Bubbling AdS black holes

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ABSTRACT: We explore the non-BPS analog of 'AdS bubbles', which are regular spherically symmetric 1/2 BPS geometries in type IIB supergravity. They have regular horizons and can be thought of as bubbling generalizations of non-extremal AdS black hole solutions in five-dimensional gauged supergravity. Due to the appearance of the Heun equation even at the linearized level, various approximation and numerical methods are needed in order to extract information about this system. We study how the vacuum expectation value and mass of a particular dimension two chiral primary operator depend on the temperature and chemical potential of the thermal Yang-Mills theory. In addition, the mass of the bubbling AdS black holes is computed. As is shown numerically, there are also non-BPS solitonic bubbles which are completely regular and arise from continuous deformations of BPS AdS bubbles.

Contents

1. Introduction

Black holes in five dimensions have been extensively studied in the framework of both ungauged and gauged supergravity theories. Furthermore, many of these explicit studies have been performed in the context of the STU model, which corresponds to $\mathcal{N} = 2$ supergravity coupled to two vector multiplets. Because of the presence of the graviphoton and two vector fields, the STU model generally admits three-charge black holes with up to two rotation parameters. In fact, in the ungauged context, many solutions have been constructed, and their explicit forms have often been proven to be useful, especially in the context of black hole thermodynamics and stringy microstate counting.

At the same time, AdS black holes (and their variants) in gauged supergravity theories have found widespread application in the study of the AdS/CFT correspondence. For instance, an R-charged black hole in global AdS$_5$ geometry corresponds to equilibrium non-zero temperature $\mathcal{N} = 4$ SU(N) supersymmetric Yang-Mills theory on $S^3 \times \mathbb{R}$ with

finite chemical potential. Five-dimensional BPS 'black holes' were first constructed in [1], although there it was also realized that they are in fact naked singularities. Unlike the ungauged case, angular momentum must be turned on in order to obtain true BPS black holes in gauged supergravity. This was done in [3, 4] for one rotation parameter, and subsequently generalized to two independent rotations. As in the ungauged case, these black holes admit non-extremal generalizations [2], which can be further generalized to include rotations [5 − 9].

BPS objects play an important rôle in AdS/CFT, regardless of their precise nature, since they necessarily survive in both the strong and weak coupling regimes of the duality. Along these lines, the BPS naked singularities were shown in [10] to correspond to a distribution of giant gravitons in $AdS_5 \times S^5$, where they were also denoted 'superstars.' In terms of the Lin, Lunin and Maldacena (LLM) boundary conditions for 1/2 BPS configurations [11], these superstars correspond to a disk in the phase space of free fermions with a uniform shade of gray, which may be interpreted as a coarse graining of an underlying spacetime foam picture of gravity [13].

The investigation of smooth 1/2 BPS geometries in [11] also led to the construction of a new set of 1/2 BPS 'AdS bubbles' which eliminate the naked singularity of the singular black holes– not through angular momentum but rather by turning on additional scalar fields φ_i. These scalars are present in the full $\mathcal{N} = 8$ gauged supergravity but lie outside of the $\mathcal{N} = 2$ truncation [11, 14]. These AdS bubbles are in fact completely regular geometries without horizons and, as such, admit a description in the LLM language as a deformation of the AdS disk into an ellipse in the phase space of free fermions.

In order to generalize the above five-dimensional single-charge AdS bubbles to include three charges, the authors of [14] considered a consistent truncation of five-dimensional $\mathcal{N} = 8$ gauged supergravity that retains the three $U(1)$ gauge fields in the maximal torus of the $SO(6)$ gauge group along with five scalar fields. Although this truncated system is not itself the bosonic sector of any supersymmetric theory, it nevertheless allows the construction and lifting of BPS solutions to yield supersymmetric configurations of the full $\mathcal{N} = 8$ theory and hence the original IIB supergravity as well.

Although LLM configurations are by their very nature pure states, and not thermal ones, it is natural to investigate the effect of turning on non-zero temperature starting from a particular LLM background. The main goal of this paper is to initiate such an exploration by constructing AdS bubbles away from the BPS limit. In particular, we will look for non-extremal AdS black holes with regular horizon and carrying φ_i scalar deformations of the same sort encountered in the BPS bubbles of [11, 14]. We will refer to these solutions as 'bubbling AdS black holes,' which are a subset of thermal AdS bubbles. Though not the focus of our paper, we will demonstrate numerically that there are also non-BPS solitonic AdS bubbles which are completely regular and horizon-free.

We note that a coarse-graining of LLM geometries was considered in [12], and expanded upon in [13, 15], where the free fermion configuration was given an equilibrium non-zero temperature. Using the LLM correspondence, this configuration can then be mapped into a 'hyperstar' supergravity background by appropriate transformation of the Fermi-Dirac distribution into the LLM $z(x_1, x_2, y)$ function. It is not clear, however, whether

this background actually describes a non-zero temperature field theory; it is by no means obvious that simply giving the matrix eigenvalues (the 'free fermions') a Fermi-Dirac profile would correspond to turning on non-zero temperature in $\mathcal{N} = 4$ super Yang-Mills. If this were the case, then one would expect the corresponding supergravity background to have a horizon and to break supersymmetry. In [12], it was pointed out that the coarse-graining has been taken over only the half-BPS sector of the full Hilbert space of type IIB supergravity, and that the result of using the complete Hilbert space should be a non-BPS background that has a horizon. Because of these considerations, we shall not pursue this direction of coarse-graining, but will look directly for bubbling AdS black holes in the supergravity itself.

For the case of the AdS black hole, the main effect of moving away from the extremal limit is to introduce a 'renormalization' of the charge. One might hope that the AdS bubble might be generalized away from the BPS limit in a similar way, in which much of the original structure survives in a 'renormalized' form. Unfortunately, upon closer inspection of the equations of motion, it does not appear that the AdS bubble can be made non-extremal in such a simple manner. Thus, we must content ourselves with either approximate or numerical methods.

We will proceed by performing a linearized analysis of the second-order equations of motion. Although the non-extremal AdS bubble solution is not known, explicit solutions are known in the two separate limits of either turning off non-extremality or turning off the φ_i deformation. Since we will focus on solutions for which there is a horizon, we choose to consider the initial background to be that of the non-extremal R-charged black hole. We will turn on the additional scalars φ_i at linear order, which will then backreact onto the other fields at higher order.

The linearized φ_i equations are second-order equations with four regular singular points, and hence may be mapped to the Heun equation. Unfortunately, in contrast with the hypergeometric equation which has three regular singular points, much less is known about the solutions to the Heun equation. In particular, the general two-point connection problem, i.e. relating local solutions at two regular singular points, remains unsolved.

This is of course not the first time where the Heun equation has arisen in the study of the wave equation in AdS black hole backgrounds. In particular, calculations involving quasi-normal modes for black holes have generally led to expressions related to the Heun equation. In such cases, various methods have been applied in order to obtain approximate solutions of the resulting system. A common method, also used in black hole absorption calculations, involves matching approximate solutions in two overlapping regions: an inner region containing the horizon and an outer region that includes asymptotic infinity. Provided that there is a large overlap, then essentially complete information may be reliably extracted from this approach. Higher-order corrections can be included via a perturbative approach. We will implement this approximation technique for the regime $T \gg \mu_i$, where T and μ_i are the temperature and chemical potentials of the field theory, respectively. Furthermore, there is a high temperature limit $T \gg 1$, along with $T \gg \mu_i$, for which there is an approximate solution which covers the entire region from the horizon to asymptotic infinity without the need for matching.

The asymptotic behavior of φ_i is related to perturbations away from the UV super-conformal fixed point of the dual field theory. In particular, as in the BPS case, the normalizable mode of φ_i corresponds to giving a vacuum expectation value (vev) to the dimension two chiral primary operator $\text{Tr}\, Z_i^2$, where $Z_i = \frac{1}{\sqrt{2}}(\phi_1^i + i\phi_2^i)$ and ϕ_1^i and ϕ_2^i are three pairs of real scalars of $\mathcal{N} = 4$ super Yang-Mills theory, for $i = 1, 2, 3$. On the other hand, the non-normalizable mode of φ_i corresponds to a term of the form $\text{Tr}\, Z_i^2$ in the Lagrangian of the conformal field theory. This massive term is only present for nonzero temperature, which could be indicative of a phase transition in the field theory at zero temperature. We will discuss how these field theory deformations depend on the physical parameters of the thermal field theory, namely the temperature, chemical potential and R-charge, in both the grand canonical ensemble and the canonical ensemble.

The paper is organized as follows. In section 2, we review the previously-known AdS black hole and BPS bubble solutions. In section 3, we perform a linearized analysis of the bubbling AdS black hole. In particular, we consider the linearized φ_i equations in the background of the AdS black hole. Focusing on the single-charge case, matching and perturbation techniques are used to find approximate solutions to the linearized φ_1 equation. Properties of the φ_1 solution are discussed. Specifically, the asymptotic behavior is matched with perturbations of the dual field theory as functions of temperature and chemical potential. In section 4, we consider the backreaction of φ_1 onto the metric and other matter fields, and in section 5 we then discuss the mass of these non-BPS bubbles. In section 6, we present some numerical support for the existence of bubbling AdS black holes for arbitrary values of φ_1. Lastly, we conclude in section 7.

2. Review of AdS black holes and BPS bubbles

2.1 AdS black hole

The bosonic Lagrangian for the STU model takes the form

$$\mathcal{L} = R * \mathbf{1} - \frac{1}{2} \sum_{\alpha=1}^{2} *d\phi_\alpha \wedge d\phi_\alpha - \frac{1}{2} \sum_{i=1}^{3} X_i^{-2} *F^i \wedge F^i - V * \mathbf{1} + F^1 \wedge F^2 \wedge A^3, \qquad (2.1)$$

where the X_i's are constrained scalars satisfying $X_1 X_2 X_3 = 1$, and which may be taken to be

$$X_1 = e^{-\frac{1}{\sqrt{6}}\phi_1 - \frac{1}{\sqrt{2}}\phi_2}, \qquad X_2 = e^{-\frac{1}{\sqrt{6}}\phi_1 + \frac{1}{\sqrt{2}}\phi_2}, \qquad X_3 = e^{\frac{2}{\sqrt{6}}\phi_1}. \qquad (2.2)$$

The scalar potential is given by

$$V = -4g^2 \sum_{i=1}^{3} X_i^{-1} = -4g^2 (X_2 X_3 + X_3 X_1 + X_1 X_2), \qquad (2.3)$$

and the ungauged system is recovered by setting $g = 0$.

The three-charge AdS$_5$ black hole solution is given by

$$\begin{aligned}
ds^2 &= -\mathcal{H}^{-2/3} f dt^2 + \mathcal{H}^{1/3}(f^{-1} dr^2 + r^2 d\Omega_3^2), \\
A_{(1)}^i &= -\coth \beta_i H_i^{-1} dt, \qquad X_i = \mathcal{H}^{1/3} H_i^{-1}, \qquad \varphi_i = 0, \\
f &= 1 - \frac{m}{r^2} + g^2 r^2 \mathcal{H}, \qquad \mathcal{H} = H_1 H_2 H_3,
\end{aligned} \qquad (2.4)$$

where the harmonic functions are given by

$$H_i = 1 + \frac{q_i}{r^2}, \qquad q_i \equiv m \sinh^2 \beta_i. \tag{2.5}$$

We shall focus primarily on the case of a single charge q_1, for which the roots of f are given by

$$r_\pm^2 = -\frac{1}{2g^2}(1 + g^2 q_1) \pm \frac{1}{2g^2}\sqrt{(1 + g^2 q_1)^2 + 4g^2 m}\,. \tag{2.6}$$

The event horizon is located at $r_h = r_+$. Notice that, in the one-charge case with $q_1 > 0$, any positive value of m guarantees a regular horizon. On the other hand, for the two and three-charge cases, a regular horizon may always be obtained for sufficiently large m.

This supergravity background is dual to equilibrium non-zero temperature $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory on $S^3 \times \mathbb{R}$ with chemical potentials for the $U(1)$ R-charges. The temperature of the field theory is equated with the Hawking temperature of the black hole, which is [20]

$$T = \frac{2r_h^6 + r_h^4(1 + \sum_i q_i) - \prod_i q_i}{2\pi r_h^2 \prod_i \sqrt{r_h^2 + q_i}}\,, \tag{2.7}$$

where the horizon radius r_h is the largest root of f. We have set $g = 1$ for simplicity. Likewise, the R-charge chemical potentials μ_i of the field theory are equated with the electric potentials at the horizon, which are

$$\mu_i = \frac{Q_i}{r_h^2 + q_i}\,, \tag{2.8}$$

where

$$Q_i^2 = q_i(r_h^2 + q_i)\Big[1 + \frac{1}{r_h^2}\prod_{j \neq i}(r_h^2 + q_j)\Big]. \tag{2.9}$$

Also, the physical charges Q_i of the AdS black hole correspond to R-charges in the dual field theory. These relations will be useful for expressing various results in terms of the physical quantities of the field theory.

