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$\mathrm{AdS}_3 \times S^3$ (Un)twisted and Squashed, and An $O(2,2;\mathbb{Z})$ Multiplet of Dyonic Strings

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ABSTRACT

We consider type IIB configurations carrying both NS-NS and R-R electric and magnetic 3-form charges, and whose near horizon geometry contains $\mathrm{AdS}_3 \times S^3$. Noting that S^3 is a $U(1)$ bundle over $CP^1 \sim S^2$, we construct the dual type IIA configurations by a Hopf T-duality along the $U(1)$ fibre. In the case where there are only R-R charges, the S^3 is untwisted to $S^2 \times S^1$ (in analogy with a previous treatment of $\mathrm{AdS}_5 \times S^5$). However, in the case where there are only NS-NS charges, the S^3 becomes the cyclic lens space S^3/Z_p with its round metric (and is hence invariant when $p = 1$), where p is the magnetic NS-NS charge. In the generic case with NS-NS and R-R charges, the S^3 not only becomes S^3/Z_p but is also squashed, with a squashing parameter that is related to the values of the charges. Similar results apply if we regard AdS_3 as a bundle over AdS_2 and T-dualise along the fibre. We show that Hopf T-dualities relate different black holes, and that they preserve the entropy. The $\mathrm{AdS}_3 \times S^3$ solutions arise as the near-horizon limits of dyonic strings. We construct an $O(2,2;\mathbb{Z})$ multiplet of such dyonic strings, where $O(2,2;\mathbb{Z})$ is a subgroup of the $O(5,5)$ or $O(5,21)$ six-dimensional duality groups, which captures the essence of the NS-NS/R-R and electric/magnetic dualities.

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Contents

1 Introduction

The six-dimensional space $\mathrm{AdS}_3 \times S^3$ emerges as the near horizon geometry [1, 2] of the self-dual string [3, 4] or, more generally, the dyonic string [5, 4, 6, 7]. The dyonic string admits the ten-dimensional interpretation [5] of an intersecting NS-NS 1-brane and 5-brane, which in a type II context is in turn related by U-duality to the D1-D5 brane system [8, 9, 10, 11, 12]. This geometry plays a part in recent studies of black holes [14, 15, 16, 17, 18, 19, 10, 11, 12, 20] and has attracted a good deal of attention lately [21, 22, 23, 24, 12, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 13] following the conjectured duality [21] between physics in the bulk of the anti-de Sitter spacetimes and conformal field theories on their boundary. AdS_3 is particularly interesting in this regard because the conformal field theory is then of the familiar and well-understood $1 + 1$ dimensional variety.

In a previous paper [37], devoted mainly to $\mathrm{AdS}_5 \times S^5$, we noted that odd-dimensional spheres S^{2n+1} may be regarded as $U(1)$ bundles over CP^n and that this permits an unconventional type of T-duality along the $U(1)$ fibre that we called "Hopf" duality. This Hopf duality has the effect of untwisting S^{2n+1} to $CP^n \times S^1$. Applying this to S^5, we were able to construct the duality chain: $n = 4$ Yang-Mills $\rightarrow$ type IIB string on $\mathrm{AdS}_5 \times S^5$ $\rightarrow$ type IIA string on $\mathrm{AdS}_5 \times CP^2 \times S^1$ $\rightarrow$ M-theory on $\mathrm{AdS}_5 \times CP^2 \times T^2$. In an earlier paper [38], devoted mainly to $\mathrm{AdS}_4 \times S^7$, we exhibited the duality: M-theory on $\mathrm{AdS}_4 \times S^7$ $\rightarrow$ type IIA string on $\mathrm{AdS}_4 \times CP^3$. In both contexts, similar techniques were also applied to more general spaces $\mathrm{AdS} \times M$ where M are Einstein spaces that are not necessarily spheres. These emerge as the near horizon geometries of supermembranes with fewer Killing spinors [39, 40, 41] and whose boundary conformal field theories have correspondingly less supersymmetry [42, 37, 40, 43, 44].

In the present paper, we wish to apply these techniques to find type IIA and M-theory duals of six-dimensional type IIB $\mathrm{AdS}_3 \times S^3$ configurations obtained by either T^4 or K3 compactifications. The novel ingredient is that these can be supported by both NS-NS and R-R 3-forms, in contrast to the $\mathrm{AdS}_5 \times S^5$ example where the 5-form was strictly R-R. This has some interesting and unexpected consequences. Noting that S^3 is a $U(1)$ bundle over $CP^1 \sim S^2$, we construct the dual type IIA configurations by a Hopf T-duality along the $U(1)$ fibre. In the case where there are only R-R charges, the S^3 is untwisted to $S^2 \times S^1$ (in analogy with a previous treatment of $\mathrm{AdS}_5 \times S^5$). However, in the case where there are only NS-NS charges, the S^3 becomes the cyclic lens space S^3/Z_p with its round metric (and is hence invariant when $p = 1$), where p is the magnetic NS-NS charge. In the generic case with NS-NS and R-R charges, the S^3 not only becomes S^3/Z_p but is also squashed, with a squashing parameter that is related to the values of the charges. Similar results apply if we regard AdS_3 as a bundle over AdS_2 and T-dualise along the fibre. We note that these Hopf dualities preserve the area of the horizons, and hence they preserve the black hole entropies. In particular, we show in appendix D that a Hopf reduction of a single 3-charge isotropic black hole in $D = 5$ gives a 4-charge black hole in $D = 4$, where the fourth charge is of unit strength, and comes from the magnetic charge carried by the Kaluza-Klein vector. The reduction coordinate in this case is the $U(1)$ fibre coordinate of the foliating 3-spheres in the transverse space of the $D = 5$ black hole. In general Hopf reduction and Hopf T-duality not only relate the near-horizon limits of black holes, but also relate the full solutions themselves. We show also that this statement extends to all p-branes, including non-extremal ones, that have 4-dimensional overall transverse spaces. In all cases, the Hopf reduction maps an N-charge solution in $D + 1$ dimensions to an $(N + 1)$-charge solution in D dimensions.

Studying the T-duality and the U-duality multiplets of BPS solitons in the full theories obtained from the T^4 or K3 reductions of ten-dimensional supergravities is a complicated matter, owing to the large number of fields, and the size of the global symmetry groups in $D = 6$. For this reason, it is helpful to make truncations of the six-dimensional theories, to more manageable subsectors that capture the essential features that we wish to study. We therefore begin, in section 2, by making a consistent truncation of six-dimensional maximal supergravity, to a subsector of bosonic fields that includes two 2-form potentials, one NS-NS, and the other R-R. We show that this theory has an $O(2,2) \sim SL(2,\mathbb{R})_1 \times SL(2,\mathbb{R})_2$ global symmetry. The $SL(2,\mathbb{R})_1$ describes an S-duality symmetry that interchanges the NS-NS and R-R 2-form potentials, while the $SL(2,\mathbb{R})_2$ is an electric/magnetic duality symmetry of the 3-form field strengths, which acts locally only at the level of the equations of motion. This consistent truncation is most conveniently constructed from the toroidal reduction in the type IIB field variables. We then consider a different consistent truncation, which is most conveniently obtained from the toroidal reduction in the type IIA fields variables. In fact the two truncations are characterised by the feature that the six-dimensional fields that are retained are precisely the original ten-dimensional ones, with the spacetime indices simply restricted to run over the six-dimensional range, and in addition the breathing-mode scalar parameterising the volume of the 4-torus. For this reason, the truncated theories can equally well be obtained by compactifying the type IIB and type IIA theories on K3, and following the identical prescription for which fields to retain.

The two truncated theories in $D = 6$ are related by a T-duality transformation upon reduction on S^1 to $D = 5$. In section 2, supplemented by appendix A, we obtain the above truncations and perform the S^1 reductions on the two theories. We then derive the explicit T-duality transformations relating them. In appendix B, we study the $O(2,2)$ symmetry in detail, and give the explicit transformation rules. In appendix C, we use these transformation rules to construct an $O(2,2;\mathbb{Z})$ multiplet of dyonic strings carrying four independent charges, namely electric and magnetic charges for each of the NS-NS and R-R 3-forms. We then extend these results to boosted and twisted dyonic strings. Since all the solutions are obtained as solutions in a consistent truncation, the dyonic strings are therefore solutions of the original supersymmetric theories, and in fact they are, as usual, BPS states preserving some fraction of the supersymmetry.

In section 3, we study the near-horizon $\mathrm{AdS}_3 \times S^3$ limits of the dyonic strings. Noting that S^3 can be described as a $U(1)$ bundle over S^2, we perform a Hopf T-duality transformation on the $U(1)$ fibres, and show that the S^3 can be untwisted or squashed, as described previously. In the case of solutions supported purely by NS-NS fields, we also supply a CFT proof that strings on S^3/Z_m with 3-form flux n are dual to strings on S^3/Z_n with 3-form flux m.

In section 4, we perform a similar Hopf T-duality transformation on the AdS_3 instead, exploiting the fact that AdS_3 can analogously be written in the form of a bundle over AdS_2. In section 5 we perform simultaneous Hopf T-duality transformations on the fibres of S^3 and AdS_3. Section 6 addresses the issue of supersymmetry and the Hopf T-duality transformations. We construct the Killing spinors on AdS_3 and S^3 explicitly, in coordinates appropriate to the bundle descriptions, and show that Hopf T-duality on AdS_3 or S^3 either preserves all or none of the supersymmetry, at the level of the massless Kaluza-Klein modes in supergravity, depending on the orientation of the fibration. We also discuss the supersymmetry in the context of the full string theory.

In section 7, we list all the non-dilatonic black holes in $D = 5$ and $D = 4$, and study their near-horizon limits when they are oxidised to $D = 6$. We show that all the near-horizon limits can be obtained by Hopf T-duality on $\mathrm{AdS}_3 \times S^3$.

2 $O(2,2)$ truncation of maximal supergravity in $D = 6$

We begin from the Lagrangian in $D = 6$ obtained by dimensional reduction of type IIB on a 4-torus. Since we want to consider $\mathrm{AdS}_3 \times S^3$ solutions that carry both NS-NS and R-R 3-form charges, we first make a consistent truncation to a subset of the fields that includes the necessary pair of 3-forms. We can do this by just retaining the subset of fields comprising the reductions of the original $F^{\rm NS}_{(3)}$ and $F^{\rm RR}_{(3)}$ fields themselves, together with the axion χ_1 and the dilaton $\hat\phi$ of $D=10$ type IIB, and the axion χ_2 coming from the dualisation of the potential $B_{(4)}$. (All details of the full reduction are given in [45].) In $D=6$, we obtain

$$\begin{aligned} e^{-1}\,\mathcal{L}_6 &= R-\tfrac12(\partial\hat\phi)^2-\tfrac12(\partial\vec\varphi)^2-\tfrac12 e^{2\hat\phi}\,(\partial\chi_1)^2-\tfrac12 e^{-2\vec a\cdot\vec\varphi}\,(\partial\chi_2)^2\\ &\quad -\tfrac1{12}e^{-\hat\phi+\vec a\cdot\vec\varphi}\,(F^{\rm NS}_{(3)})^2-\tfrac1{12}e^{\hat\phi+\vec a\cdot\vec\varphi}\,(F^{\rm RR}_{(3)})^2+\chi_2\,dA^{\rm NS}_{(2)}\wedge dA^{\rm RR}_{(2)}\ , \end{aligned}\tag{2.1}$$

where $F^{\rm NS}_{(3)}=dA^{\rm NS}_{(2)}$ and $F^{\rm RR}_{(3)}=dA^{\rm RR}_{(2)}+\chi_1\,dA^{\rm NS}_{(2)}$. Here $\vec\varphi$ denotes the set of 4 dilatonic scalars coming from the reduction on T^4, and $\vec a$ is a constant vector that can be found in [45], characterising the couplings of the field strengths to the dilatonic scalars $\vec\varphi$. The combinations of the dilatons $\vec\varphi$ that are perpendicular to $\vec a$ can also be consistently truncated, resulting in the Lagrangian

$$\begin{aligned} e^{-1}\,\mathcal{L}_{6B} &= R-\tfrac12(\partial\phi_1)^2-\tfrac12(\partial\phi_2)^2-\tfrac12 e^{2\phi_1}\,(\partial\chi_1)^2-\tfrac12 e^{2\phi_2}\,(\partial\chi_2)^2\\ &\quad -\tfrac1{12}e^{-\phi_1-\phi_2}\,(F^{\rm NS}_{(3)})^2-\tfrac1{12}e^{\phi_1-\phi_2}\,(F^{\rm RR}_{(3)})^2+\chi_2\,dA^{\rm NS}_{(2)}\wedge dA^{\rm RR}_{(2)}\ , \end{aligned}\tag{2.2}$$

where $\phi_1=\phi$ and $\phi_2=-\vec a\cdot\vec\varphi$. This truncated Lagrangian is characterised by the fact that it follows from the truncation of the T^4 reduction of the type IIB theory in which all the original fields are retained, but with their indices now running only over the remaining six dimensions. (Note that the potential for the self-dual 5-form is now dualised to give the axion χ_2.) In addition, the breathing-mode scalar ϕ_2 that parameterises the volume of T^4 is also included, but all other fields with indices internal to the 4-torus are set to zero. For this reason, (2.2) can also be obtained by making a consistent truncation of the $N=2$ theory obtained by compactifying type IIB on K3, following the identical prescription for which fields are to be retained. (One cannot tell what it is that is "breathing" if all other modes are truncated.)

The Lagrangian (2.2) has an $O(2,2)\sim SL(2,\mathbb{R})_1\times SL(2,\mathbb{R})_2$ global symmetry which is a subgroup of the original $O(5,5)$ Cremmer-Julia symmetry of maximal supergravity in $D=6$. Note that $SL(2,\mathbb{R})_1$, realised by (ϕ_1,χ_1) in the scalar sector, is a symmetry of the Lagrangian, whilst $SL(2,\mathbb{R})_2$, realised by (ϕ_2,χ_2), is a symmetry of the equations of motion. We give the explicit transformation rules in appendix B. It should emphasised that this $O(2,2)$-invariant theory is not itself the bosonic sector of any supergravity theory; rather, it is a convenient *consistent* truncation of $D=6$ maximal supergravity that contains all the fields necessary for describing the (supersymmetric) solutions of $D=6$ maximal supergravity that we are going to discuss in this paper.[1] Because of the consistency of the truncation, all the solutions in the theory (2.2) are solutions of the original untruncated supersymmetric theory. This provides a powerful tool for studying the BPS states of the original theory, since it reduces the original $O(5,5)$ or $O(5,21)$ global symmetry to a more manageable $O(2,2)$ global symmetry that nevertheless captures the essence of the NS-NS/R-R and the electric/magnetic dualities.

