then use the estimated state to design a linear output feedback. More precisely, we propose an observer-based controller as follows

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + f(\hat{x}(t)) + K(y - C\hat{x}(t)) \\ u(t) = F\hat{x}(t), \quad x(0) = \hat{x}_0 \end{cases} \tag{6}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$, $\hat{x}(0) = \hat{x}_0$ is the initial value of the estimate, $K \in \mathbb{R}^{n \times p}$ is the state observer gain, and $F \in \mathbb{R}^{m \times n}$ is the output feedback gain. Here, K and F are the two real matrices to be determined later.

Denote the estimation error by $e(t) := x(t) - \hat{x}(t)$. From equations (1) and (6), we have

$$\dot{e}(t) = (A - KC)e(t) + f - \hat{f} \tag{7}$$

where $f := f(x)$ and $\hat{f} := f(\hat{x})$. Moreover, with equation (6), system (1) becomes

$$\dot{x} = (A + BF)x - BFe + f \tag{8}$$

In view of equations (7) and (8), the closed-loop system can be rewritten as

$$\dot{\overbrace{\begin{bmatrix} x \\ e \end{bmatrix}}} = \begin{bmatrix} A + BF & -BF & I & 0 \\ 0 & A - KC & 0 & I \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \tag{9}$$

where $\tilde{f} := f - \hat{f}$.

For system (9), let us consider the following Lyapunov function candidate

$$V(x, e) = \begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} = x^T P x + e^T \mathrm{Re} \tag{10}$$

Consequently, calculating the derivative of V along the state trajectories of equation (9) gives

$$\dot{V}(x,e) = \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \underbrace{\begin{bmatrix} \Sigma_{11} & -PBF & P & 0 \\ * & \Sigma_{22} & 0 & R \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}}_{\Sigma} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \tag{11}$$

where

$$\Sigma_{11} = (A + BF)^T P + P(A + BF)$$
$$\Sigma_{22} = (A - KC)^T R + R(A - KC)$$

Notice that $\dot{V}(x,e) \leq 0$ if the matrix inequality $\Sigma \leq 0$ holds. However, as pointed out in Kheloufi et al.,[2] the matrix inequality $\Sigma \leq 0$ is a BMI since it involves the PBF term. As previously mentioned, up to now there is no efficient numerical algorithm to solve the BMI problem. In the recent literature, many research efforts have been provided to overcome this problem. Inspired by Kheloufi et al.,[2] here we address the observer-based stabilization problem for system (1) by employing Young's relation. More precisely, we have the following conclusion.

Theorem 1. Consider system (1) satisfying the conditions (3) and (4). Let the observer-based output feedback controller be constructed in the form of equation (6). Then the closed-loop system (9) is asymptotically stable if there exist matrices $Q>0$, $R>0$, $\hat{K}$, and $\hat{F}$ with appropriate dimensions and scalars $\varepsilon_1>0$, $\varepsilon_2>0$, $\varepsilon_3>0$, $\varepsilon_4>0$, and $\phi_1>0$ such that

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix} < 0 \tag{12}$$

where

$$S_1 = \begin{bmatrix} \tilde{\Sigma}_{11} & 0 & I & 0 \\ * & \tilde{\Sigma}_{22} & 0 & R + (\varepsilon_4\gamma - \varepsilon_3)I \\ * & * & -2\varepsilon_2 I & 0 \\ * & * & * & -2\varepsilon_4 I \end{bmatrix} \tag{13}$$

$$S_2 = \begin{bmatrix} -B\hat{F}^T & 0 & Q & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

$$S_3 = diag\left\{-\frac{Q}{\phi_1}, -\phi_1 Q, -\psi(\varepsilon_1,\varepsilon_2)I, -\zeta(\varepsilon_1,\varepsilon_2)I\right\} \tag{15}$$

$$\tilde{\Sigma}_{11} = QA^T + AQ + \hat{F}B^T + B\hat{F}^T$$

$$\tilde{\Sigma}_{22} = A^T R + RA - C^T\hat{K} - \hat{K}^T C + 2(\varepsilon_3\rho + \varepsilon_4\beta)I$$

$$\psi(\varepsilon_1,\varepsilon_2) = \frac{1}{[(2\rho - 1)\varepsilon_1 + (2\beta + \gamma)\varepsilon_2]}$$

$$\zeta(\varepsilon_1,\varepsilon_2) = \frac{1}{(\varepsilon_2\gamma - \varepsilon_1)}$$

Furthermore, the resulting observer gain matrix K and the output feedback gain matrix F are, respectively, given by $K = R^{-1}\hat{K}^T$ and $F = \hat{F}^T Q^{-1}$.

