# The Six-Dimensional Self-Dual Tensor 

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#### Abstract

The equations of motion for a self-interacting self-dual tensor in six dimensions are extracted from the equations describing the $M$-theory five-brane. These equations are presented in a self-contained, six-dimensional Lorentz-covariant form. In particular, it is shown that the fieldstrength tensor satisfies a non-linear generalised self-duality constraint. The self-duality equation is rewritten in five-dimensional notation and shown to be identical to the corresponding equation in the non-covariant formalism.


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## 1 Introduction

Super $p$-branes play a central rôle in string duality and in $M$-theory, and it is therefore important to understand their properties. One aspect of super $p$-branes which can be studied with available mathematical tools is the construction of the worldsurface actions that describes their dynamics. A well-known property of all the existing super $p$-brane actions is that in addition to being worldsurface reparametrization invariant, they possess a fermionic gauge symmetry, called $\kappa$ symmetry. Gauge-fixing of these symmetries leads to worldsurface supersymmetric field theories that describe highly nonlinear self-interactions of matter supermultiplets.

The focus of this paper will be on the $M$-theory five-brane for which the resulting worldsurface multiplet contains a chiral two-form. The full equations of motion for this object were given in [1], using a concise superspace language. More recently, full actions for the same object have been constructed in [2, 3] in different approaches, and the component version of the results given in [1] has been obtained [4]. An interesting aspect of these results is the manner in which the self-duality equation for the interacting chiral two-form arises. In an earlier work that led to [2], Perry and Schwarz [5] constructed a self-duality equation with manifest fivedimensional covariance, and they suggested that it was impossible to make manifest the hidden six-dimensional covariance. On the other hand, the superspace formalism of [1] and the explicit component results of [4] involved just such a system.

In view of these developments, it is of some interest to study the self-dual tensor in six dimensions by itself, i.e. extracted from the brane context. This is what we shall do in this paper. We begin in the next section by extracting these equations in terms of a self-dual three-form field of the type introduced in [6] and then show that there is a new three-form which satisfies the Bianchi identity and a generalised, but still manifestly six-dimensional covariant, duality constraint. In the following section we rewrite our equations with only manifest five-dimensional covariance in order to study the relationship between our results [1, (4] and those of [5]. The result we find is that our six-dimensional covariant self-duality equation, when expressed in terms of fivedimensional fields, reduces precisely to that of Perry and Schwarz. Section 4 contains some concluding remarks.

## 2 The 6D Covariant Self-Duality Equation

In the superspace approach to the five-brane in eleven dimensions the antisymmetric tensor makes its first appearance as a self-dual tensor $h_{a b c}$ which occurs directly in the embedding. This field (or rather its leading component in a $\theta$ expansion) is not directly related to a twoform potential, but it can be shown that there is superspace three-form $H_{3}$ which is; in fact this three-form satisfies

$$
\begin{equation*}
d H_{3}=-\frac{1}{4} H_{4} \tag{1}
\end{equation*}
$$

where $H_{4}$ is the pull-back of the four-form of eleven-dimensional supergravity. The purely vectorial component of this three-form, which we denote by $H_{a b c}^{\prime}$, is related to $h$ by

$$
\begin{equation*}
H_{a b c}^{\prime}=m_{a}{ }^{d} m_{b}{ }^{e} h_{c d e} \tag{2}
\end{equation*}
$$

where

$$
\begin{equation*}
m_{a}{ }^{b}:=\delta_{a}^{b}-2 k_{a}{ }^{b}, \tag{3}
\end{equation*}
$$

and where

$$
\begin{equation*}
k_{a}{ }^{b}:=h_{a c d} h^{b c d} . \tag{4}
\end{equation*}
$$

We shall extract from the brane equations the equations describing a self-interacting self-dual tensor field in six-dimensional flat spacetime by setting all the other fields equal to their flat space values. The equations are self-duality,

$$
\begin{equation*}
h^{a b c}=\frac{1}{3!} \epsilon^{a b c d e f} h_{d e f} . \tag{5}
\end{equation*}
$$

with $\epsilon^{012345}=+1$ and $\eta_{a b}=\operatorname{diag}(-1,+1, \ldots,+1)$, and the equation of motion for $h$ which becomes

$$
\begin{equation*}
m^{a b} \partial_{a} h_{b c d}=0 . \tag{6}
\end{equation*}
$$

Equations (5) and (6) are equations in ordinary flat six-dimensional spacetime as are all subsequent equations in this paper. The goal is to rewrite these equations in terms of the more familiar $H$ field. In fact equation (2) has been written in a slightly unusual basis and it is necessary to correct this in order to find the relation between the components of $H$ in a coordinate basis and the components of $h$. This turns out to be

$$
\begin{equation*}
H_{a b c}=\left(m^{-1}\right)_{a}{ }^{d} h_{b c d} . \tag{7}
\end{equation*}
$$

Through this relation the self duality of $h_{b c d}$ imposes a self duality condition on $H_{a b c}$ which, as we will show below, is a rather complicated condition.