The six-dimensional Lagrangian (2.2) is related by T-duality in $D=5$ to a different six-dimensional theory that is also a consistent truncation of maximal six-dimensional supergravity. This theory is most conveniently obtained by truncating maximal six-dimensional supergravity described in terms of the T^4 reduction in the type IIA field variables. In this description, it corresponds again to retaining the original ten-dimensional fields, with their indices running only over the remaining six dimensions, and including in addition the breathing-mode scalar parameterising the volume of the 4-torus. Again, as with (2.2), all other six-dimensional fields, with indices internal to the 4-torus, are set to zero. This truncated Lagrangian can therefore also be obtained as a consistent truncation of the $N=2$ theory obtained by K3 compactification of type IIA, where the same truncation prescription is applied. It is given by

$$\begin{aligned} e^{-1}\,\mathcal{L}_{6A} &= R - \tfrac{1}{2}(\partial\phi_1)^2 - \tfrac{1}{2}(\partial\phi_2)^2 - \tfrac{1}{48}e^{\frac{1}{2}\phi_1-\frac{3}{2}\phi_2}\,(F_{(4)})^2 \\ &\quad -\tfrac{1}{12}e^{-\phi_1-\phi_2}\,(F_{(3)})^2 - \tfrac{1}{4}e^{\frac{3}{2}\phi_1-\frac{1}{2}\phi_2}\,(F_{(2)})^2\,, \end{aligned} \tag{2.3}$$

where, in the notation of [48, 49], $F_{(3)}$ means the NS-NS 3-form $F_{(3)1}$, and $F_{(2)}$ means the R-R 2-form $\mathcal{F}^1_{(2)}$, with the index "1" here denoting the reduction step from $D=11$ to $D=10$. (In the rest of the paper, an index "1" will be used exclusively to denote a reduction step from 6 to 5 dimensions.) We have again performed a consistent truncation and orthogonal transformation on the dilatons. To be precise, ϕ_1 is the original ten-dimensional dilaton, and ϕ_2 is the breathing mode of the 4-torus or K3. Note that $F_{(4)}=dA_{(3)}-dA_{(2)}\wedge A_{(1)}$, while $F_{(3)}=dA_{(2)}$ and $F_{(2)}=dA_{(1)}$. The $D=6$ string coupling constant in both (2.2) and (2.3) is given by $g_6=e^{\frac{1}{2}(\phi_1+\phi_2)}$. The Lagrangian (2.3) is in fact simply the dimensional

[1] A similar use of a non-supersymmetric truncation was made in [46], where the 5-form field strength of the type IIB theory was set to zero in order to simplify the discussion of the multiplet of supersymmetric NS-NS and R-R string solitons. In that case the truncation was actually *inconsistent*, since generic solutions of the truncated theory would be configurations which, in the full type IIB theory, would provide sources that would force the 5-form field strength to be non-zero. However, these source terms are actually zero for the class of solutions considered in [46], and so the truncation there was consistent only in this restricted sense.

reduction of

$$e^{-1}\,\mathcal{L}_7 = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{48} e^{\sqrt{\frac{8}{5}}\phi}\, F_{(4)}^2\ . \tag{2.4}$$

This provides another way of verifying a statement we made previously, that the six-dimensional Lagrangians (2.3) and (2.2) are consistent truncations of six-dimensional maximal supergravity. We do this by noting that (2.4) is itself a consistent truncation of seven-dimensional maximal supergravity. This can be easily seen from the form of the $D=7$ theory given in [48], and by noting that $F_{(4)}$ cannot act as a source for any of the other fields that are being truncated. (Recall that for a truncation to be consistent, every solution of the truncated theory must be a solution of the untruncated theory.) Having established that (2.4) is a consistent truncation of seven-dimensional maximal supergravity, it follows that (2.3) and (2.2) are consistent truncations of six-dimensional maximal supergravity. (For the latter case, one has to invoke the T-duality relating the two theories in $D=5$.)

To see the T-duality relating (2.2) and (2.3), we dimensionally reduce the two six-dimensional theories to $D=5$; the details are given in appendix A. The T-duality relations between the field strengths in the five-dimensional theories are indicated in Table 1 below, which also defines our notation for the dimensional reductions of the fields. Note that we present the identifications at the level of the field strengths because it is necessary to perform some dualisations in $D=5$ in order to implement the identifications.[2] (The precise statement of the identifications is given in appendix A.) Note also that we are now, and henceforth, using a "1" subscript on a field strength to denote the reduction step from $D=6$ to $D=5$.

[2]Note that perturbative T-duality does not require dualisations in order to relate two theories. The reason why it is necessary to make a dualisation here is that we have already dualised the 4-form potential in the type IIB picture to an axion. Had we not done so, then the identification of five-dimensional fields would not have required any dualisation.

	IIA			IIB	
	$D=6$	$D=5$	T-duality	$D=5$	$D=6$
R-R fields	$F_{(4)}$	$F_{(4)}$	$\longleftrightarrow$	$d\chi_2$	$d\chi_2$
		$F_{(3)1}$	$\longleftrightarrow$	$F^{\rm RR}_{(3)}$	$F^{\rm RR}_{(3)}$
	$F_{(2)}$	$F_{(2)}$	$\longleftrightarrow$	$F^{\rm RR}_{(2)1}$	
		$F_{(1)1}$	$\longleftrightarrow$	$d\chi_1$	$d\chi_1$
NS-NS fields	$G_{\mu\nu}$	$\mathcal{F}_{(2)}$	$\longleftrightarrow$	$F^{\rm NS}_{(2)1}$	$F^{\rm NS}_{(3)}$
	$F_{(3)}$	$F_{(3)}$	$\longleftrightarrow$	$F^{\rm NS}_{(3)}$	
		$F_{(2)1}$	$\longleftrightarrow$	$\mathcal{F}_{(2)}$	$G_{\mu\nu}$

Table 1: Fields of the truncated type II theories in $D=6$ and $D=5$

Our ansatz for the reduction of the metric from $D=6$ to $D=5$ is

$$ds_6^2 = e^{-\varphi/\sqrt{6}}\, ds_5^2 + e^{\sqrt{\frac{3}{2}}\varphi}\,(dz + \mathcal{A}_{(1)})^2\ . \tag{2.5}$$

We find that the dimensional reductions of the two Lagrangians (2.2) and (2.3) become equivalent, after making the identifications given in Table 1, provided that the dilatons of the two theories are related by the orthogonal transformation

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \varphi \end{pmatrix}_{IIA} = \Lambda \begin{pmatrix} \phi_1 \\ \phi_2 \\ \varphi \end{pmatrix}_{IIB} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\sqrt{\frac{3}{8}} \\ -\frac{1}{4} & \frac{3}{4} & -\sqrt{\frac{3}{8}} \\ -\sqrt{\frac{3}{8}} & -\sqrt{\frac{3}{8}} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \varphi \end{pmatrix}_{IIB}\ . \tag{2.6}$$

Note that this matrix satisfies $\Lambda = \Lambda^{-1}$. In terms of the string metrics, the radius of the compactifying circle is given by $R = e^{\frac{1}{4}\phi_1 + \frac{1}{4}\phi_2 + \sqrt{\frac{3}{8}}\varphi}$. It is easily seen that under the transformation (2.6) we have $R_{IIA} = 1/R_{IIB}$.

It is worth remarking that the Lagrangians (2.2) and (2.3) are both consistent truncations of maximal supergravity. It follows that their respective solutions that are related by T-duality transformation are all solutions of the untruncated maximal supergravity. If instead we consider the two Lagrangians (2.2) and (2.3) as consistent truncations of the K3 compactifications of the type IIB and type IIA supergravities, then their solutions that are related by T-duality remain as distinct solutions of the original type IIB and type IIA supergravities.

In the rest of the paper, we shall refer to the two Lagrangians (2.2) and (2.3) as the type IIB and the type IIA descriptions of the six-dimensional truncated theories.

3 S^3 (un)twisted and squashed

In the previous section, we obtained truncated six-dimensional type IIB and type IIA Lagrangians, and their T-duality relation. We are now in a position to consider the $\mathrm{AdS}_3\times S^3$ solution. The Lagrangian (2.2) admits dyonic string solutions supported either by the NS-NS 3-form $F_{(3)}^{\rm NS}$ or the R-R 3-form $F_{(3)}^{\rm RR}$. More general solutions can be obtained by acting with the $O(2,2)$ symmetry of the theory, allowing us, in particular, to find solutions for dyonic strings carrying both NS-NS and R-R charges. We do this in detail in appendix C, obtaining an $O(2,2;\mathbb{Z})$ multiplet of dyonic strings.

Near the horizon, even though the above dyonic solutions carry four independent charges, the 3-forms $F_{(3)}^{\rm NS}$ and $F_{(3)}^{\rm RR}$ become self-dual, and the metric approaches that of $\mathrm{AdS}_3\times S^3$. In fact it is more convenient to construct these solutions directly. The dilatons ϕ_1 and ϕ_2 and the axions χ_1 and χ_2 are constant in the solution, and for simplicity we shall take them to be zero. The remaining equations are solved by taking the metric and 3-forms to be

$$\begin{aligned} ds_6^2 &= ds^2(\mathrm{AdS}) + ds^2(S^3)\ , \\ F_{(3)}^{\rm NS} &= \lambda\,\epsilon(\mathrm{AdS}) + \lambda\,\epsilon(S^3)\ , \\ F_{(3)}^{\rm RR} &= \mu\,\epsilon(\mathrm{AdS}) + \mu\,\epsilon(S^3)\ , \end{aligned} \tag{3.1}$$

where λ and μ are constants, and the metrics on the AdS_3 and S^3 have Ricci tensors given by

$$R_{\mu\nu} = -\tfrac12(\lambda^2+\mu^2)\,g_{\mu\nu}\ ,\qquad R_{mn} = \tfrac12(\lambda^2+\mu^2)\,g_{mn} \tag{3.2}$$

respectively. (It is easy to see from (2.2) that the equations of motion will only be satisfied by taking the dilatons to be constant if the coefficients in front of the volume forms $\epsilon(\mathrm{AdS})$ and $\epsilon(S^3)$ for the AdS_3 and S^3 metrics are equal, and hence the 3-form field strengths are self-dual.) The constants λ and μ are related to the magnetic charges as follows:

$$Q_{\rm NS}\equiv \tfrac{1}{16\pi^2}\int F_{(3)}^{\rm NS} = \frac{\lambda}{(\lambda^2+\mu^2)^{3/2}}\ ,\qquad Q_{\rm RR}\equiv \tfrac{1}{16\pi^2}\int F_{(3)}^{\rm RR} = \frac{\mu}{(\lambda^2+\mu^2)^{3/2}}\ . \tag{3.3}$$

In calculating the charges, we have made use of the fact that a 3-sphere whose Ricci tensor is given by R_{mn} in (3.2) has volume $16\pi^2(\lambda^2+\mu^2)^{-3/2}$, and in fact its metric can be written as

$$ds^2(S^3) = \frac{4}{\lambda^2+\mu^2}\,d\Omega_3^2\ , \tag{3.4}$$

where $d\Omega_3^2$ is the metric on a unit 3-sphere. Note that since we are taking the constant values of the two dilatons ϕ_1 and ϕ_2 to be zero for simplicity, this means that the electric charges (whose calculation we have not explicitly presented above) are equal to the magnetic charges. If one chooses non-zero values for the constants ϕ_1 and ϕ_2 then the field strengths are still self-dual, but the electric and magnetic charges will be unequal. Note that the charges will all be integers.

We now make use of the fact that the metric $d\Omega_3^2$ can be written as a $U(1)$ bundle over $CP^1 \sim S^2$ as follows:

$$d\Omega_3^2 = \tfrac{1}{4} d\Omega_2^2 + \tfrac{1}{4}(dz + B)^2 \ , \tag{3.5}$$

where $d\Omega_2^2$ is the metric on the unit 2-sphere, whose volume form $\Omega_{(2)}$ is given by $\Omega_{(2)} = dB$. (If $d\Omega_2^2$ is written in spherical polar coordinates as $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, then we can write B as $B = \cos\theta\, d\phi$.) The fibre coordinate z has period 4π. Thus the six-dimensional metric given in (3.1) can be written as

$$ds_6^2 = ds^2(\mathrm{AdS}) + \frac{1}{\lambda^2+\mu^2}\, d\Omega_2^2 + \frac{1}{\lambda^2+\mu^2}\,(dz+B)^2 \ . \tag{3.6}$$

The four-dimensional area of the horizon is given by

$$A \sim L\,(\lambda^2+\mu^2)^{-3/2} \ , \tag{3.7}$$

where L is the contribution from $ds^2(\mathrm{AdS})$ at the boundary at constant time. The field strengths in (3.1) can now be written as

$$\begin{aligned} F_{(3)}^{\rm NS} &= \lambda\,\epsilon(\mathrm{AdS}) + \frac{\lambda}{(\lambda^2+\mu^2)^{3/2}}\,\Omega_{(2)}\wedge(dz+B) \ , \\ F_{(3)}^{\rm RR} &= \mu\,\epsilon(\mathrm{AdS}) + \frac{\mu}{(\lambda^2+\mu^2)^{3/2}}\,\Omega_{(2)}\wedge(dz+B) \ . \end{aligned} \tag{3.8}$$

Comparing (3.6) with the general reduction ansatz (2.5), we see that if we dimensionally reduced on the fibre coordinate we obtain the 5-dimensional metric

$$ds_5^2 = (\lambda^2+\mu^2)^{-1/3}\, ds^2(\mathrm{AdS}) + (\lambda^2+\mu^2)^{-4/3}\, d\Omega_2^2 \ , \tag{3.9}$$

while the new dilaton φ is a constant, given by

$$e^{\varphi/\sqrt{6}} = (\lambda^2+\mu^2)^{-1/3} \ . \tag{3.10}$$

Comparing (3.8) with the reduction ansätze $F_{(n)} \to F_{(n)} + F_{(n-1)}\wedge(dz+B)$ for the field strengths, we find that in $D=5$ we have

$$\begin{aligned} &F_{(3)}^{\rm NS} = \lambda\,\epsilon(\mathrm{AdS}) \ , \qquad F_{(2)1}^{\rm NS} = \frac{\lambda}{(\lambda^2+\mu^2)^{3/2}}\,\Omega_{(2)} \ , \\ &F_{(3)}^{\rm RR} = \mu\,\epsilon(\mathrm{AdS}) \ , \qquad F_{(2)1}^{\rm RR} = \frac{\mu}{(\lambda^2+\mu^2)^{3/2}}\,\Omega_{(2)} \ , \\ &\mathcal{F}_{(2)} = dB = \Omega_{(2)} \ . \end{aligned} \tag{3.11}$$

We are now in a position to implement the T-duality transformation from the type IIB description to the type IIA description in $D = 5$. From appendix A, and (3.11), we see that the field strengths in the type IIA picture will be

$$\begin{aligned} &F_{(3)} = \lambda\,\epsilon(\mathrm{AdS})\ , \qquad \mathcal{F}_{(2)} = \frac{\lambda}{(\lambda^2+\mu^2)^{3/2}}\,\Omega_{(2)}\ ,\\ &F_{(3)1} = -\mu\,\epsilon(\mathrm{AdS})\ , \qquad F_{(2)} = \frac{\mu}{(\lambda^2+\mu^2)^{3/2}}\,\Omega_{(2)}\ ,\\ &F_{(2)1} = \Omega_{(2)}\ . \end{aligned} \tag{3.12}$$

From (2.6) and (3.10), together with the fact that we are taking $\phi_1 = \phi_2 = 0$ in the original type IIB solution, it follows that the dilatons in the type IIA picture will be given by

$$e^{\varphi} = (\lambda^2+\mu^2)^{1/\sqrt{6}}\ , \qquad e^{\phi_1} = e^{\phi_2} = (\lambda^2+\mu^2)^{1/2}\ . \tag{3.13}$$

Finally, we can oxidise the type IIA solution that we have just obtained back to $D = 6$, by retracing the standard Kaluza-Klein reduction steps. Doing so, we find that the six-dimensional metric in the type IIA picture is

$$ds_6^2 = (\lambda^2+\mu^2)^{-1/2}\,ds^2(\mathrm{AdS}) + (\lambda^2+\mu^2)^{-3/2}\Big[d\Omega_2^2 + \frac{\lambda^2}{\lambda^2+\mu^2}\,(dz'+B)^2\Big]\ , \tag{3.14}$$

where B is a potential such that $\Omega_{(2)} = dB$, and the coordinate z' is related to z by

$$z = \frac{\lambda}{(\lambda^2+\mu^2)^{3/2}}\,z' = Q_{\mathrm{NS}}\,z'\ . \tag{3.15}$$

(The last equality follows from (3.3).) It is straightforward to verify that the area of the horizon of the metric (3.14) is the same as that before the Hopf T-duality transformation, given by (3.7). The type IIA field strengths in $D = 6$ are given by

$$\begin{aligned} &F_{(4)} = -\mu\,\epsilon(\mathrm{AdS})\wedge(dz+\mathcal{A}_{(1)})\ , \qquad F_{(3)} = \lambda\,\epsilon(\mathrm{AdS}) + \Omega_{(2)}\wedge(dz+\mathcal{A}_{(1)})\ ,\\ &F_{(2)} = \frac{\mu}{(\lambda^2+\mu^2)^{3/2}}\,\Omega_{(2)}\ , \end{aligned} \tag{3.16}$$

where

$$\mathcal{A}_{(1)} = \frac{\lambda}{(\lambda^2+\mu^2)^{3/2}}\,B = Q_{\mathrm{NS}}\,B\ . \tag{3.17}$$

Note that in the above T-duality transformation the Buscher rules [50] are insufficient, since we have R-R fields as well as NS-NS fields involved in the solution. For this reason, we have constructed the two low-energy effective actions and explicitly derived the T-duality transformations that relate them.

