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Spacelike strings and jet quenching from a Wilson loop

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Abstract

We investigate stationary string solutions with spacelike worldsheet in a five-dimensional AdS black hole background, and find that there are many branches of such solutions. Using a non-perturbative definition of the jet quenching parameter proposed by Liu *et. al.*, hep-ph/0605178, we take the lightlike limit of these solutions to evaluate the jet quenching parameter in an $\mathcal{N} = 4$ super Yang-Mills thermal bath. We show that this proposed definition gives zero jet quenching parameter, independent of how the lightlike limit is taken. In particular, the minimum-action solution giving the dominant contribution to the Wilson loop has a leading behavior that is linear, rather than quadratic, in the quark separation.

1 Introduction, summary and conclusions

Results coming from RHIC have raised the issue of how to calculate transport properties of ultra-relativistic partons in a strongly coupled gauge theory plasma. For example, one would like to calculate the friction coefficient and jet quenching parameter, which are measures of the rate at which partons lose energy to the surrounding plasma [1–9]. With conventional quantum field theoretic tools, one can calculate these parameters only when the partons are interacting perturbatively with the surrounding plasma. The AdS/CFT correspondence [10] may be a suitable framework in which to study strongly coupled QCD-like plasmas. Attempts to use the AdS/CFT correspondence to calculate these quantities have been made in [11–14] and were generalized in various ways in [15–43].

The most-studied example of the AdS/CFT correspondence is that of the large N, large 't Hooft coupling limit of four-dimensional $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills (SYM) theory and type IIB supergravity on $AdS_5 \times S^5$. At finite temperature, this SYM theory is equivalent to type IIB supergravity on the background of the near-horizon region of a large number N of non-extremal D3-branes. From the perspective of five-dimensional gauged supergravity, this is the background of a neutral AdS black hole whose Hawking temperature equals the temperature of the gauge theory [44]. Since at finite temperature the superconformal invariance of this theory is broken, and since fundamental matter can be added by introducing D7-branes [45], it is thought that this model may shed light on certain aspects of strongly coupled QCD plasmas.

According to the AdS/CFT dictionary, the endpoints of open strings on this background can correspond to quarks and antiquarks in the SYM thermal bath [46–49]. For example, a stationary single quark can be described by a string that stretches from the probe D7-brane to the black hole horizon. A semi-infinite string which drags behind a steadily-moving endpoint and asymptotically approaches the horizon has been proposed as the configuration dual to a steadily-moving quark in the $\mathcal{N} = 4$ plasma, and was used to calculate the drag force on the quark [12, 13]. A quark-antiquark pair or "meson", on the other hand, corresponds to a string with both endpoints ending on the D7-brane. The static limit of this string solution has been used to calculate the inter-quark potential in SYM plasmas [48, 49]. Smooth, stationary solutions for steadily-moving quark-antiquark pairs exist [27, 29, 31, 32, 34, 35, 42] but are not unique and do not "drag" behind the string endpoints as in the single quark configuration. This lack of drag has been interpreted to mean that color-singlet states such as mesons are invisible to the SYM plasma and experience no drag (to leading order in large N) even though the string shape is dependent on the velocity of the meson with respect to the plasma. Nevertheless, a particular no-drag string configuration with spacelike worldsheet [27, 42] has been used to evaluate a lightlike Wilson loop in the field theory [11, 15, 17, 19, 21–23, 25, 29, 33, 39]. It has been proposed that this Wilson loop can be used for a non-perturbative definition of the jet quenching parameter $\hat{q}$ [11].

The purpose of this paper is to do a detailed analysis of the evaluation of this Wilson loop using no-drag spacelike string configurations in the simplest case of finite-temperature $\mathcal{N} = 4$ $SU(N)$ SYM theory.

Summary. We use the Nambu-Goto action to describe the classical dynamics of a smooth stationary string in the background of a five-dimensional AdS black hole. We put the endpoints of the string on a probe D7-brane with boundary conditions which describe a quark-antiquark pair with constant separation moving with constant velocity either perpendicular or parallel to their separation.

In section 2, we present the string embeddings describing smooth and stationary quark-antiquark configurations, and we derive their equations of motion. In section 3, we discuss spacelike solutions of these equations. We find that there can be an infinite number of spacelike solutions for given boundary conditions, although there is always a minimum-length solution.

In section 4, we apply these solutions to the calculation of the lightlike Wilson loop observable proposed by [11] to calculate the jet quenching parameter $\hat{q}$, by taking the lightlike limit of spacelike string worldsheets [27, 42]. We discuss the ambiguities in the evaluation of this Wilson loop engendered by how the lightlike limit is taken, and by how self-energy subtractions are performed. Technical aspects of the calculations needed in section 4 are collected in two appendices. We also do the calculation for Euclidean-signature strings for the purpose of comparison.

Conclusions. We find that the lightlike limit of the spacelike string configuration used in [11, 42] to calculate the jet quenching parameter $\hat{q}$ is not the solution with minimum action for given boundary conditions, and therefore gives an exponentially suppressed contribution to the path integral. Regardless of how the lightlike limit is taken, the minimum-action solution giving the dominant contribution to the Wilson loop has a leading behavior that is linear in its width, L. Quadratic behavior in L is associated with radiative energy loss by gluons in perturbative QCD, and the coefficient of the L^2 term is taken as the definition of the jet-quenching parameter $\hat{q}$ [11]. In the strongly coupled $\mathcal{N} = 4$ SYM theory in which we are computing, we find $\hat{q} = 0$.

We now discuss a few technical issues related to the validity of the dominant spacelike string solution which gives rise to the linear behavior in L.

Depending on whether the velocity parameter approaches unity from above or below, the minimum-action string lies below ("down string") or above ("up string") the probe D7-brane, respectively. The down string worldsheet is spacelike regardless of the region of the bulk space in which it lies. On the other hand, in order for the up string worldsheet to be spacelike, it must lie within a region bounded by a certain maximum radius which is related to the position of the black hole horizon. The lightlike limit of the up string involves taking the maximal radius and the radius of the string endpoints to infinity simultaneously, such that the string always lies within the maximal radius. Therefore, even though the string is getting far from the black hole, its dynamics are still sensitive to the black hole through this maximal radius.

In the lightlike limit, the up and down strings with minimal action both approach a straight string connecting the two endpoints. This is the "trivial" solution discarded in [11], though we do not find a compelling physical or mathematical reason for doing so. If the D7-brane radius were regarded as a UV cut-off, then one might presume that the dominant up string solution should be discarded, since it probes the region above the cut-off. However, this is unconvincing for two reasons. First, if one approaches the lightlike limit from $v > 1$, then the dominant solution is a down string, and so evades this objection. Second, and more fundamentally, in a model which treats the D7-brane radius as a cut-off one does not know how to compute accurately in the AdS/CFT correspondence. For this reason we deal only with the $\mathcal{N} = 4$ SYM theory and a probe brane D7-brane, for which the AdS/CFT correspondence is precise.

A spacelike string lying straight along a constant radius is discussed briefly in [42]. This string also approaches the "trivial" lightlike solution in [11] as the radius is taken to infinity. As pointed out in [42], this straight string at finite radius is not a solution of the (full, second order) equations of motion, and should be rejected. We emphasize that our dominant string solutions are *not* this straight string, even though they approach the straight string as the D7-brane radius goes to infinity, and are genuine solutions to the full equations of motion.

To conclude, the results in this paper show these solutions to be robust, in the sense that they give the same contribution to the path integral independently of how the lightlike limit is taken. Furthermore, though not a compelling argument, the fact that these solutions do not exhibit any drag is consistent with the fact that they give $\hat{q} = 0$. Therefore, for the non-perturbative definition of $\hat{q}$ given in [11], direct computation of $\hat{q} \neq 0$ within the AdS/CFT correspondence for $\mathcal{N} = 4$ $SU(N)$ SYM would require either a compelling argument for discarding the leading contribution to the path integral, or a different class of string solutions giving the dominant contribution. On the other hand, this computation may simply imply that at large N and strong 't Hooft coupling, the mechanism for relativistic parton energy loss in the SYM thermal bath gives a linear rather than quadratic dependence on the Wilson loop width L.

2 String embeddings and equations of motion

We consider a smooth and stationary string in the background of a five-dimensional AdS black hole with the metric

$$ds_5^2 = h_{\mu\nu} dx^\mu dx^\nu = -\frac{r^4 - r_0^4}{r^2 R^2}\, dx_0^2 + \frac{r^2}{R^2}(dx_1^2 + dx_2^2 + dx_3^2) + \frac{r^2 R^2}{r^4 - r_0^4}\, dr^2. \quad (2.1)$$

R is the curvature radius of the AdS space, and the black hole horizon is located at $r = r_0$. We put the endpoints of the string at the minimal radius r_7 that is reached by a probe D7-brane. The classical dynamics of the string in this background is described by the Nambu-Goto action

$$S = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{-G}, \quad (2.2)$$

with

$$G = \det[h_{\mu\nu}(\partial X^\mu/\partial \xi^\alpha)(\partial X^\nu/\partial \xi^\beta)], \quad (2.3)$$

where $\xi^\alpha = \{\tau, \sigma\}$ and $X^\mu = \{x_0, x_1, x_2, x_3, r\}$.

The steady state of a quark-antiquark pair with constant separation and moving with constant velocity either perpendicular or parallel to the separation of the quarks can be described (up to worldsheet reparametrizations), respectively, by the worldsheet embeddings

$$\begin{aligned} [v_\perp]: &\qquad x_0 = \tau, \quad x_1 = v\tau, \qquad\quad x_2 = \sigma, \quad x_3 = 0, \quad r = r(\sigma), \\ [v_{||}]: &\qquad x_0 = \tau, \quad x_1 = v\tau + \sigma, \quad x_2 = 0, \quad x_3 = 0, \quad r = r(\sigma). \end{aligned} \tag{2.4}$$

For both cases, we take boundary conditions

$$0 \le \tau \le T, \qquad -L/2 \le \sigma \le L/2, \qquad r(\pm L/2) = r_7, \tag{2.5}$$

where $r(\sigma)$ is a smooth embedding.

The endpoints of strings on D-branes satisfy Neumann boundary conditions in the directions along the D-brane, whereas the above boundary conditions are Dirichlet, constraining the string endpoints to lie along fixed worldlines a distance L apart on the D7-brane. The correct way to impose these boundary conditions is to turn on a worldvolume background $U(1)$ field strength on the D7-brane [12] to keep the string endpoints a distance L apart. Thus at finite r_7, it is physically more sensible to describe string solutions for a fixed force on the endpoints instead of a fixed endpoint separation L.[1] Our discussion of spacelike string solutions in the next section will describe both the force-dependence and the L-dependence of our solutions. In the application to evaluating a Wilson loop in section 4, though, we are interested in string solutions (in the $r_7 \to \infty$ limit) with endpoints lying along the given loop, *i.e.*, at fixed L.