Since these equations were derived from the superspace system the Bianchi identity (11) will ensure that $d H=0$ in the truncated theory, where $H$ is now the spacetime three-form. However, to be completely self-contained we shall show that the Bianchi identity for $H$ can be derived from the basic equations for $h$. It follows from self-duality that the tensor $k$ introduced in (4) is traceless and that its square is proportional to the unit tensor, i.e., in matrix notation,

$$
\begin{equation*}
k^{2}=\frac{1}{6} \operatorname{tr} k^{2}, \tag{8}
\end{equation*}
$$

and that

$$
\begin{equation*}
h_{a b e} h^{c d e}=-\delta_{[a}{ }^{[c} k_{b]}^{d]}, \tag{9}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
m^{-1}=Q^{-1}(1+2 k)=Q^{-1}(2-m), \tag{10}
\end{equation*}
$$

where we have introduced

$$
\begin{equation*}
Q=1-\frac{2}{3} \operatorname{tr} k^{2} . \tag{11}
\end{equation*}
$$

We note in passing that the above expession for $m^{-1}$ shows that $H$ defined by (7) is indeed totally antisymmetric since duality implies that $k_{a}{ }^{d} h_{b c d}$ is totally antisymmetric and anti-selfdual (while $k_{a}{ }^{d} k_{b}{ }^{e} h_{c d e}$ is totally antisymmetric and self-dual). Using this expression for $m^{-1}$ we find

$$
\begin{align*}
\epsilon_{a b c d e f} \partial^{c} H^{\text {def }} & =6 \partial^{c}\left(Q^{-1} m_{c}{ }^{d} h_{a b d}\right)  \tag{12}\\
& =6 \partial^{c}\left(Q^{-1} m_{c}{ }^{d}\right) h_{a b d} \tag{13}
\end{align*}
$$

where in the second step we have used the equation of motion (6). It is not difficult to show, again using the equations of motion, that

$$
\begin{equation*}
m^{a b} \partial_{a} m_{b c}=0 . \tag{14}
\end{equation*}
$$

Using (10) in this equation we find

$$
\begin{equation*}
m^{a b} \partial_{a}\left(Q\left(m^{-1}\right)_{b c}\right)=0, \tag{15}
\end{equation*}
$$

and this leads, after a little algebra, to

$$
\begin{equation*}
\partial^{a}\left(Q^{-1} m_{a b}\right)=0 . \tag{16}
\end{equation*}
$$

Hence we have established

$$
\begin{equation*}
\epsilon^{a b c d e f} \partial_{c} H_{d e f}=0, \tag{17}
\end{equation*}
$$

as required.
We next translate the self-duality condition on $h$ into a generalised self-duality condition for $H$. Splitting $H$ into its self-dual and anti-self-dual parts, $H^{+}$and $H^{-}$, one finds from (7) that

$$
\begin{align*}
H_{a b c}^{+} & =Q^{-1} h_{a b c},  \tag{18}\\
H_{a b c}^{-} & =Q^{-1} k_{a}{ }^{d} h_{d b c} . \tag{19}
\end{align*}
$$

If we define

$$
\begin{equation*}
K_{a}{ }^{b}=H_{a c d}^{+} H^{+b c d} \tag{20}
\end{equation*}
$$

then

$$
\begin{equation*}
K_{a b}=Q^{-2} k_{a b} . \tag{21}
\end{equation*}
$$

Now

$$
\begin{equation*}
H^{+} \cdot H^{-}:=H_{a b c}^{+} H^{-a b c}=Q^{-2} \operatorname{tr} k^{2}=Q^{2} \operatorname{tr} K^{2}, \tag{22}
\end{equation*}
$$

and

$$
\begin{equation*}
H^{-}=Q^{2} K_{a}{ }^{d} H_{b c d}^{+}, \tag{23}
\end{equation*}
$$

so that we finally derive

$$
\begin{equation*}
H^{-}=\left(\operatorname{tr} K^{2}\right)^{-1}\left(H^{+} \cdot H^{-}\right) K_{a}{ }^{d} H_{b c d}^{+}, \tag{24}
\end{equation*}
$$

where we recall that $K_{a}{ }^{d} H_{b c d}^{+}$is totally antisymmetric and anti-self-dual as a consequence of the self-duality of $H^{+}$. This is the self-duality condition we were looking for. At first sight it does not appear to determine $H^{-}$in terms of $H^{+}$since it is clearly invariant under rescalings of $H^{-}$. However, the equation has been derived by purely algebraic manipulations and does not take into account the Bianchi identity which must also be satisfied. When one does this one finds that the freedom to rescale $H^{-}$disappears.