2.2 BPS AdS bubble

The authors of [14] considered a consistent truncation of five-dimensional $\mathcal{N} = 8$ gauged supergravity retaining the three $U(1)$ gauge fields in the maximal torus of the $SO(6)$ gauge group along with five scalar fields. In $\mathcal{N} = 2$ language, this corresponds to taking the bosonic sector of the STU model and coupling it to three additional scalars φ_i, which are not described by special geometry. The Lagrangian is given by [14]

$$\mathcal{L} = R * \mathbf{1} - \frac{1}{2}\sum_{\alpha=1}^{2} *d\phi_\alpha \wedge d\phi_\alpha - \frac{1}{2}\sum_{i=1}^{3} X_i^{-2} * F^i \wedge F^i - V * \mathbf{1} + F^1 \wedge F^2 \wedge A^3$$

$$-\frac{1}{2}\sum_{i=1}^{3} *d\varphi_i \wedge d\varphi_i - 2g^2 \sum_{i=1}^{3} \sinh^2 \varphi_i * A^i \wedge A^i, \tag{2.10}$$

where the scalar potential has the modified form

$$V = 2g^2 \left(\sum_{i=1}^{3} X_1^2 \sinh^2 \varphi_i - 2 \sum_{i<j}^{3} X_i X_j \cosh \varphi_i \cosh \varphi_j \right). \qquad (2.11)$$

The original STU model scalars satisfy the same constraints as above, and in particular may be given by (2.2).

In order to ensure a supersymmetric solution, the regular three-charge AdS bubble configuration then takes the form [14]

$$\begin{aligned}
ds^2 &= -\mathcal{H}^{-2/3} f \, dt^2 + \mathcal{H}^{1/3}(f^{-1}dr^2 + r^2 d\Omega_3^2), \\
A_{(1)}^i &= -H_i^{-1}dt, \qquad X_i = \mathcal{H}^{1/3}H_i^{-1}, \qquad \cosh \varphi_i = (xH_i)', \\
f &= 1 + g^2 x \mathcal{H}, \qquad \mathcal{H} = H_1 H_2 H_3,
\end{aligned} \qquad (2.12)$$

where $x \equiv r^2$, and a prime denotes a derivative with respect to x. The above solution is fully determined up to the functions H_i, which must satisfy the conditions

$$f(xH_i)'' = -g^2[(xH_i)'^2 - 1]\mathcal{H}H_i^{-1}, \qquad (2.13)$$

to ensure that the equations of motion are satisfied.

For the one-charge case, corresponding to a 1/2 BPS configuration, we may take $H_2 = H_3 = 1$. Then the equation of motion (2.13) reduces to

$$[(1 + g^2 x H_1)^2]'' = 2g^4, \qquad (2.14)$$

which admits a general solution of the form

$$H_1 = \sqrt{1 + \frac{2(1 + g^2 q_1)}{g^2 x} + \frac{\overline{c}^2}{g^4 x^2}} - \frac{1}{g^2 x}. \qquad (2.15)$$

Here q_1 is the R-charge, and $\overline{c}$ is a constant related to the φ_1 scalar deformation. Note that, at large distances, H_1 admits the expansion

$$H_1 \sim 1 + \frac{q_1}{x} + \cdots, \qquad (2.16)$$

while regularity of the AdS bubble at short distances demands $\overline{c} = 1$. In addition, the BPS naked singularity of [1, 2] is recovered by taking $\overline{c} = 1 + g^2 q_1$, in which case H_1 reduces to the standard 'harmonic function' form $H_1 = 1 + q_1/x$.

In general, as we will demonstrate below, the scalar φ_1 carries $E_0 = 2$, and hence has an asymptotic expansion of the form

$$\varphi_1 \sim \frac{c_1 + c_2 \log x}{x} + \cdots, \qquad (2.17)$$

However, based on the explicit solution (2.15), we see that the log term vanishes, $c_2 = 0$, while

$$c_1 = \sqrt{(1 + q_1)^2 - \overline{c}^2} = \sqrt{q_1(q_1 + 2)}, \qquad (2.18)$$

where the second expression is for the regular AdS bubble solution. We have set $g = 1$. Applying the AdS/CFT dictionary (see e.g. [19]), this implies that the dimension two chiral primary operator $\text{Tr} \, Z_1^2$ gets a vev c_1, where $Z_1 = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and ϕ_1 and ϕ_2 are two of the six real scalars of $\mathcal{N} = 4$ super Yang-Mills theory.

Although we are not aware of any closed form expressions, it can be seen that regular two and three-charge solutions to (2.13) also exist. These solutions correspond to 1/4 and 1/8 BPS AdS bubbles, and can be realized in the framework of LLM configurations with fewer supersymmetries [16 – 18].

3. Linearized analysis of the bubbling AdS black hole

Given the relatively simple non-extremal generalization (2.4) of the BPS AdS black hole solution, we have been led to look for a corresponding non-extremal version of the AdS bubble (2.12) where the φ_i scalars are present. In the absence of supersymmetry, we no longer have the benefit of working with first-order Killing spinor equations. However, the form of the non-extremal black hole (2.4) is curiously close to that of the BPS limit; the primary difference is that the charge parameters q_i in the harmonic functions H_i are 'renormalized' as follows: $q_i \rightarrow m \sinh^2 \beta_i$. This suggests that perhaps a similarly straightforward generalization may be obtained for the AdS bubble, where much of the BPS structure might survive, except perhaps in 'renormalized' form.

Unfortunately, closer inspection of the equations of motion arising from the Lagrangian (2.10) does not suggest any simple manner in which the AdS bubble solution may be made non-extremal. In particular, retaining the BPS-like relation $\cosh \varphi_i = (xH_i)'$ leads to either the possibility that $\varphi_i = 0$, in which case H_i takes the form (2.5), or to a constrained system of equations which only appear to admit a natural solution of the BPS bubble form given by (2.13). As a result, we must use either approximate or numerical methods when moving away from extremality.

3.1 Linearized φ_i equations

In this section, we explore the basic features of the non-extremal AdS bubble by performing a linearized analysis of the second-order equations of motion obtained from (2.10). We proceed by noting that, although it is not clear how to write down the complete solution for a non-extremal AdS bubble, explicit forms for the solutions are known in the two separate limits of either turning off non-extremality or turning off the φ_i deformation. Since we are mainly interested in solutions with a horizon, we choose to start from the non-extremal R-charged black hole background of (2.4) and (2.5) and then turn on the additional scalars φ_i at linear order. Turning on these scalars will then backreact onto the other fields. However this backreaction occurs at the next order, and may be ignored in the initial analysis. (We will return to the backreaction in another section.)

At linearized order we are only concerned with the linearized equation of motion for

φ_i, which takes the form

$$\left[\Box - 4g^2 X_i (2X_i - \sum_{j=1}^{3} X_j) - 4g^2 (A_\mu^i)^2 \right] \varphi_i = 0. \tag{3.1}$$

Here the scalars X_i take on the background values given above in (2.4) and (2.5). There is a slight subtlety for the background gauge fields, however, related to the 'gauge fixed' form of the action (2.10). In particular, while the standard black hole solution (2.4) is invariant under gauge transformations of A^i, the above equation of motion is not. Taking this into account, we allow a constant term in the expression for the background electric potential

$$A^i = (b^i - \coth \beta_i H_i^{-1}) dt \equiv A_t^i \, dt. \tag{3.2}$$

We shall see below that this constant must be chosen to make the potential vanish at the horizon. The motivation for this requirement can already be seen by noting that the invariant square of the electric potential, $(A_\mu^i)^2$, which acts as a source in (3.1), blows up at the horizon unless A_t^i is arranged to vanish there.

By substituting the background fields into (3.1), we obtain the scalar equation

$$(x^2 f \varphi_i')' + g^2 \left[\sum_{j \neq i}^{3} (x + q_j) + (x^2 f)^{-1} \prod_{j \neq i}^{3} (x + q_j) \left(-g^2 \prod_{j \neq i}^{3} (x + q_j) \right. \right.$$
$$\left. \left. + \frac{mx}{q_i} - 2b^i \sqrt{1 + m/q_i} \, x + (b^i)^2 (x + q_i) \right) \right] \varphi_i = 0, \tag{3.3}$$

where

$$x^2 f = g^2 \prod_{j=1}^{3} (x + q_j) + x^2 - mx, \tag{3.4}$$

is a cubic polynomial, and where $q_i \equiv m \sinh^2 \beta_i$. Note that, in the event $q_i = 0$, the above equation is replaced by the considerably simpler expression

$$(x^2 f \varphi_i')' + \frac{g^2}{x} \left(x^2 - \prod_{j \neq i}^{3} q_j \right) \varphi_i = 0, \tag{3.5}$$

which may be obtained by directly taking $A^i = 0$ in (3.1).

Since $x^2 f$ is cubic, it can be seen that (3.3) is a second-order equation with three regular singular points at the roots of $x^2 f$. Including the singular point at infinity, which is also regular, this equation in fact has four regular singular points, and hence may be mapped into the general Heun equation. Unfortunately, the general two-point connection problem is as yet unsolved. However, general features of the linearized φ_i deformation may be extracted from the second-order equation (3.3).

Recalling that the goal of the linearized analysis is to turn on a φ_i deformation starting from the R-charged black hole background, we demand that the solution to (3.3) be regular and bounded in the entire region from the horizon to the spatial boundary at infinity. Before examining the solution, we find it convenient to trade the non-extremality parameter m

with the horizon location x_h, defined to be the largest positive root of $f(x)$. Note that, in terms of x_h, we have

$$m = x_h + \frac{g^2}{x_h} \prod_{i=1}^{3} (x_h + q_i), \tag{3.6}$$

along with the factorization

$$x^2 f = g^2 (x - x_h)(x^2 + Qx - R), \tag{3.7}$$

where

$$Q = x_h + \frac{1}{g^2} + q_1 + q_2 + q_3, \qquad R = \frac{q_1 q_2 q_3}{x_h}. \tag{3.8}$$

Examination of the indicial equation around the horizon x_h (which is a regular singular point) yields the characteristic exponents $\pm \zeta_i$ where

$$\zeta_i^2 = -\frac{\prod_{j=1}^{3}(x_h + q_j)}{(x_h^2 + Qx_h - R)^2} \left(b^i - \sqrt{1 + \frac{m}{q_i} \frac{x_h}{x_h + q_i}} \right)^2. \tag{3.9}$$

Noting that the expression above is non-positive (since $x_h + q_j$ must be positive to avoid naked singularities), we immediately see that the characteristic exponents are purely imaginary, except for the case when they vanish. Since imaginary exponents give rise to undesirable oscillatory solutions of the form $\varphi_i \sim \sin(|\zeta_i| \log(x - x_h))$, we conclude that the constant b^i in the electric potential must be adjusted to satisfy

$$b^i = \sqrt{1 + \frac{m}{q_i} \frac{x_h}{x_h + q_i}} = \frac{\coth \beta_i}{H_i(x_h)}. \tag{3.10}$$

This ensures that the potentials given by (3.2) indeed vanish at the horizon, $A^i(x_h) = 0$, thus confirming what we had alluded to above.

Demanding that the electric potentials vanish at the horizon, the φ_i equation can now be put into the form

$$((x - x_h)(x^2 + Qx - R)\varphi_i')' + \left[\sum_{j \neq i}^{3} (x + q_j) - (x^2 + Qx - R)^{-1} \prod_{j \neq i}^{3} (x + q_j) \right.$$
$$\left. \times \left(\frac{x_h^2 + Qx_h - R}{x_h + q_i} + (x - x_h) \right) \right] \varphi_i = 0. \tag{3.11}$$

The solutions to this equation may be characterized by the Riemann P-symbol

$$P \left\{ \begin{matrix} x_h & x_1 & x_2 & \infty \\ 0 & \alpha_1 & \alpha_2 & 1 \\ 0 & -\alpha_1 & -\alpha_2 & 1 \end{matrix} \; ; x \right\}, \tag{3.12}$$

where x_1 and x_2 are the two roots of the quadratic equation $x^2 + Qx - R = 0$. Their associated exponents, α_1 and α_2, may easily be obtained, although we have no particular need for their explicit forms. We do note, however, that the characteristic exponents sum to 2, which is in agreement with the general theory of the Heun equation.

The single-charge case. Since we are primarily interested in the non-extremal generalization of the single-charge 1/2 BPS bubble, we now focus on the case where $q_2 = q_3 = 0$. In this case, we assume the corresponding scalars φ_2 and φ_3 ought not to be excited. As a result, we are left with a single equation for φ_1, which now takes the form

$$(x(x - x_h)(x + Q)\varphi_1')' + x\left[1 + \frac{x_h + Q}{x + Q}\frac{q_1}{x_h + q_1}\right]\varphi_1 = 0. \tag{3.13}$$

This equation still contains four regular singular points, and as such can be brought into Heun form. In particular, we may map the singular points $\{0, x_h, -Q\}$ to $\{0, 1, a\}$ by introducing

$$x = zx_h, \qquad Q = -ax_h, \tag{3.14}$$

after which the scalar equation becomes

$$(z(z - 1)(z - a)\varphi_1')' + z\left[1 + \zeta\frac{1 - a}{z - a}\right]\varphi_1 = 0. \tag{3.15}$$

Here the prime denotes a derivative with respect to z. We have also defined the dimensionless parameter

$$\zeta = \frac{q_1}{x_h + q_1}. \tag{3.16}$$

Note that $\zeta \to 1$ in the limit of large charge, while $\zeta \to 0$ in the limit of vanishing charge.