According to the AdS/CFT correspondence, strings ending on the D7-brane are equivalent to quarks in a thermal bath in four-dimensional finite-temperature $\mathcal{N}$=4 $SU(N)$ super Yang-Mills (SYM) theory. The standard gauge/gravity dictionary is that $N = R^4/(4\pi\alpha'^2 g_s)$ and $\lambda = R^4/\alpha'^2$ where g_s is the string coupling, $\lambda := g^2_{\rm YM}N$ is the 't Hooft coupling of the SYM theory. In the semiclassical string limit, *i.e.*, $g_s \to 0$ and $N \to \infty$, the supergravity approximation in the gauge/gravity correspondence holds when the curvatures are much greater than the string length $\ell_s := \sqrt{\alpha'}$. Furthermore, in this limit, one identifies

$$\beta = \pi R^2/r_0, \qquad m_0 = r_7/(2\pi\alpha'), \tag{2.6}$$

[1]We thank A. Karch for discussions on this point.

where β is the (inverse) temperature of the SYM thermal bath, and m_0 is the quark mass at zero temperature.

It will be important to note that the velocity parameter v entering in the string worldsheet embeddings (2.4) is *not* the proper velocity of the string endpoints. Indeed, from (2.1) it is easy to compute that the string endpoints at $r = r_7$ move with proper velocity

$$V = \frac{r_7^2}{\sqrt{r_7^4 - r_0^4}}\, v. \tag{2.7}$$

We will see shortly that real string solutions must have the same signature everywhere on the worldsheet. Thus a string wroldsheet will be timelike or spacelike depending on whether V, rather than v, is greater or less than 1. Thus, translating $V \lesseqgtr 1$ into corresponding inequalities for the velocity parameter v, we have

$$\begin{array}{ccl} \begin{array}{c}\text{timelike}\\ \text{string worldsheet}\end{array} & \Leftrightarrow & \quad \text{both} \quad v < 1 \;\; (\gamma^2 > 1) \text{ and } z_7 > \sqrt{\gamma}, \\ & & \\ \begin{array}{c}\text{spacelike}\\ \text{string worldsheet}\end{array} & \Leftrightarrow & \left\{ \begin{array}{ll} \text{either} & v \geq 1 \;\; (\gamma^2 < 0) \text{ and any } z_7, \\ \text{or} & v < 1 \;\; (\gamma^2 > 1) \text{ and } z_7 < \sqrt{\gamma}. \end{array} \right. \end{array} \tag{2.8}$$

Here we have defined the dimensionless ratio of the D7-brane radial position to the horizon radius,

$$z_7 := \frac{r_7}{r_0}, \qquad \text{and} \qquad \gamma^2 := \frac{1}{1 - v^2}. \tag{2.9}$$

Furthermore, since the worldsheet has the same signature everywhere, this implies that timelike strings can only exist for $r > \sqrt{\gamma}\, r_0$, but spacelike strings may exist at all r, as illustrated in figure 1. In this respect, $r = \sqrt{\gamma}\, r_0$ plays a role analogous to that of the ergosphere of a Kerr black hole, although in this case it is not actually an intrinsic feature of the background geometry but instead a property of certain string configurations (2.4) in the background geometry (2.1).

With the embeddings (2.4) and boundary conditions (2.5), the string

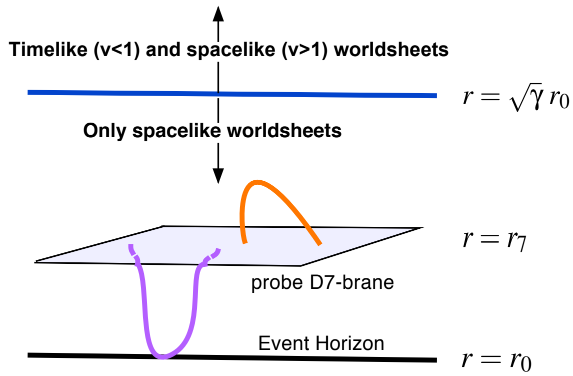


Figure 1: Both timelike and spacelike worldsheets can exist above the radius $r = \sqrt{\gamma} r_0$ (blue line) for $v < 1$ and $v > 1$, respectively. On the other hand, only spacelike worldsheets exist in the region between the blue line and the event horizon, given by $r_0 < r < \sqrt{\gamma} r_0$.

action becomes[2]

$$\begin{aligned}
[v_\perp] : \qquad & S = \frac{-T}{\gamma\, \pi\alpha'} \int_0^{L/2} d\sigma \sqrt{\frac{r^4 - \gamma^2 r_0^4}{R^4} + \frac{r^4 - \gamma^2 r_0^4}{r^4 - r_0^4} r'^2}, \\
[v_{||}] : \qquad & S = \frac{-T}{\gamma\, \pi\alpha'} \int_0^{L/2} d\sigma \sqrt{\gamma^2 \frac{r^4 - r_0^4}{R^4} + \frac{r^4 - \gamma^2 r_0^4}{r^4 - r_0^4} r'^2}, \qquad (2.10)
\end{aligned}$$

where $r' := \partial r / \partial\sigma$. The resulting equations of motion are

$$\begin{aligned}
[v_\perp] : \qquad & r'^2 = \frac{1}{\gamma^2\, a^2 r_0^4 R^4} (r^4 - r_0^4)(r^4 - \gamma^2[1 + a^2] r_0^4), \\
[v_{||}] : \qquad & r'^2 = \frac{\gamma^2}{a^2 r_0^4 R^4} (r^4 - r_0^4)^2 \frac{(r^4 - [1 + a^2] r_0^4)}{(r^4 - \gamma^2 r_0^4)}, \qquad (2.11)
\end{aligned}$$

where a^2 is a real integration constant. Here we have taken the first integral of the second order equations of motion which follows from the existence of a conserved momentum in the direction along the separation of the string endpoints. Since a is associated with this conserved momentum, $|a|$ is proportional to the force applied (via a constant background $U(1)$ field strength on the D7 brane) to the string endpoints in this direction [12].

[2]These expressions for the string action are good only when there is a single turning point around which the string is symmetric. We will later see that for $[v_\perp]$ there exist solutions with multiple turns. For such solutions the limits of integration in (2.10) are changed, and appropriate terms for each turn of the string are summed.

Although we have written a^2 as a square, it can be either positive or negative. Using (2.11), the determinant of the induced worldsheet metric can be written as

$$
\begin{aligned}
[v_\perp]: \qquad & G = -\frac{1}{\gamma^4\, a^2 r_0^4 R^4}(r^4 - \gamma^2 r_0^4)^2, \\
[v_{||}]: \qquad & G = -\frac{1}{a^2 r_0^4 R^4}(r^4 - r_0^4)^2.
\end{aligned}
\tag{2.12}
$$

Thus, the sign of G is the same as that of $-a^2$ (since the other factors are squares of real quantities). In particular, the worldsheet is timelike ($G < 0$) for $a^2 > 0$ and spacelike ($G > 0$) for $a^2 < 0$.

The reality of r' implies that the right sides of (2.11) must be positive in all these different cases, which implies certain allowed ranges of r. Therefore, there can only be real string solutions when the ends of the string, at $r = r_7$, are within this range. The edges of this range are (typically) the possible turning points r_t for the string, whose possible values will be analyzed in the next section.

Given these turning points, (2.11) can be integrated to give

$$
\begin{aligned}
[v_\perp]: \qquad & \frac{L}{\beta} = \frac{2|a\gamma|}{\pi}\left|\int_{z_t}^{z_7} \frac{dz}{\sqrt{(z^4-1)(z^4-\gamma^2[1+a^2])}}\right|, \\
[v_{||}]: \qquad & \frac{L}{\beta} = \frac{2|a|}{\pi|\gamma|}\left|\int_{z_t}^{z_7} \frac{dz\sqrt{z^4-\gamma^2}}{(z^4-1)\sqrt{z^4-[1+a^2]}}\right|,
\end{aligned}
\tag{2.13}
$$

where we have used $r_0 = \pi R^2/\beta$. Also, in (2.13) we have rescaled $z = r/r_0$ and likewise $z_t := r_t/r_0$ and $z_7 := r_7/r_0$. (The absolute value takes care of cases where $z_7 < z_t$.) These integral expressions determine the integration constant a^2 in terms of L/β and v.

Also, we can evaluate the action for the solutions of (2.11):

$$
\begin{aligned}
[v_\perp]: \qquad & S = \pm\frac{T\sqrt{\lambda}}{\gamma\beta}\int_{z_t}^{z_7} \frac{(z^4-\gamma^2)\, dz}{\sqrt{(z^4-1)(z^4-\gamma^2[1+a^2])}}, \\
[v_{||}]: \qquad & S = \pm\frac{T\sqrt{\lambda}}{\gamma\beta}\int_{z_t}^{z_7} dz\sqrt{\frac{z^4-\gamma^2}{z^4-[1+a^2]}},
\end{aligned}
\tag{2.14}
$$

where we have used $R^2/\alpha' = \sqrt{\lambda}$. The plus or minus signs are to be chosen depending on the relative sizes of z_7, z_t, and γ^2, and will be discussed in specific cases below. For finite z_7, these integrals are convergent. They diverge when $z_7 \to \infty$ and need to be regularized by subtracting the self-energy of the quark and the antiquark [46, 47], which will be discussed in more detail in section 4.

Note that, in writing (2.13) and (2.14), we have assumed that the string goes from z_7 to the turning point z_t and back only once. We will see that more complicated solutions with multiple turning points are possible. For these cases, one must simply add an appropriate term, as in (2.13) and (2.14), for each turn of the string.

3 Spacelike solutions

Positivity of the determinant of the induced worldsheet metric (2.12) implies that the integration constant $a^2 < 0$ for spacelike configurations. It is convenient to define a real integration constant α by

$$\alpha^2 := -a^2 > 0. \tag{3.1}$$

As remorked above, α is proportional to the magnitude of a background $U(1)$ field strength on the D7 brane. We will now classify the allowed ranges of r for which r'^2 is positive in the equations of motion (2.11). These ranges, as well as the associated possible turning points of the string depend on the relative values of α, v and 1.

3.1 Perpendicular velocity

The configurations of main interest to us are those for which the string endpoints move in a direction perpendicular to their separation. As we will now see, the resulting solutions have markedly different behavior depending on whether the velocity parameter is greater or less than 1.