## 3 The Self-dual Tensor in 5D Notation

In this section we will use hatted indices for six dimensions, and unhatted ones for five dimensions. For clarity we shall also put a hat on the six-dimensional $m$-matrix. Our main purpose is to analyse (7) in five-dimensional language. We begin by defining

$$
\begin{align*}
f_{a b} & :=h_{a b 5},  \tag{25}\\
F_{a b} & :=H_{a b 5} . \tag{26}
\end{align*}
$$

Hence

$$
\begin{equation*}
h_{a b c}=\frac{1}{2} \epsilon_{a b c d e} f^{d e} . \tag{27}
\end{equation*}
$$

Inverting (7) we get

$$
\begin{equation*}
h_{\hat{a} \hat{b} \hat{c}}=\hat{m}_{\hat{a}} \hat{d} H_{\hat{b} \hat{c} \hat{d}} \tag{28}
\end{equation*}
$$

from which it follows that

$$
\begin{equation*}
f_{a b}=m_{a}^{c} F_{c b} \tag{29}
\end{equation*}
$$

Finally, let us define

$$
\begin{equation*}
\tilde{H}_{a b}=\frac{1}{3!} \epsilon_{a b c d e} H^{c d e} . \tag{30}
\end{equation*}
$$

Our goal is now to express $\tilde{H}$ in terms of $H_{a b 5}:=F_{a b}$. To this end, we begin by expressing the matrix $m$ and its inverse in five-dimensional notation. Recalling the definition (3), one finds

$$
\begin{align*}
m_{a}{ }^{b} & =\delta_{a}{ }^{b}\left(1-2 \operatorname{tr} f^{2}\right)+8\left(f^{2}\right)_{a}{ }^{b},  \tag{31}\\
m_{a}{ }^{5} & =-\epsilon_{a b c d e} f^{b c} f^{d e},  \tag{32}\\
m_{5}{ }^{5} & =\left(1+2 \operatorname{tr} f^{2}\right) . \tag{33}
\end{align*}
$$

The components of the inverse matrix can be calculated from (10):

$$
\begin{align*}
\left(m^{-1}\right)_{a}{ }^{b} & =Q^{-1}\left[\delta_{a}{ }^{b}\left(1+2 \operatorname{tr} f^{2}\right)-8\left(f^{2}\right)_{a}^{b}\right]  \tag{34}\\
\left(m^{-1}\right)_{a}{ }^{5} & =Q^{-1} \epsilon_{a b c d e} f^{b c} f^{d e}  \tag{35}\\
\left(m^{-1}\right)_{5}{ }^{5} & =Q^{-1}\left(1-2 \operatorname{tr} f^{2}\right) \tag{36}
\end{align*}
$$

where

$$
\begin{equation*}
Q=1+4\left(\operatorname{tr} f^{2}\right)^{2}-16 \operatorname{tr} f^{4} \tag{37}
\end{equation*}
$$

From (7) we have

$$
\begin{equation*}
H_{a b c}=\left(m^{-1}\right)_{a}{ }^{d} h_{d b c}+\left(m^{-1}\right)_{a}{ }^{5} f_{b c} . \tag{38}
\end{equation*}
$$

Taking the dual of this equation, and using the definitions and formulae above, one finds

$$
\begin{equation*}
\tilde{H}_{a b}=-Q^{-1} m_{a}^{c} f_{c b}, \tag{39}
\end{equation*}
$$

or, in matrix notation,

$$
\begin{equation*}
\tilde{H}=-Q^{-1}\left[\left(1-2 \operatorname{tr} f^{2}\right) f+8 f^{3}\right] \tag{40}
\end{equation*}
$$