Proof. Consider the closed-loop system (9) with equations (3) and (4). Based on Lyapunov stability theory, we will prove that equation (9) is asymptotically stable if the conditions of Theorem 1 are satisfied. To begin with, let the Lyapunov function candidate $V(x,e)$ be defined in the form of equation (10).

Notice that $f(0) = 0$. Then from conditions (3) and (4), for any positive scalars ε_1 and ε_2, we can obtain

$$\varepsilon_1 \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} 2\rho I & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \geq 0 \tag{16}$$

and

$$\varepsilon_2 \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} 2\beta I & 0 & \gamma I & 0 \\ 0 & 0 & 0 & 0 \\ \gamma I & 0 & -2I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \geq 0 \tag{17}$$

Similarly, from condition (4), we get

$$\varepsilon_3 \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\rho I & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \geq 0 \tag{18}$$

and

$$\varepsilon_4 \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\beta I & 0 & \gamma I \\ 0 & 0 & 0 & 0 \\ 0 & \gamma I & 0 & -2I \end{bmatrix} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \geq 0 \tag{19}$$

where ε_3 and ε_4 are the two positive scalars. Consequently, adding the left sides of equations (16)–(19) to the right side of equation (11) gives

$$\dot{V} \leq \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix}^T \underbrace{\begin{bmatrix} \Sigma_{11} + \eta_1 I & -PBF & P + \eta_2 I & 0 \\ * & \bar{\Sigma}_{22} & 0 & R + \eta_3 I \\ * & * & -2\varepsilon_2 I & 0 \\ * & * & * & -2\varepsilon_4 I \end{bmatrix}}_{\bar{\Sigma}} \begin{bmatrix} x \\ e \\ f \\ \tilde{f} \end{bmatrix} \tag{20}$$

where

$$\bar{\Sigma}_{22} = A^T R + RA - C^T\hat{K} - \hat{K}^T C + 2(\varepsilon_3\rho + \varepsilon_4\beta)I$$

$$\hat{K} = K^T R, \quad \eta_1 = 2(\varepsilon_1\rho + \varepsilon_2\beta)$$

$$\eta_2 = \varepsilon_2\gamma - \varepsilon_1, \quad \eta_3 = \varepsilon_4\gamma - \varepsilon_3$$

From equation (20), we know that $\dot{V} < 0$ if the condition $\bar{\Sigma} < 0$ holds.

Let us define $Q := P^{-1}$. Pre- and post- multiplying $\bar{\Sigma} < 0$ by matrix $diag(Q, I, I, I)$ yields

$$\tilde{\Sigma} = \begin{bmatrix} \tilde{\Sigma}_{11} + \eta_1 QQ & -BF & I + \eta_2 Q & 0 \\ * & \tilde{\Sigma}_{22} & 0 & R + \eta_3 I \\ * & * & -2\varepsilon_2 I & 0 \\ * & * & * & -2\varepsilon_4 I \end{bmatrix} < 0 \tag{21}$$

where

$$\tilde{\Sigma}_{11} = QA^T + AQ + \hat{F}B^T + B\hat{F}^T, \quad \hat{F} = QF^T, \quad \tilde{\Sigma}_{22} = \bar{\Sigma}_{22}$$

Consequently, by developing $\tilde{\Sigma} < 0$, we get

$$\tilde{\Sigma} = \hat{\Sigma} + \eta_1 \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} -BF \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -BF \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \eta_2 \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T + \eta_2 \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0$$

where

$$\hat{\Sigma} = \begin{bmatrix} \tilde{\Sigma}_{11} & 0 & I & 0 \\ * & \tilde{\Sigma}_{22} & 0 & R + \eta_3 I \\ * & * & -2\varepsilon_2 I & 0 \\ * & * & * & -2\varepsilon_4 I \end{bmatrix}$$

Using Lemma 1, we get the following inequality

$$\tilde{\Sigma} \leq \hat{\Sigma} + \begin{bmatrix} -BF^T \\ 0 \\ 0 \\ 0 \end{bmatrix} (\phi_1 Q) \begin{bmatrix} -BF^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} (\phi_1 Q)^{-1} \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}^T + \eta_1 \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \eta_2 \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \eta_2 \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T < 0 \tag{22}$$

where $\phi_1 > 0$ is a positive scalar.