3.1.1 $\sqrt{1-z_7^{-4}} < v < 1$

If $v < 1$, we have seen that the string worldsheet can be spacelike as long as $v > \sqrt{1-z_7^{-4}}$. A case-by-case classification of the possible turning points of the $v_\perp$ equation in (2.11) gives the following table of possibilities:

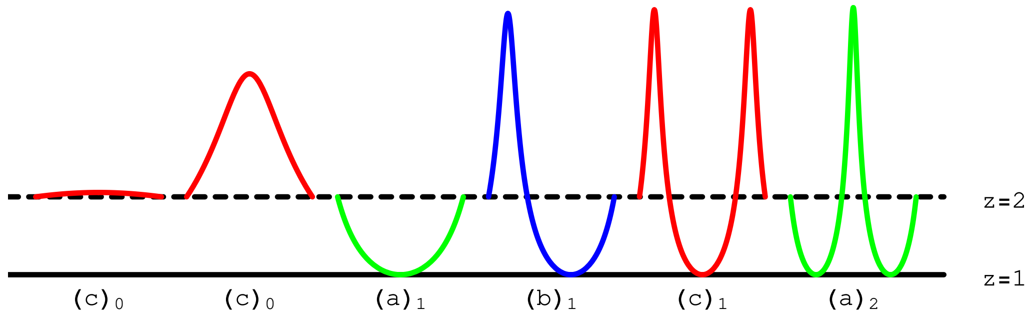


Figure 2: Spacelike string solutions with fixed $L/\beta = 0.25$, $\gamma = 20$ ($v \approx 0.99875$), $z_7 = 2$, and with low values of n (the number of turns at the horizon). The horizon is the solid line at $z = 1$, and the minimum radius of the D7-brane is the dashed line at $z = 2$.

parameters	allowed ranges	
$0 < \alpha < v < 1$		$1 \leq\ z^4 \leq \gamma^2(1-\alpha^2)$
$0 < v < \alpha < 1$	$\gamma^2(1-\alpha^2) \leq\ z^4 \leq 1$	
$0 < v < 1 < \alpha$	$0 \leq\ z^4 \leq 1$	

The left column of allowed ranges are those that lie inside the horizon and the right one are the allowed ranges outside the horizon.

At the horizon, $r' = 0$ and the string becomes tangent to $z = 1$ at finite transverse distance giving a smooth turning point for the string. In the last entry in the above table, "$0 \leq z^4$" indicates that there is no turning point before meeting the singularity at $z = 0$ and the string necessarily meets the singularity.

Since only string solutions that extend into the $z > 1$ region can reliably describe quarks, this eliminates the left column of allowed ranges. Thus, the only viable configurations are those in the right column with $\alpha < v$, which all have turning points at 1 or $\gamma^2(1-\alpha^2)$. This restricts the D7-brane minimum radius to lie between these two turning points which, in turn, gives rise to string configurations with multiple turns.

In order for the D7-brane to be within the allowed range $1 < z_7^4 < \gamma^2(1-\alpha^2)$, the parameter v must be at least $v^2 > 1 - z_7^{-4}$. For a given z_7, v, and α satisfying these inequalities, we integrate (2.13) to obtain L/β. There are two choices for the range of integration: $[1, z_7]$ and $[z_7, \gamma^2(1-\alpha^2)]$. The first one is appropriate for a string which decends down to the horizon and then turns back up to the D7-brane; we will call this a "down string". The second range describes a string which ascends to larger radius and then turns back down to the D7-brane; we will call this an "up string". Given these two behaviors, it is clear that we can equally well construct infinitely many other solutions by simply alternating segments of up and down strings. In particular, there are three possible series of string configurations, which we will call the $(\mathrm{a})_n$, $(\mathrm{b})_n$, and $(\mathrm{c})_n$ series. An $(\mathrm{a})_n$ string starts with a down string then adds $n-1$ pairs of up and down strings, thus ending with a down string; a $(\mathrm{b})_n$ string concatenates n pairs of up and down strings—for example, starts with an up string and ends with a down string; and a $(\mathrm{c})_n$ string starts with an up string and then adds n pairs of down and up strings, thus ending with an up string. n counts the number of turns the string makes at the horizon, $z=1$. In particular, for the $(\mathrm{a})_n$ and $(\mathrm{b})_n$ series, n is an integer $n \geq 1$, while for the $(\mathrm{c})_n$ series, $n \geq 0$. Examples of these string configurations appear in Figure 2. If the separation of the ends of the up and down strings are L_{up} and L_{down}, respectively, then the possible total separations of the strings fall into three classes of lengths

$$
\begin{aligned}
L_{(\mathrm{a}),n} &= nL_{\mathrm{down}} + (n-1)L_{\mathrm{up}}, \\
L_{(\mathrm{b}),n} &= nL_{\mathrm{down}} + nL_{\mathrm{up}}, \\
L_{(\mathrm{c}),n} &= nL_{\mathrm{down}} + (n+1)L_{\mathrm{up}}. \qquad (3.2)
\end{aligned}
$$

Figure 3 illustrates the systematics of the $L_{(\mathrm{a,b,c}),n}$ dependence on α. Here we have chosen $z_7 = 2$, so the minimum value of v for the solutions to exist has $\gamma = 4$. The leftmost plot illustrates that, for small γ, $L_{(\mathrm{c}),0} = L_{\mathrm{up}} \ll L_{\mathrm{down}}$ for all α. Thus, for each $n \geq 1$, $L_{(\mathrm{a}),n} \approx L_{(\mathrm{b}),n} \approx L_{(\mathrm{c}),n}$, and are virtually indistinguishable in the figure. As γ increases, L_{up} and L_{down} begin to approach each other for most α, except for α near $\alpha_{\max} := \sqrt{1 - z_7^4/\gamma^2}$, where L_{up} decreases sharply back to zero.

This behavior implies that, for every fixed L and v, there is a very large number of solutions[3] in each series but that the minimum value of n that occurs decreases as v increases. In detail, it is not too hard to show that the pattern of appearance of solutions as v increases for fixed L is as follows: if for a given v there is one solution (*i.e.*, value of α) for each $(\mathrm{a,b,c})_n$–string with $n > n_0$, then as v increases first *two* $(\mathrm{c})_{n_0}$ solutions will appear, then the $(\mathrm{c})_{n_0}$ solution with the greater α will disappear just as a $(\mathrm{b})_{n_0}$ and an $(\mathrm{a})_{n_0}$ solution appear. Also, $\alpha((\mathrm{a})_n) < \alpha((\mathrm{b})_n) < \alpha((\mathrm{c})_n)$. For example, in Figure 3, when $L/\beta = 0.25$ and $\gamma = 4.2$, there are $(\mathrm{a,b,c})_n$ solutions for $n \geq 2$.

[3]Although n does not formally have an upper bound, as n increases the turns of the string become sharper and denser. Therefore, for large enough n the one can no longer ignore the backreaction of the string on the background.

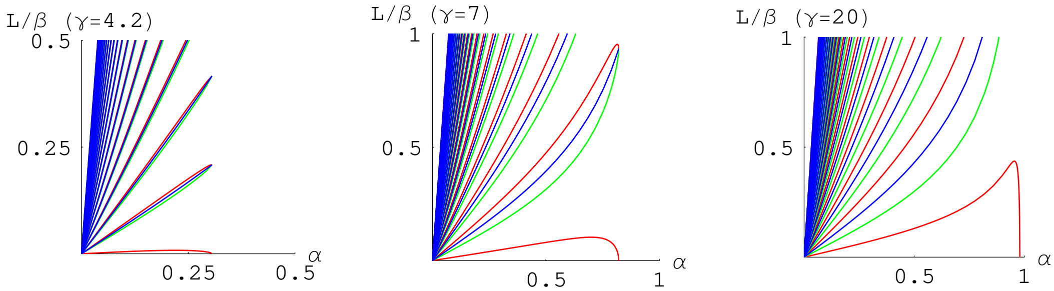


Figure 3: L/β as a function of α for spacelike string configurations with perpendicular velocity, $z_7 = 2$ and $\gamma = 4.2$, 7, and 10. Green curves correspond to the (a)–series, blue to (b)–series, and red to (c)–series. Only the series up to $n = 20$ are shown; the rest would fill the empty wedge near the L/β axis. Note that the scale of the $\gamma = 4.2$ plot is half that of the other two.

Increasing v to $\gamma = 7$ (for the same L), there are now (a,b,c)$_n$ solutions for $n \geq 1$. Increasing v further to $\gamma = 20$, there are now in addition two (c)$_0$ (*i.e.*, up string) solutions. Figure 2 plots the string solutions when $z_7 = 2$, $L/\beta = 0.25$, and $\gamma = 20$, for low values of n.

Note that if one keeps the D7-brane $U(1)$ field strength, α, constant instead of the endpoint separation, L, then there will still be an infinite sequence of string solutions qualitatively similar to that shown in Figure 2. In this case the endpoint separation L increases with the number of turns.

The action for spacelike configurations is imaginary because the Nambu-Goto Lagrangian is $\sqrt{-G} = \pm i\sqrt{G}$. Ignoring the $\pm i$ factor (which we will return to in the next section), the integral of $\sqrt{G}$ just gives the area of the worldsheet. Dividing by the "time" parameter T in (2.14) then gives the length of the string: $\ell = \pm iS/T$. Figure 4 plots the lengths of the various series of string configurations for increasing values of the velocity parameter. There are negative lengths because the length of a pair of straight strings stretched between the D7-brane and the horizon has been subtracted, for comparison purposes. It is clear from the figure that the (c)$_0$ up strings are the shortest for any given L less than a velocity-dependent critical value. Furthermore, for L small enough, they are also shorter than the straight strings.

In particular, the shorter (larger α) of the two up strings has the smallest ℓ of all. As $v \to 1$, the critical value of L below which the up string is the

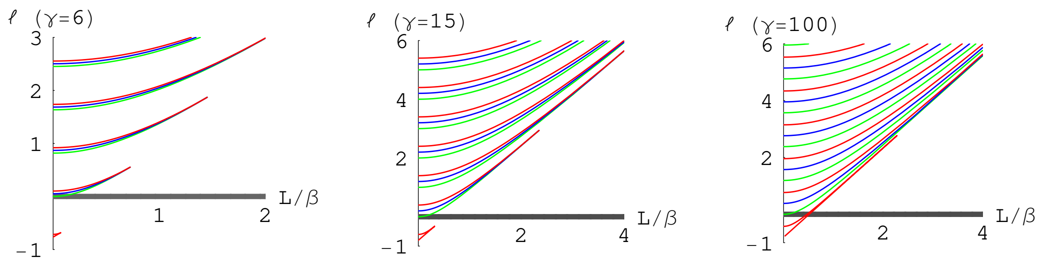


Figure 4: Spacelike string lengths ℓ in units of $\sqrt{\lambda}/\beta$ as a function of endpoint separation L/β and $z_7 = 2$, for $\gamma = 6, 15$ and 100. The gray line along the L/β axis is the (subtracted) length of a pair of straight strings stretched between the D7-brane and the horizon. Note that the scale of the $\gamma = 6$ plot is half that of the others.

solution with the minimum action increases without bound. In this case, any of the other spacelike strings will decay to this minimum-action configuration. Therefore, it is this configuration which must be used for any calculations of physical quantities, such as the jet quenching parameter $\hat{q}$.