We find that the charges carried by these field strengths are as follows:

$$
\begin{aligned}
Q^{(3)}_{\rm elec} &\equiv \tfrac{1}{16\pi^2}\int_{S^3} e^{-\phi_1-\phi_2}\,{*F_{(3)}} = Q_{\rm NS}\ ,\\
Q^{(3)}_{\rm mag} &\equiv \tfrac{1}{16\pi^2}\int_{S^3} F_{(3)} = 1\ ,\\
Q^{(4)}_{\rm elec} &\equiv \tfrac{1}{4\pi}\int_{S^2} e^{\frac12\phi_1-\frac32\phi_2}\,{*F_{(4)}} = -Q_{\rm RR}\ ,\\
Q^{(2)}_{\rm mag} &\equiv \tfrac{1}{4\pi}\int_{S^2} F_{(2)} = Q_{\rm RR}\ .
\end{aligned} \tag{3.18}
$$

If the fibre coordinate z' in (3.14) had had the period 4π, then the topology of the compact 3-space would have been S^3. Since it is related to z as given in (3.15), and z has period 4π, it follows that z' has period $4\pi/Q_{\rm NS}$, and hence the topology of the compact 3-space is $S^3/Z_{Q_{\rm NS}}$, the cyclic lens space of order $Q_{\rm NS}$.

On the other hand the magnetic charge carried by the field strength $F_{(3)}$ is equal to 1, having started, in the original solution, as $Q_{\rm NS}$. This is because the T-duality has exchanged the original charge $Q_{\rm NS}$ with the unit charge corresponding to the unit topological charge of the $U(1)$ bundle over S^2 that describes the 3-sphere. Furthermore, we can see from (3.14) that the metric on the lens space is not in general the "round" one, but is instead squashed along the $U(1)$ fibre direction, with a squashing factor ν given by

$$
\nu = \frac{\lambda}{\sqrt{\lambda^2+\mu^2}} = \frac{Q_{\rm NS}}{\sqrt{Q_{\rm NS}^2+Q_{\rm RR}^2}}\ . \tag{3.19}
$$

In other words, the metric on the compact 3-space is proportional to

$$
ds^2 = \tfrac14 d\Omega_2^2 + \tfrac14\nu^2\,(dz+B)^2\ , \tag{3.20}
$$

whose Ricci tensor, in the natural orthonormal basis, is given by

$$
R_{ab} = {\rm diag}\,(4-2\nu^2, 4-2\nu^2, 2\nu^2)\ . \tag{3.21}
$$

When $\nu=1$, the metric is Einstein and the 3-sphere or lens space is "round."

As we mentioned earlier, we could have considered original solutions in which the constant dilatons ϕ_1 and ϕ_2 were non-zero, in which case the original electric and magnetic charges need not have been equal. The lens space after the Hopf T-duality transformation will then be $S^3/Z_{Q_{\rm NS}^{\rm mag}}$. Also, we can generalise the starting point further by consider a solution on the product of AdS$_3$ and the lens space S^3/Z_n, rather than simply AdS$_3\times S^3$. (From the lower-dimensional point of view, this corresponds to giving the Kaluza-Klein vector a magnetic charge n rather than 1.) If we do this, then we find that a solution

$\mathrm{AdS}_3 \times S^3/Z_n$ for (2.2), with charges $Q^{\rm elec}_{\rm NS}$, $Q^{\rm mag}_{\rm NS}$, $Q^{\rm elec}_{\rm RR}$ and $Q^{\rm mag}_{\rm RR}$ will result, after the T-duality transformation, in a solution $\mathrm{AdS}_3 \times S^3/Z_{Q^{\rm NS}_{\rm mag}}$ for (2.3) with charges

$$Q^{(3)}_{\rm elec} = Q^{\rm elec}_{\rm NS} \ , \qquad Q^{(3)}_{\rm mag} = n \ , \qquad Q^{(4)}_{\rm elec} = -Q^{\rm elec}_{\rm RR} \ , \qquad Q^{(2)}_{\rm mag} = Q^{\rm mag}_{\rm RR} \ . \tag{3.22}$$

To summarise, we see that if we consider the case where the original $\mathrm{AdS}_3 \times S^3/Z_n$ solution in the type IIB description carries only an NS-NS charge whose magnetic component is $Q^{\rm NS}_{\rm mag} = m$, and so $Q_{\rm RR} = 0$, then after the Hopf T-duality transformation the solution for (2.3) will be of the form $\mathrm{AdS}_3 \times S^3/Z_m$, where the metric on the cyclic lens space is still the "round" one, and the magnetic charge becomes $Q^{\rm NS}_{\rm mag} = n$. In the special case where $n = m$ and $Q_{\rm RR} = 0$, the $\mathrm{AdS}_3 \times S^3/Z_n$ solution is invariant under the Hopf T-duality. If, on the other hand, we start with an $\mathrm{AdS}_3 \times S^3/Z_n$ solution with only R-R charges, then after the Hopf T-duality transformation the S^3/Z_n is completely "untwisted," and the solution will be of the form $\mathrm{AdS}_3 \times S^1 \times S^2$. (This is analogous to the untwisting of $\mathrm{AdS}_5 \times S^5$ discussed in [37] and the untwisting of $\mathrm{AdS}_4 \times S^7$ discussed in [38]. The untwisting of S^3 to $S^2 \times S^1$ was also discussed in [47].) In the generic case where the original $\mathrm{AdS}_3 \times S^3/Z_n$ solution carries both NS-NS and R-R charges, then after the T-duality transformation the solution will be of the form $\mathrm{AdS}_3 \times S^3/Z_m$, where the metric on the compact 3-space space is not only factored now by Z_m, but it is also squashed.

Although the construction of conformal field theories with background R-R charges is problematical, there is an exact CFT duality statement[3] in the case of pure NS-NS charge, *i.e.* when $\mu = 0$. This can be seen by looking at the original solution (3.1), and the final Hopf-dualised solution (3.14, 3.16), in the string-frame metric $ds_6^2(\text{string})$. This is related to the six-dimensional Einstein-frame metric ds_6^2 by $ds_6^2(\text{string}) = e^{\frac12(\phi_1+\phi_2)}\, ds_6^2$. Thus we find that the original solution can be written as

$$\begin{aligned} ds_6^2(\text{string}) &= ds^2(AdS) + \lambda^{-2}\left[d\Omega_2^2 + (dz+B)^2\right] \ , \\ F_{(3)} &= \lambda^{-2}\,\Sigma_{(3)} + \lambda^{-2}\,\Omega_2 \wedge (dz + B) \ , \end{aligned} \tag{3.23}$$

where $\Sigma_{(3)}$ is the volume form of the "unit" AdS_3, and that correspondingly in the final Hopf-dualised solution we have

$$\begin{aligned} ds_6^2(\text{string}) &= ds^2(AdS) + \lambda^{-2}\left[d\Omega_2^2 + (dz'+B)^2\right] \ , \\ F_{(3)} &= \lambda^{-2}\,\Sigma_{(3)} + \lambda^{-2}\,\Omega_2 \wedge (dz' + B) \ , \end{aligned} \tag{3.24}$$

[3]We are grateful to Costas Bachas for urging us to provide a CFT proof. A CFT discussion of the $D = 10$ superstring compactification on $S^3 \times S^3$ down to $\mathrm{AdS}_2 \times S^2$ may be found in [71]

where $z' = \lambda^2\, z = Q_{\rm NS}^{-1}\, z$. The T-duality can be understood in this case from the standard Buscher rules [50] applied to the $SU(2)$ WZW model. In general, one has the statement that a solution on S^3/Z_m with 3-form flux n is Hopf dual to a solution on S^3/Z_n with 3-form flux m. This is because the Kaluza-Klein vector in the S^3/Z_m solution carries m units of charge, whereas the winding mode vector coming from the 3-form carries n units of charge. The invariance under T-duality in the special case $n=m=1$ is discussed in [70].

4 AdS$_3$ (un)twisted and squashed

In the same way as odd-dimensional spheres can be viewed as $U(1)$ bundles over complex projective spaces, so we can view odd-dimensional AdS spacetimes as bundles over certain Lorentzianisations of the complex projective spaces. The case of AdS$_3$ is particularly simple, since in this case the base space is nothing but AdS$_2$.

Let us begin by considering the unit S^3, written as a $U(1)$ bundle over S^2:

$$ds^2 = \tfrac14 d\theta^2 + \tfrac14 \sin^2\theta\, d\phi^2 + \tfrac14 (d\psi + \cos\theta d\phi)^2\ . \tag{4.1}$$

Now perform an analytic continuation to a Lorentzian signature, by sending:

$$\theta \longrightarrow \tfrac12\pi - i\,\rho\ ,\qquad \psi \longrightarrow i\,x\ ,\qquad \phi \longrightarrow t\ . \tag{4.2}$$

This gives us the metric (after making an overall sign change to go from east-coast to west-coast notation)

$$ds^2 = -\tfrac14 \cosh^2\rho\, dt^2 + \tfrac14 d\rho^2 + \tfrac14 (dx + \sinh\rho\, dt)^2\ . \tag{4.3}$$

It is not hard to calculate the curvature for this metric, and to verify that it has Ricci tensor $R_{\mu\nu} = -2\, g_{\mu\nu}$. Thus it is AdS$_3$, since the cosmological constant is negative. Note that t is periodic, $0 \le t \le 2\pi$, but ρ and x both range over the entire real line. The parameterisation of points in AdS$_3$, viewed as the $O(2,2)$-invariant hyperboloid $X_1^2 + X_2^2 - X_3^2 - X_4^2 = 1$ in $\mathbb{R}^4$, can be given in terms of t, ρ and x as follows:

$$\begin{aligned}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &= \frac{1}{\sqrt2}\begin{pmatrix} \cos\frac12 t & \sin\frac12 t \\ -\sin\frac12 t & \cos\frac12 t \end{pmatrix}\begin{pmatrix} \cosh x_- \\ \cosh x_+ \end{pmatrix}\ ,\\ \begin{pmatrix} X_3 \\ X_4 \end{pmatrix} &= \frac{1}{\sqrt2}\begin{pmatrix} \cos\frac12 t & \sin\frac12 t \\ -\sin\frac12 t & \cos\frac12 t \end{pmatrix}\begin{pmatrix} \sinh x_- \\ \sinh x_+ \end{pmatrix}\ ,\end{aligned} \tag{4.4}$$

where $x_\pm \equiv \frac12(x \pm \rho)$. It can be shown that the coordinates t, ρ and x give a 1-1 mapping to points on the hyperboloid.

It is evident that we can view (4.3) as a bundle over a base space which is AdS_2, for which the unit-radius metric is $d\Sigma_2^2 = -\cosh^2\rho\, dt^2 + d\rho^2$. (We shall in general use $d\Sigma_n^2$ to denote the "unit" AdS_n metric, which is defined to be the one whose Ricci tensor has the form $R_{\mu\nu} = -(n-1)\, g_{\mu\nu}$. This is analogous to the definition of the unit n-sphere.) Thus we have

$$d\Sigma_3^2 = \tfrac{1}{4} d\Sigma_2^2 + \tfrac{1}{4}(dx + \widetilde{B})^2 \ , \tag{4.5}$$

where $d\widetilde{B} = \Sigma_{(2)}$. (We are defining $\Sigma_{(n)}$ to be the volume form on the unit AdS_n.)

Let us note here that there is also another closely related metric on AdS_3, which also allows one to do a reduction on the fibre coordinate x, namely

$$ds^2 = -\tfrac{1}{4} e^{2\rho}\, dt^2 + \tfrac{1}{4} d\rho^2 + \tfrac{1}{4}(dx + e^{\rho}\, dt)^2 \ . \tag{4.6}$$

This can be shown to be an AdS_3 metric with $R_{\mu\nu} = -2\, g_{\mu\nu}$. It is interesting because it arises by taking the near-horizon limit of the boosted dyonic string in $D = 6$ (see appendix C). (In other words, the intersection of the dyonic string with a wave. If reduced to $D = 5$, this boosted dyonic string becomes a 3-charge black hole; *i.e.* the Reissner-Nordstrøm black hole.)

The coordinate transformation that relates the two AdS_3 metrics (4.3) and (4.6) is the following:

$$\begin{aligned} e^{\tilde{\rho}} &= \sinh\rho + \cosh\rho\, \cos t \ , \\ \tilde{t}\, e^{\tilde{\rho}} &= \cosh\rho\, \sin t \ , \\ e^{\tilde{x}} &= \frac{(e^{\rho}\, \cot \tfrac{1}{2} t - 1)\, e^{x}}{(e^{\rho}\, \cot\tfrac{1}{2} t + 1)} \ , \end{aligned} \tag{4.7}$$

where the tilded coordinates here denote the coordinates in the metric (4.6). An important feature of both (4.3) and (4.6) is that there is no coordinate-dependent function multiplying the vielbein in the fibre direction, and so the circle in the $U(1)$ reduction will have a constant length.

This Hopf T-duality on the fibres of AdS_3 should be contrasted with the T-duality discussed in [52, 27], or with a T-duality on one of the horospherical coordinates of AdS. (See also [53].) In these two cases the metric is already diagonal, and the size of the compactifying circle is not constant, but instead depends on other coordinates of the AdS. It follows that after T-dualisation, the dual theory has a dilaton that is singular on the horizon, and hence so also is the metric. By contrast, in the Hopf dualisation of AdS_3 discussed above, the constant radius of the circle implies that the dilaton is non-singular, and the metric has no local curvature singularity. The difference is further highlighted by the analysis of the supersymmetry. In the case of the horospherical T-duality or the T-duality in [52, 27], supersymmetry is always (at least partially) broken. In the case of Hopf T-duality on AdS_3, however, the supersymmetry is either all preserved or all broken, depending on an orientation choice in the Hopf fibering. We shall discuss this, and make detailed comparisons between the various T-dualities, in section 6.