It is straightforward even at this stage to write the matrix part of the above equation in terms of $F$. Using the relation $f=m F$, the identity $F_{a}{ }^{c} m_{c}{ }^{5}=0$, which again follows directly from equation (7) and the above expressions for $m$, we can write

$$
\begin{equation*}
\tilde{H}_{a b}=-Q^{-1} F_{a}^{c} G_{c b} \tag{41}
\end{equation*}
$$

where

$$
\begin{align*}
G_{a b} & :=m_{a}{ }^{\hat{c}} m_{\hat{c} b}  \tag{42}\\
& =\left(2-4 \operatorname{tr} f^{2}-Q\right) \eta_{a b}+16\left(f^{2}\right)_{a b} \tag{43}
\end{align*}
$$

which follows from (31), (32) and (37). Multiplying (42) with $F^{2}$, using $F_{a}{ }^{c} m_{c}{ }^{5}=0$ and recalling that $f=m F$, we derive $G F^{2}=f^{2}$. Using this relation in (43) we can solve for $G$. Substituting the result in (41) we find

$$
\begin{equation*}
\tilde{H}=-Q^{-1}\left(2-4 \operatorname{tr} f^{2}-Q\right)\left(\frac{F}{1-16 F^{2}}\right) \tag{44}
\end{equation*}
$$

To find the final result we must express the traces of $f$ in terms of $F$. At this point, it is important to recall a well known identity that holds for any antisymmetric $5 \times 5$ matrix $X$ :

$$
\begin{equation*}
X^{5}=\frac{1}{4}\left[\operatorname{tr} X^{4}-\frac{1}{2}\left(\operatorname{tr} X^{2}\right)^{2}\right] X+\frac{1}{2}\left(\operatorname{tr} X^{2}\right) X^{3} . \tag{45}
\end{equation*}
$$

Applying this identity to the matrix $F$, we write

$$
\begin{equation*}
F^{5}=\left(y_{2}-\frac{1}{2} y_{1}^{2}\right) F+y_{1} F^{3}, \tag{46}
\end{equation*}
$$

where

$$
\begin{align*}
& y_{1}=\frac{1}{2} \operatorname{tr} F^{2}, \\
& y_{2}=\frac{1}{4} \operatorname{tr} F^{4} . \tag{47}
\end{align*}
$$

The next step in the analysis of (40) is to express $f$ in terms of $F$. From the main duality equation (29), and the general identity (45), it follows that $f$ necessarily has the form

$$
\begin{equation*}
f=a F+b F^{3}, \tag{48}
\end{equation*}
$$

where $a$ and $b$ are functions of $F$ that can be determined from the duality equation (29), reproduced here for the reader's convenience:

$$
\begin{equation*}
f=\left(1-2 \operatorname{tr} f^{2}+8 f^{2}\right) F \tag{49}
\end{equation*}
$$

The $f^{2}$ term can easily be computed from (48) and (46):

$$
\begin{equation*}
f^{2}=\left[a^{2}+b^{2}\left(y_{2}-\frac{1}{2} y_{1}^{2}\right)\right] F^{2}+\left(2 a b+b^{2} y_{1}\right) F^{4} \tag{50}
\end{equation*}
$$

All the terms in (49) can be computed in a similar manner. At the end, comparing the coefficients of the terms proportional to $F$ and $F^{3}$, one finds

$$
\begin{align*}
& a+\frac{1}{2} b y_{1}=1  \tag{51}\\
& a^{2}+2 a b y_{1}+b^{2}\left(y_{2}+\frac{1}{2} y_{1}^{2}\right)=\frac{1}{8} b \tag{52}
\end{align*}
$$

The solution of these equations is given by

$$
\begin{equation*}
a_{ \pm}=\frac{32-y_{1} \pm y_{1} Z}{32 y_{2}-8 y_{1}^{2}}, \quad b_{ \pm}=\frac{1-8 y_{1} \pm Z}{16 y_{2}-4 y_{1}^{2}}, \tag{53}
\end{equation*}
$$

where

$$
\begin{equation*}
Z=\sqrt{1-16 y_{1}+128 y_{1}^{2}-256 y_{2}} \tag{54}
\end{equation*}
$$

The equations (51) and (52) will be used frequently in the calculations to simplify the resulting expressions. It turns out that we will not need to use (53) and therefore the final result is independent of the sign ambiguity in this equation.
Having determined $f$ explicitly in terms of $F$, we next evaluate $\operatorname{tr} f^{2}, f^{3}$ and $Q$, in order to express (40) in terms of $F$ alone. An expression for $\operatorname{tr} f^{2}$ is easily obtained from (50). The calculation of $f^{3}$ requires a bit of work, but is straighforward. With the aid of (51) and (52), we find

$$
\begin{equation*}
f^{3}=\left(2 y_{2}-y_{1}^{2}+\frac{1}{8} b\right) F+\frac{1}{8}\left[a+b\left(y_{1}-y_{1}^{2}+2 y_{2}\right)\right] F^{3} . \tag{55}
\end{equation*}
$$