3.1.2 $v > 1$

A case-by-case classification of the possible turning points of the $v_\perp$ equation in (2.11) when $v > 1$ gives the following table of possibilities:

parameters	allowed ranges	
$0 < \alpha < 1 < v$		$1 \leq z^4 < \infty$
$0 < 1 < \alpha < v$	$0 \leq z^4 \leq \gamma^2(1-\alpha^2)$	$1 \leq z^4 < \infty$
$0 < 1 < v < \alpha$	$0 \leq z^4 \leq 1$	$\gamma^2(1-\alpha^2) \leq z^4 < \infty$

The left column of allowed ranges are those that lie inside the horizon, while the right one lists the allowed ranges which are outside the horizon. At the horizon, $r' = 0$ and the string becomes tangent to $z = 1$ at finite transverse distance. This can be either a smooth turning point for the string or, if there is an allowed region on the other side of the horizon, then the string can have an inflection point at the horizon and continue through it. From the table, we see that this can only happen at the crossover between the last two lines—in other words, when $\alpha = v > 1$. In the above table we have written "$0 \leq z^4$" when there is no turning point in an allowed region before the singularity at $z = 0$. In these cases, a string extending towards smaller z will necessarily meet the singularity. As before, we are only interested in string

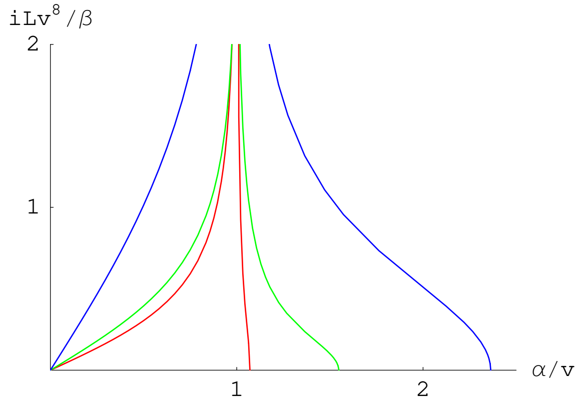


Figure 5: Lv^8/β as a function of α/v for spacelike string configurations with perpendicular velocity, $z_7 = 2$, and $v = 1.005$ (red), 1.05 (green), and 1.2 (blue).

solutions that extend into the $z > 1$ region. This eliminates the left column of allowed ranges, with the possible exception of the $v = \alpha > 1$ crossover case, for which the string might inflect at the horizon and then extend inside. However, if it does extend inside, then it will hit the singularity. Therefore, we can also discard this possibility as being outside the regime of validity of our approximation. Thus, the only viable configurations are those given in the right column, which all have turning points at either 1 or $\gamma^2(1-\alpha^2)$, or else go off to infinity. Since we want to identify the quarks with the ends of the strings on the D7-brane, we are only interested in string configurations that begin and end at $z_7 > 1$, and so discard configurations which go off to $z \to \infty$ instead of turning. Thus the $v > 1$ ranges compatible with these conditions all have only one turning point, describing strings dipping down from the D7-brane and either turning at the horizon or above it, depending on α versus v.

Indeed, it is straightforward to check that for any L there are two $v > 1$ solutions, one with $\alpha > v$ and one with $\alpha < v$. L/β as a function of α is plotted in Figure 5. (The rescalings by powers of v are just so the curves will nest nicely in the figure.) The $\alpha < v$ solutions are long strings which turn at the horizon, while the $\alpha > v$ solutions are short strings with turning point $z_t^4 = (\alpha^2-1)/(v^2-1)$. The norm of the action for these configurations (which is proportional to the length of the strings) is likewise greater for the $\alpha < v$ solutions than for the $\alpha > v$ ones.

If, instead, one keeps the D7-brane $U(1)$ field strength α constant, then there is at most a single string solution with a given velocity v.

3.2 Parallel velocity

For completenes, though we will not be using these confurations in the rest of the paper, we briefly outline the set of solutions for suspended spacelike string configurations with velocity parallel to the endpoint separation. A case-by-case analysis of the equations of motion (2.11) gives the following table of allowed ranges for string solutions:

parameters	allowed ranges	
$0 < (\alpha, v) < 1$	$1 - \alpha^2 \leq z^4 < 1$	$1 < z^4 \leq \gamma^2$
$0 < v < 1 < \alpha$	$0 \leq z^4 < 1$	$1 < z^4 \leq \gamma^2$
$0 < \alpha < 1 < v$	$1 - \alpha^2 \leq z^4 < 1$	$1 < z^4 < \infty$
$0 < 1 < (\alpha, v)$	$0 \leq z^4 < 1$	$1 < z^4 < \infty$

Although $z = 1$ is always included in the allowed ranges, in the table we have split each range into two regions: one inside the horizon and one outside. The reason is that the string equation of motion (2.11) near $r = r_0$ is $r'^2 \sim (r - r_0)^2$, whose solutions are of the form $r - r_0 \sim \pm e^{\pm\sigma}$. This implies that these solutions asymptote to the horizon and never turn. Thus, the parallel spacelike strings can never cross the horizon.

As always, we only look at solutions that extend into the $z > 1$ region, since that is where we can reliably put D7-branes. This eliminates the left-hand column of configurations. Recall that the signature of the worldsheet metric for a string with $v < 1$ changes at $z = \sqrt{\gamma}$. A string that reaches this radius will have a cusp there. This is qualitatively similar to the timelike parallel solutions with cusps described in [32], except that in the spacelike case the strings extend away from the horizon (towards greater z). Thus, the string solutions corresponding to the ranges in the right-hand column all either asymptote to $z = 1$, go off to infinity or have a cusp at $z = \sqrt{\gamma}$. The first two cases do not give strings with two endpoints on the D7-brane at $z = z_7$. Therefore, the only potentially interesting configurations for our purposes are those with $1 < z_7 < z < \sqrt{\gamma}$, which occur for $v < 1$ and any α. However, since these configurations have cusps, their description in terms of the Nambu-Goto action is no longer complete. That is, there must be additional boundary conditions specified, which govern discontinuities in the first derivatives of the string shape. As discussed in [32] for the analogous timelike strings, these cusps cannot be avoided by extending the string to include a smooth but self-intersecting closed loop, since real string solutions cannot change their worldsheet signature.

4 Application to jet quenching

We will now apply the results of the last two sections to the computation of the expectation value of a certain Wilson loop $W[\mathcal{C}]$ in the SYM theory. The interest of this Wilson loop is that it has been proposed [11] as a non-perturbative definition of the jet quenching parameter $\hat{q}$. This medium-dependent quantity measures the rate per unit distance traveled at which the average transverse momentum-squared is lost by a parton moving in plasma [1].

In particular, [11] considered a rectangular loop $\mathcal{C}$ with parallel lightlike edges a distance L apart which extend for a time duration T. Motivated by a weak-coupling argument, the leading behavior of $W[\mathcal{C}]$ (after self-energy subtractions) for large T and $L\beta \ll 1$ is claimed to be

$$\langle W^A(\mathcal{C})\rangle = \exp[-\frac{1}{4}\ \hat{q}\ TL^2], \tag{4.1}$$

where $\langle W^A(\mathcal{C})\rangle$ is the thermal expectation value of the Wilson loop in the adjoint representation. We will simply view this as a definition of $\hat{q}$.[4] Note that exponentiating the Nambu-Goto action gives rise to the thermal expectation value of the Wilson loop in the fundamental representation $\langle W^F(\mathcal{C})\rangle$. Therefore, we will make use of the relation $\langle W^A(\mathcal{C})\rangle \approx \langle W^F(\mathcal{C})\rangle^2$, which is valid at large N.

Self-energy contributions are expected to contribute on the order of TL^0 and, since this is independent of L, their subtraction does not affect the L-dependence of the results. The subtraction is chosen to remove infinite constant contributions, but are ambiguous up to finite terms.[5] However, there may be other leading contributions of order TL^{-1} or TL. For example, as we discussed in the conclusions, a term linear in L would be consistent with energy loss by elastic scattering. Therefore, one requires a subtraction

[4] This differs by a constant factor from the definition written in [11] since here it is expressed in the reference frame of the plasma rather than that of the parton.

[5] Note that [50] shows that the correct treatment of the Wilson loop boundary conditions should automatically and uniquely subtract divergent contributions; it would be interesting to evaluate our WIlson loop using this prescription instead of the more *ad hoc* one used here and throughout the literature.

prescription. We will assume the following one: extract $2^{-3/2}\hat{q}$ as the coefficient of L^2 in a Laurent expansion of the action around $L = 0$. Thus, concretely,

$$W[\mathcal{C}] \sim \exp\left\{-T\left(\cdots + \frac{\alpha_{-1}}{L} + \alpha_0 + \alpha_1 L + \frac{\hat{q}}{4}L^2 + \cdots\right)\right\}. \tag{4.2}$$

Implicit in this is a choice of finite parts of leading terms to be subtracted, which could affect the value of $\hat{q}$; we have no justification for this prescription beyond its simplicity. We will see that this issue of L-dependent leading terms indeed arises in the computation of $\hat{q}$ using the AdS/CFT correspondence.

There is a second subtlety in the definition of $\hat{q}$ given in (4.2), which involves how the lightlike limit of $\mathcal{C}$ is approached. In the AdS/CFT correspondence, we evaluate the expectation value of the Wilson loop as the exponential $\exp\{iS\}$ of the Nambu-Goto action for a string with boundary conditions corresponding to the Wilson loop $\mathcal{C}$. If we treat $\mathcal{C}$ as the lightlike limit of a sequence of timelike loops, then the string worldsheet will be timelike and the exponential will be oscillatory, instead of exponentially suppressed in T as in (4.2). The exponential suppression requires either an imaginary action (of the correct sign) or a Wick rotation to Euclidean signature.

The authors of [11] advocate the use of the lightlike limit of spacelike strings to evaluate the Wilson loop [27, 42]. Below, we will evaluate the Wilson loop using both the spacelike prescription and the Euclidean one. Our interest in the Euclidean Wilson loop is mainly for comparison purposes and to help elucidate some subtleties in the calculation; we emphasize that it is *not* the one proposed by the authors of [11] to evaluate $\hat{q}$. (Though the Euclidean prescription is the usual one for evaluating static thermodynamic quantities, we are here evaluating a non-static property of the SYM plasma and so the usual prescription may not apply.)

In both cases we will find that, regardless of the manner in which the above ambiguities are resolved, the computed value of $\hat{q}$ is zero.

4.1 Euclidean Wilson loop

Euclidean string solutions [32] are reviewed in appendix A. Here we just note their salient properties. In Euclidean signature, nothing special happens in the limit $V \to 1$ ($v \to \sqrt{1 - z_7^{-4}}$). When $V = 1$ there are always only two Euclidean string solutions: the "long string", with turning point at the horizon $z = 1$, and the "short string", with turning point above the horizon. The one which gives the dominant contribution to the path integral is the one with smallest Euclidean action. For endpoint separation L less than a critical value, the dominant solution is the short string. This is the string configuration that remains the furthest from the black hole horizon [32].