Note that we can also have the concept of "squashing" for AdS_3 where the length of the fibres is rescaled relative to the size of the base AdS_2. As in the case of the sphere, this will be an homogeneous squashing. Thus we may consider a squashed AdS_3 metric

$$ds^2 = \tfrac{1}{4}d\Sigma_2^2 + \tfrac{1}{4}\,\nu^2\,(dx + \widetilde{B})^2\ , \tag{4.8}$$

where ν is a constant squashing parameter. The vielbein components of the Ricci tensor in the natural orthonormal basis are

$$R_{ab} = \mathrm{diag}\,(4 - 2\nu^2, 2\nu^2 - 4, -2\nu^2)\ , \tag{4.9}$$

where the first entry is the R_{00} component. More generally, there also exist squashed versions of AdS_{2n+1} for any n, of the form $ds^2(\mathrm{AdS}_{2n+1}) = ds^2(\widetilde{CP}^n) + \nu^2\,(dx + \widetilde{B})^2$, where $\widetilde{CP}^n$ denotes a Lorentianisation of CP^n, and $d\widetilde{B}$ is the volume form on $\widetilde{CP}^n$.

All the previous steps of dimensional reduction on the fibres, which we did for the S^3 factor in $\mathrm{AdS}_3 \times S^3$, can now be repeated for the AdS_3 itself. The computations are essentially identical, since the dimension of the AdS_3 is the same as that of the S^3, with the exception of the details of the field strengths. The final result is that the solution $\mathrm{AdS}_3 \times S^3$ given by (3.1) becomes, after performing a T-duality transformation on the AdS_3 fibre coordinate x,

$$ds_6^2 = (\lambda^2 + \mu^2)^{-3/2}\left[d\Sigma_2^2 + \frac{\lambda^2}{\lambda^2 + \mu^2}\,(dx' + \widetilde{B})^2\right] + (\lambda^2 + \mu^2)^{-1/2}\,ds^2(S^3)\ , \tag{4.10}$$

where $\widetilde{B}$ is a potential such that $\Sigma_{(2)} = d\widetilde{B}$, and the coordinate x' is related to x by

$$x = \frac{\lambda}{(\lambda^2 + \mu^2)^{3/2}}\,x' = Q_{\mathrm{NS}}\,z'\ . \tag{4.11}$$

The type IIA field strengths in $D = 6$ are given by

$$\begin{aligned} &F_{(4)} = -\mu\,Q_{\mathrm{NS}}\,\epsilon(S^3) \wedge (dx' + \widetilde{B})\ , \qquad F_{(3)} = \lambda\,\epsilon(S^3) + Q_{\mathrm{NS}}\,\Sigma_{(2)} \wedge (dx' + \widetilde{B})\ , \\ &F_{(2)} = Q_{\mathrm{RR}}\,\Sigma_{(2)}\ . \end{aligned} \tag{4.12}$$

In particular, we see that in the case where the solution is supported purely by R-R 3-form charges, then after doing the T-duality transformation we arrive at an "untwisted" solution

$\mathrm{AdS}_2 \times S^1 \times S^3$. If the solution instead carries only NS-NS charges, then the structure of the T-duality transformed solution is essentially unchanged, in the sense that there are just overall rescalings $1/\lambda$ and $1/\lambda^3$ of the 3-sphere and AdS_3 factors in the metric. Since the fibre coordinate x here ranges over the entire real line, we do not really have the notion of a "lens space" for AdS. We could consider instead the spacetime that one obtains by taking the fibre coordinate x to be periodic. This will no longer be globally AdS_3, but instead only a part of it. If we define a spacetime U_n to be AdS_3 with x having a period L/n, for some specified L, then one could say that the T-dual of the solution $U_1 \times S^3$ carrying purely NS-NS charge $Q_{\rm NS}$ is the solution $U_{Q_{\rm NS}} \times S^3$. As in the case of the Hopf T-duality for S^3, all of the above discussion generalises straightforwardly to the case where we allow the dilaton moduli to be non-zero, so that there can be independent electric and magnetic charges for each field strength.

Note that the area of the horizon is preserved under the Hopf T-duality on the fibre coordinate x of the AdS_3. It is given by $A \sim (\lambda^2 + \mu^2)^{-2}$.

5 AdS$_3$ and S^3 (un)twisted and squashed

We may now put together the results of the previous two sections, by starting with an $\mathrm{AdS}_3 \times S^3$ solution (3.1), performing a T-duality transformation first using the $U(1)$ fibres of S^3, and then performing a second T-duality transformation using the fibres of AdS_3. After doing this, we find that the final metric is given by

$$ds_6^2 = \frac{1}{(\lambda^2+\mu^2)^2} \Big[d\Sigma_2^2 + \frac{\lambda^2}{\lambda^2+\mu^2}\, (dx' + \widetilde B)^2 + d\Omega_2^2 + \frac{\lambda^2}{\lambda^2+\mu^2}\, (dz' + B)^2 \Big]\,, \tag{5.1}$$

where z' is given by (3.15) and x' is given by (4.11). The field strengths in the final solution are given by

$$\begin{aligned} F_{(3)}^{\rm NS} &= \Omega_{(2)} \wedge (dz + \mathcal{A}_{(1)}) + \Sigma_{(2)} \wedge (dx + \widetilde{\mathcal{A}}_{(1)})\,, \\ F_{(3)}^{\rm RR} &= Q_{\rm RR}\, \Omega_{(2)} \wedge (dx + \widetilde{\mathcal{A}}_{(1)}) - Q_{\rm RR}\, \Sigma_{(2)} \wedge (dz + \mathcal{A}_{(1)})\,, \end{aligned} \tag{5.2}$$

where $\mathcal{A}_{(1)}$ is given by (3.17) and

$$\widetilde{\mathcal{A}}_{(1)} = \frac{\lambda}{(\lambda^2+\mu^2)^{3/2}}\, \widetilde B = Q_{\rm NS}\, \widetilde B\,. \tag{5.3}$$

If the NS-NS charge is zero, then in this doubly-transformed solution both the S^3 and the AdS_3 are untwisted, giving $(\mathrm{AdS}_2 \times S^1) \times (S^2 \times S^1)$. If instead the R-R charge is zero, the solution becomes twisted to $U_{Q_{\rm NS}} \times (S^3/Z_{Q_{\rm NS}})$. When the NS-NS and R-R charges are both present, the solution is squashed and twisted in both the AdS_3 and S^3 factors. Note that the squashing parameter is the same for both the AdS_3 and the S^3.

Again in this case, the Hopf T-duality transformation preserves the area of horizon, given by $A \sim (\lambda^2 + \mu^2)^{-2}$. It follows that the black hole entropy, which is a quarter of the area, is also invariant under the Hopf T-duality. In the case of a 5-dimensional black hole that oxidises to a boosted dyonic NS-NS string in $D = 6$, this invariance can be easily understood: It follows from (C.17) in this case that the black hole entropy is given by $S \sim \sqrt{Q_w\, q\, p} = \sqrt{n\, Q_e^{\rm NS}\, Q_m^{\rm NS}}$. As we have seen earlier, Hopf T-duality has the effect of interchanging n and $Q_e^{\rm NS}$, and hence the entropy is left invariant. In this paper, we have shown that in general the entropy of the black hole is preserved under Hopf T-duality, even when it is supported by R-R as well as NS-NS charges. Thus even though the metric on the AdS_3 or S^3 may be (un)twisted and squashed by the transformation, the area of the horizon is preserved.

6 Supersymmetry and Killing spinors

It is well known that T-duality transformations can break supersymmetries of p-brane solutions, at the level of the low-energy effective supergravity. For example, the near-horizon limit of the D3-brane in ten dimensions, which is of the form $\mathrm{AdS}_5 \times S^5$ and hence preserves all the supersymmetry, is T-dual to the near-horizon limit of a D4-brane, which is a product of a domain wall and a sphere (rather than an AdS and a sphere), and which breaks half of the supersymmetry. The phenomenon of supersymmetry breaking, at the level of supergravity, has also been seen in the Hopf T-duality transformations discussed in [37]. For example, the Hopf dualisation of $\mathrm{AdS}_5 \times S^5$ to $\mathrm{AdS}_5 \times CP^2 \times S^1$ breaks all the supersymmetry at the supergravity level [37]. In another example, the Hopf reduction of the $\mathrm{AdS}_4 \times S^7$ solution of $D = 11$ supergravity to the $\mathrm{AdS}_4 \times CP^3$ of $D = 10$ type IIA supergravity breaks either all of the eight supersymmetries, or two out of the eight, depending on the orientation of the internal manifold [51, 38].

Of course all these statements about supersymmetry breaking are made at the level of the massless Kaluza-Klein modes in the supergravity theory. In some cases, the supersymmetry will be restored when one includes the Kaluza-Klein massive modes or the string winding modes. To analyse the behaviour of supersymmetry under T-duality, one should separate the discussion into two parts: firstly, compactification on a circle, and secondly, the T-duality transformation itself. Supersymmetry breaking, if it occurs at all, is a consequence of the compactification on the circle. The T-duality transformation in the full string theory always preserves whatever supersymmetry has survived the circle compactification. An example where compactification breaks supersymmetry is an AdS spacetime, written in horospherical coordinates $ds^2 = d\rho^2 + e^{2\rho}\,\eta_{\mu\nu}\,dx^\mu\,dx^\nu$, and compactified on one of the spatial x^μ coordinates. Although there is a translational isometry, thus allowing a circle compactification, one finds that half of the Killing spinors on the AdS spacetime depend linearly on x^μ, and thus the supersymmetries associated with these Killing spinors will be broken once the chosen x^μ coordinate is taken to be periodic [54]. The other half of the Killing spinors are independent of x^μ, and so half of the supersymmetries survive the compactification. This statement is true even after including all the massive Kaluza-Klein modes. A T-duality transformation will not result in any further breaking (or restoring) of supersymmetry.

A contrasting example is provided by the Hopf reduction on the $U(1)$ fibres of S^5 in the $\mathrm{AdS}_5 \times S^5$ solution of the type IIB theory. The fibre coordinate here is naturally periodic. Although at the level of the massless Kaluza-Klein modes the compactification on the $U(1)$ fibres breaks all the supersymmetry, it is restored once the massive Kaluza-Klein modes are included. If we perform a T-duality transformation on the fibre coordinate, the $\mathrm{AdS}_5 \times S^5$ solution is mapped to an $\mathrm{AdS}_5 \times CP^2 \times S^1$ solution in the type IIA theory. At the level of supergravity, even when the massive Kaluza-Klein modes are included, this type IIA solution breaks all the supersymmetries. However in the full string theory, where the string winding modes are also included, the full supersymmetry is reinstated [37].

Thus, for example, we see that there are two different kinds of T-duality transformations that can be performed on a D3-brane. One of these maps the D3-brane to a D4-brane. In this case, the near-horizon $\mathrm{AdS}_5 \times S^5$ limit of the D3-brane is mapped to a product of a domain wall and a 4-sphere, which is the near-horizon limit of the D4-brane. In this case it is one of the horospherical coordinates x^μ in AdS_5 that is periodically identified and used in the T-duality transformation. (Since the x^μ coordinates are in fact the world-volume coordinates of the D3-brane.) Thus half of the supersymmetry is already broken in the process of making the identification. In other words, the apparent discrepancy between the "enhanced supersymmetry" seen in the near horizon limit of an unwrapped D3-brane and the usual 1/2 supersymmetry of the near-horizon limit of the D4-brane is not the result of any breaking of supersymmetry by T-duality. Rather, it is simply a consequence of the fact that there *is* no "supersymmetry enhancement" in the near-horizon limit of a wrapped D3-brane [54]. The other kind of T-duality that can be performed on a D3-brane is on the fibre coordinate of the foliating 5-spheres in its 6-dimensional transverse space. In this case, the fibre coordinate is naturally periodic, and the "supersymmetry breaking" is a pure artifact of the low-energy supergravity approximation. In the full string theory, when the Kaluza-Klein and winding modes are included, there is no supersymmetry breaking at all [37].

At the level of the massless modes in the reduction of the supergravity theory, the Hopf T-duality transformations on $\mathrm{AdS}_3\times S^3$ that we have been considering in this paper either preserve all the supersymmetry, or they break all the supersymmetry, depending upon an orientation associated with the Hopf reduction. In this section, we shall demonstrate this by giving an explicit construction of the Killing spinors on AdS_3 and S^3. These have been given previously [54, 55, 56], but in coordinate systems that are not convenient for our present purposes. The relation between the coordinates used in [56], in which the metric on the unit 3-sphere is $d\Omega_3^2 = d\theta_3^2 + \sin^2\theta_3\,(d\theta_2^2+\sin^2\theta_2\,d\theta_1^2)$, and the coordinates in (4.1), is $\theta_1=\frac{1}{2}(\psi+\phi)$, $\cot\theta_2=\tan\frac{1}{2}\theta\,\cos\frac{1}{2}(\psi-\phi)$, $\cos\theta_3=\sin\frac{1}{2}\theta\,\sin\frac{1}{2}(\psi-\phi)$. In principle, the Killing spinors obtained in [56] can be re-expressed in terms of the coordinates of the metric (4.1) using these relations, but the result will be in an inconvenient local Lorentz frame.

We begin by constructing the Killing spinors for the unit S^3, using the metric given in (4.1). The vielbein and spin connection are given by

$$
\begin{aligned}
&e^1=\tfrac{1}{2}d\theta\ ,\qquad e^2=\tfrac{1}{2}\sin\theta\,d\phi\ ,\qquad e^3=\tfrac{1}{2}(d\psi+\cos\theta\,d\phi)\ ,\\
&\omega_{23}=-e^1\ ,\qquad \omega_{31}=-e^2\ ,\qquad \omega_{12}=-2\cot\theta\,e^2+e^3\ .
\end{aligned}
\tag{6.1}
$$

The Killing spinor equation is $D_\mu\,\epsilon^\pm=\pm\frac{\mathrm{i}}{2}\,\Gamma_\mu\,\epsilon^\pm$. In a given choice of conventions in the $\mathrm{AdS}_3\times S^3$ supergravity solution, either the ϵ^+ Killing spinors or the ϵ^- Killing spinors will be the ones that are associated with unbroken supersymmetries. Since the round S^3 is invariant under orientation reversal, equal numbers of Killing spinors ϵ^+ and ϵ^- exist (namely two of each). Substituting (6.1) into the Killing spinor equation, we find, in the basis where $\Gamma_1=\sigma_3$, $\Gamma_2=\sigma_1$ and $\Gamma_3=\sigma_2$, with σ_i denoting the standard Pauli matrices, that the two sets of Killing spinors are

$$
\epsilon_1^+=\begin{pmatrix} e^{\frac{\mathrm{i}}{2}(\phi+\theta)}\\ \mathrm{i}\,e^{\frac{\mathrm{i}}{2}(\phi-\theta)}\end{pmatrix}\ ,\qquad \epsilon_2^+=\begin{pmatrix} e^{\frac{\mathrm{i}}{2}(\theta-\phi)}\\ -\mathrm{i}\,e^{-\frac{\mathrm{i}}{2}(\theta+\phi)}\end{pmatrix}\ ,\tag{6.2}
$$

$$
\epsilon_1^-=\begin{pmatrix} e^{\frac{\mathrm{i}}{2}\psi}\\ -\mathrm{i}\,e^{\frac{\mathrm{i}}{2}\psi}\end{pmatrix}\ ,\qquad \epsilon_2^-=\begin{pmatrix} e^{-\frac{\mathrm{i}}{2}\psi}\\ \mathrm{i}\,e^{-\frac{\mathrm{i}}{2}\psi}\end{pmatrix}\ ,\tag{6.3}
$$

Thus we see that if the conventions have been chosen so that the ϵ^+ Killing spinors are associated with unbroken supersymmetries, then they survive the dimensional reduction to $D = 5$ since they are independent of ψ. Under these circumstances, the supergravity solution after the Hopf T-duality transformation on the ψ coordinate will still be fully supersymmetric. If, on the other hand, the conventions have been chosen so that the ϵ^- Killing spinors are associated with unbroken supersymmetries, then they will not survive the dimensional reduction process since they both depend on ψ. In this case, all the supersymmetry will be broken in the supergravity solution after the Hopf T-duality transformation. Note that in our discussion here, we have made a fixed orientation choice for our Hopf reduction (implicit in our definition of the vielbeins in (6.1), and so the two supersymmetry possibilities arise from two possible convention choices for the original supergravity solution. Equivalently, one could think of making a fixed convention choice in the original solution, and the two supersymmetry possibilities would then arise from the two possible orientations in the Hopf reduction.