The scalar equation (3.15) can be brought into the canonical form of the Heun equation through the substitution $\varphi_1 = (z - a)^{\pm\sqrt{\zeta}}\tilde{\varphi}_1$. As a result, the solution may be written in terms of a local Heun function

$$\varphi_1 = \frac{1 - a}{z - a}Hl\left(\frac{1}{a}, -\frac{\zeta}{a}; 1 - \sqrt{\zeta}, 1 + \sqrt{\zeta}, 1, 1; \frac{z - 1}{z - a}\right). \tag{3.17}$$

Here we have imposed the boundary condition that φ_1 is regular at the horizon. In particular, the expansion of the local Heun function gives

$$\varphi_1 = 1 + (1 + \zeta)\frac{z - 1}{a - 1} + \frac{4 - a + (6 - a)\zeta + \zeta^2}{4}\frac{(z - 1)^2}{(a - 1)^2} + \cdots. \tag{3.18}$$

where ζ is given in (3.16), and where

$$a = -\left(1 + \frac{q_1 + 1/g^2}{x_h}\right). \tag{3.19}$$

In principle, we would also like to obtain the expansion of φ_1 in the asymptotic regime $z \to \infty$ in order to extract the boundary behavior (3.20). However, the two-point connection problem for the Heun equation is in general a difficult task, and there is as yet no straightforward way to connect the behavior of (3.17) near the horizon with (3.20) near the boundary. In order to match the horizon and boundary behaviors of φ_1, we need to utilize further approximation techniques. Before turning to this, we shall first consider the general behavior of φ_i.

General behavior. The asymptotic behavior of the solution for φ_i is governed by the point at infinity. In particular, given the repeated characteristic exponent of 1, it may be shown that the solution near the boundary has the form

$$\varphi_i(x) \sim \frac{c_1^i + c_2^i \log x}{x} + \cdots, \qquad x \to \infty, \tag{3.20}$$

which is consistent with φ_i being an $E_0 = 2$ scalar (or equivalently being associated with conformal dimension $\Delta = 2$ in the dual gauge theory). Although the two independent solutions encoded in (3.20) admit different interpretations in the dual gauge theory, they are both allowed to be present in the supergravity solution.

In particular, as in BPS case, c_1^i can be interpreted as the vev of the dimension 2 chiral primary operator $\mathrm{Tr}\, Z_i^2$, where $Z_i = \frac{1}{\sqrt{2}}(\phi_1^i + \mathrm{i}\phi_2^i)$ and ϕ_1^i and ϕ_2^i are three pairs of real scalars of $\mathcal{N} = 4$ super Yang-Mills theory, for $i = 1, 2, 3$. On the other hand, the c_2^i term corresponds to adding the relevant deformation $c_2^i \,\mathrm{Tr}\, Z_i^2$ to the Lagrangian of the conformal field theory. The c_2^i term is only present when one moves away from extremality, which indicates that there may be a phase transition in the field theory at zero temperature.

Turning to the horizon, we see that, with the choice of b^i in (3.10), the repeated characteristic exponent of 0 at $x = x_h$ indicates that the solution near the horizon has the form

$$\varphi_i(x) \sim d_1^i + d_2^i \log(x - x_h) + \cdots, \qquad x \to x_h. \tag{3.21}$$

Clearly, we must set $d_2^i = 0$ to avoid a logarithmic divergence of φ_i at the horizon.

In principle, this boundary condition now fixes the complete solution, in the sense that the coefficients c_1^i and c_2^i in the asymptotic expansion may be determined directly from d_1^i and the physical parameters of the background solution. However, in the absence of any general connection formulae for the Heun equation, there is no straightforward way to make this relation explicit. (We note that, even if the generic connection matrix were known, it might not be applicable to this solution as it has repeated characteristic exponents.) In order to match the horizon and boundary behaviors of φ_1 for the single-charge case, we instead turn to approximate solutions of equation (3.15).

3.2 Matching approximate solutions

We first consider an approximation technique which involves matching approximate solutions in two overlapping regions. In particular, one region contains the horizon while the other includes asymptotic infinity. Provided these two regions overlap, the solutions may then be matched up in the overlap region. A drawback of this approach is that it is not always possible to ensure a large overlap region, depending on the physical parameters of the system. However, if such a large overlap exists, then essentially complete information may be reliably extracted using this matching.

Noting that a introduces a new scale into the problem, we may consider solving (3.15) in the two regions: i) the asymptotic region where $z \gg 1$, and ii) the horizon region where $z \ll |a|$. Provided $|a| \gg 1$, these two regions will have a large overlap ($1 \ll z \ll |a|$) where reliable matching may be performed. From (3.19), we see that overlap is ensured for either $q_1 \gg x_h$ or $x_h \ll 1/g^2$.

3.2.1 The asymptotic region

To highlight the asymptotic region, $z \gg 1$, we may rewrite the scalar equation (3.15) as

$$\varphi_1'' + \left(\frac{2}{w} + \frac{1}{w-1}\right)\varphi_1' + \frac{1}{w(w-1)}\left(1 - \frac{\zeta}{w-1}\right)\varphi_1 = -\frac{1}{a}\left[\frac{1}{w\left(w-\frac{1}{a}\right)}\left(\varphi_1' + \frac{1+\zeta}{w-1}\varphi_1\right)\right], \qquad (3.22)$$

where we have introduced the rescaled coordinate $w = z/a$. In this case, the additional factor of $1/a$ on the right-hand side of (3.22) allows us formally to develop a solution for φ_1 as a perturbative expansion in $1/a$

$$\varphi_1 = \varphi^{(0)} + \frac{1}{a}\varphi^{(1)} + \frac{1}{a^2}\varphi^{(2)} \cdots, \qquad (3.23)$$

where $\varphi^{(0)}$ solves the homogeneous equation corresponding to the left-hand side of (3.22). Since this can be put into hypergeometric form, the solution is essentially known. In practice, however, matching of the asymptotic and horizon expansions is facilitated by introducing yet another expansion, this time in ζ. Examination of (3.16) indicates that there are two relevant limits to consider, namely the large and small charge limits.

The large charge limit corresponds to $q_1 \gg x_h$, or equivalently $\zeta \approx 1$. In this case, we let $\zeta = 1 + \hat{\zeta}$ and rearrange (3.22) to read

$$\varphi_1'' + \left(\frac{2}{w} + \frac{1}{w-1}\right)\varphi_1' + \frac{w-2}{w(w-1)^2}\varphi_1 = \frac{\hat{\zeta}}{w(w-1)^2}\varphi_1 - \frac{1}{a}\left[\frac{1}{w\left(w-\frac{1}{a}\right)}\left(\varphi_1' + \frac{2+\hat{\zeta}}{w-1}\varphi_1\right)\right]. \quad (3.24)$$

Solutions to this equation can now be developed as a double expansion in $\hat{\zeta}$ and $1/a$. Although this may seem to be only a slight rearrangement of (3.22), the main simplification here is that the homogeneous equation can now be solved in terms of elementary functions, $\varphi_1^{(0)} = c_1 u_1 + c_2 u_2$ where

$$u_1 = \frac{1}{w-1}, \qquad u_2 = \frac{1}{w-1}\left(\log(-w) + \frac{1}{w}\right). \qquad (3.25)$$

At each successive order in the perturbation, the lower order solutions feed in as sources on the right-hand side of (3.24). However, since the homogeneous solutions are elementary, the inhomogeneous system has a straightforward solution which can be developed, e.g through variation of parameters.

Up to first order in both $\hat{\zeta}$ and $1/a$, we find that the two linearly independent solutions for the outside function can be expressed as

$$\varphi_{\text{out}}^1 = \frac{1}{w-1}\left[1 - \frac{\hat{\zeta}}{2}\left(\log\left(1 - \frac{1}{w}\right) - \frac{1}{w}\right) - \frac{1}{aw}\left(1 - \frac{\hat{\zeta}}{2}\left(\log\left(1 - \frac{1}{w}\right) - 2 + \frac{1}{2w}\right)\right) + \cdots\right],$$

$$\varphi_{\text{out}}^2 = \frac{1}{w-1}\left[\log(-w) + \frac{1}{w} - \frac{\hat{\zeta}}{2}\left(\log(-w)\left(\log\left(1 - \frac{1}{w}\right) - \frac{1}{w}\right) + \left(2 - \frac{1}{w}\right)\log\left(1 - \frac{1}{w}\right)\right.\right.$$

$$\left.\left. + \log^2\left(1 - \frac{1}{w}\right) + 2\text{Li}_2\left(\frac{1}{1-w}\right)\right) + \frac{1}{aw}\left(-3 - \log(-w) + \frac{1}{2w} + \mathcal{O}(\hat{\zeta})\right) + \cdots\right], \quad (3.26)$$

where we recall that $w = z/a$. We have organized these two solutions according to the asymptotic behavior

$$\varphi_{\text{out}}^1 \sim \frac{a}{z} + \cdots , \qquad \varphi_{\text{out}}^2 \sim \frac{a}{z} \log z + \cdots , \qquad (3.27)$$

corresponding to the normalizable and non-normalizable $E_0 = 2$ modes of φ_1, as in (3.20).

3.2.2 The horizon region

Turning next to the horizon region, $z \ll |a|$, we now choose to write the scalar equation (3.15) as

$$\varphi_1'' + \left(\frac{1}{z} + \frac{1}{z-1} \right) \varphi_1' = \frac{1}{a} \left(\frac{1}{1 - z/a} \right) \left[\varphi_1' + \frac{1}{z-1} \left(1 + \zeta \frac{1 - 1/a}{1 - z/a} \right) \varphi_1 \right] . \qquad (3.28)$$

This again allows us to develop an expansion in $1/a$, where the independent solutions to the homogeneous equation on the left-hand side are simply

$$u_1 = 1, \qquad u_2 = \log \frac{z-1}{z} . \qquad (3.29)$$

These two solutions correspond directly to the near horizon behavior given in (3.21); in particular, we see that only u_1 is well behaved at the horizon. Developing this solution to the first few orders in $1/a$ gives

$$\varphi_{\text{in}} = 1 + \frac{1}{2a}(1 + \zeta)(z - 1 + \log z) + \frac{1}{24a^2} \Big[(z-1)\big(9(1 - \zeta^2) + 2(\zeta^2 + 7\zeta + 4)z\big)$$
$$- \big(5\zeta^2 + 14\zeta + 5 - 6(1 + \zeta)^2 z\big) \log z + 3(1 + \zeta)^2 \log^2 z + 6(1 + \zeta)^2 \text{Li}_2 \left(1 - \frac{1}{z} \right) \Big] + \cdots .$$
$$(3.30)$$

Note that this is purely an expansion in $1/a$, and in particular it is valid for arbitrary ζ. Furthermore, it can be seen that this expression for φ_{in} agrees with the near horizon expansion given in (3.18) in the overlapping region of validity $z \to 1$ and $a \to \infty$. The advantage of (3.30) over (3.18), however, is that here φ_{in} remains valid even for z away from the horizon (provided $z \ll |a|$). This is precisely what is needed in order to match the horizon expression with the asymptotic forms of the solution given above in (3.26).

3.2.3 Matching

While the asymptotic and horizon solutions (3.26) and (3.30) were derived under the independent conditions of $z \gg 1$ and $z \ll |a|$, they are both valid in the overlap region $1 \ll z \ll |a|$, so long as $|a| \gg 1$. In order to match the solutions in this overlap region, it is convenient to rewrite the horizon solution (3.30) in terms of w.

After suitable rearrangement, φ_{in} then takes the form of a series solution in w as well as an expansion in $1/a$

$$\varphi_{\text{in}} = \left[1 + \frac{w}{2}(1 + \zeta) + \frac{w^2}{12}(\zeta^2 + 7\zeta + 4) + \mathcal{O}(w^3) \right]$$
$$+ \frac{1}{a} \left[\frac{1}{2}(1 + \zeta)(\log(wa) - 1) + \frac{w}{24}(-11\zeta^2 - 14\zeta + 1 + 6(1 + \zeta)^2 \log(wa)) + \mathcal{O}(w^2) \right]$$
$$+ \mathcal{O}\left(\frac{1}{a^2} \right). \tag{3.31}$$

Here it is important to realize that, although the expressions are no longer complete at each new order in $1/a$, the resulting series in w are in principle well behaved for $|w| < 1$. This is what allows a consistent matching with the asymptotic solution (3.26), with φ_{out} similarly expanded as a series in w.

