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A note on $W_{2,s}$ strings

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ABSTRACT

BRST operators for two-dimensional theories with spin-2 and spin-s currents, generalising the W_3 BRST operator of Thierry-Mieg, have previously been obtained. The construction was based on demanding nilpotence of the BRST operators, making no reference to whether or not an underlying W algebra exists. In this paper, we analyse the known cases ($s = 3$, 4, 5 and 6), showing that the two $s = 4$ BRST operators are associated with the WB_2 algebra, and that two of the four $s = 6$ BRST operators are associated with the WG_2 algebra. We discuss the cohomology of all the known higher-spin BRST operators, the Weyl symmetry of their physical states, and their relation with certain minimal models. We also obtain the BRST operator for the case $s = 7$.

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1 Introduction

In 1987 Thierry-Mieg [1] constructed a BRST operator for the Zamolodchikov W_3 algebra [2]. The new feature of this BRST operator, as compared with the one for the standard Virasoro algebra, is that the W_3 algebra is nonlinear. This nonlinearity is reflected in the fact that the matter currents are present in the ghost currents. Subsequently it was shown by Romans [3] that all scalar realisations of the W_3 algebra can be expressed in terms of an energy-momentum tensor $T^{\rm eff}$ together with a scalar field φ which commutes with $T^{\rm eff}$. Later it was shown that by performing a nonlinear redefinition involving φ and the spin-2 and spin-3 ghost fields, the BRST operator could be simplified and written in the form $Q_B = Q_0 + Q_1$, where Q_0 has grade $(1,0)$ and Q_1 has grade $(0,1)$, with (p,q) denoting the grading of an operator with ghost number p for the redefined spin-2 (b, c) ghost system and ghost number q for the redefined spin-3 (β, γ) ghost system [4]. In particular Q_1 only involves φ, β, γ. This leads to an immediate generalisation [5], in which the BRST operator has similar form except that the (β, γ) system is that for a spin-s current rather than a spin-3 current. In [5], two different BRST operators were found in the case $s = 4$; one in the case $s = 5$; and four in the case $s = 6$. In this paper we shall investigate the properties of these various BRST operators in some detail, including a discussion of their cohomologies and their relation with certain minimal models. We also extend the previous results to the case $s = 7$, and find that there is just one BRST operator for this case.

The BRST operator for the spin-2 plus spin-s string takes the form [5]:

$$
\begin{aligned}
Q_B &= Q_0 + Q_1, &(1)\\
Q_0 &= \oint dz\, c\Big(T^{\rm eff} + T_\varphi + T_{\gamma,\beta} + \tfrac{1}{2}T_{c,b}\Big), &(2)\\
Q_1 &= \oint dz\, \gamma\, F(\varphi,\beta,\gamma), &(3)
\end{aligned}
$$

where the energy-momentum tensors are given by

$$
\begin{aligned}
T_\varphi &= -\tfrac{1}{2}(\partial\varphi)^2 - \alpha\,\partial^2\varphi, &(4)\\
T_{\gamma,\beta} &= -s\,\beta\,\partial\gamma - (s-1)\,\partial\beta\,\gamma, &(5)\\
T_{c,b} &= -2\,b\,\partial c - \partial b\, c, &(6)\\
T^{\rm eff} &= -\tfrac{1}{2}\eta_{\mu\nu}\,\partial X^\mu\,\partial X^\nu \; - i a_\mu\,\partial^2 X^\mu. &(7)
\end{aligned}
$$

The operator $F(\varphi,\beta,\gamma)$ has spin s and ghost number zero. Because of the grading discussed above, it follows that one will have the nilpotency conditions $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. The first of these conditions is satisfied provided that the total central charge vanishes, *i.e.*

$$
0 = -26 - 2(6s^2 - 6s + 1) + 1 + 12\alpha^2 + c^{\rm eff}\ , \qquad (8)
$$

where $c^{\rm eff}$ is the central charge for $T^{\rm eff}$. The remaining two nilpotency conditions determine the precise form of the operator $F(\varphi,\beta,\gamma)$ appearing in (7). Solutions for s =4, 5 and 6 were found in [5]. Together with the $s = 7$ case which we find in this paper, the list of known $W_{2,s}$ BRST operators is

	α^2	a^2	$c_{\rm min}$
$W_{2,3}$	$\frac{49}{8}$	$\frac{49}{24}$	$\frac{1}{2}$
$W_{2,4}$	$\frac{243}{20}$	$\frac{121}{60}$	$\frac{4}{5}$
	$\frac{361}{30}$	$\frac{32}{15}$	$-\frac{3}{5}$
$W_{2,5}$	$\frac{121}{6}$	2	1
$W_{2,6}$	$\frac{845}{28}$	$\frac{167}{84}$	$\frac{8}{7}$
	$\frac{1681}{56}$	$\frac{361}{168}$	$-\frac{11}{14}$
	$\frac{5041}{168}$	$\frac{121}{56}$	$-\frac{13}{14}$
	$\frac{361}{12}$	$\frac{25}{12}$	0
$W_{2,7}$	$\frac{675}{16}$	$\frac{95}{48}$	$\frac{5}{4}$