In order to discuss the supersymmetry in the case of the T-duality transformations on the Hopf fibres of AdS_3, we need first to give an analogous construction of the Killing spinors in AdS_3. On the unit AdS_3, these satisfy $D_\mu\, \epsilon^\pm = \pm\frac{1}{2}\Gamma_\mu\, \epsilon^\pm$. Let us first consider the metric (4.3), for which the vielbein and spin connection will be

$$\begin{aligned} &e^0 = \tfrac{1}{2}\cosh\rho\, dt\ , \qquad e^1 = \tfrac{1}{2} d\rho\ , \qquad e^2 = \tfrac{1}{2}(dx + \sinh\rho\, dt)\ , \\ &\omega_{01} = -2\tanh\rho\, e^0 + e^2\ , \qquad \omega_{02} = e^1\ , \qquad \omega_{12} = -e^0\ . \end{aligned} \tag{6.4}$$

Taking the Dirac matrices to be $\Gamma_0 = -\mathrm{i}\,\sigma_1$, $\Gamma_1 = \sigma_3$ and $\Gamma_2 = \sigma_2$, we find that the solutions for the Killing spinors are

$$\epsilon_1^+ = \begin{pmatrix} e^{\frac{1}{2}(\rho+\mathrm{i}t)} \\ -e^{\frac{1}{2}(-\rho+\mathrm{i}t)} \end{pmatrix}\ , \qquad \epsilon_2^+ = \begin{pmatrix} e^{\frac{1}{2}(\rho-\mathrm{i}t)} \\ e^{-\frac{1}{2}(\rho+\mathrm{i}t)} \end{pmatrix}\ , \tag{6.5}$$

$$\epsilon_1^- = \begin{pmatrix} e^{\frac{1}{2}x} \\ -\mathrm{i}\, e^{\frac{1}{2}x} \end{pmatrix}\ , \qquad \epsilon_2^- = \begin{pmatrix} e^{-\frac{1}{2}x} \\ \mathrm{i}\, e^{-\frac{1}{2}x} \end{pmatrix}\ . \tag{6.6}$$

Again we see that the Hopf T-duality transformation on the fibre coordinate x will preserve either all or none of the supersymmetry, depending upon the orientation.

It is also instructive to construct the Killing spinors for the AdS_3 metric (4.6), since this is the one that arises as the near-horizon limit of the boosted dyonic string. For this, the vielbein and spin connection will be:

$$\begin{aligned} &e^0 = \tfrac{1}{2} e^\rho\, dt\ , \qquad e^1 = \tfrac{1}{2} d\rho\ , \qquad e^2 = \tfrac{1}{2}(dx + e^\rho\, dt)\ , \\ &\omega_{01} = -2\, e^0 + e^2\ , \qquad \omega_{02} = e^1\ , \qquad \omega_{12} = -e^0\ . \end{aligned} \tag{6.7}$$

Elementary calculations then show that the Killing spinors $\epsilon^\pm$, satisfying $D_\mu\,\epsilon^\pm = \pm\frac{1}{2}\,\Gamma_\mu\,\epsilon^\pm$, are given by

$$\epsilon_1^+ = \begin{pmatrix} t\,e^{\frac{1}{2}\rho} \\ \mathrm{i}\,e^{-\frac{1}{2}\rho} \end{pmatrix}\,, \qquad \epsilon_2^+ = \begin{pmatrix} e^{\frac{1}{2}\rho} \\ 0 \end{pmatrix}\,, \tag{6.8}$$

$$\epsilon_1^- = \begin{pmatrix} e^{\frac{1}{2}x} \\ -\mathrm{i}\,e^{\frac{1}{2}x} \end{pmatrix}\,, \qquad \epsilon_2^- = \begin{pmatrix} e^{-\frac{1}{2}x} \\ \mathrm{i}\,e^{-\frac{1}{2}x} \end{pmatrix}\,. \tag{6.9}$$

Here also, we see that the two ϵ^+ Killing spinors are independent of the fibre coordinate x, while the two ϵ^- Killing spinors depend on x. It is worth remarking that the metric (4.6) is in some ways reminiscent of the horospherical metric on AdS_3, and indeed this reflects itself in certain similarities in the form of the solutions for the Killing spinors [54].

The orientation dependence of the supersymmetry is echoed in $D = 4$, where 4-charge black hole solutions, can either preserve 1/8 of the supersymmetry or break it entirely, depending on the relative signs of the charges [18, 57]. These black holes are all related by U-duality to a black hole in which two of the charges, one electric and one magnetic, are carried by the two Kaluza-Klein vectors coming from the reduction from $D = 6$. Upon oxidation to $D = 6$, this gives a metric whose near-horizon limit is $\mathrm{AdS}_3 \times S^3$, with the two Kaluza-Klein vectors providing the twisting of the S^3 and the AdS_3 fibres. Thus the sign of the fibre orientations is precisely related to the signs of the charges in $D = 4$.

The above discussion is at the level of the massless Kaluza-Klein modes in the supergravity theory. If the orientation for the Hopf fibration is such that the relevant Killing spinors do not depend on the fibre coordinate, then statement of preservation of full supersymmetry under the Hopf T-duality extends to the full string theory. If the orientation is opposite, so that all the relevant Killing spinors depend on the fibre coordinate, then the discussion bifurcates into two categories in the full string theory. If the Hopf reduction is on the (naturally periodic) $U(1)$ fibres of the S^3, then, as we see from (6.3), the Killing spinors will be included once the non-zero modes in the Kaluza-Klein Fourier expansions are taken into account. In this case, the full supersymmetry preservation under Hopf T-duality is reinstated by including all the Kaluza-Klein modes. On the other hand, if the Hopf reduction is on the fibre coordinate of AdS_3, then, as we can see from (6.6) or (6.9), the Killing spinors will still be excluded from the spectrum even after the Kaluza-Klein non-zero modes are included. In this case, therefore, the supersymmetry remains broken even in the full string theory.

7 Non-dilatonic black holes, and Hopf T-duality

Non-dilatonic black holes (for which the dilatons are finite on the horizon) in maximal supergravity arise in $D = 5$ and $D = 4$. In $D = 5$, they are supported by three field strengths, and associated with each is an independent harmonic function. There are a total of 45 possible field configurations that can give rise to five-dimensional non-dilatonic black holes [58], *viz.*

$$D = 5: \quad \{F_{(2)ij}, F_{(2)k\ell}, F_{(2)mn}\}_{15} \ , \quad \{*F_{(3)i}, \mathcal{F}^j_{(2)}, F_{(2)ij}\}_{30} \ . \tag{7.1}$$

(We are using the notation of [48, 49, 58] here. The subscripts on each set of field strengths denotes the multiplicities of the solutions. The Hodge duals indicate that the associated fields carry electric charges if the fields without duals carry magnetic charges, and *vice versa*.) In $D = 4$, non-dilatonic black holes are supported by four field strengths, and can arise with the following possible field-strength configurations [58]:

$$\begin{aligned} N = 4: \quad & \{F_{(2)ij}, F_{(2)k\ell}, F_{(2)mn}, *\mathcal{F}^p_{(2)}\}_{105+105} \ , \quad \{F_{(2)ij}, *F_{(2)ik}, \mathcal{F}^j_{(2)}, *\mathcal{F}^k_{(2)}\}_{210} \ , \\ & \{F_{(2)ij}, F_{(2)k\ell}, *F_{(2)ik}, *F_{(2)j\ell}\}_{210} \ . \end{aligned} \tag{7.2}$$

The near-horizon regions of these $D = 5$ and $D = 4$ black holes are $\mathrm{AdS}_2 \times S^3$ and $\mathrm{AdS}_2 \times S^2$ respectively. If we dimensionally oxidise these solutions to $D = 6$, they will describe the intersections of p-branes, waves and NUTS. There are four possible near-horizon limits that can arise for these intersections, namely

$$\mathrm{AdS}_3 \times S^3,\ \mathrm{AdS}_3 \times (S^2 \times S^1),\ (\mathrm{AdS}_2 \times S^1) \times S^3,\ (\mathrm{AdS}_2 \times S^1) \times (S^2 \times S^1) \ . \tag{7.3}$$

(To be precise, the AdS_3 and the S^3 can in general be factored by cyclic groups, in the manner discussed previously.) If we oxidise these near-horizon solutions further, to $D = 10$ or $D = 11$, then the additional dimensions provide additional factors of T^4 or T^5 respectively.

For example, the four-charge black hole solution using the field strengths of the last entry in the list (7.2) becomes an intersection of p-branes in $D = 6$, and its near-horizon region is $(\mathrm{AdS}_2 \times S^1) \times (S^2 \times S^1)$. This is because the Kaluza-Klein vector is not involved in the solution, and so the six-dimensional metric is diagonal. If the solution is built using the set of field strengths in the first entry of the list (7.2), two possibilities can arise. If the Kaluza-Klein field $\mathcal{F}^p_{(2)}$ carries a magnetic charge (implying that the other three field strengths carry electric charges), then the solution becomes $(\mathrm{AdS}_2 \times S^1) \times S^3$. In other words, the Kaluza-Klein vector describes a magnetic monopole which corresponds, from the six-dimensional point of view, to a NUT charge that twists the $S^2 \times S^1$ product to give

S^3. On other hand, if the Kaluza-Klein field carries an electric charge (and so the other three field strengths carry magnetic charges), then the solution describes $\mathrm{AdS}_3 \times (S^2 \times S^1)$. This is because in this case the Kaluza-Klein vector has a configuration which, from the six-dimensional point of view, corresponds to a wave which twists the $\mathrm{AdS}_2 \times S^1$ product to give AdS_3. The oxidation of the solutions for the configurations of field strengths listed in the second entry in (7.2) describe $\mathrm{AdS}_3 \times S^3$ in $D = 6$. This is because there are two Kaluza-Klein fields in this case, *viz.* $\mathcal{F}^j_{(2)}$ and $\mathcal{F}^k_{(2)}$. One of them carries a magnetic charge and hence twists the $S^2 \times S^1$ product, while the other carries an electric charge, and hence twists the $\mathrm{AdS}_2 \times S^1$ product.

We showed in sections 3, 4 and 5 that the near-horizon structures (7.3) of the six-dimensional intersections that come from the oxidations of the non-dilatonic black holes in $D = 5$ and $D = 4$ are related to each other through Hopf T-duality. Furthermore, even the near-horizon limits of the 4-charge solutions which are, owing to sign choices, non-supersymmetric, are related by Hopf T-duality to the supersymmetric ones. As we show in appendix D, the Hopf T-duality not only relates the near-horizon limits listed in and below (7.3), but also relates the associated full solutions. Thus in particular, a Hopf reduction and T-duality has the effect of mapping the solutions (7.1) and (7.2) among each other. The harmonic function associated with the Kaluza-Klein vector coming from the Hopf reduction lacks a constant term; however, it can be introduced by performing an appropriate U-duality transformation [59, 8, 60, 61].

Appendices

A T-duality of the truncated six-dimensional theories

In section 2, we obtained two different consistent truncations of six-dimensional maximal supergravity. One of them, given by (2.2), naturally arose as a consistent truncation of six-dimensional maximal supergravity in the type IIB picture. This truncated theory has an $O(2, 2)$ global symmetry. The other theory, given by (2.3), naturally arose as a consistent truncation of six-dimensional maximal supergravity in the type IIA picture. This theory has only an $\mathbb{R} \times \mathbb{R}$ global symmetry. Interestingly, it can be obtained as the dimensional reduction of the seven-dimensional Lagrangian (2.4), which itself can easily be shown to be a consistent truncation of seven-dimensional maximal supergravity. In this appendix, we shall show that the two different six-dimensional Lagrangians (2.2) and (2.3) are T-dual to each other, in the sense that upon dimensional reduction they give rise to the same five-dimensional theory, up to field redefinitions.

We begin by obtaining the dimensional reduction of the Lagrangian (2.2), coming from the consistent truncation in the type IIB picture. We find

$$\begin{aligned} e^{-1}\,\mathcal{L}_{5B} &= R-\tfrac12(\partial\phi_1)^2-\tfrac12(\partial\phi_2)^2-\tfrac12(\partial\varphi)^2-\tfrac12 e^{2\phi_1}\,(\partial\chi_1)^2-\tfrac12 e^{2\phi_2}\,(\partial\chi_2)^2\\ &\quad -\tfrac{1}{12}e^{-\phi_1-\phi_2+\sqrt{\frac23}\varphi}\,(F^{\rm NS}_{(3)})^2-\tfrac{1}{12}e^{\phi_1-\phi_2+\sqrt{\frac23}\varphi}\,(F^{\rm RR}_{(3)})^2\\ &\quad -\tfrac14 e^{-\phi_1-\phi_2-\sqrt{\frac23}\varphi}\,(F^{\rm NS}_{(2)1})^2-\tfrac14 e^{\phi_1-\phi_2-\sqrt{\frac23}\varphi}\,(F^{\rm RR}_{(2)1})^2\\ &\quad -\tfrac14 e^{\sqrt{\frac83}\varphi}\,(\mathcal{F}_{(2)})^2+\chi_2\,dA^{\rm NS}_{(2)}\wedge dA^{\rm RR}_{(1)1}-\chi_2\,dA^{\rm RR}_{(2)}\wedge dA^{\rm NS}_{(1)1}\,. \end{aligned} \tag{A.1}$$

Here, the field strengths are given by

$$\begin{aligned} &F^{\rm NS}_{(3)}=dA^{\rm NS}_{(2)}-dA^{\rm NS}_{(1)1}\wedge\mathcal{A}_{(1)}\,,\qquad F^{\rm NS}_{(2)1}=dA^{\rm NS}_{(1)1}\,,\qquad \mathcal{F}_{(2)}=d\mathcal{A}_{(1)}\,,\\ &F^{\rm RR}_{(3)}=dA^{\rm RR}_{(2)}-dA^{\rm RR}_{(1)1}\mathcal{A}_1+\chi_1(dA^{\rm NS}_{(2)}-dA^{\rm NS}_{(1)1}\wedge\mathcal{A}_{(1)})\,,\quad F^{\rm RR}_{(2)1}=dA^{\rm RR}_{(1)1}+\chi_1 dA^{\rm NS}_{(1)1}\,. \end{aligned} \tag{A.2}$$