We are interested in evaluating the Euclidean string action for the short string in the small L limit (the so-called "dipole approximation"). However, there is a subtlety associated with taking this limit since it does not commute with taking the $z_7 \to \infty$ limit, which corresponds to infinite quark mass. Recall that the quark mass scales as r_7 in string units; introduce a rescaled length parameter

$$\epsilon := \frac{1}{z_7} = \frac{r_0}{r_7} \tag{4.3}$$

associated with the Compton wavelength of the quark. Then the behavior of the Wilson loop depends on how we parametrically take the $L \to 0$ and $\epsilon \to 0$ limits. For instance, if one keeps the mass (ϵ^{-1}) fixed and takes $L \to 0$ first, then the Wilson loop will reflect the overlap of the quark wave functions. On the other hand, if one takes $\epsilon \to 0$ before L, then the Wilson loop should reflect the response of the plasma to classical sources. The second limit is presumably the more physically relevant one for extracting the $\hat{q}$ parameter. We perform the calculation in both limits in appendix A to verify this intuition.

In the $L \to 0$ limit at fixed (small) ϵ, the action of the short string as a function of L and ϵ is found in appendix A to be

$$S = \frac{\pi T \sqrt{\lambda}}{\sqrt{2}\beta^2} \left\{ \frac{L}{\epsilon^2} \left[1 + \tfrac{1}{4}\epsilon^4 + \mathcal{O}(\epsilon^8) \right] - \frac{\pi^2 L^3}{\beta^2 \epsilon^4} \left[\tfrac{1}{3} - \tfrac{1}{6}\epsilon^4 + \mathcal{O}(\epsilon^8) \right] + \mathcal{O}\left(\tfrac{L^5}{\beta^4 \epsilon^6} \right) \right\}. \tag{4.4}$$

(In fact, this result is valid as long as $L \to 0$ as $L \propto \epsilon$ or faster.) The main thing to note about this expression is that it is divergent as $\epsilon \to 0$. This is *not* a self-energy divergence that we failed to subtract, since any self-energy subtraction (*e.g.*, subtracting the action of two straight strings extending radially from $z = z_7$ to $z = 1$) will be independent of the quark separation and so cannot cancel the divergences in (4.4). (In fact, it inevitably adds an $\epsilon^{-1}L^0$ divergent piece.) This divergence as the quark mass is taken infinite is a signal of the unphysical nature of this order of limits.

The other order of limits, in which $\epsilon \to 0$ at fixed (small) L, is expected to reflect more physical behavior. Indeed, appendix A gives

$$\frac{\beta \hat{S}}{T\sqrt{\lambda}} = -0.32\,\frac{\beta}{L} + 1.08 - 0.76\,\frac{L^3}{\beta^3} + \mathcal{O}(L^7). \tag{4.5}$$

Here $\hat{S}$ is the action with self-energy subtractions. This result (which is in the large mass, or $\epsilon \to 0$, limit) is finite for finite quark separation L. The L^{-1} term recovers the expected Coulombic interaction. Since there is no L^2 term in (4.5), the subtraction prescription (4.2) implies that the Euclidean analog of the jet quenching parameter vanishes.

For the sake of comparison, we also compute the long string action in this limit with the same regularization in Appendix A, giving

$$\frac{\hat{S}_{\rm long}}{T\sqrt{\lambda}} = +2.39\,\frac{L^2}{\beta^3} + \mathcal{O}(L^4). \tag{4.6}$$

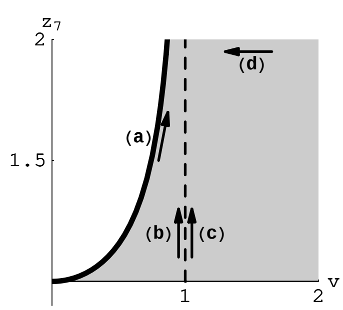
This does have the leading L^2 dependence, giving rise to an unambiguous nonzero $\hat{q}$. But it is exponentially suppressed compared to the short string contribution (4.5), and so gives no contribution to the effective $\hat{q}$ in the $T \to \infty$ limit.

4.2 Spacelike Wilson loop

We now turn to the spacelike prescription for calculating the Wilson loop. We will show that a similar qualitative behavior to that of the Euclidean path integral shown in (4.5) and (4.6) also holds for spacelike strings. In particular, the leading contribution is dominated by a confining-like (L) behavior with no jet quenching-like (L^2) subleading term, and only an exponentially suppressed longer-string contribution has a leading jet quenching-like behavior. The analogous results are recorded in (4.7) and (4.8), below.

Since $-G < 0$ for spacelike worldsheets, the Nambu-Goto action is imaginary and so $\exp\{iS\} = \exp\{\pm A\}$, where A is the positive real area of the string worldsheet. The sign ambiguity comes from the square root in the Nambu-Goto action. For our stationary string solutions, the worldsheet area is the time of propagation T times the length of the string. Thus, with the choice of the plus sign in the exponent, the longest string length exponentially dominates the path integral, while for the minus sign, the shortest string length dominates. Only the minus sign is physically sensible, though, since we have seen in section 3 that the length of the spacelike string solutions is unbounded from above (since there are solutions with arbitrarily many turns). Thus we must pick the minus sign, and, as in the Euclidean case, the solution with shortest string length exponentially dominates the path integral.

As we illustrated in our discussion of the Euclidean Wilson loop, the physically sensible limit is to take the quarks infinitely massive ($z_7 \to \infty$) at fixed quark separation L. In the spacelike case, however, there is a new subtlety: *a priori* it is not obvious that the lightlike limit $V \to 1$ will commute with the $z_7 \to \infty$ limit. Since $V = v(1 - z_7^{-4})^{-1/2}$, the lightlike limit is $v \to 1$ when $z_7 \to \infty$. We will examine four different approaches to this limit, shown in figure 6.[6]



(a) $\lim_{z_7\to\infty} \lim_{V\to 1^+}$

(b) $\lim_{z_7\to\infty} \lim_{v\to 1^-}$

(c) $\lim_{z_7\to\infty} \lim_{v\to 1^+}$

(d) $\lim_{v\to 1^+} \lim_{z_7\to\infty}$

Figure 6: The shaded region is the set of (v, z_7) for which the string worldsheet is spacelike and outside the horizon. The curved boundary corresponds to lightlike worldsheets. The various approaches to the lightlike $z_7 = \infty$ limit discussed in the text are shown.

Limit (a): $\lim_{\mathbf{z_7}\to\infty} \lim_{\mathbf{V}\to\mathbf{1}^+}$.

This is the limit in which we take the lightlike limit at fixed z_7, then take the mass to infinity. Recall from (2.8) that a spacelike worldsheet requires either $v \geq 1$ ($\gamma^2 < 0$) for any z_7, or $v < 1$ ($\gamma^2 > 1$) and $z_7 < \sqrt{\gamma}$. Since, at fixed z_7, $V = 1$ corresponds to $\gamma = z_7^2$, we necessarily have $v < 1$. Thus, only the $v < 1$ spacelike solutions discussed in section 3.1.1 will contribute. Recall that for these solutions $1 \leq z^4 \leq \gamma^2(1 - \alpha^2)$, where the integration constant is in the range $0 < \alpha < v$. In particular, we must keep $\gamma^2(1 - \alpha^2) \geq z_7^4$

[6] See [29] for a related discussion of the lightlike limit.

and $\gamma^2 \geq z_7^4$ while taking the $\gamma^2 \to z_7^4$ limit. This implies that we must take solutions with $\alpha \to 0$. However, such solutions necessarily have $L \to 0$ (see figure 3), which contradicts our prescription of keeping L fixed. Therefore, this limit is not interesting.

Limit (b): $\mathbf{\lim_{z_7\to\infty}\lim_{v\to 1^-}}$.

Another approach to the lightlike limit takes $v \to 1$ from below and then takes $z_7 \to \infty$. Then the conditions for a spacelike worldsheet are automatically satisfied. Again, only the $v < 1$ spacelike solutions discussed in section 3.1.1 contribute, but now the $\gamma^2(1-\alpha^2) \geq z_7^4$ condition places no restrictions on α. In particular, this limit will exist at fixed L. The behavior of figure 3 as $v \to 1$ suggests that, at any given (small) L, all the series of string solutions illustrated in figure 2 occur. The lengths of these strings follow the pattern plotted in figure 4. Actually, the analysis given in appendix B shows that the short $(c)_0$ "up" string does not exist in the limit with fixed L. Thus the long $(c)_0$ "up" string dominates the path integral, with the $(a)_1$ "down" string and all longer strings relatively exponentially suppressed.

The result from appendix B.1 for the action of the $(c)_0$ string as a function of L is

$$\frac{\beta \hat{S}}{T\sqrt{\lambda}} = -1.31 + \frac{\pi}{2}\,\frac{L}{\beta}. \tag{4.7}$$

This result is exact, in the sense that no higher powers of L enter. The constant term is from the straight string subtraction. The linear term is consistent with energy loss by elastic scattering.

For comparison, the next shortest string is the $(a)_1$ down string solution. Appendix A computes its action to be

$$\frac{\beta \hat{S}_{\rm long}}{T\sqrt{\lambda}} = 0.941\,\frac{L^2}{\beta^2} + \mathcal{O}(L^4), \tag{4.8}$$

which shows the jet-quenching behavior found in [11]. However, since the contribution from this configuration to the path integral is exponentially suppressed, the actual jet quenching parameter is zero.

Limit (c): $\mathbf{\lim_{z_7\to\infty}\lim_{v\to 1^+}}$.

When $v > 1$, the string worldsheet is spacelike regardless of the value of z_7. Thus, we are free to take the order of limits in many ways. Limit (c) takes

$v \to 1$ from above at fixed z_7 and then takes $z_7 \to \infty$. We saw in section 3.1.2 that there are always two string solutions for $v > 1$: a short one with $\alpha > v$, which turns at $z = z_t := \gamma^2(1-\alpha^2)$, and a long one with $\alpha < v$, which turns at the horizon $z = 1$. Appendix B.2 shows that in the (c) limit, the short string gives precisely the same contribution as the $(c)_0$ up string did in the (b) limit. Similarly, the long string contirbution coincides with the $(a)_1$ down string. This agreement is reassuring, showing that the path integral does not jump discontinuously between the (b) and (c) limits even though they are evaluated on qualitatively different string configurations. (The (b) and (c) limits approach the lightlike limit in the same way, see figure 6.)

Limit (d): $\lim_{\mathbf{v}\to\mathbf{1}^+} \lim_{\mathbf{z_7}\to\infty}$.

Limit (d) approaches the lightlike limit in the opposite order to the (c) limit. Somewhat unexpectedly, the results for the string action in the (d) limit are numerically the same as those found in the (b) and (c) limits. This is unexpected since the details of evaluating the integrals in the (c) and (d) limits are substantially different. We take this agreement as evidence that the result is independent of how the lightlike limit is taken. (Note that there are, in principle, many different lightlike limits intermediate between the (c) and (d) limts.)