Putting together the results obtained so far, we find that the self-duality equation (40) takes the simple form

$$
\begin{equation*}
\tilde{H}=-\frac{1}{8} b Q^{-1}\left[\left(1-16 y_{1}\right) F+16 F^{3}\right] . \tag{56}
\end{equation*}
$$

There remains the evaluation of $Q$. Starting from (37), and using (50) to calculate $\left(\operatorname{tr} f^{2}\right)^{2}$ and $\operatorname{tr} f^{4}$, we find, with the aid of (51) and (52), the result

$$
\begin{equation*}
Q=\frac{1}{8} b Z . \tag{57}
\end{equation*}
$$

Therefore, recalling the definition of $Z$, and rescaling $F \rightarrow \frac{1}{4} F, H \rightarrow-\frac{1}{4} H$, we deduce the final result

$$
\begin{equation*}
\tilde{H}=\frac{\left(1-y_{1}\right) F+F^{3}}{\sqrt{1-y_{1}+\frac{1}{2} y_{1}^{2}-y_{2}}}=\sqrt{1-y_{1}+\frac{1}{2} y_{1}^{2}-y_{2}}\left(\frac{F}{1-F^{2}}\right) . \tag{58}
\end{equation*}
$$

This formula is precisely the self-duality equation of Perry and Schwarz [5] (their equation (51)).

## 4 Conclusions

Our main result is that there does indeed exist a self-interacting self-dual tensor in six dimensions for which the generalised duality relation and the equations of motion can be written in manifestly six-dimensional covariant form. Furthermore, the self-duality equation which arises naturally in the description of the M-theory five-brane reduces precisely to the self-duality equation of Perry and Schwarz, when expressed in five-dimensional language. Starting from the PerrySchwarz version, it may not have been obvious in the past how to obtain manifest six-dimensional covariance. However, we now know the answer: first, one combines $H_{a b c}=-\frac{1}{2} \epsilon_{a b c d e} \tilde{H}^{d e}$ and $F_{a b}$ into a six-dimensional field strength as

$$
\begin{equation*}
\left(H_{a b c}, F_{a b}\right) \rightarrow H_{\hat{a} \hat{b} \hat{c}}, \tag{59}
\end{equation*}
$$

where $\hat{a}$ is now a six dimensional index, $\hat{a}=(a, 5)$, and $F_{a b}:=H_{a b 5}$. Next, one defines three-form $h_{\hat{a} \hat{b} \hat{c}}$ by

$$
\begin{equation*}
h_{\hat{a} \hat{b} \hat{c}}=\left(\delta_{\hat{a}}^{\hat{d}}-2 h_{\hat{a} \hat{e} \hat{f}} h^{\hat{d} \hat{e} \hat{f}}\right) H_{\hat{b} \hat{c} \hat{d}}, \tag{60}
\end{equation*}
$$

then, as a consequence of the five-dimensional duality relation, the derived field $h_{\hat{a} \hat{b} \hat{c}}$ is self-dual. It is interesting that this equation arises naturally in the supersurface embedding approach to the formulation of super $p$-branes. The approach is by no means new, and it has been applied to various $p$-branes in the past, with varying degrees of success. We refer the reader to [6] for a brief review of the extensive literature on the subject. The point we wish to emphasize here is that the approach is very natural, and its power becomes apparent when one gets to terms with the idea of searching for the equations of motion first and later deriving an action, when possible.
The beauty of the superembedding approach is that not only it treats the worldsurface and target space supersymmetry on an equal footing by considering the embedding of a world supersurface into the target superspace, but it applies to all super $p$-branes, regardless of the precise nature of the worldsurface multiplet, in a universal way that has a natural geometrical interpretation. Indeed, new kinds of super $p$-branes, e.g. $L$-branes, which have matter supermultiplets other than those considered so far have been suggested in [6], and we have already preliminary results
which show that things work in much the same way they do in the case of M-theory five-brane. We should expect, therefore, the uncovering of interesting new results on nonlinear self-couplings of matter supermultiplets in diverse dimensions which have not been explored so far.
Finally, it would be interesting to investigate various aspects of the $M$-theory five-brane selfduality equation discussed in this paper equation further. We have already exhibited the sixdimensional self-dual tensor equations as part of the full set of equations for the five-brane in an eleven-dimensional supergravity background [1. [4. A lot remains to be done, however, and we will report elsewhere $\sqrt{7}$ on further aspects of the emerging picture of the $M$-theory five-brane which seems to possess many interesting features.

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