Table 1: The $W_{2,3}$–$W_{2,7}$ BRST operators

In the above table, $c_{\rm min}$ is the central charge of $T_\varphi + T_{\gamma,\beta}$, and is related to $c^{\rm eff}$ by $c_{\rm min} = 26 - c^{\rm eff}$. The column labelled by a^2 gives the value of the background charge for $T^{\rm eff}$ in the case where it is realised in terms of a single scalar field X:

$$T^{\rm eff} = -\tfrac{1}{2}(\partial X)^2 - a\,\partial^2 X\ . \tag{9}$$

For each s, the first BRST operator listed in the above table corresponds to a general sequence of BRST operators with $\alpha^2 = (s-1)(2s+1)^2/(4(s+1))$ and $c_{\rm min} = 2(s-2)/(s+1)$. In these cases, as discussed in [6], the (φ, β, γ) system provides a realisation of the lowest non-trivial unitary minimal model for the W_{s-1} algebra. In the case of a multi-scalar realisation for $T_{\rm eff}$, this sequence of BRST operators is associated with a unitary string theory. In this paper, the principal focus will be on the remaining exceptional BRST operators. Although in a multi-scalar realisation these would in general correspond to non-unitary string theories, they nevertheless provide explicit realisations for certain minimal models.

In section 2 we study the $W_{2,4}$ BRST operators, paying particular attention to the second case with $\alpha^2 = \frac{361}{30}$. We shall study its relation to the (3,5) Virasoro minimal model, the Weyl symmetry of its physical states, and the complete cohomology of its BRST operator, using the method developed in [7]. In section 3 we study the $s = 6$ BRST operators, especially the last three $W_{2,6}$ BRST operators in table 1. We shall study their connection to minimal models, the Weyl symmetry of the physical states, and comment on their physical spectra. In section 4 we bosonise the spin-s ghosts, and discuss the new features of the BRST operators. The paper ends with a summary.

2 The $W_{2,4}$ BRST operators

In general, the $W_{2,s}$ BRST operators are not associated with closed algebras of spin-2 and spin-s currents at the quantum level. However, there is such an association in certain special cases, such as $W_{2,3}$ (*i.e.* the usual W_3 algebra). There are two further examples where there are underlying W algebras, namely for $s = 4$, which we shall discuss in this section, and $s = 6$, which will be discussed in section 3. These BRST operators were all obtained by imposing the requirement of nilpotence on the ansatz in equations (1)–(7), without making any reference to any possible underlying W algebra. It is in fact not easy to extract the currents of such an algebra from this form of the BRST operator, and so in order to uncover the relation with the W algebra we find it simpler to adopt a more indirect approach. This was done already for the $W_{2,4}$ BRST operator with $\alpha^2 = \frac{243}{20}$ in [8], where we showed that the physical states in the two-scalar case form representations under the B_2 Weyl group. This indicates that the BRST operator is one for the WB_2 algebra, which is generated by an energy-momentum tensor and a spin-4 primary current.

We shall now examine the second $W_{2,4}$ BRST operator, with $\alpha^2 = \frac{361}{30}$. This is given by (1)–(7), with $s = 4$ and

$$\begin{aligned} F(\beta,\gamma,\varphi) \quad = \quad & (\partial\varphi)^4 + 4\alpha\,\partial^2\varphi\,(\partial\varphi)^2 + \tfrac{253}{30}(\partial^2\varphi)^2 + \tfrac{39}{5}\partial^3\varphi\,\partial\varphi + \tfrac{41}{570}\alpha\,\partial^4\varphi \\ & +8(\partial\varphi)^2\,\beta\,\partial\gamma - \tfrac{26}{19}\alpha\,\partial^2\varphi\,\beta\,\partial\gamma - \tfrac{66}{19}\alpha\,\partial\varphi\,\beta\,\partial^2\gamma \\ & -\tfrac{26}{15}\beta\,\partial^3\gamma + \tfrac{29}{5}\partial^2\beta\,\partial\gamma. \end{aligned} \tag{10}$$

Physical states are defined in the usual way by the requirement that they be BRST closed, but not exact. Using the method developed in [7], we shall now discuss this cohomology problem for the two-scalar case.

From the pattern of the explicit low-level physical states that we have obtained, we have observed that, just as in the case of the two-scalar W_3 string, the momenta $(p_1,\, p_2)$ of all physical states are quantised in rational multiples of the background charges $(\alpha,\, a)$ for the scalars $(\varphi,\, X)$. Specifically, in this case, we find

$$p_1 = \frac{k_1}{19}\,\alpha, \qquad\qquad p_2 = \frac{k_2}{8}\,a\,, \tag{11}$$

where k_1 and k_2 are integers. All physical states satisfy a mass-shell condition,

$$5(12\ell + 1) = (k_1 + 19)^2 + (k_2 + 8)^2\,, \tag{12}$$

where ℓ is the level number.