A Euclidean action

Euclidean string solutions

Wick rotate $x_0 \to ix_4$ in (2.1), and adopt the rotated boundary conditions (2.5) with $x_0 \to x_4$. Then the Euclidean version of the $[v_\perp]$ embedding (2.4) becomes

$$x_4 = \tau, \qquad x_1 = v\tau, \qquad x_2 = \sigma, \qquad x_3 = 0, \qquad r = r(\sigma). \tag{A.1}$$

One then finds [32] that the integral expressions (2.13) and (2.14) for the quark separation and string action stay the same except for the replacement

$$\gamma^2 \to \gamma_E^2 := \frac{1}{1+v^2}. \tag{A.2}$$

Real Euclidean string configurations must have the integration constant a^2 be positive, to have positive G. Then an analysis of the Euclidean string equations of motion [32] shows that real solutions can exist for any v as long as the string is at radii satisfying

$$z^4 \ > \ \max\left\{1,\ \gamma_E^2(1+a^2)\right\}. \tag{A.3}$$

(These are for the string configurations with endpoint "velocity" perpendicular to their separation.)

For $a > v$ and v sufficiently large, there is a unique Euclidean solution with turning point at $z^4 = z_t^4 := (1+a^2)/(1+v^2) > 1$. We call these the "short string" solutions. For $a < v$ there is a branch of Euclidean solutions which have the radial turning point on the black hole horizon $z = 1$. These are the "long string" solutions. The solution with the smallest energy dominates the path integral. The energy of the Euclidean string configurations is given by $E = S/T$, where S is the Nambu-Goto action and T is the time interval. For L less than a critical value, the energetically favorable state is the short string [32].

The Euclidean rotation of strings whose endpoints have lightlike worldlines are those with Euclidean worldsheet (A.1) with $V = 1$. By (2.7) this is when $v = \sqrt{1 - z_7^{-4}}$. But since nothing special happens to the Euclidean string configurations at this velocity, we will do our computations below at arbitrary v, and specialize to the lightlike value at the end.

We are interested in evaluating the action for this string in the small L and small $\epsilon := z_7^{-1}$ (large mass) limit. These two limits do not commute, so we evaluate them separately in the two different orders.

$L \to 0$ at fixed (small) ϵ

From (2.13) with $\gamma \to \gamma_E$, the $L \to 0$ limit corresponds to taking $z_t \to z_7$. So introduce a small parameter δ defined by

$$z_t^4 := \frac{z_7^4}{(1+\delta)^4} = \frac{1}{\epsilon^4(1+\delta)^4}. \tag{A.4}$$

Thus δ replaces the parameter a.

Changing variables to $y = \epsilon(1+\delta)z$, (2.13) and (2.14) can be rewritten in terms of δ as

$$\begin{aligned} \frac{L}{\beta} &= \frac{2}{\pi}\epsilon(1+\delta)\int_1^{1+\delta}\frac{dy\,\sqrt{1-\gamma_E^2\epsilon^4(1+\delta)^4}}{\sqrt{(y^4-1)(y^4-\epsilon^4(1+\delta)^4)}}, \\ \frac{\beta S}{T\sqrt{\lambda}} &= \frac{1}{\gamma_E\epsilon(1+\delta)}\int_1^{1+\delta}\frac{dy\,\left[y^4-\gamma_E^2\epsilon^4(1+\delta)^4\right]}{\sqrt{(y^4-1)(y^4-\epsilon^4(1+\delta)^4)}}. \end{aligned} \tag{A.5}$$

Systematically expanding in small δ gives series expressions in terms of integrals of the form

$$J_{nm} := \int_1^{1+\delta} \frac{y^{4m} dy}{(y^4-1)^{\frac{1}{2}} (y^4-\epsilon^4)^{\frac{1}{2}+n}}, \tag{A.6}$$

which have a series expansion of the form $\sum_{n=0}^{\infty} c_n \delta^{n+\frac{1}{2}}$, but whose coefficients $c_n(\epsilon)$ lack closed-form expressions. Nevertheless, the J_{mn} are uniformly convergent for $\epsilon < 1$, so we can expand the integrands in power series in small ϵ to find

$$\begin{aligned}
\frac{L}{\beta} &= \frac{2\epsilon\delta^{1/2}}{\pi} \Big\{ \left[1 + \tfrac{1}{2}(1-\gamma_E^2)\epsilon^4 + \mathcal{O}(\epsilon^8)\right] \\
&\qquad + \delta \left[\tfrac{1}{12} + \tfrac{1}{24}(33 - 49\gamma_E^2)\epsilon^4 + \mathcal{O}(\epsilon^8)\right] + \mathcal{O}(\delta^2) \Big\}, \\
\frac{\beta S}{T\sqrt{\lambda}} &= \frac{\delta^{1/2}}{\gamma_E \epsilon} \Big\{ \left[1 + \tfrac{1}{2}(1-\gamma_E^2)\epsilon^4 + \mathcal{O}(\epsilon^8)\right] \\
&\qquad + \delta \left[-\tfrac{5}{4} + \tfrac{17}{24}(1 - \gamma_E^2)\epsilon^4 + \mathcal{O}(\epsilon^8)\right] + \mathcal{O}(\delta^2) \Big\}.
\end{aligned}$$

Eliminating δ between these two expressions order-by-order in δ then gives the action as a function of L and ϵ:

$$\frac{\beta S}{T\sqrt{\lambda}} = \frac{\pi}{2\gamma_E} \frac{L}{\beta} \left[\frac{1}{\epsilon^2} + \mathcal{O}(\epsilon^6)\right] - \frac{\pi^3}{6\gamma_E} \frac{L^3}{\beta^3} \left[\frac{1}{\epsilon^4} - \frac{1}{2}(2-\gamma_E^2) + \mathcal{O}(\epsilon^4)\right] + \mathcal{O}(L^5). \tag{A.7}$$

The lightlike limit corresponds to taking $\gamma_E = (2+\epsilon^4)^{-1/2}$, giving

$$\frac{\beta S}{T\sqrt{\lambda}} = \frac{\pi}{\sqrt{2}} \frac{L}{\beta} \left[\frac{1}{\epsilon^2} + \frac{1}{4}\epsilon^2 + \mathcal{O}(\epsilon^6)\right] - \frac{\pi^3}{3\sqrt{2}} \frac{L^3}{\beta^3} \left[\frac{1}{\epsilon^4} - \frac{1}{2} + \mathcal{O}(\epsilon^4)\right] + \mathcal{O}(L^5). \tag{A.8}$$

Note that because of the nice convergence properties of the integrals in (A.6), the order of limits as $\epsilon \to 0$ and $\delta \to 0$ does not affect this result. The limiting case where $\epsilon \to 0$ with δ fixed (and small) corresponds to taking $L \propto \epsilon$. Thus the result (A.8) is valid for all limits $\epsilon \to 0$ with $L \to 0$ as $L \propto \epsilon$ or faster.

$\epsilon \to 0$ limit at fixed L

To keep L fixed, examination of (A.5) shows that we need to scale $\delta \to \infty$ as $\epsilon \to 0$ keeping $\epsilon(1+\delta)$ fixed. So change variables in (A.5) from δ to

$\ell := \epsilon(1+\delta)$. Since for fixed $\ell < 1$ the integral is convergent, we can take the $\epsilon \to 0$ limit directly, and then expand in powers of ℓ to get

$$\frac{L}{\beta} = \frac{2}{\pi}\ell \int_1^\infty \frac{dy\,\sqrt{1-\gamma_E^2\ell^4}}{\sqrt{(y^4-1)(y^4-\ell^4)}} = \frac{2}{\sqrt{\pi}}\frac{\Gamma[\frac{3}{4}]}{\Gamma[\frac{1}{4}]}\,\ell\left(1+\frac{3-5\gamma_E^2}{10}\ell^4+\mathcal{O}(\ell^8)\right). \tag{A.9}$$

Similarly, the S integral is

$$\begin{aligned}\frac{\beta S}{T\sqrt{\lambda}} &= \lim_{\epsilon\to 0}\frac{1}{\gamma_E \ell}\int_1^{\ell/\epsilon}\frac{dy\,[y^4-\gamma_E^2\ell^4]}{\sqrt{(y^4-1)(y^4-\ell^4)}} \\ &= \frac{1}{\epsilon\gamma_E} - \frac{\sqrt{\pi}}{\ell\gamma_E}\frac{\Gamma[\frac{3}{4}]}{\Gamma[\frac{1}{4}]}\left\{1-\frac{1-2\gamma_E^2}{2}\ell^4+\mathcal{O}(\ell^8)\right\},\end{aligned} \tag{A.10}$$

where we used the fact that only the leading term at small ℓ diverges as $1/\epsilon$, and so the $\epsilon \to 0$ limit can be taken directly in all the other terms.

Since S is divergent as $\epsilon \to 0$, we regulate the action by subtracting the action of a pair of straight strings with the same boundary conditions. (See, however, [29] for a discussion of an alternative regularization procedure.) The straight string solutions have embeddings

$$x_4 = \tau, \qquad x_1 = v\tau, \qquad x_2 = \pm L/2, \qquad x_3 = 0, \qquad r = \sigma, \tag{A.11}$$

with boundary conditions $0 \le \tau \le T$ and $r_0 \le \sigma \le r_7$. The action evaluated on these two solutions is then

$$\frac{\beta S_0}{T\sqrt{\lambda}} = \lim_{\epsilon\to 0}\frac{1}{\gamma_E}\int_1^{1/\epsilon} dz\,\sqrt{\frac{z^4-\gamma_E^2}{z^4-1}} = \frac{1}{\epsilon\gamma_E} - \frac{\sqrt{\pi}}{\gamma_E}\frac{\Gamma[\frac{3}{4}]}{\Gamma[\frac{1}{4}]}{}_2F_1[-\tfrac{1}{2},-\tfrac{1}{4},\tfrac{1}{4},\gamma_E^2],$$

where ${}_2F_1$ is a hypergeometric function. Thus the regularized action $\hat{S} := S - S_0$ is

$$\frac{\beta\hat{S}}{T\sqrt{\lambda}} = \frac{\sqrt{\pi}}{\gamma_E}\frac{\Gamma[\frac{3}{4}]}{\Gamma[\frac{1}{4}]}\left\{{}_2F_1[-\tfrac{1}{2},-\tfrac{1}{4},\tfrac{1}{4},\gamma_E^2]-\frac{1}{\ell}+\frac{1-2\gamma_E^2}{2}\ell^3+\mathcal{O}(\ell^7)\right\}. \tag{A.12}$$

Eliminating ℓ order-by-order between (A.9) and (A.12) gives

$$\frac{\beta\hat{S}}{T\sqrt{\lambda}} = -\frac{\Gamma[\frac{3}{4}]^4}{\pi^2\gamma_E}\frac{\beta}{L} + \frac{\Gamma[\frac{3}{4}]^2}{\sqrt{2\pi}\gamma_E}{}_2F_1[-\tfrac{1}{2},-\tfrac{1}{4},\tfrac{1}{4},\gamma_E^2] + \Gamma[\tfrac{1}{4}]^4(2-5\gamma_E^2)\frac{L^3}{\beta^3}+\mathcal{O}(L^7). \tag{A.13}$$