The five-dimensional Lagrangian coming from the dimensional reduction of the truncated theory (2.3) is given by

$$\begin{aligned} e^{-1}\,\mathcal{L}_{5A} &= R-\tfrac12(\partial\phi_1)^2-\tfrac12(\partial\phi_2)^2-\tfrac12(\partial\varphi)^2\\ &\quad -\tfrac12 e^{-\frac12\phi_1+\frac32\phi_2-\sqrt{\frac32}\varphi}(\partial\chi')^2-\tfrac12 e^{\frac32\phi_1-\frac12\phi_2-\sqrt{\frac32}\varphi}(\partial A_{(0)1})^2\\ &\quad -\tfrac{1}{12}e^{-\phi_1-\phi_2+\sqrt{\frac23}\varphi}\,(F_{(3)})^2-\tfrac{1}{12}e^{\frac12\phi_1-\frac32\phi_2-\frac{1}{\sqrt6}\varphi}\,(F_{(3)1})^2\\ &\quad -\tfrac14 e^{-\phi_1-\phi_2-\sqrt{\frac23}\varphi}\,(F_{(2)1})^2-\tfrac14 e^{\frac32\phi_1-\frac12\phi_2+\frac{1}{\sqrt6}\varphi}\,(F_{(2)})^2\\ &\quad -\tfrac14 e^{\sqrt{\frac83}\varphi}(\mathcal{F}_{(2)})^2+\chi'\,F_{(3)1}\wedge\mathcal{F}_2+\chi'\,F_{(3)}\wedge F_{(2)}\,, \end{aligned} \tag{A.3}$$

where we have dualised the 3-form potential $A_{(3)}$ to an axion χ'. The field strengths in (A.4) are given given by

$$\begin{aligned} &F_{(3)1}=dA_{(2)1}+dA_{(1)1}\wedge dA_{(1)}-A_{(0)1}\,dA_{(2)}\,,\qquad F_{(2)1}=dA_{(1)1}\,,\qquad F_{(1)1}=dA_{(0)1}\,,\\ &F_{(3)}=dA_{(2)}-dA_{(1)1}\wedge\mathcal{A}_{(1)}\,,\qquad F_{(2)}=dA_{(1)}-dA_{(0)1}\wedge\mathcal{A}_{(1)}\,. \end{aligned} \tag{A.4}$$

It can be verified that the two 5-dimensional Lagrangians (A.1) and (A.3) are related to each other by the field redefinition described in Table 1 and (2.6). To be precise, we first make the following field redefinitions:

$$A'_{(1)}=A_{(1)}-A_{(0)1}\,\mathcal{A}_{(1)}\,,\qquad A'_{(2)}=A_{(2)}-A_{(1)1}\wedge\mathcal{A}_{(1)}\,,\qquad A'_{(2)1}=A_{(2)1}+A_{(1)1}\wedge A'_{(1)}\,. \tag{A.5}$$

After doing this we find that the Lagrangian (A.3) can be mapped to the Lagrangian (A.1) by the following transformations

$$\begin{aligned}
&A'_{(2)} \longrightarrow A^{\rm NS}_{(2)}\ , \quad A'_{(2)1} \longrightarrow -A^{\rm RR}_{(2)}\ , \quad \mathcal{A}_1 \longrightarrow A^{\rm NS}_{(1)1}\ , \quad A'_{(1)} \longrightarrow A^{\rm RR}_{(1)1}\ ,\\
&A_{(1)1} \longrightarrow \mathcal{A}_{(1)}\ , \quad A_{(0)1} \longrightarrow \chi_1\ , \quad \chi' \longrightarrow \chi_2\ ,
\end{aligned} \tag{A.6}$$

together with the transformation of the dilatons given by (2.6).

B $O(2,2)$ symmetry of the truncated six-dimensional theory

Here, we give the explicit $O(2,2) \sim SL(2,\mathbb{R})_1 \times SL(2,\mathbb{R})_2$ global symmetry transformations for the truncated six-dimensional theory (2.2). The factor $SL(2,\mathbb{R})_1$ is an S-duality symmetry that maps between the NS-NS and R-R 2-form potentials, and is realised at the level of the Lagrangian. The factor $SL(2,\mathbb{R})_2$ is an electric/magnetic duality symmetry between the NS-NS and R-R 3-form field strengths, which is realised only at the level of the equations of motion.

To present the global transformation rules, it is useful first to define the two complex scalar fields

$$\tau_1 \equiv \chi_1 + {\rm i}\, e^{-\phi_1}\ , \qquad \tau_2 \equiv \chi_2 + {\rm i}\, e^{-\phi_2}\ . \tag{B.1}$$

The two $SL(2,\mathbb{R})$ transformations act non-linearly on the scalar manifold as follows:

$$\begin{aligned}
SL(2,\mathbb{R})_1: &\qquad \tau_1 \longrightarrow \Lambda_1\cdot\tau_1 \equiv \frac{a_1\,\tau_1 + b_1}{c_1\,\tau_1 + d_1}\ , \qquad \tau_2 \longrightarrow \tau_2\ ,\\
SL(2,\mathbb{R})_2: &\qquad \tau_2 \longrightarrow \Lambda_2\cdot\tau_2 \equiv \frac{a_2\,\tau_2 + b_2}{c_2\,\tau_2 + d_2}\ , \qquad \tau_1 \longrightarrow \tau_1\ ,
\end{aligned} \tag{B.2}$$

where $a_1\,d_1 - b_1\,c_1 = 1 = a_2\,d_2 - b_2\,c_2$, and we may define the $SL(2,\mathbb{R})$ matrices Λ_1 and Λ_2 in the standard way:

$$\Lambda_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\ , \qquad \Lambda_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\ . \tag{B.3}$$

$SL(2,\mathbb{R})_1$ is a symmetry of the Lagrangian, and it acts linearly on the 2-form potentials:

$$A_{(2)} \equiv \begin{pmatrix} A^{\rm NS}_{(2)} \\ A^{\rm RR}_{(2)} \end{pmatrix} \longrightarrow (\Lambda_1^T)^{-1}\, A_{(2)}\ . \tag{B.4}$$

$SL(2,\mathbb{R})_2$ is a symmetry only at the level of the equations of motion, and it acts locally only on the field strengths. We shall also, for convenience, present the action of $SL(2,\mathbb{R})_1$ on the field strengths. We begin by defining two field-strength doublets, $H^1_{(3)}$ and $H^2_{(3)}$:

$$H^1_{(3)} = \begin{pmatrix} e^{-\phi_1}\, F^{\rm NS}_{(3)} + \chi_1\, e^{\phi_1}\, F^{\rm RR}_{(3)} \\ e^{\phi_1}\, F^{\rm RR}_{(3)} \end{pmatrix}\ , \qquad H^2_{(3)} = \begin{pmatrix} e^{-\phi_2}\, {*F^{\rm NS}_{(3)}} + \chi_2\, F^{\rm RR}_{(3)} \\ F^{\rm RR}_{(3)} \end{pmatrix}\ . \tag{B.5}$$

Their $SL(2,\mathbb{R})$ transformations are:

$$\begin{aligned} SL(2,\mathbb{R})_1: &\qquad H^1_{(3)} \longrightarrow \Lambda_1\, H^1_{(3)}\ , \\ SL(2,\mathbb{R})_2: &\qquad H^2_{(3)} \longrightarrow \Lambda_2\, H^2_{(3)}\ . \end{aligned} \tag{B.6}$$

Note that the two fields $H^1_{(3)}$ and $H^2_{(3)}$ are not independent, and so given the transformation on one, the transformation on the other is in principle determined. Each was introduced for the specific purpose of encoding one or other of the two $SL(2,\mathbb{R})$ transformations in a simple way, as indicated in (B.6), and we do not need to give the associated transformations on the other $H_{(3)}$ field.

The theory can describe strings that carry four independent charges, namely the electric and the magnetic charges for both the NS-NS and the R-R 3-forms. It is easily seen from the equations of motion following from (2.2), and from the Bianchi identities for $F^{\rm NS}_{(3)}$ and $F^{\rm RR}_{(3)}$, that they are given by

$$\begin{aligned} Q^{\rm NS}_e &= \tfrac{1}{16\pi^2}\int \Big\{ e^{-\phi_1-\phi_2} {*F^{\rm NS}_{(3)}} + \chi_2\, F^{\rm RR}_{(3)} + \chi_1 \Big(e^{\phi_1-\phi_2} {*F^{\rm RR}_{(3)}} - \chi_2\, F^{\rm NS}_{(3)}\Big)\Big\}\ , \\ Q^{\rm NS}_m &= \tfrac{1}{16\pi^2}\int F^{\rm NS}_{(3)}\ , \\ Q^{\rm RR}_e &= \tfrac{1}{16\pi^2}\int \Big\{ e^{\phi_1-\phi_2} {*F^{\rm RR}_{(3)}} - \chi_2\, F^{\rm NS}_{(3)}\Big\}\ , \\ Q^{\rm RR}_m &= \tfrac{1}{16\pi^2}\int \Big\{ F^{\rm RR}_{(3)} - \chi_1\, F^{\rm NS}_{(3)}\Big\}\ . \end{aligned} \tag{B.7}$$

¿From the $SL(2,\mathbb{R})_1$ and $SL(2,\mathbb{R})_2$ transformations rules (B.2) and (B.6), we find that these charges transform as:

$$\begin{aligned} SL(2,\mathbb{R})_1: &\qquad \vec Q_{ee} \equiv \begin{pmatrix} Q^{\rm NS}_e \\ Q^{\rm RR}_e \end{pmatrix} \longrightarrow \Lambda_1\, \vec Q_{ee}\ , \quad \vec Q_{mm} \equiv \begin{pmatrix} Q^{\rm NS}_m \\ Q^{\rm RR}_m \end{pmatrix} \longrightarrow (\Lambda_1^T)^{-1}\, \vec Q_{mm}\ , \\ SL(2,\mathbb{R})_2: &\qquad \vec Q_{em} \equiv \begin{pmatrix} Q^{\rm NS}_e \\ Q^{\rm RR}_m \end{pmatrix} \longrightarrow \Lambda_2\, \vec Q_{em}\ , \quad \vec Q_{me} \equiv \begin{pmatrix} Q^{\rm NS}_m \\ Q^{\rm RR}_e \end{pmatrix} \longrightarrow (\Lambda_2^T)^{-1}\, \vec Q_{me}\ . \end{aligned} \tag{B.8}$$

Here we are introducing the notation that $\vec Q_{xy}$ is a two-component charge vector, whose upper component is the electric ($x=e$) or magnetic ($x=m$) NS-NS charge, and whose lower component is the electric ($y=e$) or magnetic ($y=m$) R-R charge.

It is worthwhile pausing at this point, to understand why the charge vectors $\vec Q_{mm}$ and $\vec Q_{me}$ transform contragrediently in comparison to the transformations of $\vec Q_{ee}$ and $\vec Q_{em}$. If we introduce an index notation for the two fields, so that $F^i_{(3)}$ denotes the NS-NS field when $i=1$, and the R-R field when $i=2$, then the dual fields $\widetilde F_{(3)1} = e^{-\phi_1-\phi_2} {*F^1_{(3)}}$ and $\widetilde F_{(3)2} = e^{\phi_1-\phi_2} {*F^2_{(3)}}$ actually correspond to the first and second components of a doublet

$\widetilde{F}_{(3)i}$ with a *downstairs* index i. (This can be seen from the fact that the relevant $SL(2,{\rm I\!R})_1$-invariant kinetic terms in the Lagrangian have the form $-\frac{1}{2}\widetilde{F}_{(3)i}\wedge F^i_{(3)}$ [62].) The electric charges therefore can be labelled as $Q_{e\,i}=\{Q_e^{\rm NS},Q_e^{\rm RR}\}$, while the magnetic charges can be labelled as $Q^i_m=\{Q_m^{\rm NS},Q_m^{\rm RR}\}$. Since the electric charges transform directly with Λ, we can see that the suppressed matrix indices on Λ_1, if made explicit, are located as follows: $(\Lambda_1)_i{}^j$. It is now clear why it is the contragedient representation that acts on the magnetic charges Q^i_m. An analogous comment applies to the electric and magnetic indices associated with the $SL(2,{\rm I\!R})_2$ transformations.

It is convenient to give a 4×4 matrix representation for the $SL(2,{\rm I\!R})_1\times SL(2,{\rm I\!R})_2$ transformations, by defining the tensor product of 2×2 matrices, and 2-component vectors, as follows:

$$
\begin{aligned}
M\otimes N &\equiv \begin{pmatrix} m_{11}\,N & m_{12}\,N \\ m_{21}\,N & m_{22}\,N \end{pmatrix}\,,\\
\begin{pmatrix} x \\ y \end{pmatrix}\otimes\begin{pmatrix} u \\ v \end{pmatrix} &\equiv \begin{pmatrix} x\,u \\ x\,v \\ y\,u \\ y\,v \end{pmatrix}\,,
\end{aligned}
\tag{B.9}
$$

where m_{ij} denotes the components of M. In view of the remarks in the previous paragraph, it follows that we can represent the action of the two $SL(2,{\rm I\!R})$'s on the charges as $\vec Q\longrightarrow\Lambda\,\vec Q$, where $\Lambda=\Lambda_2\otimes\Lambda_1$, and

$$
\vec Q=\begin{pmatrix} Q_e^{\rm NS} \\ Q_e^{\rm RR} \\ Q_m^{\rm RR} \\ -Q_m^{\rm NS} \end{pmatrix}\,. \tag{B.10}
$$

For later convenience, we may also now introduce an alternative parameterisation of the scalar coset manifold. It is sufficient for this purpose to consider a generic $SL(2,{\rm I\!R})/O(2)$ coset, and the formalism can then be applied to both $SL(2,{\rm I\!R})_1$ and $SL(2,{\rm I\!R})_2$. We define the upper-triangular matrix

$$
\mathcal{V}=\begin{pmatrix} e^{-\frac{1}{2}\phi} & \chi\,e^{\frac{1}{2}\phi} \\ 0 & e^{\frac{1}{2}\phi} \end{pmatrix}\,. \tag{B.11}
$$

This gives a Borel parameterisation of the coset, whose Lagrangian can now be written as $\frac{1}{4}{\rm tr}(\partial_\mu(\mathcal{M}^{-1})\,\partial^\mu\mathcal{M})$, where

$$
\mathcal{M}=\mathcal{V}\,\mathcal{V}^T=\begin{pmatrix} e^{-\phi}+\chi^2\,e^{\phi} & \chi\,e^{\phi} \\ \chi\,e^{\phi} & e^{\phi} \end{pmatrix}\,. \tag{B.12}
$$

The $SL(2,\mathbb{R})$ transformation (B.2) of the scalar fields can then be expressed as $\mathcal{V} \longrightarrow \Lambda\,\mathcal{V}\,\mathcal{O}$, where $\mathcal{O}$ is a field-dependent compensating $O(2)$ transformation that restores $\mathcal{V}$ to the Borel gauge. On the matrix $\mathcal{M}$, the $SL(2,\mathbb{R})$ transformation is simply $\mathcal{M} \longrightarrow \Lambda\,\mathcal{M}\,\Lambda^T$. Note that $H^1_{(3)}$ in (B.5) can now be written as $H^1_{(3)} = \mathcal{M}_1\, dA_{(2)}$.