The method used in [7] to solve the cohomology consists of finding two particular physical operators, which we shall call x and y, which have the property that they have inverses, x^{-1} and y^{-1}, in the sense that the normal-ordered products $(x^{-1}\,x)$ and $(y^{-1}\,y)$ are non-vanishing constants. These operators have the further property that they have well-defined normal-ordered products with all physical operators, and so they map physical operators into other BRST non-trivial physical operators. They may therefore be used to generate the entire cohomology from a basic set of physical operators whose momenta lie in a fundamental unit cell.

The simplest candidates for the x and y operators in the present case have momenta $(k_1,\, k_2)$ given by

$$\begin{aligned} x \ &:\ (15,\, 15) \qquad\qquad x^{-1}:\ (-15,\, -15) \\ y \ &:\ (15,\, -15) \qquad\qquad y^{-1}:\ (-15,\, 15) \end{aligned} \tag{13}$$

It is easy to see from the mass-shell condition (12) that operators with these momenta have well-defined normal-ordered products with those for all integer solutions of (12), and hence in particular with all physical operators. One can also see from (12) that x and x^{-1} have level numbers $\ell = 28$ and $\ell = 1$; whilst y and y^{-1} have level numbers $\ell = 20$ and $\ell = 9$. We have been able to construct the operators x^{-1} and y explicitly. We find that the x^{-1} operator has ghost number $G = 3$, which implies that the x operator has $G = -3$. The y operator has ghost number $G = -1$, which implies that y^{-1} has $G = 1$. Although we have been unable to construct the x and y^{-1} operators explicitly, owing to their complexity, we have accumulated considerable evidence that supports their existence and invertibility. Assuming for now that this is the case, one can easily see that the fundamental cell may be taken as a 30×15 rectangle in the (k_1, k_2)-plane. There are 40 integer solutions to the mass-shell condition (12) in any such rectangle, of which 36 turn out to correspond to physical states. The remaining four, which can be characterised by $k_1 = 1, 6 \bmod 15$, do not correspond to physical states. For our calculations we chose the fundamental cell to be the rectangle with $-34 \le k_1 \le -5$, $-15 \le k_2 \le -1$. All the physical operators in this cell have level numbers $\ell \le 4$ and it is straightforward to find them explicitly. The complete cohomology can be obtained by normal-ordering $x^m\, y^n$ with all the physical operators in this fundamental cell, for arbitrary integers m and n.

Although, as we remarked above, we were unable to construct explicitly the x and y^{-1} operators, we have found numerous examples of pairs of physical states whose momenta and ghost numbers are consistent with their existence as cohomology generating operators.

We now turn to the consideration of the B_2 Weyl group and its action on physical states. For the $W_{2,4}$ BRST operator with $\alpha^2 = \frac{243}{20}$, this was discussed extensively in [8]. For the present case, with $\alpha^2 = \frac{361}{30}$, we first define the shifted momenta $\hat{k}_1 = k_1 + 19$, and $\hat{k}_2 = k_2 + 8$. The B_2 Weyl group is generated by

$$\begin{aligned} S_1 \quad &: \qquad (\hat{k}_1,\, \hat{k}_2) \longrightarrow (\hat{k}_1,\, -\hat{k}_2)\ , \\ S_2 \quad &: \qquad (\hat{k}_1,\, \hat{k}_2) \longrightarrow (\hat{k}_2,\, \hat{k}_1)\ . \end{aligned} \tag{14}$$

The eight elements of the Weyl group are given by 1, S_1, S_2, S_1S_2, S_2S_1, $S_1S_2S_1$, $S_2S_1S_2$, $S_1S_2S_1S_2$. It is clear that the Weyl group leaves the mass-shell condition (12) invariant. Furthermore it maps the momentum of any physical operator into the momentum of another physical operator [7]. For example, the eight tachyons $c\partial^2\gamma\partial\gamma\,\gamma e^{p_1\varphi+p_2X}$ have momenta $(\hat{k}_1, \hat{k}_2) = (\pm 1, \pm 2)$, $(\pm 2, \pm 1)$, where the $\pm$ signs are independent. These manifestly form a multiplet under the Weyl group.

The existence of this Weyl group symmetry indicates that the BRST operator is based on the WB_2 algebra. The explicit realisation of the WB_2 algebra is given in [9, 8]. Owing to the fact that B_2 is a non-simply-laced algebra, there are two inequivalent choices of the background charge for φ that give the same critical value of the central charge. As was shown in [8], these two values are $\alpha^2 = \frac{243}{20}$ and $\alpha^2 = \frac{361}{30}$; *i.e.* precisely the two values that were found for the $W_{2,4}$ BRST operators.