Next, in order to retrieve the variable QF^T, we apply Young's relation given in Boyd et al.[25] Notice that $\hat{F} = QF^T$. Consequently, from equation (22) we get the following inequality

$$\tilde{\Sigma} \leq \hat{\Sigma} - \begin{bmatrix} -B\hat{F}^T \\ 0 \\ 0 \\ 0 \end{bmatrix} (-\phi_1 Q^{-1}) \begin{bmatrix} -B\hat{F}^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} \left(-\frac{Q^{-1}}{\phi_1}\right) \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}^T - (-\eta_1 - \eta_2) \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} Q \\ 0 \\ 0 \\ 0 \end{bmatrix}^T - (-\eta_2) \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T < 0 \tag{23}$$

Notice that the right side of equation (23) has the following form

$$S_1 - S_2 S_3^{-1} S_2^T < 0 \tag{24}$$

provided that $\eta_1 + \eta_2 > 0$, $\eta_2 > 0$, $Q^{-1} > 0$, and $\phi_1 > 0$, where S_1, S_2, and S_3 are defined by equations (13)–(15), respectively. By Lemma 2, we know that equation (24) is satisfied if the matrix inequality (12) has a feasible solution. This completes the proof.

It should be mentioned that equation (12) is not a standard LMI form because in its blocks there exist some terms like Q/ϕ_1 and $\phi_1 Q$. Next, we will discuss how to formulate equation (12) into an LMI using a similar technique as used in Kheloufi et al.[2]

Remark 1. In order to transfer the matrix inequality (12) into an LMI form, we need to make a suitable choice of ϕ_1. We can employ the following additional constraints, that is, $Q > \alpha_1 I$ and $-I/\phi_1 \leq -(2 - \phi_1)I$. Thus, equation (12) can be formulated into an LMI with respect to α_1 and $\beta_1 := \alpha_1\phi_1$, where $\alpha_1 > 0$ is a positive scalar. In fact, we have

$$-\frac{Q}{\phi_1} \leq -(2 - \phi_1)Q \leq -(2 - \phi_1)\alpha_1 I = -(2\alpha_1 - \beta_1)I \tag{25}$$

and

$$-\phi_1 Q \leq -\phi_1\alpha_1 I = -\beta_1 I \tag{26}$$

Hence, equations (25) and (27) can replace the blocks Q/ϕ_1 and $\phi_1 Q$ in equation (23) by $(2\alpha_1 - \beta_1)I$ and $\beta_1 I$ with $2\alpha_1 - \beta_1 > 0$ and $\beta_1 > 0$, respectively.

Corollary 1. Consider system (1) satisfying the Lipschitz condition (2). Let the observer-based output feedback controller be constructed in the form of equation (6). Then the closed-loop system (9) is asymptotically stable if there exist some matrices $Q > 0$, $R > 0$, $\hat{K}$, and $\hat{F}$ with appropriate dimensions and scalars $\mu_1 > 0$, $\mu_2 > 0$, and $\phi_2 > 0$, such that

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_3 \end{bmatrix} < 0 \tag{27}$$

where

$$W_1 = \begin{bmatrix} W_{11} & 0 & I & 0 \\ * & W_{22} & 0 & R \\ * & * & -\mu_1 I & 0 \\ * & * & * & -\mu_2 I \end{bmatrix} \tag{28}$$

$$W_2 = \begin{bmatrix} Q & -B\hat{F}^T & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{29}$$

$$W_3 = diag\left\{ -\frac{I}{(\mu_1 \alpha^2)}, -\frac{Q}{\phi_2}, -\phi_2 Q \right\} \tag{30}$$

$$W_{11} = QA^T + AQ + \hat{F}B^T + B\hat{F}^T \tag{31}$$

$$W_{22} = A^T R + RA - C^T \hat{K} - \hat{K}^T C + \mu_2 \alpha^2 I \tag{32}$$

Furthermore, the resulting observer gain matrix K and the output feedback control gain matrix F are, respectively, given by $K = R^{-1}\hat{K}^T$ and $F = \hat{F}^T Q^{-1}$, respectively.

It should be mentioned that there exist some terms like Q/ϕ_2, $\phi_2 Q$, $\mu_1 I$, and $I/(\mu_1\alpha^2)$ in condition (27). Hence, equation (27) is not a standard LMI. Then how to check this condition is not an easy task. Here, we provide Remarks 2 and 3 to address this problem.

Remark 2. Using a similar technique as in Remark 1, we can formulate equation (27) into an LMI. Indeed, if $Q > \alpha_2 I$ and $\beta_2 = \alpha_2\phi_2$, we can replace the blocks Q/ϕ_2 and $\phi_2 Q$ in equation (27) by $(2\alpha_2 - \beta_2)I$ and $\beta_2 I$, respectively.