The asymptotic solution to the scalar equation is in general a linear combination of the two solutions given in (3.26):

$$\varphi_{\text{out}}(x) = c_1 \varphi_{\text{out}}^1 + c_2 \varphi_{\text{out}}^2 \sim \frac{a x_h (c_1 + c_2 \log(-x/a x_h))}{x}, \tag{3.32}$$

where we have transformed back to the coordinate $x = r^2$. By matching this with the normalized horizon solution (3.31), we find

$$c_1 = -1 + \frac{1}{a} \left(1 - \log(-a) - \frac{\hat{\zeta}}{2} \left(\frac{\pi^2}{3} + 1 + \log(-a) \right) \right) + \mathcal{O}(\hat{\zeta}^2) + \mathcal{O}\left(\frac{1}{a^2} \right),$$
$$c_2 = \frac{\hat{\zeta}}{2} + \frac{1}{a} \left(-1 + \frac{\hat{\zeta}}{2} \log(-a) \right) + \mathcal{O}(\hat{\zeta}^2) + \mathcal{O}\left(\frac{1}{a^2} \right), \tag{3.33}$$

where

$$\hat{\zeta} = -\frac{x_h}{x_h + q_1}, \tag{3.34}$$

and a is given by (3.19). Note that both $\hat{\zeta}$ and a are negative for physical values of the charge and horizon radius.

In order to more readily apply the AdS/CFT dictionary for examining the asymptotic behavior, we express (3.32) in terms of the original r coordinate:

$$\varphi_{\text{out}}(r) = \frac{a x_h [c_1 - c_2 \log(-a x_h)]}{r^2} + 2 a x_h c_2 \frac{\log r}{r^2}. \tag{3.35}$$

Note that we can always include factors of g to ensure that the logarithms have dimensionless arguments. However, we have set $g = 1$ for simplicity. The above $1/r^2$ term corresponds to giving a vev to the chiral primary operator $\text{Tr} \, Z^2$. The $(\log r)/r^2$ term, on the other hand, corresponds to adding a massive deformation $-\frac{1}{2}\bar{m}^2 \, \text{Tr} \, Z^2$ to the Lagrangian of the conformal field theory. A priori, there is an ambiguity in the normalization of φ_1. Recall that, for the BPS AdS bubble, the thermal mass $\bar{m} = 0$ while, from (2.18), $\text{Tr} \, Z^2$ gets a vev $v_1 \equiv \sqrt{q_1(q_1 + 2)}$. Therefore, since we are interested in a thermalization of $\mathcal{N} = 4$ super-Yang Mills on top of the 1/2 BPS sector specified by a given vev v_1, we

may fix the normalization by taking $\langle \operatorname{Tr} Z^2 \rangle = v_1$ for the bubbling AdS black hole. Then, taking the limit $q_1 \gg x_h$, we find

$$\bar{m}^2 \approx 4x_h \left(\frac{1}{q_1 + 1} - \frac{1}{2q_1} \right) \sqrt{q_1(q_1 + 2)}. \tag{3.36}$$

As x_h vanishes, the thermal mass goes to zero. This is certainly not unexpected, since the thermal mass vanishes in the BPS limit, though at the same time it would not have been surprising if $\bar{m}$ did not behave in a smooth manner in this limit. Interestingly enough, $\bar{m}$ also vanishes for $q_1 \to 1$. As we will see, this could be due to a phase transition in the field theory that is associated with the Hawking-Page transition. In particular, for $q_1 < 1$, the operator $\operatorname{Tr} Z^2$ could become tachyonic, thereby destabilizing the moduli space of the field theory.

We would now like to express $\bar{m}$ in terms of the physical parameters of the thermal field theory, namely the temperature T, chemical potential μ_1 and R-charge Q_1. These quantities are all expressed in terms of the AdS black hole parameters q_i and $x_h = r_h^2$ in (2.7), (2.8) and (2.9). In the limit $q_1 \gg x_h$ for a single charge,

$$T = \frac{1 + q_1}{2\pi \sqrt{q_1}}, \qquad \mu_1 = \sqrt{1 + x_h}, \qquad Q_1 = q_1 \sqrt{1 + x_h}. \tag{3.37}$$

We therefore have a choice of expressing x_h and q_1 in terms of two out of the three physical parameters of the field theory, which corresponds to different ensembles. For example, the R-charge Q_1 is held fixed in the canonical ensemble. Therefore, the thermal mass is a function of temperature and chemical potential in the grand canonical ensemble, while it depends on temperature and R-charge in the canonical ensemble.

We first work in the grand canonical ensemble. We can express x_h and q_1 in terms of T and μ_1 in the limit $q_1 \gg x_h$ as $x_h \approx \mu_1^2 - 1$ and $q_1 \approx 2\pi T(\pi T \pm \sqrt{\pi^2 T^2 - 1}) - 1$. Notice that black holes with two different values of q_1 correspond to the same temperature. However, since the entropy is given by $S \approx 2\pi x_h \sqrt{q_1}$, the black hole with larger q_1 is entropically favorable and so we shall take the $+$ sign in the expression for q_1. Since $x_h \geq 0$, in this regime $\mu_1^2 \geq 1$. Also, $\pi T > 1$ in order for q_1 to be real.

One way in which $q_1 \gg x_h$ is if $T \gg \mu_1$. This implies that $T \gg 1$ and $q_1 \approx 4\pi^2 T^2$. We can then express the thermal mass as $\bar{m}^2 \approx 2(\mu_1^2 - 1)$. Thus, for a given chemical potential, the thermal mass vanishes at zero temperature and approaches a constant in this high temperature limit, which may imply a sort of saturation taking place. Also, the thermal mass gets enhanced by increased chemical potential. Notice that this high temperature regime does not include the point at which the thermal mass vanishes at a finite temperature.

Alternatively, $q_1 \gg x_h$ can be satisfied without having to take T to be large by taking $\mu_1 \approx 1$. Then after expressing the thermal mass in (3.36) in terms of T and μ_1, we find that $\bar{m} = 0$ for $\pi T = 1$, which corresponds to $q_1 = 1$. As T increases, the thermal mass asymptotically approaches the constant value discussed in the previous large temperature limit, that is $\bar{m}^2 \to 2(\mu_1^2 - 1)$.

We will now look at the situation in the canonical ensemble, for which the R-charge Q_1 is held fixed. One way to satisfy $q_1 \gg x_h$ is to consider $T^3 \gg Q_1$ and $T \gg 1$. Then we

have

$$\bar{m}^2 \approx 2\left(\frac{Q_1^2}{16\pi^4 T^4} - 1\right), \tag{3.38}$$

where $\bar{m}^2 \geq 0$ since $x_h \geq 0$. We see that the thermal mass decreases with temperature and increases with the R-charge Q_1. Note that in order for there to be a horizon at $x_h > 0$, $\bar{m}^2 > 0$. This requires that $Q_1 > 4\pi^2 T^2$.

We will now briefly discuss the conditions for the regime $q_1 \gg x_h$ to be consistent with local thermodynamical stability constraints and dynamical considerations for the charged AdS black hole that were given in [20], in both the canonical ensemble and grand canonical ensemble. At the level of linearized φ_1, we do not need to consider the backreaction on this background, which would then alter these thermodynamical relations.

In the grand canonical ensemble, in order for the AdS black hole to be dynamically preferred over pure AdS, $\pi T > \pi T_c \approx 1$, where T_c is the temperature of the Hawking-Page transition. Furthermore, the local thermodynamical stability constraint is satisfied only for $\pi T \approx 1$. Thus, we only have a small window for which the above calculations are consistent with stability. The temperature at which $\bar{m}$ vanishes lies within this window; in particular, this occurs at the Hawking-Page transition and presumably signifies the corresponding phase transition in the field theory. On the other hand, the large temperature regime where $\bar{m}$ saturates does not lie within this window. For the canonical ensemble, the local stability constraint is satisfied for $\pi T \approx \sqrt{2}$, which also satisfies the condition for the AdS black hole to be dynamically preferred over pure AdS.

We would now like to recall the approximations that have been made, namely that $|\hat{\zeta}| \ll 1$ and $|a| \gg 1$ correspond to the large charge limit $q_1 \gg x_h$, which is equivalent to $T \gg \mu_1$. We could also match approximate solutions in the small charge limit $q_1 \ll x_h$, which means that $|\zeta| \ll 1$. Then in order to be consistent with the condition that the asymptotic and horizon regions have a large overlap, we require that $|a| \gg 1$, which further implies that $x_h \ll 1$. We can use (2.7) and (2.8) to express these conditions as $\mu_1 \ll 1 \ll T$. Therefore, this is the regime of high temperature and small chemical potential. However, as we will see in the next section, we do not have to rely on the matching technique if $T \gg 1$ and $T \gg \mu_1$, which encompasses the above high temperature regime.

3.3 Perturbative approach for a second high temperature regime

We will now consider the linearized φ_1 equation (3.15) for the case of $x_h \gg q_1$ and $x_h \gg 1$, where we have taken $g = 1$ for simplicity. As we will see, this is a second high temperature regime. Note that φ_1 can be expanded for small x_h^{-1} as

$$\varphi_1 = u_0\left(1 + x_h^{-1} u_1 + \mathcal{O}(x_h^{-2})\right). \tag{3.39}$$

The function u_0 satisfies the following linear differential equation

$$((z^2 - 1)z\, u_0')' + z\, u_0 = 0, \tag{3.40}$$

which has the general solution

$$u_0 = \frac{2c_1}{\pi}\, K(1 - z^2) + \frac{2c_2}{\pi}\, K(z^2). \tag{3.41}$$

where $K(x)$ denotes the complete elliptic integral of the first kind. For the solution to be regular at the horizon $z = 1$, it is necessary to set $c_2 = 0$. Furthermore, without loss of generality we can set the scaling factor $c_1 = 1$, so that near the horizon u_0 behaves like

$$u_0 = 1 - \frac{1}{2}(z-1) + \frac{5}{16}(z-1)^2 + \mathcal{O}((z-1)^3). \tag{3.42}$$

Asymptotically we find that

$$u_0 \to \frac{2[\log z - \gamma - \psi(1/2)]}{\pi z}, \tag{3.43}$$

where γ is Euler's constant and ψ is the digamma function. Approximate numerical values are $\gamma \approx .58$ and $\psi(1/2) \approx -1.96$.

We can now consider the x_h^{-1} corrections in φ_1. The function u_1 can be expressed in terms of a functional integral as

$$u_1 = \int_1^z \frac{v(y)}{y(y^2-1)u_0^2(y)}\,dy,$$

$$v = I_+(z) + q_1 I_-(z), \quad I_\pm(z) \equiv \int_1^z \frac{1}{y+1}\Big(y(y-1)u_0(y)u_0'(y) \pm y\,u_0^2(y)\Big)dy. \tag{3.44}$$

Setting the lower bound of the v integral to unity ensures that the solution remains regular at the horizon. Moreover, the chosen lower bound of the u_1 integral guarantees that u_1 vanishes at $z = 1$, so that the boundary condition on φ_1 remains unchanged.

To compute the asymptotic behavior in u_1, evaluate $I_\pm(\infty)$ in order to take the large z expansion of the u_1 integral. The result gives

$$\varphi_1 \to u_0 - \frac{\pi(I_+(\infty) + q_1 I_-(\infty))}{2x_h z}, \tag{3.45}$$

for large z. The approximate numerical values are $I_+(\infty) \approx 1.03$ and $I_-(\infty) \approx -3.51$. In terms of the coordinate r, the asymptotic behavior is roughly given by

$$\varphi_1 \to \frac{4x_h \log r}{\pi r^2} + \frac{c}{r^2}, \qquad c \equiv -\frac{2x_h}{\pi}[\log x_h + \gamma + \psi(1/2)] - \frac{\pi}{2}[I_+(\infty) + q_1 I_-(\infty)]. \tag{3.46}$$

Once again, the $1/r^2$ term corresponds to giving a vev to the chiral primary operator $\mathrm{Tr}\,Z^2$. We normalize φ_1 by equating this vev to the value $v_1 = \sqrt{q_1(q_1+2)}$ for the case of the BPS AdS bubble, as we did in the previous section. This enables us to find the thermal mass of $\mathrm{Tr}\,Z^2$ to be

$$\bar{m}^2 \approx \frac{\sqrt{q_1(q_1+2)}}{\log x_h}, \tag{3.47}$$

in the limit $x_h \gg q_1$ and $x_h \gg 1$. We would like to express this in terms of the temperature, chemical potential and R-charge of the field theory. From (2.7) and (2.8) and (2.9), we can express the temperature, chemical potential and R-charge Q_1 in terms of the black hole parameters x_h and q_1 in the regime $x_h \gg q_1$ and $x_h \gg 1$ as

$$T \approx \frac{\sqrt{x_h}}{\pi}, \qquad \mu_1 \approx \sqrt{q_1}, \qquad Q_1 \approx \sqrt{q_1}\,x_h. \tag{3.48}$$

In the grand canonical ensemble, this regime corresponds to $T \gg \mu_1$ and $T \gg 1$ and the thermal mass can be expressed as

$$\bar{m}^2 \approx \frac{\sqrt{\mu_1^2(\mu_1^2 + 2)}}{2 \log T} \,. \tag{3.49}$$

Thus, the thermal mass increases with chemical potential and decreases with temperature.