In previous studies of W string theories, it has been found that the physical states in the case of a multi-scalar realisation can be associated with the primary fields of certain minimal models. To be more specific, the physical operators take the form

$$c\,U(\varphi, \beta, \gamma)\, R(X^\mu)\ , \tag{15}$$

where $U(\varphi, \beta, \gamma)$ is a primary field of the minimal model, and $R(X^\mu)$ is a highest weight state under $T^{\rm eff}$. The cohomology of a multi-scalar W string can be associated with a subset of the cohomology of the two-scalar W string in a way that has been extensively discussed in [7, 8]. In the case of the $W_{2,4}$ string with $\alpha^2 = \frac{361}{30}$, we find that the weights of primary operators $U(\varphi, \beta, \gamma)$ are $h = (-\frac{1}{20},\, 0,\, \frac{1}{5},\, \frac{3}{4})$. Noting that the central charge of the (φ, β, γ) system is $c_{\rm min} = -\frac{3}{5}$, we recognise that the minimal model in this case is the (3, 5) Virasoro minimal model.

3 The $W_{2,6}$ BRST operators

There is just one further $W_{2,s}$ algebra that exists for arbitrary central charge (and, in particular, for the value that is needed for criticality), namely WG_2, which is generated by an energy-momentum tensor and a spin-6 primary current [10]. Since G_2 is non-simply laced, there will again be two inequivalent realisations, with different background charges, that give rise to nilpotent BRST operators. The form of the energy-momentum tensor is

$$T = -\tfrac{1}{2}(\partial\vec{\varphi})^2 - (t\,\vec{\rho} + \frac{1}{t}\,\vec{\rho}^{\vee}) \cdot \partial^2\vec{\varphi}\,, \tag{16}$$

where $\vec{\rho} = (\frac{3}{2},\, \frac{\sqrt{3}}{6})$ is the G_2 Weyl vector and $\vec{\rho}^{\vee} = (5,\, \sqrt{3})$ is the co-Weyl vector. Solving for the critical central charge $c = 388$ gives two solutions for t^2, namely $t^2 = \frac{7}{2}$ and $t^2 = \frac{24}{7}$. These correspond to a background charge α given by $\alpha^2 = \frac{1681}{56}$ and $\alpha^2 = \frac{5041}{168}$. Thus we see that the second and third $W_{2,6}$ BRST operators listed in table 1 correspond to BRST operators of the WG_2 algebra. We shall demonstrate below that indeed the physical states of these two BRST operators display a symmetry under the action of the Weyl group of G_2. The other two $W_{2,6}$ BRST operators, with $\alpha^2 = \frac{845}{28}$ and $\alpha^2 = \frac{361}{12}$, are not associated with any underlying W algebra, and in fact as we shall show, there is no Weyl-group symmetry in these cases.

First, we shall consider the two BRST operators associated with the WG_2 algebra. The low-level physical spectra of the two-scalar realisation indicate that the momenta of all physical states are quantised in rational multiples of the background charges α and a. For the case $\alpha^2 = \frac{5041}{168}$, we find that the momenta have the form

$$p_1 = \Big(\frac{\hat{k}_1}{71} - 1\Big)\,\alpha\ , \qquad\qquad p_2 = \Big(\frac{\hat{k}_2}{11} - 1\Big)\,a\ , \tag{17}$$

where $\hat{k}_1$ and $\hat{k}_2$ are integers. (We find it convenient to work directly with the shifted momentum variables here.) The mass-shell condition in this case is

$$28(12\ell + 1) = \hat{k}_1^2 + 3\hat{k}_2^2\,. \tag{18}$$

It is easy to see that this equation is invariant under the twelve-element G_2 Weyl group generated by

$$\begin{aligned} S_1 &: \ (\hat{k}_1,\, \hat{k}_2) \longrightarrow (\hat{k}_1,\, -\hat{k}_2)\,, \\ S_2 &: \ (\hat{k}_1,\, \hat{k}_2) \longrightarrow (\tfrac{1}{2}(-\hat{k}_1 + 3\hat{k}_2),\, \tfrac{1}{2}(\hat{k}_1 + \hat{k}_2))\,. \end{aligned} \tag{19}$$

We find that the tachyons, which are given by $c\,\partial^4\gamma\,\partial^3\gamma\,\partial^2\gamma\,\partial\gamma\,\gamma\,e^{p_1\varphi+p_2 X}$, have momenta $(\hat{k}_1,\,\hat{k}_2)=(\pm 1,\,\pm 3)$, $(\pm 4,\,\pm 2)$ and $(\pm 5,\,\pm 1)$, where the $\pm$ signs are independent. These twelve momenta of tachyons do indeed form a multiplet under the Weyl group.