With these preliminaries, we are now in a position to construct, in the next section, an $O(2,2)$ multiplet of dyonic strings

C An $O(2,2;\mathbb{Z})$ multiplet of dyonic strings

We may construct general dyonic string solutions with arbitrary electric and magnetic NS-NS and R-R charges $(Q_e^{\rm NS}, Q_m^{\rm NS}, Q_e^{\rm RR}, Q_m^{\rm RR})$ at an arbitrary modulus point specified by τ_1^0 and τ_2^0, by starting with the simple case of an NS-NS dyonic string in the $\tau_1^0 = \tau_2^0 = {\rm i}$ vacuum, with electric and magnetic NS-NS charges q and p. We then act with $O(2)_1$ and $O(2)_2$ transformations

$$\Lambda_1(\theta_1) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}\,, \qquad \Lambda_2(\theta_2) = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}\,, \tag{C.1}$$

which lie in the stability subgroups of the $SL(2,\mathbb{R})$'s that rotate the charges while leaving the $\tau^0 = {\rm i}$ modulus points fixed. Next, we act with the Borel transformations

$$\Lambda_1(\tau_1^0) = \begin{pmatrix} e^{-\frac12\phi_1^0} & \chi_1^0\, e^{\frac12\phi_1^0} \\ 0 & e^{\frac12\phi_1^0} \end{pmatrix}\,, \qquad \Lambda_2(\tau_2^0) = \begin{pmatrix} e^{-\frac12\phi_2^0} & \chi_2^0\, e^{\frac12\phi_2^0} \\ 0 & e^{\frac12\phi_2^0} \end{pmatrix}\,, \tag{C.2}$$

which map the original modulus points $\tau_1^0 = \tau_2^0 = {\rm i}$ to the arbitrary points $\tau_1^0 = \chi_1^0 + {\rm i}\, e^{-\phi_1^0}$ and $\tau_2^0 = \chi_2^0 + {\rm i}\, e^{-\phi_2^0}$. The combined effect of the stability-subgroup and Borel-subgroup transformations is to map the original charges $\vec Q = (Q_e^{\rm NS}, Q_e^{\rm RR}, Q_m^{\rm RR}, -Q_m^{\rm NS}) = (q,0,0,-p)$ to arbitrary charges, which are related to the parameters θ_1, θ_2, q and p (at the given modulus point $(\phi_1^0, \chi_1^0, \phi_2^0, \chi_2^0)$) in a manner that we shall determine below. Thus we can obtain strings with their four charges lying at arbitrary points on the charge lattice (that satisfy the Dirac quantisation condition) by appropriately choosing the four parameters. (The spirit of this construction is similar to that used in [63] in the discussion of the charge lattice for the $O(6,22;\mathbb{Z})$ multiplet of black holes in the $D=4$ heterotic string, and in [46] for the construction of the $SL(2,\mathbb{Z})$ multiplet of type IIB strings. A general procedure for generating the U-duality multiplets for all p-brane solitons was given in [64]. A group theoretic approach was also introduced in [64], using the homogeneous scaling symmetries of the equations of motion that arise in theories such as the maximal supergravities, in order to give a construction of genuine spectrum-generating groups for BPS states. An explicit construction of U-duality multiplets for BPS states in eight dimensions was given in [65].)

C.1 Unboosted isotropic dyonic strings

Moving now to the details, let us consider an isotropic unboosted dyonic string, supported by the NS-NS 3-form field [5];

$$\begin{aligned}
ds^2 &= (H_e\,H_m)^{-1/2}\,dx^\mu\,dx_\mu + (H_e\,H_m)^{1/2}\,d\vec{y}\cdot d\vec{y}\ ,\\
F_{(3)}^{\rm NS} &= 8\,p\,\Omega_{(3)} + 8\,q\,H_m\,H_e^{-1}\,{*Omega}_{(3)}\ , \qquad F_{(3)}^{\rm RR} = 0\ ,\\
\tau_1 &= \chi_1 + {\rm i}\,e^{-\phi_1} = \tau_2 = \chi_2 + {\rm i}\,e^{-\phi_2} = {\rm i}\,(H_e/H_m)^{1/2}\ ,
\end{aligned} \tag{C.3}$$

where $H_e = 1 + 4q/r^2$ and $H_m = 1 + 4p/r^2$, q and p are the electric and magnetic charges (following the normalisations given in (B.7)), and $\Omega_{(3)}$ is the volume form on the unit S^3. After some algebra, we find, following the steps outlined above, that the solution after performing the stability-subgroup and Borel-subgroup transformations becomes

$$\begin{aligned}
ds^2 &= (H_e\,H_m)^{-1/2}\,dx^\mu\,dx_\mu + (H_e\,H_m)^{1/2}\,d\vec{y}\cdot d\vec{y}\ ,\\
\tau_1 &= \chi_1^0 + {\rm i}\,e^{-\phi_1^0}\,\frac{\sqrt{H_e} - {\rm i}\,\tan\theta_1\,\sqrt{H_m}}{\sqrt{H_m} - {\rm i}\,\tan\theta_1\,\sqrt{H_e}}\ ,\\
\tau_2 &= \chi_2^0 + {\rm i}\,e^{-\phi_2^0}\,\frac{\sqrt{H_e} - {\rm i}\,\tan\theta_2\,\sqrt{H_m}}{\sqrt{H_m} - {\rm i}\tan\theta_2\,\sqrt{H_e}}\ ,\\
F_{(3)}^{\rm NS} &= e^{\frac12(\phi_1^0+\phi_2^0)}\,\Big(\cos\theta_1\,\cos\theta_2\,\Theta - \sin\theta_1\,\sin\theta_2\,\frac{H_e}{H_m}\,{*\Theta}\Big)\ ,\\
F_{(3)}^{\rm RR} &= \frac{-e^{\frac12(\phi_2^0-\phi_1^0)}\,H_e}{\sin^2\theta_1\,H_e + \cos^2\theta_1\,H_m}\,\Big(\sin\theta_1\,\cos\theta_2\,\Theta + \cos\theta_1\,\sin\theta_2\,{*\Theta}\Big)\ ,
\end{aligned} \tag{C.4}$$

where $\Theta \equiv 8\,p\,\Omega_{(3)} + 8\,q\,{*\Omega}_{(3)}$.

Under the stability-subgroup and Borel-subgroup transformations (C.1) and (C.2), the initial charge 4-vector $\vec{Q}_0 = \{q, 0, 0, -p\}$ is mapped to the final charge vector $\vec{Q}_f$, given by

$$\vec{Q}_f = \Big(\Lambda_2(\tau_2^0)\otimes\Lambda_1(\tau_1^0)\Big)\,\Big(\Lambda_2(\theta_2)\otimes\Lambda_1(\theta_1)\Big)\,\vec{Q}_0\ , \tag{C.5}$$

which implies

$$\Big(\Lambda_2(\theta_2)\otimes\Lambda_1(\theta_1)\Big)\,\vec{Q}_0 = \Big(\Lambda_2(\tau_2^0)^{-1}\otimes\Lambda_1(\tau_1^0)^{-1}\Big)\,\vec{Q}_f \equiv \Lambda(\tau_0)^{-1}\,\vec{Q}_f\ . \tag{C.6}$$

This equation provides the relation between the four parameters $(\theta_1, \theta_2, q, p)$ and the four final charges $\vec{Q}_f$, for any given values of the scalar moduli $\tau_0 = (\tau_1^0, \tau_2^0)$. To obtain the explicit solution, we first note that the stability-subgroup $O(2)$ rotations (C.1) can be written as $\Lambda(\theta) = e^{{\rm i}\theta\sigma}$, where $\sigma = \begin{pmatrix} 0 & -{\rm i} \\ {\rm i} & 0 \end{pmatrix}$. Equation (C.6) can now be written as

$$e^{{\rm i}\theta_2\sigma}\otimes e^{{\rm i}\theta_1\sigma}\,\vec{Q}_0 = \Lambda(\tau_0)^{-1}\,\vec{Q}_f\ . \tag{C.7}$$

This implies that $e^{\mathrm{i}\theta_2\sigma}\otimes e^{\mathrm{i}\theta_1\sigma}\,U_\pm = V_\pm$, where

$$U_\pm = (\mathbb{1}+\sigma)\otimes(\mathbb{1}\pm\sigma)\,\vec{Q}_0\ ,\qquad V_\pm = (\mathbb{1}+\sigma)\otimes(\mathbb{1}\pm\sigma)\,\vec{Q}_f\ . \tag{C.8}$$

After elementary algebra, it now follows that

$$\begin{aligned}
(q+p)\,e^{\mathrm{i}(\theta_2+\theta_1)} &= e^{\frac12(\phi_1^0+\phi_2^0)}\,(Q_e^{\rm NS}-\tau_1^0\,Q_e^{\rm RR}-\tau_2^0\,Q_m^{\rm RR}-\tau_1^0\,\tau_2^0\,Q_e^{\rm NS})\equiv\Delta_+\ ,\\
(q-p)\,e^{\mathrm{i}(\theta_2-\theta_1)} &= e^{\frac12(\phi_1^0+\phi_2^0)}\,(Q_e^{\rm NS}-\bar\tau_1^0\,Q_e^{\rm RR}-\tau_2^0\,Q_m^{\rm RR}-\bar\tau_1^0\,\tau_2^0\,Q_e^{\rm NS})\equiv\Delta_-\ .
\end{aligned} \tag{C.9}$$

¿From here, the solutions for (θ_1,θ_2,q,p) immediately follow. Note that the factor $e^{\frac12(\phi_1^0+\phi_2^0)}$ is precisely the six-dimensional effective string coupling constant.

It is now easy to determine the formula for the mass m per unit length for the general 4-charge dyonic string. To do this, we note that in the original NS-NS dyonic string solution (C.3), the mass is simply given by $m=q+p$, which we can write in the $O(2,2)$-invariant form

$$m^2 = q^2+p^2+2q\,p = \vec{Q}_0^T\,\vec{Q}_0-\vec{Q}_0^T\,\hat\Omega\,\vec{Q}_0\ , \tag{C.10}$$

where $\hat\Omega\equiv\Omega\otimes\Omega$, and $\Omega=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. In terms of the final charges $\vec{Q}_f$ of the generic 4-charge dyonic string, the mass is therefore given by

$$\begin{aligned}
m^2 &= \vec{Q}_f^T\,(\Lambda(\tau^0)^T)^{-1}\,\Lambda(\tau^0)^{-1}\,\vec{Q}_0-\vec{Q}_f^T\,(\Lambda(\tau^0)^T)^{-1}\,\hat\Omega\,\Lambda(\tau^0)^{-1}\,\vec{Q}_0\\
&= \vec{Q}_f^T\,\Big((\Lambda(\tau^0)\,\Lambda(\tau^0)^T)^{-1}-\hat\Omega\Big)\,\vec{Q}_f\ .\\
&= |\Delta_+|^2\ .
\end{aligned} \tag{C.11}$$

(The expression for m^2 in the last line follows directly from (C.10) and (C.9).)

Note that the second line of the mass formula (C.11) is composed of two independent $O(2,2)$-invariant quantities. The second term is precisely the quantity that appears in the Dirac quantisation condition [66], namely

$$\vec{Q}_f^T\,\hat\Omega\,\vec{Q}_f = \text{integer}\ . \tag{C.12}$$

This condition implies that the four charges $\vec{Q}_f=(Q_e^{\rm NS},Q_e^{\rm RR},Q_m^{\rm RR},-Q_m^{\rm NS})$ must lie on a discrete lattice, which, for simplicity, may be taken to be the square integer lattice. In our construction of the multiplet of integer-charge solutions, we of course allowed the initial charge parameters q and p and the rotation angles θ_1 and θ_2 to be unrestricted by any quantisation condition. For a given modulus point $\tau^0=(\tau_1^0,\tau_2^0)$, after restricting the final charges to lie on the Dirac charge lattice, the initial parameters will themselves fill out only a discrete lattice of values.

The points in a given charge lattice, for example the integer square lattice, can be filled out by a discrete $O(2,2;\mathbb{Z})$ spectrum-generating group that leaves the scalar modulus point τ^0 invariant. It is therefore not itself the discretised form of the of the $O(2,2)$ global symmetry transformations that we originally discussed, since this transforms the scalar moduli at the same time as it moves the charges on their lattice. The difference is highlighted by the fact that the spectrum-generating group must, of course, in particular generate strings of different masses, whilst the original global symmetry group leaves the metric, and hence the mass, invariant. This issue was extensively discussed in [64], where it was shown that to get the true spectrum-generating group it is necessary to make use also of an homogeneous scaling "trombone" symmetry of the theory. In [64], the example of the $SL(2,\mathbb{Z})$ multiplet of type IIB strings was discussed in detail. However, in the $O(2,2)$ case we are considering here there is an added subtlety. This can be seen most easily at the classical level. In the case of the type IIB NS-NS and R-R strings, the charge space is two-dimensional, and can be completely spanned, for any given modulus point, by the action of the $O(2)$ denominator group together with the trombone rescaling symmetry. In our present case, however, the charge space is four-dimensional, while the modulus-preserving denominator group is $O(2) \times O(2)$. Together with the trombone symmetry this gives only three parameters, and hence this is insufficient to span the charge vector space. In our construction, we were nevertheless able to construct the complete charge lattice of dyonic strings. This is because we started with the solution (C.3) that had the two free parameters q and p. The trombone symmetry is responsible for the existence of the 1-parameter sub-family of solutions where q and p are uniformly rescaled by the same factor, which has the effect of also rescaling the mass. It is less clear what is the symmetry that is responsible for allowing solutions with different relative ratios between the electric and magnetic charge parameters q and p.

It was argued in [64] that one resolution to this puzzle might be that since the dyonic string can be viewed as a bound state with *zero* binding energy, it is less fundamental than the individual electric and magnetic building blocks. Thus it would be unnecessary to find a symmetry to relate the bound-state solutions with different ratios for the electric and magnetic charge. (Another example where there is no homogeneous scaling symmetry to account for the existence of arbitrary-charge solutions is in the heterotic string. This is because Yang-Mills fields $F \sim dA + A \wedge A$ do not scale uniformly under $A \to \lambda A$. (One requires a scaling symmetry that rescales the metric, so that the mass can be changed, while leaving the scalars invariant, so that the scalar moduli are unchanged.))

In this section, we constructed an $O(2,2;\mathbb{Z})$ multiplet of dyonic strings in $D=6$. This subgroup of the full $O(5,5;\mathbb{Z})$ or $O(5,21;\mathbb{Z})$ duality groups captures the essence of the complete groups, in that it describes both the NS-NS/R-R duality and the electric/magnetic duality. The extension to the full duality group is straightfoward in principle, using, for example, the techniques described in [64], but its detailed implementation would be tedious and rather unrewarding. The most interesting aspect of the U-duality multiplets is to see how the charges transform under the full U-duality group. The classification of the U-duality orbits of the charges in all dimensions can be found in [67, 68].