In the canonical ensemble, this regime corresponds to $T^3 \gg Q_1$ and $T \gg 1$ and

$$\bar{m}^2 \approx \frac{Q_1 \sqrt{Q_1^2 + 2\pi^4 T^4}}{2\pi^4 T^4 \log T} \,. \tag{3.50}$$

The thermal mass now increases with R-charge Q_1 and still decreases with temperature. Note that, in this regime, the results for the thermal mass are identical for the case of three equal charges.

Recall that there was only a small window for which the regime discussed in the previous section satisfied the local thermodynamic stability constraints and entropic considerations. On the other hand, for the regime $x_h \gg q_1$ and $x_h \gg 1$, all of these constraints are satisfied. This provides a large range of temperatures for which this system can be reliably discussed. Of course, backreaction and non-linear effects would also have to be taken into account, should we desire a more detailed treatment of the thermodynamics.

4. Taking backreaction into account

In the previous section, we have explored the linearized equations of motion for the deformation scalars φ_i. Going beyond linear order, these scalars will backreact on the metric through the Einstein equation, as well as on the other matter fields through the couplings implicit in the Lagrangian (2.10). In particular, the field φ_i acts as a source for the A^i gauge fields through the $\sinh^2 \varphi_i * A^i \wedge A^i$ couplings, and the X_i (or equivalently ϕ_1 and ϕ_2) scalars through the modified scalar potential (2.11).

While the generalized backreaction equations are straightforward to obtain, we restrict the analysis to the non-extremal generalization of the single-charge AdS bubble. In this case, the natural way to parameterize the metric backreaction is to start from the (one-charge) black hole solution (2.4) and to write

$$ds^2 = -H^{-2/3} f dt^2 + H^{1/3}(f^{-1} dr^2 + r^2 d\Omega_3^2), \tag{4.1}$$

where the metric functions are, to second order in the linearization parameter ϵ,

$$
\begin{aligned}
H &= 1 + \frac{q_1}{x} + \epsilon^2 h_2(x) + \cdots, \\
f &= 1 - \frac{m}{x} + g^2 x H + \epsilon^2 f_2(x) + \cdots = 1 + g^2 q_1 - \frac{m}{x} + g^2 x + \epsilon^2 (f_2 + g^2 x h_2) + \cdots. \tag{4.2}
\end{aligned}
$$

Here we recall that $x = r^2$ and $q_1 = m \sinh^2 \beta_1$, as indicated in (2.5). Note that we have continued to write the function f in the natural combination of $f = 1 - m/x + g^2 xH$ plus corrections.

We now turn to the matter fields. For the electric potential, we follow (3.2) and write

$$
\begin{aligned}
A_t^1 &= (b^1 - \coth\beta_1 H^{-1}) + \epsilon^2 a_2(x) + \cdots \\
&= \coth\beta_1 \left(\frac{x_h}{x_h + q_1} - \frac{x}{x + q_1 + x\epsilon^2 h_2} \right) + \epsilon^2 a_2 + \cdots \\
&= -q_1 \coth\beta_1 \frac{x - x_h}{(x_h + q_1)(x + q_1)} + \epsilon^2 \left(\coth\beta_1 \frac{x^2}{(x + q_1)^2} h_2 + a_2 \right) + \cdots , \quad (4.3)
\end{aligned}
$$

where $\coth\beta_1 = \sqrt{1 + m/q_1} = \sqrt{(g^2/q_1)(x_h + 1/g^2)(x_h + q_1)}$. Finally, for the backreaction of φ_1 on the X_i scalars, we note that it is consistent to set $X_2 = X_3 = 1/\sqrt{X_1}$, since φ_1 sources X_2 and X_3 in an identical manner. We then take a multiplicative parameterization of X_1:

$$
X_1 = H^{-2/3}(1 + \epsilon^2 \chi_2(x)) = \left(1 + \frac{q_1}{x} \right)^{-2/3} \left(1 + \epsilon^2 \left(\chi_2 - \frac{2}{3}\frac{x}{x + q_1} h_2 \right) \right) + \cdots . \quad (4.4)
$$

For φ_1 of order ϵ, its backreaction on the metric fields h_2 and f_2, the electric potential a_2 and scalar χ_2 is then governed by the set of inhomogeneous second-order equations

$$
[x^2 h_2' + 2q_1 \chi_2]' = -x(x + q_1)\varphi_1'^2 - \frac{q_1 x(x_h + 1/g^2)}{(x_h + q_1)(x + x_h + 1/g^2 + q_1)^2}\varphi_1^2, \quad (4.5)
$$

$$
\begin{aligned}
(x^2 f_2)'' &+ g^2[x^3 h_2' + 2x^2 h_2]' + g^2[x^2 h_2' + 2q_1 \chi_2] \\
&= \frac{g^2 x(x + x_h + 1/g^2 + 2q_1 + q_1(x_h + 1/g^2)/(x_h + q_1))}{x + x_h + 1/g^2 + q_1}\varphi_1^2, \quad (4.6)
\end{aligned}
$$

$$
((x + q_1)^2 a_2')' = \coth\beta_1 x(x + q_1) \left(\varphi_1'^2 - \frac{q_1}{(x_h + q_1)(x + x_h + 1/g^2 + q_1)^2}\varphi_1^2 \right), \quad (4.7)
$$

$$
[(x(x - x_h)(x + x_h + 1/g^2 + q_1)\chi_2')' + x\chi_2] + [2xf_2/g^2 + x^2 h_2]' + [x^2 h_2' + 2q_1 \chi_2] = x\varphi_1^2. \quad (4.8)
$$

In addition, the constraint equation coming from the Einstein equations gives rise to a rather cumbersome first-order equation

$$
\begin{aligned}
&\frac{2g^2 q_1}{x + q_1}(x - x_h)\left(x + x_h + 1/g^2 + q_1 \right)\chi_2' + 2g^2 q_1 \left(\frac{(x_h + 1/g^2)(x_h + q_1)}{(x + q_1)^2} - 1 \right)\chi_2 \\
&- \frac{x(3x + 2q_1)}{x + q_1}f_2' + \left(\frac{q_1^2}{(x + q_1)^2} - 3 \right)f_2 + 2q_1 \coth\beta_1 a_2' \\
&- \left(4g^2 x^2 - \frac{xx_h(1 + g^2(x_h + q_1))}{x + q_1} \right)h_2' - \left(4g^2 x - \frac{q_1 x_h(1 + g^2(x_h + q_1))}{(x + q_1)^2} \right)h_2 \\
&\quad = -g^2 x(x - x_h)\left(x + x_h + 1/g^2 + q_1 \right)\varphi_1'^2 - \frac{g^2 x\left(x + x_h + 1/g^2 + 2q_1 + q_1\frac{x_h + 1/g^2}{x_h + q_1} \right)}{(x + x_h + 1/g^2 + q_1)}\varphi_1^2. \quad (4.9)
\end{aligned}
$$

We observe that the metric backreaction equations (4.5) and (4.6), along with the X_1 equation (4.8), are coupled in a non-trivial manner. On the other hand, the electric potential equation (4.7) is independent, and can be solved by quadratures:

$$
\begin{aligned}
a_2(x) = \coth\beta_1 \int^x \frac{dx'}{(x' + q_1)^2} \int^{x'} dx'' \, x''(x'' + q_1) \times \\
\left(\varphi_1'^2(x'') - \frac{q_1}{(x_h + q_1)(x'' + x_h + 1/g^2 + q_1)^2}\varphi_1^2(x'') \right). \quad (4.10)
\end{aligned}
$$

Note that the indefinite integrals allow for the addition of an arbitrary homogeneous solution

$$a_2 = k_1 + k_2 \frac{1}{x + q_1} + \cdots . \tag{4.11}$$

These constants may be fixed by demanding that the leading asymptotic behavior of the solution be unchanged by the backreaction. In particular, for the electric potential we demand the vanishing of any $1/x$ correction to A_t^1 (which would modify the R charge). We also insist that the potential continues to vanish at the horizon, $A_t^1(x_h) = 0$, so that A^1 remains normalizable at the horizon. Note, however, that these two conditions do not fix k_1 and k_2 directly, since h_2 also enters into the correction to the potential, as indicated in (4.3). Assuming (as we demonstrate below) that h_2 falls off faster than $1/x$ at infinity, the above requirements then lead to

$$a_2(x) = -\coth\beta_1 \left[\int_{X_H}^x \frac{dx'}{(x'+q_1)^2} \int_{x'}^\infty dx'' \, x''(x''+q_1) \times \right.$$
$$\left. \left(\varphi_1'^2(x'') - \frac{q_1}{(x_h+q_1)(x''+x_h+1/g^2+q_1)^2} \varphi_1^2(x'') \right) - \frac{x_h^2}{(x_h+q)^2} h_2(x_h) \right]. \tag{4.12}$$

From the asymptotic behavior of φ_1 as given in (3.20)

$$\varphi_1 \sim \frac{c_1 + c_2 \log x}{x} + \cdots , \qquad x \to \infty, \tag{4.13}$$

we obtain

$$a_2 \sim a_2(\infty) + \frac{\coth\beta_1}{x^2} \left[-c_2(c_1+c_2) + \frac{x_h}{x_h+q_1} \left(\frac{1}{2}c_1^2 + \frac{3}{2}c_1 c_2 + \frac{7}{4}c_2^2 \right) \right.$$
$$\left. + \left(-c_2^2 + \frac{x_h}{x_h+q_1} c_2 \left(c_1 + \frac{3}{2}c_2 \right) \right) \log x + \frac{1}{2} \frac{x_h}{x_h+q_1} c_2^2 \log^2 x + \cdots \right]. \tag{4.14}$$

The constant $a_2(\infty)$ contributes to a shift in the value of the electric potential (or, equivalently, the R charge chemical potential) compared to the zeroth-order solution.

Turning to the other fields $\{h_2, f_2, \chi_2\}$, the coupled system of backreaction equations (4.5), (4.6) and (4.8) do not appear to admit a straightforward solution. Nevertheless, some general properties are evident. Firstly, in the limit of vanishing electric charge, $q_1 \to 0$, the metric equations (4.5) and (4.6) become self-contained. In this limit, (4.5) may be integrated twice to obtain h_2 and, in turn, knowledge of h_2 allows (4.6) to be integrated for f_2. Secondly, even with $q_1 \neq 0$, the combination of the metric equations admit first integrals, thus reducing the order of the coupled system. In fact, using the first-order equation, (4.10), along with the backreaction equations for $\{h_2, f_2, a_2\}$, we may obtain the homogeneous equation

$$g^2 x_h(x_h + 1/g^2 + q_1) \left(\frac{x}{x+q_1} h_2 \right)' + g^2 x_h (x^2 h_2')' + \left(x^2 f_2' - \frac{x^2}{x+q_1} f_2 \right)'$$
$$+ q_1 \coth\beta_1 \left(a_2 + \frac{(x+q_1)(x-x_h)}{x_h+q_1} a_2' \right)' + 2g^2 q_1(x_h + 1/g^2) \left(\frac{x-x_h}{x+q_1} \chi_2 \right)' = 0, \tag{4.15}$$

which yields yet another constant of integration (which is presumably related to the conserved energy of the static gravitational system). Finally, we note from (4.8) that the χ_2 scalar fluctuations satisfy an equation of motion which is of similar form to the φ_i equation (3.3). This should not be of much surprise, as small scalar fluctuations are naturally governed by the Klein-Gordon equation, which in the present background takes on the form

$$(x^2 f \Phi')' - \frac{1}{4} M^2 x H^{1/3} \Phi = 0, \tag{4.16}$$

for a scalar Φ. (Note that, in a background with non-zero R charge, the supergravity potential (2.11) leads to position-dependent 'masses' for both the φ_i and X_i scalars.) Since φ_i and X_i both originate from the T_{ij} tensor of the sphere reduction [21], they all have $E_0 = 2$, and hence share a common value of mass $M^2 = -4g^2$ (at least when the charges are turned off). Regardless of the details, we expect that the χ_2 fluctuations are likewise described by a Heun equation. Hence similar difficulties to those we encountered in the previous section arise when obtaining explicit solutions for the backreaction of φ_1 on χ_2.