For the $W_{2,6}$ BRST operator with $\alpha^2=\frac{1681}{56}$, we find that the momenta of all physical states take the form

$$p_1=\Big(\frac{\hat{k}_1}{41}-1\Big)\,\alpha\ ,\qquad\qquad p_2=\Big(\frac{\hat{k}_2}{19}-1\Big)\,a\ ,\tag{20}$$

where $\hat{k}_1$ and $\hat{k}_2$ are again integers. The mass-shell condition is

$$28(12\ell+1)=3\hat{k}_1^2+\hat{k}_2^2\,.\tag{21}$$

This is invariant under the twelve-element G_2 Weyl group generated by

$$\begin{aligned} S_1 &\ :\ (\hat{k}_1,\,\hat{k}_2)\longrightarrow(\hat{k}_1,\,-\hat{k}_2)\,,\\ S_2 &\ :\ (\hat{k}_1,\,\hat{k}_2)\longrightarrow(\tfrac{1}{2}(\hat{k}_1+\hat{k}_2),\,\tfrac{1}{2}(3\hat{k}_1-\hat{k}_2))\,. \end{aligned}\tag{22}$$

The tachyons here have momenta $(\hat{k}_1,\,\hat{k}_2)=(\pm 1,\,\pm 5)$, $(\pm 2,\,\pm 4)$ and $(\pm 3,\,\pm 1)$, where the $\pm$ signs are independent. The momenta of these twelve tachyons again form a multiplet under the Weyl group.

We expect that these results should generalise to higher-level physical states. In order to investigate this completely, one would like to find the cohomology-generating operators x, x^{-1}, y and y^{-1} in each case. We have not attempted to do this here, because of the complexity of the BRST operators. However, it is very plausible that they exist.

Like the W strings previously studied, we would expect that the physical states of the multi-scalar WG_2 strings have the form given in (15). The physical states of this form are relatively easy to construct, and we have studied all the low-level examples for both the BRST operators of WG_2. For the case with $\alpha^2=\frac{5041}{168}$, we find that the weights of the primary operators $U(\varphi,\beta,\gamma)$ are given by $h=(-\frac{1}{14},\,-\frac{5}{112},\,0,\,\frac{1}{7},\,\frac{27}{112},\,\frac{9}{14},\,\frac{13}{16},\,\frac{10}{7},\,\frac{5}{2})$. The central charge of the (φ,β,γ) system is $c_{\rm min}=-\frac{13}{14}$. These weights and central charge are precisely those of the $(4,7)$ Virasoro minimal model. It is interesting to note that this value of central charge is exactly the one found by Zamolodchikov to be necessary for the associativity of the $W_{2,\frac{5}{2}}$ algebra, which is generated by the energy-momentum tensor and a spin-$\frac{5}{2}$ current [2].

For the case with $\alpha^2=\frac{1681}{56}$, the central charge of the (φ,β,γ) system is $c_{\rm min}=-\frac{11}{14}$. We can expect in this case that the operators $U(\varphi,\beta,\gamma)$ should correspond to the primary fields of the $(7,12)$ Virasoro minimal model. In fact this central charge is also equal to that for one of the WB_2 minimal models. At level $\ell=19$ and $G=1$, there exists a physical state with momentum $p_1=0$, whose operator $U(\varphi,\beta,\gamma)$ has weight $h=4$, given by

$$\begin{aligned} U\ =\ & (\partial\varphi)^4+4\alpha\,\partial^2\varphi(\partial\varphi)^2+\tfrac{21557}{532}(\partial^2\varphi)^2+\tfrac{425}{38}\partial\varphi\partial^3\varphi+\tfrac{727}{9348}\alpha\,\partial^4\varphi\\ & +24(\partial\varphi)^2\,\beta\,\partial\gamma+\tfrac{2154}{133}(\partial\varphi)^2\partial\beta\,\gamma+\tfrac{3036}{133}\partial\varphi\partial^2\varphi\,\beta\gamma+\tfrac{30360}{5453}\alpha\,\partial\varphi\,\partial\beta\,\partial\gamma\\ & +\tfrac{18975}{5453}\alpha\,\partial\varphi\,\partial^2\beta\gamma+\tfrac{79584}{5453}\alpha\,\partial^2\varphi\beta\,\partial\gamma+\tfrac{11679}{779}\alpha\,\partial^2\varphi\,\partial\gamma\,\gamma+\tfrac{1518}{779}\alpha\,\partial^3\varphi\,\beta\gamma\\ & -\tfrac{1325}{133}\beta\,\partial^3\gamma+\tfrac{2650}{133}\partial\beta\,\partial^2\gamma+\tfrac{21729}{931}\partial^2\beta\,\partial\gamma+\tfrac{11205}{1064}\partial^3\beta\,\gamma+\tfrac{7632}{133}\partial\beta\,\beta\,\partial\gamma\,\gamma \end{aligned}\tag{23}$$