The lightlike limit corresponds to $\gamma_E^2 = 1/2$, giving

$$\frac{\beta\hat{S}}{T\sqrt{\lambda}} = -0.32\,\frac{\beta}{L} + 1.08 - 0.76\,\frac{L^3}{\beta^3}+\mathcal{O}(L^7). \tag{A.14}$$

$\epsilon \to 0$ **limit at fixed L: long string**

For comparison purposes, we also compute the contribution to the Wilson loop from the long Euclidean string solution. This is the solution with turning point at the horizon, $z_t = 1$. The integral expression for L is convergent as $\epsilon \to \infty$ and, to keep L fixed and small, we just need to keep a fixed and small. So expanding in small a gives

$$\begin{aligned}\frac{L}{\beta} &= \frac{2a\gamma_E}{\pi}\int_1^\infty \frac{dz}{\sqrt{(z^4-1)(z^4-\gamma_E^2[1+a^2])}}, \qquad \text{(A.15)}\\ &= a\gamma_E\, 2^{3/2}\pi^{1/2}\Gamma[\tfrac{1}{4}]^2\Big[{}_2F_1[\tfrac{3}{4},\tfrac{3}{2},\tfrac{5}{4},\gamma_E^2] - \tfrac{3}{5}\gamma_E^2\,{}_2F_1[\tfrac{3}{2},\tfrac{7}{4},\tfrac{9}{4},\gamma_E^2]\\ &\qquad + \tfrac{3}{10}a^2\gamma_E^2\,{}_2F_1[\tfrac{3}{2},\tfrac{7}{4},\tfrac{9}{4},\gamma_E^2] + \mathcal{O}(a^4)\Big].\end{aligned}$$

The same expansion of the regularized action gives

$$\begin{aligned}\frac{\beta\hat{S}_{\rm long}}{T\sqrt{\lambda}} &= \frac{1}{\gamma_E}\int_1^\infty \frac{dz\,\sqrt{z^4-\gamma_E^2}}{\sqrt{z^4-1}}\left(\frac{\sqrt{z^4-\gamma_E^2}}{\sqrt{z^4-\gamma_E^2[1+a^2]}}-1\right) \qquad \text{(A.16)}\\ &= a^2\gamma_E\,2^{-1/2}\pi^{3/2}\Gamma[\tfrac{1}{4}]^{-2}\Big[{}_2F_1[\tfrac{3}{4},\tfrac{3}{2},\tfrac{5}{4},\gamma_E^2] - \tfrac{3}{5}\gamma_E^2\,{}_2F_1[\tfrac{3}{2},\tfrac{7}{4},\tfrac{9}{4},\gamma_E^2]\\ &\qquad + \tfrac{9}{20}a^2\gamma_E^2\,{}_2F_1[\tfrac{3}{2},\tfrac{7}{4},\tfrac{9}{4},\gamma_E^2] + \mathcal{O}(a^4)\Big].\end{aligned}$$

Eliminating a order-by-order between these two expressions gives

$$\frac{\beta\hat{S}_{\rm long}}{T\sqrt{\lambda}} = \frac{L^2}{\beta^2}\frac{5\sqrt{\pi}}{8\sqrt{2}\gamma_E}\Gamma[\tfrac{1}{4}]^2\left(5\,{}_2F_1[\tfrac{3}{2},\tfrac{3}{4},\tfrac{5}{4},\gamma_E^2] - 3\,\gamma_E^2\,{}_2F_1[\tfrac{3}{2},\tfrac{7}{4},\tfrac{9}{4},\gamma_E^2]\right)^{-1} + \mathcal{O}(L^4). \qquad \text{(A.17)}$$

The lightlike limit is $\gamma_E^2 = 1/2$, giving

$$\frac{\beta\hat{S}_{\rm long}}{T\sqrt{\lambda}} = +2.39\,\frac{L^2}{\beta^2} + \mathcal{O}(L^4). \qquad \text{(A.18)}$$

B Spacelike action

Here we calculate the regulated action of spacelike strings in the various limits described in section 4, keeping L fixed and small. Specifically, this requires expanding the quark separation (2.13) and string action (2.14) in terms of the appropriate small parameter, and then eliminating that parameter to obtain $\hat{S}$ as a function of L. Recall from the discussion in section 3 that for spacelike strings the integration constant a is imaginary, so we replace $a^2 = -\alpha^2$ with $\alpha^2 > 0$. Also, recall that $\gamma^2 := (1-v^2)^{-1}$ and $\epsilon := z_7^{-1}$. Then our main equations (2.13) and (2.14) become

$$\frac{L}{\beta} = \left| \frac{2\alpha\gamma}{\pi} \int_{z_t}^{1/\epsilon} \frac{dz}{\sqrt{(z^4-1)(z^4-\gamma^2(1-\alpha^2))}} \right|, \tag{B.1}$$

and

$$\frac{\beta\hat{S}}{T\sqrt{\lambda}} = \frac{1}{|\gamma|} \left\{ \left| \int_{z_t}^{1/\epsilon} \frac{(z^4-\gamma^2)\,dz}{\sqrt{(z^4-1)(z^4-\gamma^2(1-\alpha^2))}} \right| - \left| \int_1^{1/\epsilon} dz \sqrt{\frac{z^4-\gamma^2}{z^4-1}} \right| \right\}, \tag{B.2}$$

where z_t is either $z_t = 1$ or $z_t^4 := \gamma^2(1-\alpha^2)$, depending on which configuration we are considering, and we have subtracted the action of two straight strings to regulate the action. These expressions are valid for string solutions which have a single turning point, which will be the only ones we evaluate.

B.1 The (b) limit

The (c)$_0$ "up string" solutions.

The (c)$_0$ string solutions have a turning point at $z_t^4 = \gamma^2(1-\alpha^2) > z_7^4$. Since in this limit we first take $v \to 1^-$ ($\gamma \to +\infty$) before taking $\epsilon \to 0$, (B.1) becomes

$$\frac{L}{\beta} = \frac{2\alpha\gamma}{\pi} \int_{1/\epsilon}^{\gamma^{1/2}(1-\alpha^2)^{1/4}} \frac{dz}{\sqrt{(z^4-1)(\gamma^2(1-\alpha^2)-z^4)}}. \tag{B.3}$$

The upper limit of this integral gives a contribution that scales as $\alpha(1-\alpha^2)^{-3/4}\gamma^{-1/2}$, while the lower limit gives $\alpha(1-\alpha^2)^{-1/2}\epsilon$. Therefore, to keep L fixed in this limit requires that either the contribution from the upper limit or from the lower limit remains finite and non-zero. In order for the upper limit to remain finite and non-zero, it is required $1-\alpha^2 \sim \gamma^{-2/3}$. However, then the lower limit contributes $\sim \gamma^{1/3}\epsilon$ which diverges as we take $\gamma \to \infty$. Thus, this scaling does not keep L finite. If, instead, we demand that the lower limit remain finite, we must take $1-\alpha^2 \sim \epsilon^2$. This then implies that the upper limit contributes $\sim (\gamma\epsilon^3)^{-1/2}$ which vanishes in the $\gamma \to \infty$ limit, and so L remains finite.

We have thus found that there is a single (b) limit of the $(c)_0$ up strings which keeps the quark separation finite. This limit keeps

$$\delta^2 := \frac{\epsilon^2}{1-\alpha^2} \tag{B.4}$$

fixed and gives $L \sim \delta$ for small δ. Since for fixed L and ϵ and $\gamma \to \infty$ this limit keeps α fixed away from $\alpha = 1$, then from the discussion in section 3.1.1 we see that this solution corresponds to the long $(c)_0$ string (see figure 3). In particular, the short $(c)_0$ string does not contribute.

Plugging (B.4) into (B.3), changing variables to $y = (\gamma\epsilon/\delta)^{-1/2} z$, and expanding the $(z^4-1)^{-1/2}$ factor for large z gives a series of hypergeometric integrals which are finite in the $\gamma \to \infty$ limit, giving

$$\frac{L}{\beta} = \lim_{\epsilon \to 0} \frac{2}{\pi} \sqrt{\delta^2 - \epsilon^2} \left[1 + \tfrac{1}{5}\epsilon^4 + \mathcal{O}(\epsilon^8)\right] = \frac{2}{\pi}\delta. \tag{B.5}$$

Similarly, the regularized action (B.2) becomes

$$\frac{\beta \hat{S}}{T\sqrt{\lambda}} = \frac{1}{\gamma} \left\{ \int_{1/\epsilon}^{\sqrt{\gamma\epsilon/\delta}} \frac{(\gamma^2 - z^4)\, dz}{\sqrt{(z^4-1)(\gamma^2\epsilon^2\delta^{-2} - z^4)}} - \int_1^{1/\epsilon} dz \sqrt{\frac{\gamma^2 - z^4}{z^4 - 1}} \right\}, \tag{B.6}$$

which, upon making the same change of variables and taking the large γ limit, becomes

$$\frac{\beta \hat{S}}{T\sqrt{\lambda}} = -\frac{\sqrt{\pi}}{4} \frac{\Gamma[\frac{1}{4}]}{\Gamma[\frac{3}{4}]} + \lim_{\epsilon \to 0} (\delta + \epsilon) \left[1 + \tfrac{1}{10}\epsilon^4 + \mathcal{O}(\epsilon^8)\right] = -1.31 + \delta. \tag{B.7}$$

Eliminating δ between (B.5) and (B.7) gives

$$\frac{\beta \hat{S}}{T\sqrt{\lambda}} = -1.31 + \frac{\pi}{2} \frac{L}{\beta}. \tag{B.8}$$

The $(a)_1$ "down string" solution.

The $(a)_1$ string descends from the D7-brane and turns at the horizon, so that the quark separation (B.1) is given by

$$\frac{L}{\beta} = \frac{2\alpha\gamma}{\pi} \int_1^{1/\epsilon} \frac{dz}{\sqrt{(z^4-1)(\gamma^2(1-\alpha^2) - z^4)}}. \tag{B.9}$$

In the $\gamma \to \infty$ limit, L is kept finite for finite α. Taking the limit directly gives

$$\frac{L}{\beta} = \frac{1}{2\sqrt{\pi}} \frac{\Gamma[\frac{1}{4}]}{\Gamma[\frac{3}{4}]} \frac{\alpha}{\sqrt{1-\alpha^2}}. \tag{B.10}$$

Similarly, the limit of the regularized action gives

$$\frac{\beta \hat{S}_{\rm long}}{T\sqrt{\lambda}} = \frac{\sqrt{\pi}}{4} \frac{\Gamma[\frac{1}{4}]}{\Gamma[\frac{3}{4}]} \left(\frac{1}{\sqrt{1-\alpha^2}} - 1 \right). \tag{B.11}$$

Eliminating α between these two expressions and expanding in small L yields

$$\frac{\beta \hat{S}_{\rm long}}{T\sqrt{\lambda}} = \frac{\pi^{3/2}}{2} \frac{\Gamma[\frac{3}{4}]}{\Gamma[\frac{1}{4}]} \frac{L^2}{\beta^2} + \mathcal{O}(L^4) = 0.941 \frac{L^2}{\beta^2} + \mathcal{O}(L^4). \tag{B.12}$$

B.2 The (c) limit.