Remark 3. In order to choose a suitable μ_1 in equation (27), we can further assume that

$$-\frac{1}{\mu_1} < -\xi \tag{33}$$

This implies that $\mu_1 < 1/\xi$. On the other hand, notice that μ_1 is a small positive scalar. Thus, if we further assume $0 < \mu_1 < 1$, then it follows that $0 < \xi < 1$. Hence, the block $-I/\mu_1\alpha^2$ in equation (27) can be replaced by $-\xi I/\alpha^2$, where $0 < \xi < 1$. Therefore, based on the above discussions, we can formulate equation (27) into an LMI.

Application to flexible link manipulator

In this section, we study the application of the observer-based output feedback controller proposed in section "Observer-based stabilization design." Consider a one-link flexible joint manipulator actuated by a direct current (DC) motor, whose dynamics can be described as follows (see, for example, Zhu and Han[7] and Raghavan and Hedrick[11])

Table 1. Manipulator parameters.

Manipulator parameter (units)	Value
Motor inertia, I_m (kg m^2)	3.7×10^{-3}
Link inertia, I_ℓ (kg m^2)	9.3×10^{-3}
Pointer mass, m (kg)	2.1×10^{-1}
Torsional spring constant, k (N m/rad)	1.8×10^{-1}
Viscous friction coefficient, C_{vf} (N m/V)	4.6×10^{-2}
Amplifier gain, K_τ (N m/V)	8.0×10^{-2}

$$\begin{cases} \dot{\theta}_m = \omega_m \\ \dot{\omega}_m = \dfrac{k}{I_m}(\theta_\ell - \theta_m) - \dfrac{C_{vf}}{I_m}\omega_m + \dfrac{K_\tau}{I_m}u \\ \dot{\theta}_\ell = \omega_\ell \\ \dot{\omega}_\ell = -\dfrac{k}{I_\ell}(\theta_\ell - \theta_m) - \dfrac{mgh}{I_m}\sin(\theta_\ell) \end{cases} \tag{34}$$

where I_m denotes the motor inertia and I_ℓ denotes the link inertia; θ_m and θ_ℓ are, respectively, the rotation angles of the motor and the link, while $\dot{\theta}_m$ and $\dot{\theta}_\ell$ stand for their angular velocities; $k, K_\tau, m, g, h > 0$ are the constants (see Table 1).

For such a manipulator system, we can easily measure the position and velocity of the motor, but the other states are difficult to measure directly. In practice, usually we need to design a state observer first and then use the estimated state to design the output feedback controller. Using the parameters in Table 1, system (34) can be rewritten in the form of equation (1) with

$$x = \begin{bmatrix} \theta_m \\ \omega_m \\ \theta_\ell \\ \omega_\ell \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.26 & 48.6 & 0 \\ 0 & 0 & 0 & 10 \\ 1.95 & 0 & -1.95 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix} \tag{35}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad f(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.333\sin(x_1) \end{bmatrix} \tag{36}$$

In this case, it is easy to verify that conditions (3) and (4) are satisfied with $\rho = \beta = 0.1109$ and $\gamma = 0$. Consequently, we can apply Theorem 1 to design the observer-based output feedback controller (9). After

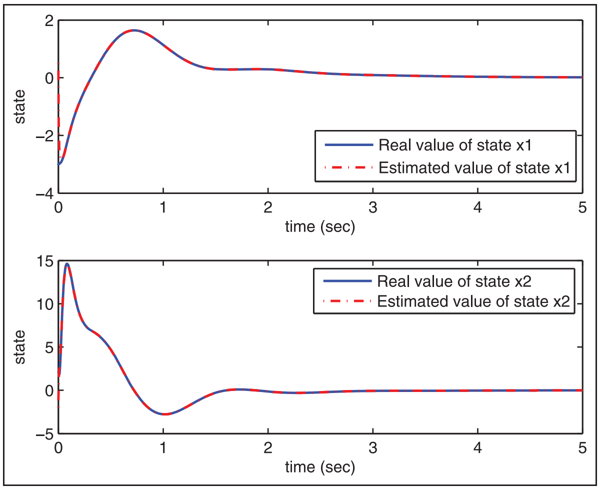

Figure 2. Simulation for states x_1 and x_2 and their estimate.