The energy-momentum tensor $T^{\rm min}$ of the (φ,β,γ) system is given by

$$T^{\rm min} = -\tfrac{1}{2}(\partial\varphi)^2 - \alpha\partial^2\varphi - 6\beta\,\partial\gamma - 5\partial\beta\,\gamma\ . \tag{24}$$

We have explicitly verified that currents $T^{\rm min}$ and U provide a realisation of the WB_2 algebra at central charge $c_{\rm min} = -\frac{11}{14}$ (up to appearance of the BRST trivial terms in the OPE of $U(z)\,U(w)$). This indicates that the $U(\varphi,\,\beta,\,\gamma)$ operators, which give a representation of the $(7,12)$ Virasoro minimal model, can be viewed as the primaries and descendants of the WB_2 minimal model with $c_{\rm min} = -\frac{11}{14}$. The $(7,12)$ Virasoro minimal model has 33 primaries, 21 of which are primaries of the WB_2 minimal model, with weights $(-\frac{1}{14},\,-\frac{1}{16},-\frac{1}{21},-\frac{3}{112},\,0,\,\frac{11}{336},\,\frac{13}{112},\,\frac{1}{6},\,\frac{25}{112},\frac{2}{7},\,\frac{3}{7},\,\frac{25}{42},\,\frac{11}{16},\,\frac{11}{14},\,\frac{299}{336},\frac{153}{112},\,\frac{3}{2},\frac{44}{21},\,\frac{125}{48},\,\frac{187}{42},\,\frac{23}{3})$. The remaining 12 are U descendants with weights $(\frac{125}{112},\,\frac{17}{7},\,\frac{333}{112},\,\frac{377}{112},4,\,\frac{25}{14},\,\frac{69}{14},\,\frac{91}{16},\,\frac{697}{112},\frac{58}{7},\,\frac{159}{16},\,\frac{25}{2})$. This is an explicit example of the phenomenon where the set of highest weight fields of a W minimal model gives rise to a larger set of highest weight fields with respect to the Virasoro subalgebra. In this example, although not in general, the set of Virasoro primaries is finite.

Finally, we turn to the last of the $W_{2,6}$ BRST operators, with $\alpha^2 = \frac{361}{12}$. In this case the $(\varphi,\,\beta,\,\gamma)$ system has central charge $c_{\rm min} = 0$. This BRST operator does not seem to correspond to any known algebra. Nevertheless the low-level examples indicate that the momenta of physical states are quantised in rational multiple of α and a. We find that p_1 and p_2 are given by

$$p_1 = \Big(\frac{\hat k_1}{95} - 1\Big)\,\alpha\ , \qquad\qquad p_2 = \Big(\frac{\hat k_2}{25} - 1\Big)\,a\ , \tag{25}$$

where $\hat k_1$ and $\hat k_2$ are integers. The mass-shell condition in this case is

$$50(12\ell+1) = \hat k_1^2 + \hat k_2^2\ . \tag{26}$$

The twelve tachyons have momenta given by $(\hat k_1,\,\hat k_2) = (\pm5,\,\pm5)$, $(\pm1,\,\pm7)$ and $(\pm7,\,\pm1)$. There is no discrete group of order 12 that maps these momenta into each other and preserves the mass-shell condition (26). Thus we conclude that there is no Weyl group in this case. (Note that (26) is similar to (12), and is therefore invariant under the B_2 Weyl group generated by (14). However, this does not act transitively on the 12 tachyons; those with momenta $(\pm5,\,\pm5)$ are mapped into one another, and the remaining 8 tachyons form a multiplet amongst themselves.) In the multi-scalar case, the weights of the primary operators $U(\varphi,\beta,\gamma)$ have values including $h = (-\frac{1}{25},\,0,\,\frac{1}{25},\,\frac{4}{25},\,\frac{11}{25},\,\frac{14}{25},\,\frac{21}{25})$ [5]. It is unclear whether these weights can be associated with any "minimal model" of the $c_{\rm min} = 0$ system realised by $(\varphi,\,\beta,\,\gamma)$.

4 Bosonisation of the β, γ ghosts

It was recently observed in [11, 12] that by bosonising the β and γ ghosts, so that they are written in terms of a free scalar field ρ, one can, after performing a non-local transformation involving ρ and φ, cast the W_3 string into the form of a parafermionic theory.

For the general sequence of $W_{2,s}$ BRST operators with $\alpha^2 = (s-1)(2s+1)^2/(4(s+1))$, one can perform the following bosonisation of β and γ:

$$\beta = e^{-i\rho}\ , \qquad\qquad \gamma = e^{i\rho}\ . \tag{27}$$

After the field redefinition

$$
\begin{aligned}
\varphi &= -i\sqrt{s^2-1}\,\phi_1 + s\,\phi_2\ , \\
\rho &= -s\,\phi_1 - i\sqrt{s^2-1}\,\phi_2\ ,
\end{aligned} \tag{28}
$$

the Q_1 BRST operator can be written as

$$
Q_1 = \oint dz\ :(\partial^s e^{i\phi_1})e^{-i(s+1)\phi_1}:\, e^{\sqrt{s^2-1}\phi_2}\ . \tag{29}
$$

The case $s=3$ was proved in [12]; we have explicitly verified the result for the cases of $s=4$, 5 and 6. In addition, we have constructed the $W_{2,7}$ BRST operator according to the general ansatz (1)–(7) and verified that in this case also, Q_1 can be written in this form. Note that in (29) we have indicated the normal ordering for the ϕ_1 terms explicitly, to emphasise that there are no contractions between the fields of the two ϕ_1–dependent exponentials.