Because the system of backreaction equations is linear, the inhomogeneous solution can in principle be obtained by variation of parameters, so long as the fundamental matrix (i.e. the complete set of linearly independent solutions to the homogeneous system) is known. More precisely, we may rewrite the second-order equations in coupled first-order form. Then the solution to the system of n first-order equations

$$\vec{f}'(x) = \vec{A}(x)x + \vec{B}(x), \tag{4.17}$$

may be formally expressed as

$$\vec{f}(x) = \Phi(x)\vec{f}_0 + \Phi(x) \int_{x_0}^{x} \Phi^{-1}(x') \vec{B}(x')\, dx', \tag{4.18}$$

where $\Phi(x)$ is the fundamental matrix satisfying $\Phi'(x) = \vec{A}(x)\Phi(x)$ and normalized according to $\Phi(x_0) = I$. This formal solution is not particularly useful, since in the present case $\Phi(x)$ would be a 7×7 matrix, corresponding to the freedom of four second-order equations (4.5) through (4.8) minus the first-order constraint (4.10). Nevertheless, we note that five out of the seven linearly independent solutions to the homogeneous system are easy to obtain:

	h_2	f_2	a_2	χ_2
1:			1	
2:	$-\dfrac{2q_1 \coth \beta_1}{mx}$		$\dfrac{1}{x+q_1}$	
3:	$\dfrac{q_1}{x}$	$-\dfrac{m}{x}$		
4:	$\dfrac{q_1}{x^2}$	$-\dfrac{m}{2x^2} - \dfrac{3m+4q_1}{2q_1 x}$		$\dfrac{1}{x}$
5:	$\dfrac{3}{2}$	$-g^2(x+q_1) + \dfrac{7m+4q_1}{2x}$		1

$$(4.19)$$

where $m = x_h(1 + g^2(x_h + q))$ as obtained from (3.6). The remaining two solutions appear to involve non-trivial Heun functions pertaining to the χ_2 system.

Our main interest in studying the backreaction is to obtain the asymptotic form of the corrections so we may discern what effects turning on φ_1 may have on the conserved quantities such as mass and R charge. For the latter, we have already seen from (4.14) that we may hold the charge q_1 fixed, even as we turn on φ_1. For the mass, we turn to an asymptotic expansion of the coupled system (4.5), (4.6) and (4.8) as $x \to \infty$. From the asymptotic φ_1 behavior (4.13), we obtain

$$
\begin{aligned}
h_2 \sim \frac{1}{x^2} \Bigg[&-\frac{1}{2}c_1^2 - \frac{1}{2}c_1 c_2 - \frac{3}{4}c_2^2 + q\left(\chi_1 + \frac{1}{2}\chi_{11}\right) \\
&+ \left(-c_2\left(c_1 + \frac{1}{2}c_2\right) + q_1\chi_{11}\right)\log x - \frac{1}{2}c_2^2 \log^2 x \Bigg] + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
f_2 \sim \frac{1}{x^2} \Bigg[&\frac{2}{3}q_1\left(\chi_1 + \frac{1}{3}\chi_{11}\right) + \frac{1}{3}c_2\left(2c_1 + \frac{5}{3}c_2\right) - \frac{x_h(1 - g^2 q_1)}{x_h + q_1}\left(\frac{1}{3}c_1^2 + \frac{8}{9}c_1 c_2 + \frac{26}{27}c_2^2\right) \\
&+ \left(\frac{2}{3}q_1\chi_{11} + \frac{2}{3}c_2^2 - \frac{2}{9}\frac{x_h(1 - g^2 q_1)}{x_h + q_1}c_2(3c_1 + 4c_2)\right)\log x \\
&- \frac{1}{3}\frac{x_h(1 - g^2 q_1)}{x_h + q_1}c_2^2 \log^2 x \Bigg] + \cdots,
\end{aligned}
$$

$$
\chi_2 \sim \frac{1}{x}\left(\chi_1 + \chi_{11}\log x\right) + \frac{1}{x^2}\left[c_2\left(c_1 + \frac{3}{2}c_2\right) + \frac{\chi_{11}}{g^2} + c_2^2 \log x\right] + \cdots. \tag{4.20}
$$

Note that here we have used the freedom expressed in (4.19) to set the leading order terms in h_2 and f_2 to zero. The content of the residual two homogeneous solutions are incorporated through the constants χ_1 and χ_{11}, which are related to the shift in the asymptotic profile of the scalar X_1 at infinity. Of course, the contribution for the full set of homogeneous solutions may have to be added back in to satisfy the desired boundary conditions. For the gauge potential, we have argued that it is natural to demand $A_t^1(x_h) = 0$. Likewise, here it would be appropriate to set $f_2(x_h) + g^2 x_h h_2(x_h) = 0$ so that $f(x)$ as given in (4.2) continues to vanish at x_h when the backreaction is included.

Although we have explored the backreaction of φ_1 on the other fields of the system, we can also see from (4.20) that the X_1 scalar may be deformed as well, through χ_1 and χ_{11}. Presumably, this would allow a wider class of thermal black hole solutions with scalar hair. However, assuming our goal is to thermalize the BPS bubble, one may presumably set $\chi_1 = \chi_{11} = 0$ directly without any major concern. At the same time, we note an interesting feature of the non-extremal bubble solution. While the R-charged black hole (2.4) and the BPS bubble (2.12) both have the X_1 scalar satisfying $X_1 = H_1^{-2/3}$ (for the one-charge case), this condition can no longer be maintained when the bubble is thermalized, as evidenced by the non-vanishing backreaction on χ_2.

We also note that, according to (4.20), it appears that the backreaction can be adjusted so that it has no effect on the mass of the black hole. This is because, at least heuristically, the mass can be read off from the $1/r^2 \equiv 1/x$ terms in the metric functions H_1 and f (since we are working in five dimensions), and both such terms are absent in (4.20). It may turn out, however, that boundary conditions at the horizon would feed in some of the

homogeneous solution of the third type in (4.19), thus leading to a possible shift in the mass. Of course, such issues cannot be properly examined in the linearized analysis, and would have to await a full (possibly numerical) solution to be addressed. Nevertheless, the heuristic concept of mass can be made rigorous, and this is what we turn to next.

5. Mass of the non-BPS bubbles

The definition, and calculation, of the mass of an asymptotically AdS spacetime is somewhat more subtle than it is in an asymptotically flat spacetime. One can no longer use the ADM definition, which assumes an asymptotically Minkowski region [22]. A generalisation of the ADM procedure, in which one decomposes the metric as the sum of an AdS background plus deviations, was introduced in [23] by Abbott and Deser. Effectively, one is making an infinite background AdS subtraction from a divergent boundary integral. The presence of scalar fields in the solution, such as one has in the supergravity black holes and bubbles, can complicate the application of this AD procedure considerably, because of the inherent ambiguities in the separation of the metric into background plus deviations. Some discussion of the AD approach, and calculations for higher-dimensional rotating black holes, can be found in [24].

A procedure for calculating the mass of asymptotically AdS spacetimes that avoids all the problems inherent in making a split into background and deviation was introduced by Ashtekar, Magnon and Das [25, 26]. This is based on a conformal definition that expresses the mass in terms of an integral of certain components of the Weyl tensor over the spatial conformal boundary at infinity. Since the metric approaches AdS asymptotically, the integrand falls off and the integral is inherently well-defined. This AMD method was applied in [27] to the calculation of the masses of higher-dimensional rotating AdS black holes in general relativity, and in [28] this was extended to the case of rotating black holes in gauged supergravities.

Alternatively, the boundary counterterm method may be used to calculate the mass of configurations in an AdS background [29 – 33]. This notion of holographic renormalization is particularly natural in the context of AdS/CFT, where the addition of boundary counterterms in AdS has a natural counterpart in the addition of renormalization counterterms in the dual field theory. Furthermore, the boundary counterterm method has the advantage that it regulates divergences not just in the mass, but also in the on-shell gravitational action which is dual to the thermodynamic potential of the CFT. The boundary counterterm method was used in [34 – 36] to investigate the mass of the (undeformed) R-charged black holes (2.4).

Here, however, we shall use the AMD method to discuss the masses of BPS bubble metrics and their non-extremal deformations. Note that, in this definition of mass, pure AdS has by construction zero energy. This is in contrast with holographic renormalization, which naturally assigns non-zero energy to the vacuum (which is viewed as dual to the Casimir energy of the CFT on $S^3 \times \mathbb{R}$). We begin by briefly summarising the AMD procedure, drawing on material presented in [27, 28].

Consider an asymptotically AdS bulk spacetime $\{X, g\}$ of dimension D, equipped with a conformal boundary $\{\partial X, \bar{h}\}$. It admits a conformal compactification $\{\bar{X}, \bar{g}\}$ if $\bar{X} = \sqcup \partial X$ is the closure of X, and the metric $\bar{g}$ extends smoothly onto $\bar{X}$ where $\bar{g} = \Omega^2 g$ for some function Ω with $\Omega > 0$ in X and $\Omega = 0$ on ∂X, with $d\Omega \neq 0$ on ∂X. One might, for example, take

$$\Omega = \frac{l}{y}, \tag{5.1}$$

where in the asymptotic region the metric approaches AdS with $R_{\mu\nu} = l^{-2} g_{\mu\nu}$. (For the solutions in gauged supergravity that we shall consider, $l = 1/g$, where g here denotes the gauge-coupling constant.) Since Ω is determined only up to a factor, $\Omega \to f \Omega$, where the function f is non-zero on ∂X, the metric $\bar{g}$ on $\bar{X}$ and its restriction $\bar{h} = \bar{g}|_{\partial X}$ are defined only up to a non-singular conformal factor. The conformal equivalence class $\{\partial \bar{X}, \bar{h}\}$ is called the conformal boundary of X. If $\bar{C}^\mu{}_{\nu\rho\sigma}$ is the Weyl tensor of the conformally rescaled metric $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, and $\bar{n}_\mu \equiv \partial_\mu \Omega$, then in D dimensions one defines

$$\bar{\mathcal{E}}^\mu{}_\nu = l^2 \Omega^{D-3} \bar{n}^\rho \bar{n}^\sigma \bar{C}^\mu{}_{\rho\nu\sigma}. \tag{5.2}$$

This is the electric part of the Weyl tensor on the conformal boundary. The conserved charge $Q[K]$ associated to the asymptotic Killing vector K is then given by

$$Q[K] = \frac{l}{8\pi (D-3)} \oint_\Sigma \bar{\mathcal{E}}^\mu{}_\nu K^\nu d\bar{\Sigma}_\mu, \tag{5.3}$$

where $d\bar{\Sigma}_\mu$ is the area element of the $(D-2)$-sphere section of the conformal boundary. (The derivation of (5.3) is discussed in [25, 26]). Note that the expression (5.3) is invariant under the non-singular conformal transformations of the boundary metric that we discuss above. Thus, one may take for Ω any conformal factor that is related to (5.1) by a non-singular multiplicative factor.

In order to define the energy, one takes $K = \partial/\partial t$, where t is the time coordinate appearing in the asymptotically AdS form

$$ds^2 = -(1 + y^2 l^{-2}) dt^2 + \frac{dy^2}{1 + y^2 l^{-2}} + y^2 d\Omega_{D-2}^2 \tag{5.4}$$

of the metric under investigation. The energy (or mass) is then given by

$$E = \frac{l}{8\pi (D-3)} \oint_\Sigma \bar{\mathcal{E}}^t{}_t d\bar{\Sigma}_t. \tag{5.5}$$

For our present discussion, we need to apply (5.5) to the class of five-dimensional metrics given by

$$ds_5^2 = -\mathcal{H}^{-2/3} f dt^2 + \mathcal{H}^{1/3}(f^{-1} dr^2 + r^2 d\Omega_3^2), \tag{5.6}$$

where $\mathcal{H}$ and f are functions only of r, and $d\Omega_3^2$ is the metric on the unit 3-sphere. It is convenient, as usual, to define $x = r^2$. From (5.5) we then find that the mass is given by

$$E = \lim_{x \to \infty} \frac{\pi g^2 x^2}{16 f \mathcal{H}^2} \left[(1 - f + xf' - 2x^2 f'')\mathcal{H}^2 + x^2(3f'\mathcal{H}\mathcal{H}' - 3f\mathcal{H}'^2 + 2f\mathcal{H}\mathcal{H}'') \right], \tag{5.7}$$

where primes denote derivatives with respect to x, and we have taken the five dimensional Newton's constant $G_5 = 1$.

Before applying this mass formula to the non-BPS bubbles that we have been investigating in this paper, it is instructive to consider some BPS bubble examples. For the spherically-symmetric 1-charge bubble constructed in [11], the metric functions in (5.6) are given by

$$
\mathcal{H} = H_1 \,, \qquad f = 1 + g^2 x H_1 \,, \qquad H_1 = \frac{\sqrt{g^4 x^2 + 2(g^2 q_1 + 1)g^2 x + 1} - 1}{g^2 x} \,, \qquad (5.8)
$$

where q_1 is the electric charge parameter. Substituting into (5.7), we find that the mass is given by

$$
E = \frac{\pi \, q_1}{4} \,. \qquad (5.9)
$$

For the more general case of 3-charge BPS bubbles, which were constructed in [14], we have

$$
\mathcal{H} = H_1 H_2 H_3 \,, \qquad f = 1 + g^2 x H_1 H_2 H_3 \,. \qquad (5.10)
$$

In this case the explicit solution for the three functions H_i is not known, but at large x they take the form

$$
H_1 = 1 + \frac{q_i}{x} + \cdots \,. \qquad (5.11)
$$

Substituting into (5.7), we find that the mass is given by

$$
E = \frac{\pi(q_1 + q_2 + q_3)}{4} \,. \qquad (5.12)
$$

As a further example, one finds from (5.7) that the 3-charge non-extremal black holes (2.4) in five-dimensional gauged supergravity have mass given by

$$
E = \frac{\pi m}{8} \sum_i \cosh 2\beta_i = \frac{\pi}{4} \left(\frac{3}{2} m + \sum_i q_i \right) \,. \qquad (5.13)
$$

In the limit when m is taken to zero, this black hole result reduces to that of the BPS bubble, (5.12).