The (c) and (d) limits take $v \to 1$ with $v > 1$. In this range it is convenient to define

$$\tilde{\gamma}^2 := -\gamma^2 = \frac{1}{v^2 - 1}, \tag{B.13}$$

so the $v \to 1^+$ limit takes $\tilde{\gamma}^2 \to +\infty$. The spacelike string solutions for $v > 1$ were discussed in section 3.1.2, where we found that there are two solutions: a short string with turning point $z_t^4 = \tilde{\gamma}^2(\alpha^2 - 1)$ and a long string with turning point at the horizon $z_t = 1$.

The short string solution.

For this solution, the integral expression for the quark separation (B.1) takes the form

$$\frac{L}{\beta} = \frac{2}{\pi} \alpha \tilde{\gamma} \int_{\tilde{\gamma}^{1/2}(\alpha^2-1)^{1/4}}^{1/\epsilon} \frac{dz}{\sqrt{(z^4-1)(z^4 - \tilde{\gamma}^2(\alpha^2-1))}}. \tag{B.14}$$

The (c) limit takes $\tilde{\gamma} \to \infty$ first before taking $\epsilon \to 0$. Examination of (B.14) shows that, in this limit, L remains finite if one takes $\alpha \to 1^+$ in such a way that

$$\delta^2 := \epsilon^6 \tilde{\gamma}^2 [1 - \epsilon^4 \tilde{\gamma}^2 (\alpha^2 - 1)] \tag{B.15}$$

remains fixed. Eliminating α in favor of δ in (B.14) and changing variables to $y = \epsilon z$ gives

$$\frac{L}{\beta} = \frac{2}{\pi}\sqrt{1-\tilde{\gamma}^{-2}\delta^2\epsilon^{-6}+\epsilon^4\tilde{\gamma}^2}\int_{(1-\tilde{\gamma}^{-2}\delta^2\epsilon^{-6})^{1/4}}^{1} \frac{\epsilon\,(1-\epsilon^4 y^{-4})^{-1/2}\,dy}{y^2\sqrt{y^4-1+\tilde{\gamma}^{-2}\delta^2\epsilon^{-6}}}. \quad \text{(B.16)}$$

For small fixed ϵ, expanding the numerator of the integrand in a power series, performing the integrals, and taking the $\tilde{\gamma} \to \infty$ limit yields

$$\frac{L}{\beta} = \lim_{\epsilon\to 0}\frac{\delta}{\pi}\left[1+\mathcal{O}(\epsilon^4)\right] = \frac{\delta}{\pi}. \quad \text{(B.17)}$$

Similarly, in the same limit, the regularized action

$$\frac{\beta\hat{S}}{T\sqrt{\lambda}} = \int_{\tilde{\gamma}^{1/2}(\alpha^2-1)^{1/4}}^{1/\epsilon} \frac{\tilde{\gamma}^{-1}(z^4+\tilde{\gamma}^2)\,dz}{\sqrt{(z^4-1)(z^4-\tilde{\gamma}^2(\alpha^2-1))}} - \int_1^{1/\epsilon}\frac{dz}{\tilde{\gamma}}\sqrt{\frac{z^4+\tilde{\gamma}^2}{z^4-1}}, \quad \text{(B.18)}$$

becomes

$$\frac{\beta\hat{S}}{T\sqrt{\lambda}} = \lim_{\epsilon\to 0}\left[\frac{\delta}{2}-\frac{\sqrt{\pi}}{4}\frac{\Gamma[\frac{1}{4}]}{\Gamma[\frac{3}{4}]}+\mathcal{O}(\epsilon)\right] = \frac{\delta}{2}-1.31. \quad \text{(B.19)}$$

Eliminating δ between (B.17) and (B.19) gives

$$\frac{\beta\hat{S}}{T\sqrt{\lambda}} = -1.31+\frac{\pi}{2}\frac{L}{\beta}. \quad \text{(B.20)}$$

The long string solution.

The turning point for the long string is at the horizon, so

$$\frac{L}{\beta} = \frac{2\alpha\tilde{\gamma}}{\pi}\int_1^{\epsilon^{-1}}\frac{dz}{\sqrt{(z^4-1)(z^4+\tilde{\gamma}^2(1-\alpha^2))}}, \quad \text{(B.21)}$$

from which it follows that L is kept finite in the $\tilde{\gamma} \to \infty$ limit for finite α. Taking the limit directly then gives

$$\frac{L}{\beta} = \frac{1}{2\sqrt{\pi}}\frac{\Gamma[\frac{1}{4}]}{\Gamma[\frac{3}{4}]}\frac{\alpha}{\sqrt{|1-\alpha^2|}}. \quad \text{(B.22)}$$

Similarly, the limit of the regularized action gives

$$\frac{\beta\hat{S}_{\rm long}}{T\sqrt{\lambda}} = \frac{\sqrt{\pi}}{4}\frac{\Gamma[\frac{1}{4}]}{\Gamma[\frac{3}{4}]}\left(\frac{1}{\sqrt{|1-\alpha^2|}}-1\right). \quad \text{(B.23)}$$

Eliminating α between (B.22) and (B.23) and expanding in powers of small L gives

$$\frac{\beta \hat{S}_{\text{long}}}{T\sqrt{\lambda}} = \frac{\pi^{3/2}}{2}\frac{\Gamma[\frac{3}{4}]}{\Gamma[\frac{1}{4}]}\frac{L^2}{\beta^2} + \mathcal{O}(L^4) = 0.941\,\frac{L^2}{\beta^2} + \mathcal{O}(L^4). \tag{B.24}$$

Note that this calculation is essentially identical to that of the $(\mathrm{a})_1$ string in the (b) limit.

B.3 The (d) limit.

The short string solution.

The (d) limit takes $\epsilon \to 0$ first, then $\tilde{\gamma} \to \infty$. Examination of (B.14) shows that, in this limit, L remains finite if one takes $\alpha \to 1^+$ in such a way that

$$\delta := \tilde{\gamma}^{-1/2}(\alpha^2 - 1)^{-3/4} \tag{B.25}$$

remains fixed. Eliminating α in favor of δ in (B.14) and changing variables to $y = \tilde{\gamma}^{-1/3}\delta^{1/3}z$, the ϵ and $\tilde{\gamma}$ limits can be taken directly to give

$$\frac{L}{\beta} = \frac{2}{\pi}\delta \int_1^\infty \frac{y^{-2}dy}{\sqrt{y^4-1}} = \frac{2}{\sqrt{\pi}}\frac{\Gamma[\frac{3}{4}]}{\Gamma[\frac{1}{4}]}\,\delta. \tag{B.26}$$

The regularized action can be written as

$$\frac{\beta \hat{S}}{T\sqrt{\lambda}} = \int_{(\tilde{\gamma}/\delta)^{1/3}}^{1/\epsilon} \frac{\tilde{\gamma}^{-1}(z^4 + \tilde{\gamma}^2)\,dz}{\sqrt{(z^4-1)(z^4-(\tilde{\gamma}/\delta)^{4/3})}} - \int_1^{1/\epsilon}\frac{dz}{\tilde{\gamma}}\sqrt{\frac{z^4+\tilde{\gamma}^2}{z^4-1}}. \tag{B.27}$$

To evaluate this expression as $\tilde{\gamma} \to \infty$ (after $\epsilon \to 0$), split the ranges of integration into $z < \tilde{\gamma}^{1/2}$ and $z > \tilde{\gamma}^{1/2}$. After the pieces of the integrals for $z > \tilde{\gamma}^{1/2}$ which are divergent at $\epsilon \to 0$ are canceled, the remainder is easily seen to vanish in the $\tilde{\gamma} \to \infty$ limit. The integrals for $z < \tilde{\gamma}^{1/2}$ are evaluated to give

$$\frac{\beta \hat{S}}{T\sqrt{\lambda}} = -\frac{\Gamma[\frac{1}{4}]^2}{4\sqrt{2\pi}} + \frac{\Gamma[\frac{3}{4}]^2}{\sqrt{2\pi}}\,\delta \tag{B.28}$$

in the $\tilde{\gamma} \to \infty$ limit. Eliminating δ between (B.26) and (B.28) gives

$$\frac{\beta \hat{S}}{T\sqrt{\lambda}} = -\frac{\Gamma[\frac{1}{4}]^2}{4\sqrt{2\pi}} + \frac{\pi}{2}\frac{L}{\beta} = -1.31 + \frac{\pi}{2}\,\frac{L}{\beta}. \tag{B.29}$$

The long string solution.

For the long strings with $v > 1$, recall that $\alpha < 1$ and the turning point is at the horizon, so that

$$\frac{L}{\beta} = \frac{2\alpha\tilde{\gamma}}{\pi}\int_1^\infty \frac{dz}{\sqrt{(z^4-1)(z^4+\tilde{\gamma}^2(1-\alpha^2))}}, \tag{B.30}$$

where we have already taken the $\epsilon \to 0$ limit since the integral is convergent. L is kept small and finite as $\tilde{\gamma} \to \infty$ if α is kept small and fixed. Then the above integral can be evaluated by splitting the range of integration into $z^4 > \tilde{\gamma}^2(1-\alpha^2)$ and $z^4 < \tilde{\gamma}^2(1-\alpha^2)$. The upper range is easily seen to give a vanishing contribution in the $\tilde{\gamma} \to \infty$ limit, while the lower range gives

$$\frac{L}{\beta} = \lim_{\tilde{\gamma}\to\infty} \frac{2\alpha}{\pi}\int_1^{\gamma^{1/2}} \frac{dz(1+z^4\tilde{\gamma}^{-2}(1-\alpha^2)^{-1})^{-1/2}}{\sqrt{1-\alpha^2}\sqrt{z^4-1}} = \frac{1}{2\sqrt{\pi}}\frac{\Gamma[\frac{1}{4}]}{\Gamma[\frac{3}{4}]}\frac{\alpha}{\sqrt{1-\alpha^2}}. \tag{B.31}$$

Similarly evaluating the integral for the action gives

$$\frac{\beta\hat{S}_{\text{long}}}{T\sqrt{\lambda}} = \frac{\sqrt{\pi}}{4}\frac{\Gamma[\frac{1}{4}]}{\Gamma[\frac{3}{4}]}\left(\frac{1}{\sqrt{1-\alpha^2}} - 1\right). \tag{B.32}$$

Eliminating α between (B.31) and (B.32) and expanding in powers of L gives

$$\frac{\beta\hat{S}_{\text{long}}}{T\sqrt{\lambda}} = \frac{\pi^{3/2}}{2}\frac{\Gamma[\frac{3}{4}]}{\Gamma[\frac{1}{4}]}\frac{L^2}{\beta^2} + \mathcal{O}(L^4) = 0.941\,\frac{L^2}{\beta^2} + \mathcal{O}(L^4). \tag{B.33}$$

Note that this calculation gives the same result as that of the $(a)_1$ (down) string in the (b) limit, and the long string in the (c) limit.

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