In [12], the following non-local transformation from ϕ_1 to $\hat\phi_1$ was then performed:

$$
e^{i\hat\phi_1} = -\partial e^{-i\phi_1}\ , \qquad\qquad e^{i\phi_1} = \partial e^{-i\hat\phi_1}\ . \tag{30}
$$

One can easily verify that under this, (29) becomes

$$
Q_1 = \oint dz\, e^{is\hat\phi_1}\, e^{\sqrt{s^2-1}\phi_2}\ . \tag{31}
$$

(In this equation, and in (29), we have dropped an unimportant overall constant factor.)

Although the transformation (30) is canonical, *i.e.* $\hat\phi_1$ satisfies the same OPE as ϕ_1, it has a significant effect on the cohomology of the BRST operator Q_B. In fact, all the original physical states become BRST trivial. Actually one can prove an even stronger result, namely that after the transformation (30) the cohomology of Q_B becomes trivial. To see this, it is convenient to perform a further canonical transformation, from the fields $(\hat\phi_1,\,\phi_2)$ to $(\tilde\phi_1,\,\tilde\phi_2)$, defined by

$$
\begin{aligned}
\tilde\phi_1 &= s\,\hat\phi_1 - i\sqrt{s^2-1}\,\phi_2\ , \\
\tilde\phi_2 &= i\sqrt{s^2-1}\,\hat\phi_1 + s\,\phi_2\ .
\end{aligned} \tag{32}
$$

The Q_1 BRST operator then becomes simply $Q_1 = \oint dz\, e^{i\tilde\phi_1}$. If we perform the fermionisation $\tilde\beta = e^{-i\tilde\phi_1}$, $\tilde\gamma = e^{i\tilde\phi_1}$, we may write Q_B as

$$
\begin{aligned}
Q_B &= Q_0 + Q_1\ , \\
Q_0 &= \oint dz\, c(T_m + \partial\tilde\beta\,\tilde\gamma + \tfrac12 T_{c,b})\ , \qquad Q_1 = \oint dz\,\tilde\gamma\ ,
\end{aligned} \tag{33}
$$

where $T_m = T^{\rm eff} + T_{\tilde\phi_2}$ is a matter energy-momentum tensor with central charge 28. Note that $\tilde\gamma$ has spin 1, and $\tilde\beta$ has spin 0.

To see that Q_B defined in (33) has no non-trivial cohomology, we write an arbitrary physical state $|\chi\rangle$ in the form $|\chi\rangle = \tilde\beta|\chi_1\rangle + |\chi_2\rangle$, where $|\chi_1\rangle$ and $|\chi_2\rangle$ contain no undifferentiated $\tilde\beta$ field. It is easy to see that the physical-state condition $Q_B|\chi\rangle = 0$ implies that $|\chi_1\rangle = -Q_0|\chi_2\rangle$, and hence that $Q_B\,\tilde\beta|\chi_2\rangle = Q_0\tilde\beta|\chi_2\rangle + Q_1\tilde\beta|\chi_2\rangle = -\tilde\beta\,Q_0|\chi_2\rangle + |\chi_2\rangle = |\chi\rangle$. Thus any state that satisfies the physical-state condition $Q_B|\chi\rangle = 0$ is BRST trivial. Since the transformation (32), and the subsequent fermionisation that we performed above, will preserve the cohomology of Q_B, it follows that the original cohomology of Q_B had already collapsed at the stage of the parafermionic formulation in (31), owing to the non-locality of the transformation (30).

It is remarkable that despite the non-locality of the transformation (30), the original physical operators can apparently all be re-expressed in terms of the $(\hat{\phi}_1, \phi_2)$ fields. We have checked this in several non-trivial examples for the W_3 string, for various levels up to $\ell = 14$. The reason why these operators, which are BRST trivial in the $(\hat{\phi}_1, \phi_2)$ parametrisation, are nevertheless BRST non-trivial prior to the transformation (30) is that they are written as Q_B acting on operators built from $(\hat{\phi}_1, \phi_2)$ that cannot themselves be re-expressed in terms of the original fields, owing to the non-locality of (30).