In general, for a gravitational background parameterized by the asymptotic behavior

$$
\begin{aligned}
\mathcal{H} &\sim 1 + \frac{\overline{h}_1}{x} + \frac{\overline{h}_2}{2x^2} + \cdots \,, \\
f &\sim 1 + g^2 x \mathcal{H} + \frac{\overline{f}_1}{x} + \frac{\overline{f}_2}{2x^2} + \cdots \,,
\end{aligned} \qquad (5.14)
$$

we find that application of the mass formula (5.7) gives simply

$$
E = \frac{\pi}{8} \left(2\overline{h}_1 - 3\overline{f}_1 \right) \,. \qquad (5.15)
$$

This demonstrates that the mass indeed receives contributions only from the $1/x$ terms in the expansion of $\mathcal{H}$ and f, as alluded to at the end of the previous section. In reality, the

expansion in (5.14) may also include log terms. However, so long as the logs are confined to the $1/x^2$ and higher terms, the mass remains finite and unchanged from (5.15).

For the linearized bubbling AdS black hole solution, the metric functions $\mathcal{H} \equiv H_1$ and f are given by the backreaction expansions (4.2). As a result, we obtain for the mass

$$E = \frac{\pi}{4} \left[\frac{3}{2}m + q_1 + \lim_{x \to \infty} x \left(h_2(x) - \frac{3}{2} f_2(x) \right) \right]. \qquad (5.16)$$

Note that we have already assumed that the backreaction functions h_2 and f_2 vanish at infinity

$$h_2(\infty) = f_2(\infty) = 0. \qquad (5.17)$$

Based on the backreaction expansion (4.20), and taking into account the possible contribution of the homogeneous solutions in (4.19), we see that the masses of the bubbling AdS black holes remain finite, at least for arbitrary linearized deformations. Unfortunately, however, the local analysis leading to (4.20) is insufficient in itself to determine how the mass varies as the φ_1 deformation is turned on while keeping, say, temperature and R-charge fixed.

6. A numerical approach

As in much of the rest of this paper, our interest is in non-BPS bubbles. However, we also numerically explore some features of the BPS bubbles of [14].

6.1 Non-BPS bubbles

There are two types of AdS bubbles. The first type can be referred to as solitonic AdS bubbles, of which the BPS bubbles are a subset. The corresponding geometry is completely regular and horizon-free. It interpolates between AdS spacetime at asymptotic infinity and Minkowskian spacetime at short distance. The second type is the thermal AdS bubble, for which the geometry contains a singular point surrounded by a horizon. In the above sections, we have mainly looked at the linearized solution of a subset of thermal AdS bubbles, which we have referred to as bubbling AdS black holes. We can demonstrate the existence of both types of AdS bubbles through numerical analysis.

Solitonic AdS bubbles. Solitonic AdS bubbles have no horizon, and may be described by a global radial coordinate $x \in [0, \infty)$. Focusing on the one-charge spherically symmetric system, the Lagrangian (2.10) gives rise to a coupled set of non-linear ordinary differential equations for the metric functions f and H_1 as well as the matter fields X_1, φ_1 and A_t^1. In general, the equations of motion, including the Einstein equations, gives rise to five second-order equations (one for each function). However, there is also a first-order 'energy' or constraint equation arising from the Einstein equations. As a result, any general solution may essentially be specified by nine constants. Not all such solutions are physically independent, however. Even with the metric given in the form (4.1)

$$ds^2 = -H(x)^{-2/3} f(x) dt^2 + H(x)^{1/3} \left(\frac{dx^2}{4xf(x)} + x d\Omega_3^2 \right), \qquad (6.1)$$

there remains a residual coordinate transformation

$$t \rightarrow \lambda^{-1}t, \qquad x \rightarrow \lambda x, \qquad H \rightarrow \lambda^{-3}H, \qquad A_t^1 \rightarrow \lambda A_t^1, \tag{6.2}$$

leaving the form of the metric invariant. This reduces the nine constants to eight physical parameters of the solution.

Of course, most of these solutions may be singular as $x \rightarrow 0^+$. For solitonic AdS bubbles, we necessarily demand regularity at the origin, and hence may set up initial conditions by obtaining the Taylor expansion of the solution near $x \rightarrow 0^+$. For small x, up to linear order, this expansion is given by

$$H_1 = H_0 - \frac{1}{2}H_0^2 A_0^2 (\cosh^2 \varphi_1^0 - 1)x + \cdots,$$

$$X_1 = X_0 + \frac{1}{3}H_0^{\frac{1}{3}}(1 - \cosh \varphi_1^0 X_0^{\frac{3}{2}} + (\cosh^2 \varphi_1^0 - 1)X_0^3)x + \cdots,$$

$$\cosh \varphi_1 = \cosh \varphi_1^0 - \frac{1}{2}(\cosh^2 \varphi_1^0 - 1)H_0^{\frac{1}{3}}(A_0^2 \cosh \varphi_1^0 H_0^{\frac{2}{3}} + 2\sqrt{X_0} - \cosh \varphi_1^0 X_0^2)x + \cdots,$$

$$A_1 = A_0 + \frac{1}{2}A_0(\cosh^2 \varphi_1^0 - 1)X_0^2 H_0^{\frac{1}{3}} x + \cdots,$$

$$f = 1 + \frac{(\cosh^2 \varphi_1^0 - 1)A_0 H_0^2 X_0 + H_0^{\frac{4}{3}}(2 + 4X_0^{\frac{3}{2}} \cosh \varphi_1^0 + (\cosh^2 \varphi_1^0 - 1)X_0^3)}{6H_0 X_0}x + \cdots. \tag{6.3}$$

Thus, by imposing regularity, we see that the solution ends up being parameterized by only four constants, namely H_0, A_0, X_0 and φ_1^0. One of the parameters is trivial, owing to the residual symmetry (6.2). Note that $f_0 \equiv f(x = 0)$ is fixed to be unity by the equations of motion, and by regularity of the spatial slice for shrinking S^3.

To obtain the BPS bubble, we may impose the first-order condition $\cosh \varphi_1 = (xH_1)'$ given in (2.12). This reduces the four constants down to one according to

$$\cosh \varphi_1^0 = H_0, \qquad A_0 = \frac{1}{H_0}, \qquad X_0 = H_0^{-\frac{2}{3}}, \tag{6.4}$$

and precisely gives rise to the BPS bubbles that preserve $\frac{1}{2}$ supersymmetry, and with charge $q = H_0 - 1$. On the other hand, general solitonic (but non-BPS) bubbles may be obtained by relaxing any or all of the constraints in (6.4). While we do not explore this parameter space in detail, we can numerically demonstrate that non-BPS solitonic bubbles can also exist. For example, taking

$$H_0 = 8, \qquad A_0 = 1/8, \qquad X_0 = 1/4, \qquad \cosh \varphi_1^0 = 2 \tag{6.5}$$

gives rise to a smooth solution with φ_1 profile given in figure 1. Taking the scaling symmetry (6.2) into account, these non-BPS bubbles are parameterized by three physical parameters, presumably the R-charge, as well as two scalar 'charges' describing s-wave excitations of the X_1 and φ_1 scalars in AdS. These solutions can be viewed as continuous non-BPS deformations of the BPS AdS bubbles.
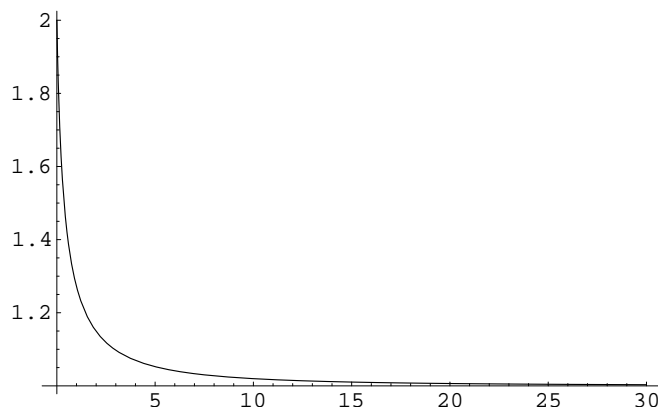
Figure 1: Plot of $\cosh\varphi_1$ as a function of x for a non-BPS soliton bubble with parameters given by (6.5).

Thermal AdS bubbles. We now turn to AdS bubbles with horizons. Since the horizon appears as a coordinate singularity, we restrict the numerical solution to cover the outside region $x \in [x_h, \infty)$. In this case, the generic regular solution near the horizon may be specified by six parameters, namely the location of the horizon $x = x_h$, the initial values of the functions $X_1(x_h)$, $H_1(x_h)$, $\cosh\varphi_1(x_h)$ and the slopes f_1 and a_1 of $f(x)$ and $A_1(x)$, which both much approach zero as $x \to x_h$. Note that, taking the scaling symmetry (6.2) into account, we may set x_h to a generic value (say $x_h = 1$). This indicates that the solutions may be specified by five physical parameters. However, we find it convenient to allow x_h to remain free in the numerical work.

If we let $\cosh\varphi_1 = 1$, then we turn off the bubble parameter and the solution reduces to that of a charged AdS black hole. It is worth remarking that the previously-known charged AdS black hole solution, in which the scalar X_1 does not have an independent charge parameter, is not the unique spherically symmetric black hole. Numerical analysis shows that the generic black hole is, in fact, characterized by three parameters: the mass, the R-charge and the scalar charge of X_1. The thermal AdS bubble solution, which also turns on φ_1, is then characterized by four parameters. (It remains unclear what the elusive fifth parameter is. Possibly it could be removed by yet another residual gauge transformation.) Using a numerical approach, we can demonstrate the existence of these solutions. In particular, we present an AdS bubble solution that can be viewed as a deformation of the previously-known AdS black hole. In this case, the initial conditions are specified by the following:

$$H_0 = 1 + \frac{q}{x_h}, \qquad X_0 = H_0^{-\frac{2}{3}}, \qquad f_1 = \frac{1 + q + 2x + h}{x_h}, \qquad a_1^2 = \frac{q(x_h + 1)}{(x_h + q)^3}. \qquad (6.6)$$

For $\cosh\varphi_1 = 1$, this leads precisely to the previously-known charged AdS black hole. We can turn on φ_1 and numerical analysis indicates that the solution exists provided that $1 \leq \cosh\varphi_1 < H_0$. In figure 2 we present a plot of $\cosh\varphi_1$ for $q = 10$, $x_h = 1$ and $\cosh\varphi_1^0 = 5$.
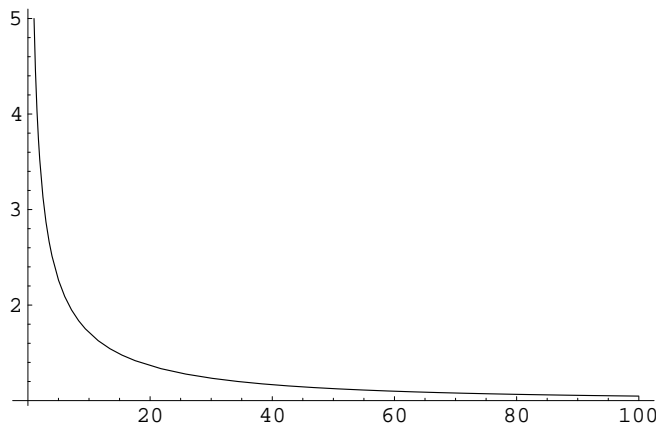
Figure 2: Plot of $\cosh\varphi_1$ as a function of x for a thermal bubble. We have taken $q = 10$, $x_h = 1$ and $\cosh\varphi_1^0 = 5$.

6.2 BPS bubbles

In [14], defining equations for multi-charge BPS bubbles were obtained in $D = 4, 5, 6$ and 7 AdS gauged supergravities. Those equations in general do not admit analytical solutions. Here we shall report that numerical analysis indicates that smooth bubble solutions exist in all of these cases. In all of these solutions, the coordinate x (which is denoted as R in [14]) runs from 0, where the metric is Minkowskian, to asymptotic AdS spacetime at $x = \infty$.