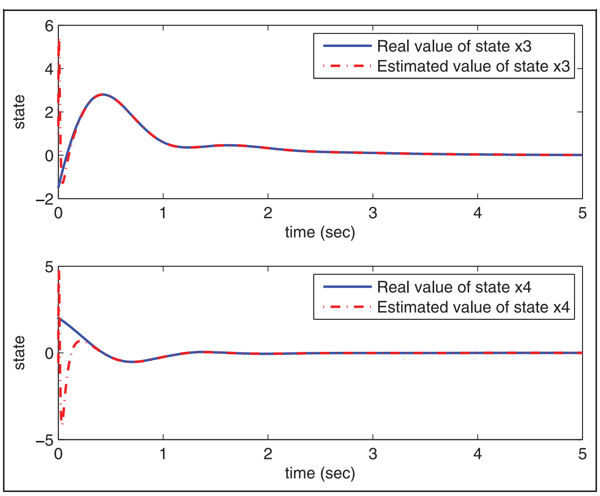

Figure 3. Simulation for states x_3 and x_4 and their estimate.

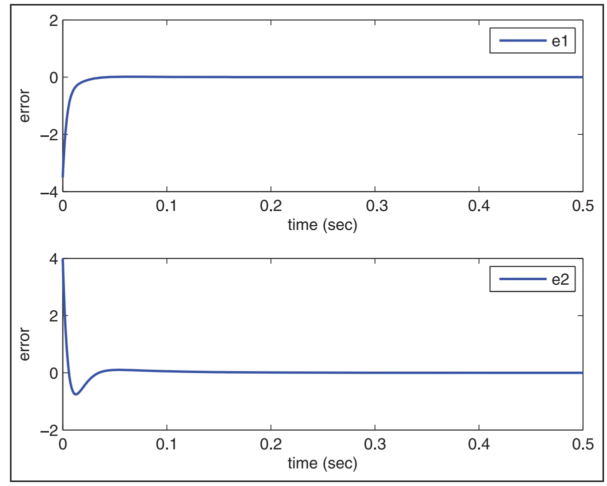

Figure 4. Simulation for the estimation errors of x_1 and x_2.

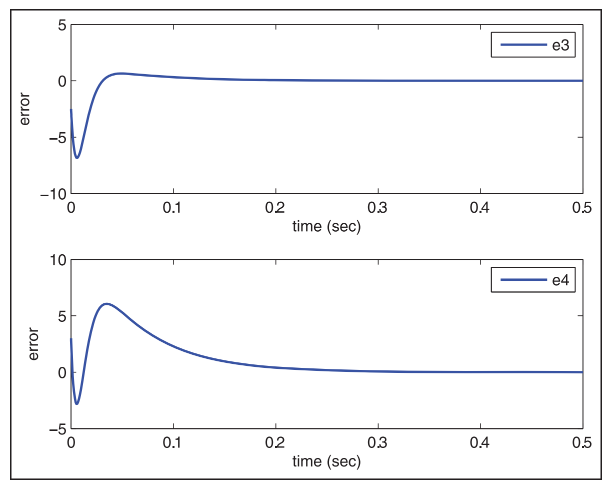

Figure 5. Simulation for the estimation errors of x_3 and x_4.

solving equation (12) via the MATLAB LMI control toolbox, we can obtain the observer gain K and the output feedback gain F as follows

$$K = \begin{bmatrix} 1.2907 & -23.0311 & 1.3713 & 5.8967 \\ -24.3525 & 2.6718 & 53.2993 & 6.7679 \end{bmatrix}^T \tag{37}$$

and

$$F = [\,-11.7561 \quad -1.8562 \quad 6.4643 \quad -14.2175\,] \tag{38}$$

Let the initial conditions be $x(0) = (-2 \quad -2 \quad -1.5 \quad 2)^T$ and $\hat{x}(0) = (0.5 \quad 1 \quad 1 \quad -1)^T$, respectively. The simulation results are shown in Figures 2–5. Figures 2 and 3 show the state and its estimate of system (1) with the observer-based output feedback controller (6). Figures 4 and 5 display the dynamics of estimation errors. It can be seen that the system state is very well estimated and the whole closed-loop system is asymptotically stable as excepted.

Conclusion

In this work, we have proposed an observer-based output feedback controller for a generalized Lipschitz nonlinear system that satisfies the one-sided Lipschitz and the quadratically inner-bounded conditions. With the help of the LMI-based observer design approach and using Young's relation, we have established some LMI-type synthesis conditions that ensure the existence of observer-based output feedback controller design. Moreover, we applied the proposed observer-based control design to a flexible link manipulator system. We used MATLAB platform to simulate the dynamic response of the closed-loop system. The simulation results showed that the whole system has good performance under the developed control design scheme. The further work should investigate the digital implementation of the observer-based output feedback controller.

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