One of the striking features of the transformation (28) is that for the general sequence of BRST operators that we have just been discussing, one can, by adding total derivatives, express Q_1 in a form where ϕ_2 appears only in the exponential, with monomials involving derivatives only of ϕ_1 in the prefactor. Furthermore, all these terms can then be expressed in the simple form given in (29). It is of interest to see how much of this can be carried through for the remaining exceptional BRST operators for $W_{2,4}$ and $W_{2,6}$. We find that the first step, namely eliminating all the ϕ_2 dependence from the prefactor, can be achieved for all the remaining BRST operators. For example, in the case of the $W_{2,4}$ BRST operator with $\alpha^2 = \frac{361}{30}$, we can perform the following transformation:

$$\begin{aligned} \varphi &= -i\sqrt{120}\,\phi_1 + 11\,\phi_2 \ , \\ \rho &= -11\,\phi_1 - i\sqrt{120}\,\phi_2 \ , \end{aligned} \tag{34}$$

leading to

$$Q_1 = \oint dz \Big(6\,(\partial\phi_1)^4 - i\sqrt{96}\,(\partial\phi_1)^2\,\partial^2\phi_1 - 13\,(\partial^2\phi_1)^2 + \partial\phi_1\,\partial^3\phi_1 - \frac{i}{\sqrt{6}}\,\partial^4\phi_1\Big) e^{-i\sqrt{6}\phi_1 + \sqrt{5}\phi_2} \ . \tag{35}$$

It is not possible to write this in the simple form (29). It is unclear whether there is nevertheless some more general kind of non-local transformation analogous to (30) that could transform Q_1 into a parafermionic form such as (31). Our findings for the exceptional $W_{2,6}$ BRST operators are similar, in that we obtain analogous expressions to (35), which again cannot be expressed in a form such as (29).

5 Summary

In this paper, we have studied all the $W_{2,s}$ BRST operators up to $s = 7$, paying particular attention to the execeptional $W_{2,s}$ BRST operators for $s = 4$ and $s = 6$ that have not been previously investigated in any detail. These BRST operators would describe non-unitary string theories in multi-scalar realisations. However, the cohomologies of these BRST operators in multi-scalar realisation give rise to explicit realisations of certain minimal models. In two-scalar realisation, the cohomologies of these BRST operators are much richer. Since the momenta of all the physical states seem to be quantised in rational multiples of the background charges, we would expect that there exist cohomology-generating operators in all the exceptional cases, and thus one would obtain the corresponding complete cohomologies, using the method developed in [7]. In this paper, we carried out the task for the exceptional $W_{2,4}$ BRST operator.

We also identified the W symmetries that underlie certain of the $W_{2,s}$ BRST operators. The two $W_{2,4}$ BRST operators are associated with WB_2 algebra, and two of the $W_{2,6}$ BRST operators are associated with WG_2 algebra. None of the other $W_{2,s}$ BRST operators with $s \geq 5$ are associated with any underlying quantum W algebras. With the possible exception of $W_{2,6}$ with $\alpha^2 = \frac{361}{12}$, all the $W_{2,s}$ BRST operators give rise to explicit realisations of certain minimal models. In particular, $W_{2,6}$ BRST operator with $\alpha^2 = \frac{1681}{56}$ is associated with WB_2 minimal model with $c_{\rm min} = -\frac{11}{14}$. The $W_{2,6}$ BRST operator with $\alpha^2 = \frac{361}{12}$ has the central charge $c_{\rm min} = 0$. It is unclear whether this BRST operator is associated with any "minimal" model.

We also explicitly verified field redefinition proposed in [12] in order to simplify the BRST operators for the general sequence of $W_{2,s}$ BRST operators with $s = 4$, 5, 6, and 7. We showed that after the non-local field transformation (30) and some further field redefinitions, all these BRST operators can be transformed into an s-independent form (33), whose cohomology is trivial. For the exceptional $W_{2,s}$ BRST operators, it is unclear whether the above procedure can be carried out.

It was recently proposed that the bosonic string could be viewed as a special vacuum of the $N = 1$ superstring, which could in turn be viewed as a special vacuum of the $N = 4$ superstring [13]. It was also suggested that the bosonic string might be viewed as the lowest member of a hierachy of W_N strings [13, 14]. In fact, as observed in [8], there is a sense in which this is already seen in the usual multi-scalar W_3 string, where physical states are those of $c = 25\frac{1}{2}$ bosonic strings tensored with the Ising model. Another proposal was made recently [12], in which a sequence of field transformations was made in order to arrive at a theory with the cohomology of the ordinary bosonic string, starting from the W_3 string. However, one step along the sequence involved the non-local transformation (30), which yields the form (31) for Q_1 and, as we have seen, gives an empty cohomology for Q_B. Although in the subsequent transformations in [12] a theory with the cohomology of the bosonic string is obtained, it would seem that one should not interpret the result as showing that this bosonic string is a special vacuum of the original W_3 string, since at an intermediate stage all the cohomology was lost. It would be interesting to see whether there is any way to bypass the step (30), and find a non-singular projection from the states of the W string to those of the bosonic string.

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