In $D = 5$, the functions H_i describing the general three-charge BPS bubble solutions satisfy the nonlinear equation (2.13), which may be rewritten as

$$\xi_i'' = -g^2 \frac{(\xi_i'^2 - 1)\xi_j\xi_k}{x^2 + \xi_1\xi_2\xi_3}, \tag{6.7}$$

where $i \neq j \neq k$, and where we have defined $\xi_i \equiv xH_i$. The one-charge case is given by $\xi_2 = \xi_3 = x$, while the two-charge case is given by $\xi_3 = x$.

Regularity of the BPS bubble demands that the ξ_i vanish linearly (or equivalently that the H_i approach constants) as $x \to 0$. Taking this into account, we may develop a Taylor expansion around $x = 0$:

$$\xi_i \equiv xH_i = a_i x - \frac{1}{2}a_j a_k(a_i^2 - 1)\,x^2 + \frac{1}{12}a_i(a_i^2 - 1)(8a_j a_k - a_j^2 - a_k^2)\,x^3 + \mathcal{O}(x^4). \tag{6.8}$$

To ensure $\cosh\varphi_i \geq 1$, it is necessary that $a_i \geq 1$. We verify numerically that smooth solutions exist for $a_i \geq 1$, and that the functions H_i behave as the following at asymptotic infinity:

$$H_i = 1 + \frac{q_i}{x} - \frac{(c_1^i)^2}{2x^2} + \cdots . \tag{6.9}$$

Here q_i is the charge parameter for the gauge fields A_i, while c_1^i is the coefficient of the normalizable scalar mode according to

$$\varphi_i = \frac{c_1^i + c_2^i \log x}{x} + \cdots \tag{6.10}$$

at infinity. In particular, the coefficient c_2^i of the non-normalizable mode always vanishes for these BPS bubbles.

The single-charge case admits an analytic solution of the form given in (2.15). This gives the exact relation $c_1 = \sqrt{q_1(q_1 + 2)}$ highlighted in (2.18). For the generic multi-charge cases, we have been unable to attain an analytical solution to (6.7). Nevertheless, we may highlight some of the general features of any such solution. To do so, it is convenient to perform the transformation $\xi_i = x + \eta_i$. The resulting differential equation is then

$$\eta_i'' = -\eta_i'(\eta_i' + 2)\left[x + \eta_i + \left(\frac{x}{x + \eta_j}\right)\left(\frac{x}{x + \eta_k}\right)\right]^{-1}. \tag{6.11}$$

The main purpose for introducing this transformation is to arrive at the asymptotic forms

$$\eta_i \sim (a_i - 1)x + \mathcal{O}(x^2) \quad \text{as} \quad x \to 0, \qquad \eta_i \sim q_i + \mathcal{O}\left(\frac{1}{x}\right) \quad \text{as} \quad x \to \infty. \tag{6.12}$$

Note also that it is only the last two terms in (6.11) which couple the equations for the three fields together.

The AdS vacuum solution is obtained by taking the trivial solution $\eta_i = 0$. Hence, for small deformations of AdS (corresponding to small changes $q_i \ll 1$), we may expect $\eta_i \approx 0$. More precisely, by assuming $\eta_i \ll x$, the above equation may be approximated by

$$\eta_i'' = -\frac{\eta_i'(\eta_i' + 2)}{x + 1}. \tag{6.13}$$

This approximation also has the feature that the three equations completely decouple in this limit. Furthermore, this now admits an exact solution consistent with (6.12):

$$\eta_i = \lambda_i \log \frac{1 + x/(1 - \lambda_i)}{1 + x/(1 + \lambda_i)}. \tag{6.14}$$

Of course, this is only consistent with our assumption $\eta_i \ll x$ for $\lambda_i \ll 1$. In this limit, we obtain the approximate solution

$$\eta_i \approx 2\lambda_i^2 \frac{x}{1 + x} \qquad (\lambda_i \ll 1). \tag{6.15}$$

Asymptotically, we read off

$$q_i = 2\lambda_i^2, \qquad c_1^i = 2\lambda \qquad (\lambda_i \ll 1), \tag{6.16}$$

in which case we have demonstrated that

$$c_1^i \sim \sqrt{2q_i} \quad \text{for} \quad q_i \to 0. \tag{6.17}$$

Physically, we see that since small deformations decouple from each other, we are allowed to turn on any independent combination of the three commuting R-charges (q_1, q_2 and q_3) as we wish, while maintaining the form of the 1/8 BPS bubble.

Larger deformations may be treated numerically. However, it is worth noting the general feature of (6.11): since η starts with a positive slope (forced by demanding $\cosh \varphi_i \geq 1$),

it will remain positive with $\eta_i'' \leq 0$. This means η is monotonic increasing with decreasing slope, and will asymptotically approach its value at infinity, $\eta_i(x \to \infty) = q_i$. The initial slope for η_i, which is $a_i - 1$, then determines how large the final charge q_i becomes; larger a_i gives larger q_i.

With this in mind, we may approximate (6.11) by noting that the expression $x/(x + \eta_i)$ is bounded to lie between 0 and 1. This expression starts at its minimum value $1/a_i$ when $x = 0$ and increases towards 1 as $x \to \infty$. We now consider the denominator in (6.11):

$$\text{den} = x + \eta_i + \left(\frac{x}{x + \eta_j} \right) \left(\frac{x}{x + \eta_k} \right). \tag{6.18}$$

For small x, this is dominated by the initial value of the last term, $1/(a_j a_k)$, while for large x it is dominated by the first term. This suggests that we may make the approximation

$$\text{den} \approx x + \eta_i + \frac{1}{a_j a_k}, \tag{6.19}$$

in which case the second-order equation

$$\eta_i'' = -\frac{\eta_i'(\eta_i' + 2)}{x + \eta_i + (a_j a_k)^{-1}} \tag{6.20}$$

admits a simple solution:

$$\eta_i = \sqrt{x^2 + 2(q_i + (a_j a_k)^{-1})x + (a_j a_k)^{-2}} - x - (a_j a_k)^{-1}. \tag{6.21}$$

This is, in fact, exact for the one-charge case (where we take, e.g. $a_2 = a_3 = 1$ and find $\eta_1 = \sqrt{x^2 + 2(q_1 + 1)x + 1} - x - 1$). Here we have chosen the constants of integration in accordance with the asymptotic form (6.12). The R-charge is simply q_i, and the scalar vevs are

$$c_1^i = \sqrt{q_i(q_i + 2(a_j a_k)^{-1})}. \tag{6.22}$$

For these expressions to be self-consistent, we examine the behavior of η_i as $x \to 0$:

$$\eta_i \sim q_i a_j a_k x + \mathcal{O}(x^2). \tag{6.23}$$

Comparing this with (6.12), we see that self-consistency demands

$$a_i = q_i a_j a_k + 1 \qquad (i \neq j \neq k), \tag{6.24}$$

which may also be expressed as

$$q_i = (a_i - 1) a_j^{-1} a_k^{-1}, \tag{6.25}$$

where the initial slopes must satisfy $a_i \geq 1$. This condition gives a range for the allowed values of the charges. For example, with three equal charges, the above reduces to

$$q = a^{-1}(1 - a^{-1}), \tag{6.26}$$

which has a maximum value $q_{\max} = 1/4$, which occurs when $a = 1/2$.

Of course, this restriction (6.25) is entirely contingent on the validity of the approximation (6.19). Numerically, we find that the above captures the qualitative behavior of the solutions, but that the actual restrictions on the charges is different. In particular, numerically we find two possibilities for the behavior of $q_i(a_i)$. For one-charge and two equal charge bubbles, q_i can be made arbitrarily large. However, for three equal charge bubbles, the actual limiting value of the charge is given by

$$q < q_{\max}, \qquad q_{\max} \approx 0.529. \tag{6.27}$$

This cutoff on the maximum amount of R-charge that can be supported by a three equal charge BPS bubble may have implications on the nature of the corresponding 1/8 BPS configuration in the dual super Yang-Mills theory.

For the multi-charge cases where one of the q_i can become large, the scalar vev expression (6.22) applies, and we find

$$\frac{c_1^i}{q_i} \to \begin{cases} \sqrt{2/q_i} & \text{for } q \to 0; \\ 1 & \text{for } q \to \infty. \end{cases} \tag{6.28}$$

Even in regions where q_i cannot become arbitrarily large, the dominant behavior is for c_1^i to approach q_i from above as q_i increases. Similar results can also be obtained for $D = 4$, 6 and 7 dimensions, given by

$$D = 4: \quad H_i = a_i - \frac{1}{6}(a_i^2 - 1)a_j a_k a_\ell \, x^2 + \frac{1}{3}a_i(a_i^2 - 1)a_j^2 a_k^2 a_\ell^2 \, x^3 + \mathcal{O}(x^4).$$
$$D = 6: \quad H_1 = a_i - \frac{9}{10}(a_i^2 - 1)a_j \, x^{\frac{2}{3}} + \mathcal{O}(x^{\frac{4}{3}}),$$
$$D = 7: \quad H_i = a_i - \frac{2}{3}(a_i^2 - 1)a_j \, x^{\frac{1}{2}} + \frac{1}{6}a_i(a_i^2 - 1)(4a_j^2 - 1)\, x + \mathcal{O}(x^{\frac{3}{2}}), \tag{6.29}$$

where the i, j, k indices are not equal when they arise within the same equation. Using this as initial data near $x = 0$, we can find numerically that smooth BPS bubbles exist in all cases for $a_i \geq 1$.

7. Conclusions

We have investigated the non-BPS analog of 'bubbling AdS' geometries in type IIB supergravity, corresponding to a special class of non-zero temperature LLM configurations. From the five-dimensional point of view, these are solutions of the STU model coupled to three additional scalars φ_i. These solutions can be considered as the bubbling generalizations of non-extremal AdS black holes, and have regular horizons. However, unlike the previously-known AdS black holes, the bubbling AdS geometries do not have a straightforward generalization away from the BPS limit. Thus, we have had to rely on various approximation methods, as well as numerical analysis. In particular, we have considered the linearized φ_i equations on the background of the non-extremal R-charged AdS black hole. The backreaction onto the other fields occurs at higher order. Even at the linearized

level, the equations do not have closed-form solutions. Specifically, the linearized φ_i equations can be mapped to the Heun equation, for which the general two-point connection problem remains unsolved.

Nevertheless, we can find approximate solutions to the Heun equation by matching solutions in two overlapping regions. This method is reliable for a certain regime provided that there is a large overlap. In our case, the relevant regime is $T \gg \mu_i$, where T and μ_i are the temperature and chemical potentials of the dual thermal Yang-Mills theory, respectively. If, in addition to this, we consider a high temperature limit for which $T \gg 1$, then the φ_i equations can be solved without matching. Corrections of the order μ_i/T and $1/T$ can then be considered via a perturbation approach. The complete φ_i solution is fixed by the boundary conditions at the horizon. Namely, we require that φ_i is regular at the horizon. For the most part, we focus on the case of one charge and a single additional scalar field φ_1.

The behavior of φ_1 at asymptotic infinity can be related to perturbations away from the UV superconformal fixed point of the Yang-Mills theory. In particular, the normalizable mode of φ_1 goes as $1/r^2$ and corresponds to the dimension two chiral primary operator $\text{Tr}\, Z_1^2$ getting a vev. The non-normalizable mode goes as $(\log r)/r^2$ and corresponds to a massive deformation of the field theory Lagrangian of the form $\text{Tr}\, Z_1^2$. We have normalized φ_1 such that the vev is fixed at the value it has for the BPS AdS bubble. Then we discuss how the thermal mass depends on the physical parameters of the field theory, namely the temperature, chemical potential and R-charge. We have considered both the grand canonical ensemble and the canonical ensemble. The fact that the $\text{Tr}\, Z_1^2$ term is only present in the Lagrangian for thermal field theories might indicate that there is a phase transition at zero temperature. Moreover, we find the thermal mass vanishes at the point of the Hawking-Page transition.

We have gone beyond the linear order in φ_1 to take into account the backreaction on the other fields, In particular, we have used this to obtain the asymptotic form of the corrections in order to read off the mass of the bubbling AdS black hole. It would be interesting to further consider the backreaction, or else use numerical analysis, to investigate the thermodynamics of bubbling AdS black holes. In particular, it would be interesting to find out how the local thermodynamic stability constraints, as well as the Hawking-Page transition, may be altered due to φ_i.

It would be useful to understand more concretely whether these bubbling AdS black hole solutions have any relation to the hyperstar solutions considered in [12]. There, it was suggested that the hyperstar background did not have a horizon because the coarse-graining was taken over only the half-BPS sector of the full Hilbert space of type IIB theory. Perhaps these bubbling AdS black holes are the result of including the non-BPS sector in the coarse-graining.

Finally, we have shown via numerical analysis that there are actually two types of non-BPS AdS bubbles. Thermal AdS bubbles, of which the bubbling AdS black holes are a subset, have an event horizon surrounding a singularity. On the other hand, solitonic AdS bubbles are completely regular and horizon-free. The latter type of non-BPS bubbles can be obtained from the BPS bubbles by smooth deformations, and therefore correspond

to non-supersymmetric deformations of the dual field theory. It would be interesting to investigate this further.

Acknowledgments

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