C.2 Boosted and twisted dyonic strings

In the discussion above, we gave the construction of the $O(2,2;\mathbb{Z})$ multiplet of unboosted, isotropic dyonic strings. It is straightforward to boost in the worldvolume, or to twist in the transverse space, and thereby obtain $O(2,2;\mathbb{Z})$ multiplets of boosted or twisted dyonic strings. We do this by following a strategy analogous to the one we used previously, namely by starting with a boosted and twisted dyonic string supported purely by NS-NS 3-form charges, and taking the scalar moduli to vanish. The form of the solutions for the fields $(\phi_1,\chi_1,\phi_2,\chi_2,F_{(3)}^{\rm NS},F_{(3)}^{\rm RR})$ is identical to that for the unboosted, untwisted case given in (C.3), except that now the harmonic functions H_e and H_m are modified to $H_e=1+q/r$, $H_m=1+p/r$. (In other words, they are now harmonic only in the 3-dimensional overall transverse space, rather than the 4-dimensional transverse space for the dyonic strings.[4])

The metric is given by

$$\begin{aligned} ds^2 &= -(H_eH_m)^{-\frac{1}{2}}\,K_w^{-1}\,dt^2+(H_eH_m)^{-\frac{1}{2}}\,K_w\,(dz_1+(K_w^{-1}-1)dt)^2\\ &\quad +(H_eH_m)^{\frac{1}{2}}\,K_N^{-1}\,(dz_2+Q_N\,\cos\theta\,d\phi)^2\\ &\quad +(H_eH_m)^{\frac{1}{2}}\,K_N\,(dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\phi^2)\ , \end{aligned} \qquad (C.13)$$

where the extra harmonic functions for the wave and the boost charges are $K_w=1+Q_w/r$ and $K_N=1+Q_N/r$. We may now repeat the steps that we followed previously, to generate the entire $O(2,2;\mathbb{Z})$ multiplet of boosted and twisted dyonic strings. The final expressions for the dilatons, axions and 3-form fields will be identical to those given in (C.4), again with the understanding that the H_e and H_m harmonic functions are modified to those

[4]Recall that we are defining charges arising from integrals over S^2 and over S^3 by $Q=\frac{1}{4\pi}\int F_2$ and $Q=\frac{1}{16\pi^2}\int F_3$ respectively, which ensures that the charge is preserved under dimensional reduction. It follows that the harmonic functions in the two cases will be of the form $H=1+Q/r$ and $H=1+4Q/r^2$ respectively.

given above. The reason for this is that the only other change in the starting point is in the metric, which is a singlet under $O(2,2)$, and so the entire calculation of the $O(2,2;\mathbb{Z})$ multiplet proceeds identically to the one we described previously. Note that the charges p and q are still given by (C.9), as in the case of unboosted isotropic dyonic strings.

The solution (C.13) can be dimensionally reduced on the fibre coordinates z_1 and z_2, giving rise to a non-dilatonic black hole in $D=4$. The near-horizon limit in $D=4$ is $\mathrm{AdS}_2\times S^2$, and so the area of the event horizon is non-vanishing, implying that there is a non-vanishing entropy, even though the solution is extremal. The entropy is of the form

$$S \sim \sqrt{p\, q\, Q_w\, Q_N}\ . \tag{C.14}$$

¿From the six-dimensional point of view, the near-horizon limit is also locally non-singular, and is given, after a rescaling of the time coordinate, by

$$\begin{aligned} ds^2 &= \sqrt{p\,q}\, Q_N \Big\{ - e^{2\rho}\, dt^2 + d\rho^2 + \big(\sqrt{\frac{Q_w}{p\, q\, Q_N}}\, dz_1 + e^{\rho}\, dt\big)^2 \\ &\quad + d\theta^2 + \sin^2\theta\, d\phi^2 + (Q_N^{-1}\, dz_2 + \cos\theta\, d\phi)^2 \Big\}\ . \end{aligned} \tag{C.15}$$

It is straightforward to verify that the 4-volume of the spatial metric at $\rho=-\infty$ is of the form (C.14). Note that Q_w measures the momentum of the wave propagating on the world-sheet of the dyonic string, and it, together with the other charges, has the effect of rescaling the fibre coordinate z_1. Q_N, on the other hand, is the NUT charge, and it has the effect of changing the local structure of the S^3 factor to the lens space S^3/Z_{Q_N}. Thus we see that the phenomenon of S^3 being factored to become a cyclic lens space plays a rôle in the understanding of the entropy of 4-dimensional black holes.

If we consider the case where the dyonic string is only twisted, but not boosted, the solution is given by (C.13) with $Q_w=0$. In this case a dimensional reduction on the fibre coordinate z_2 gives rise to a non-dilatonic string in $D=5$, which is dual to a non-dilatonic 3-charge black hole. If, on the other hand, we instead consider a case where there is only boosting, but no twisting, of the dyonic string, the solution is given by

$$\begin{aligned} ds^2 &= -(H_e H_m)^{-\frac12}\, K_w^{-1}\, dt^2 + (H_e H_m)^{-\frac12}\, K_w\, (dz_1 + (K_w^{-1}-1)dt)^2 \\ &\quad + (H_e H_m)^{\frac12}\, (dr^2 + r^2\, d\Omega_3^2)\ , \end{aligned} \tag{C.16}$$

where now the functions H_e, H_m and K_w are harmonic in a 4-dimensional transverse space, and so $H_e = 1+4q/r^2$, $H_m = 1+4p/r^2$ and $K_w = 1+4Q_w/r^2$. Its dimensional reduction on the fibre coordinate z_1 gives a non-dilatonic 3-charge black hole in $D=5$. The near-horizon limit of the six-dimensional metric (C.16), after rescaling the time coordinate, is

$$ds^2 = \sqrt{p\,q}\,\Big\{-e^{2\rho}\,dt^2 + d\rho^2 + (\sqrt{\frac{Q_w}{p\,q}}\,dz_1 + e^{\rho}\,dt)^2 + d\theta^2 + \sin^2\theta\,d\phi^2 + (dz_2 + \cos\theta\,d\phi)^2\Big\}\,. \qquad \text{(C.17)}$$

The entropy is of the form $S \sim \sqrt{p\,q\,Q_w}$.

D Hopf reductions and isentropic mappings

We have seen already that the Hopf T-duality transformations on the fibre coordinates of AdS_3 or S^3, which can have the effect of (un)twisting and squashing the metrics, preserve the entropy of the solutions. For example, we observe that the near-horizon limit (C.17) is the same as (C.15) with $Q_N = 1$, in which case the lens space S^3/Z_{Q_N} just becomes the 3-sphere. The entropy is therefore given by setting $Q_N = 1$ in (C.14). This shows, from a six-dimensional point of view, that the near-horizon limit of a $D = 5$ isotropic 3-charge black hole is the same as the near-horizon limit of a 4-charge black hole in $D = 4$, where the magnetic Kaluza-Klein charge Q_N is set to 1. In other words, in the Hopf reduction on the $U(1)$ fibres of the 3-spheres that foliate the transverse space of the 3-charge black hole in $D = 5$, a fourth charge ($Q_N = 1$) emerges in $D = 4$, corresponding to the magnetic charge of the Kaluza-Klein 2-form that governs the curvature of the fibre bundle. This should be contrasted with the more common kind of Kaluza-Klein reduction of the 3-charge black hole in $D = 5$, where the reduction is on one of the coordinates of the transverse space. In this situation, it is is necessary first to make a line (or periodic array) of $D = 5$ black holes along the intended reduction axis. The consequence of this is that the number of charges is conserved under the reduction process, and thus one arrives at a 3-charge black hole in $D = 4$ which is singular on the horizon, and which has zero entropy. Hopf reduction, however, preserves the area of the horizon, and provides a natural "isentropic mapping" between the $D = 5$, 3-charge and $D = 4$, 4-charge black holes.

To make this property of the Hopf reductions more explicit, let us consider the entire solution for a 3-charge black hole in $D = 3$, rather than just looking at the near-horizon limit. The 3-charge solution, which can be obtained, for example, from the dimensional reduction of (C.16) on the z_1 coordinate, is

$$ds_5^2 = -(H_e H_m K_w)^{-\frac{2}{3}}\,dt^2 + (H_e H_m K_w)^{\frac{1}{3}}\,(dr^2 + r^2\,d\Omega_3^2)\,, \qquad \text{(D.1)}$$

where

$$H_e = 1 + \frac{4q}{r^2}\,, \qquad H_m = 1 + \frac{4p}{r^2}\,, \qquad K_w = 1 + \frac{4Q_w}{r^2}\,. \qquad \text{(D.2)}$$

Writing the unit 3-sphere metric $d\Omega_3^2$ in the fibre bundle form (3.5), we now perform the reduction on the fibre coordinate z, using the usual Kaluza-Klein ansatz $ds_5^2 = e^{-\varphi/\sqrt{3}}\, ds_4^2 + e^{2\varphi/\sqrt{3}}\, (dz + B)^2$, giving

$$ds_4^2 = -\tfrac{1}{2} r\, (H_e H_m K_w)^{-\frac{1}{2}}\, dt^2 + \tfrac{1}{2} r\, (H_e H_m K_w)^{\frac{1}{2}}\, (dr^2 + \tfrac{1}{4} r^2\, d\Omega_2^2)\ . \tag{D.3}$$

The new Kaluza-Klein potential B is such that $\mathcal{F}_{(2)} = dB = \Omega_{(2)}$, and hence the Kaluza-Klein magnetic charge is $Q_N = \frac{1}{4\pi} \int \mathcal{F}_{(2)} = 1$.

This is not quite like a normal black hole solution in $D = 4$ for two reasons. Firstly, there is a conformal factor of r multiplying the entire metric, and secondly, the three-dimensional transverse space has the non-standard metric $dr^2 + \frac{1}{4} r^2\, d\Omega_2^2$. This metric suffers from a diverging curvature as r approaches zero, since the foliating 2-spheres are of the wrong radius to "nest" nicely around the origin. We see that it is natural to perform a coordinate transformation in which we define a new radial coordinate $\rho = \frac{1}{4} r^2$. In terms of this, the metric (D.3) becomes

$$ds_4^2 = -\Big(\frac{1}{\rho}\, H_e H_m K_w\Big)^{-\frac{1}{2}}\, dt^2 + \Big(\frac{1}{\rho}\, H_e H_m K_w\Big)^{\frac{1}{2}}\, (d\rho^2 + \rho^2\, d\Omega_2^2)\ . \tag{D.4}$$

Note that in terms of the new radial coordinate, the original harmonic functions H_e, H_m and K_w given in (D.2) become

$$H_e = 1 + \frac{q}{\rho}\ , \qquad H_m = 1 + \frac{p}{\rho}\ , \qquad K_w = 1 + \frac{Q_w}{\rho}\ , \tag{D.5}$$

which are harmonic in the new 3-dimensional transverse space. Thus we see that the effect of performing a Hopf reduction on a 3-charge black hole in $D = 5$ is to give the solution

$$ds_4^2 = -\Big(K_N H_e H_m K_w\Big)^{-\frac{1}{2}}\, dt^2 + \Big(K_N H_e H_m K_w\Big)^{\frac{1}{2}}\, (d\rho^2 + \rho^2\, d\Omega_2^2)\ , \tag{D.6}$$

which is precisely of the form of a standard 4-charge black hole, except that the fourth harmonic function $K_N = 1/\rho$, which has charge equal to 1, is lacking a constant term. Of course if we consider the near-horizon limit where ρ approaches zero, the absence of the constant term becomes immaterial. In fact, it is also possible to introduce a constant term in the harmonic function K_N, by performing an appropriate U-duality transformation [59, 8, 60, 61]. Thus we see that the Hopf dimensional reduction provides a natural isentropic mapping between a 3-charge black hole in $D = 5$ and a 4-charge black hole in $D = 4$. The Hopf reduction preserves the supersymmetry of the solution. Although the four-dimensional black hole involves 4 charges, it still preserves the same fraction 1/8 of the supersymmetry as does the 3-charge black hole in $D = 5$. (With the opposite orientation for the Hopf reduction, all the supersymmetry is broken, which is consistent with the fact that 4-charge black holes in $D=4$ also occur with no preserved supersymmetry, if the sign of the fourth charge is reversed [18, 57].)

The crucial feature in the above discussion is that the foliating spheres in the transverse space are 3-dimensional. One can perform a Hopf reduction on the $U(1)$ fibres of any odd-dimensional sphere S^{2n+1}, but in general the base space will then be CP^n. However, it is only in the special case $n=1$ that the transverse space after the Hopf reduction can become flat, with no singularity at the origin. This is because a metric of the form $d\rho^2 + \rho^2\, d\Sigma_n^2$, where $d\Sigma_n^2$ is the metric on CP^n, is non-singular at the origin only if $d\Sigma_n^2$ is a metric on the unit sphere. Only for $n=1$ is CP^n isomorphic to a sphere ($CP^1 \sim S^2$).

In fact, we can apply the Hopf reduction to any N-charge p-brane solution whose transverse space is 4-dimensional, and thereby obtain an $(N+1)$-charge p-brane solution in one dimension less, with the strength of the extra charge being 1. (As in the previous example of $D=5$ black holes, a further U-duality transformation is needed in order to introduce a constant term in the associated harmonic function.) Note that the procedure can equally well be applied to the case of non-extremal p-branes.

Let us consider a general isotropic N-charge non-extremal p-brane solution in $D+1$ dimensions, where the transverse space has dimension 4. The metric will have the form [69]

$$ds_{D+1}^2 = \prod_{i=1}^{N} H_i^{-2/(D-1)} \Big(- e^{2f}\, dt^2 + d\vec{x}\cdot d\vec{x}\Big) + \prod_{i=1}^{N} H_i^{(D-3)/(D-1)} \Big(e^{-2f}\, dr^2 + r^2\, d\Omega_3^2\Big)\ , \quad \text{(D.7)}$$

where the "harmonic" functions take the form $H_i = 1 + 4k\,\sinh^2\mu_i\, r^{-2}$, and the "blackening function" f is given by $e^{2f} = 1 - k\, r^{-2}$. (Full details of the solutions, and the relation of the charges and the mass to the parameters μ_i and k can be found in [69].) Following the same steps as we described above for the reduction of the 3-charge $D=5$ black hole, we find that the Hopf-reduced solution in D dimensions has the metric

$$ds_D^2 = \prod_{i=1}^{N+1} H_i^{-1/(D-2)} \Big(- e^{2f}\, dt^2 + d\vec{x}\cdot d\vec{x}\Big) + \prod_{i=1}^{N+1} H_i^{(D-3)/(D-2)} \Big(e^{-2f}\, d\rho^2 + \rho^2\, d\Omega_2^2\Big)\ , \quad \text{(D.8)}$$

where $\rho = \frac{1}{4} r^2$, and we now have $e^{2f} = 1 - k/(4\rho)$ and

$$H_i = 1 + \frac{k}{\rho}\,\sinh^2\mu_i\ ,\quad (1\le i\le N)\ ,\qquad H_{N+1} = \frac{1}{\rho}\ . \quad \text{(D.9)}$$

As before, we can then perform a suitable U-duality transformation on this solution in order to introduce a constant term in the new harmonic function H_{N+1} [59, 8, 60, 61]. The metric (D.9) is then precisely of the form of an $(N+1)$-charge p-brane in D dimensions, with unit strength for the extra harmonic function. The discussion extends to any intersection of p-branes, NUTs and waves where the overall transverse space is 4-dimensional.

In general, the near-horizon limit of an N-charge extremal p-brane is singular. It follows in these cases that the Hopf reduction will give rise to a singular dilaton in the lower dimension. Thus the supersymmetry will be further broken in such cases.

An application of our discussion of Hopf reductions for 4-dimensional transverse spaces is to 5-branes in $D = 10$. There are NS-NS 5-branes, and R-R D5-branes. For the case of NS-NS 5-branes, Hopf reduction on the $U(1)$ fibres of S^3 gives rise to 2-charge harmonic 5-branes in $D = 9$. Performing a T-duality transformation, the solution can be oxidised to $D = 10$, when it again becomes an NS-NS single-charge 5-brane, this time with unit charge. At the same time, the foliating 3-spheres become lens spaces S^3/Z_n, where n is the magnetic charge of the original 5-brane. Thus a single (*i.e.* unit charge) 5-brane is invariant under the Hopf T-duality. The picture is more complicated for the case of R-R 5-branes. The Hopf reduction to $D = 9$ gives the same metric form as in the NS-NS case, but after T-duality it oxidises back to the intersection of an NS-NS 5-brane and a D6-brane. In this case, the original 3-sphere is untwisted to $S^2 \times S^1$ (but with a non-direct-product metric, since there will be different harmonic functions multiplying the two factors).

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