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Fairness and externalities

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We study equitable allocation of indivisible goods and money among agents with other-regarding preferences. First, we argue that Foley's (1967) equity test, i.e., the requirement that no agent prefers the allocation obtained by swapping her consumption with another agent, is suitable for our environment. Then we establish the existence of allocations passing this test for a general domain of preferences that accommodates prominent other-regarding preferences. Our results are relevant for equitable allocation among inequity-averse agents and in a domain with linear externalities that we introduce. Finally, we present conditions guaranteeing that these allocations are efficient.

Keywords. Equity, efficiency, other-regarding preferences, equal income competitive allocations, first welfare theorem.

JEL classification. C72, D63.

1. Introduction

The need to equitably and efficiently allocate commonly and symmetrically owned resources arises naturally in a market economy. Many economic activities that require agents' cooperation result in the division of assets, gains, losses, etc. Due to diversity of preferences, it is not trivial to determine what an equitable distribution is. Foley (1967) advanced in this endeavor and developed the following equity test: "Ask each person to imagine changing places with every other [...]. If no one is willing to change, the allocation is equitable."[1] An extensive literature has investigated the existence of allocations satisfying this property and its compatibility with efficiency (see Thomson 2010, for a survey). A common feature of these studies is that each agent's welfare is assumed to be independent of the consumption of the other agents. Under this assumption, Foley's test is referred to as the *no-envy test*, and can be simplified as the requirement that no agent prefers the consumption of any other agent to her own consumption (Varian 1974). Assuming externalities away entails a significant loss of generality. A growing literature in experimental economics has consistently documented human behavior that cannot be rationalized by self-regarding preferences, but can be rationalized by other-regarding

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[1] Kolm (1971) refers to an earlier formulation of this test by Tinbergen (1946).

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preferences (see Fehr and Schmidt 2001, for a survey). This experimental evidence validates our everyday observation of altruism, philanthropy, conspicuous consumption, and the unavoidable partially public nature of consumption. We address this issue and include other-regarding preferences in the study of equitable allocation of indivisible goods and money. First, we argue that Foley's equity test is suitable and meaningful in our environment. Then we establish the existence of allocations passing this test for a general domain of preferences that accommodates prominent other-regarding preferences. Our results are relevant for equitable allocation among inequity-averse agents (Fehr and Schmidt 1999) and in a domain with linear externalities that we introduce. Finally, we present conditions guaranteeing these allocations are efficient.

We focus on assignment problems with money. That is, there are n agents and n objects, each agent must receive one object, and individual consumption of money should add up to a given budget. Examples are the assignment of tasks and salaries among employees, the allocation of a bequest with indivisible goods, and the allocation of rooms and rent among roommates (Svensson 1983, Maskin 1987, Alkan et al. 1991). We depart from the assumption—universal in previous literature—that each agent's preferences are defined on her consumption space. We consider, instead, agents who have preferences on the set of possible allocations. Our basic assumptions on preferences are continuity and anonymity of externalities, i.e., the welfare of each agent is not affected by a permutation of the allotments among the other agents. Our results may not hold if these assumptions are not satisfied (Section 3).[2]

Foley's equity test is applicable in our environment. More importantly, the normative support that has been recognized for this test when applied to economies with self-regarding preferences is preserved in economies with anonymous other-regarding preferences. Essentially, it requires no interpersonal comparisons of utility, and implements the principles of symmetry (Varian 1974) and equal opportunity (Kolm 1971). In Section 2, we discuss these properties at length. Our analysis there gives us two further insights. First, we discover other situations, beyond equitable allocation of commonly owned resources, for which allocations passing Foley's test are desirable, e.g., the allocation of tasks and salary among employees. Second, it gives us a basis to evaluate alternative extensions of the no-envy test to economies with other-regarding preferences. We conclude that we should favor Foley's original formulation. Here we also observe that in our environment, the allocations passing Foley's test coincide with those obtained by an equal income market, which is an intuitively equitable institution (Varian 1976). Thus, by studying the existence and welfare properties of these allocations, we are also studying the existence and welfare properties of equal income market outcomes. We formalize the connection between the two approaches in Section 5.1. To avoid confusion with the multiple alternative equity tests that have been proposed in the literature, we refer to the allocations passing Foley's test as being noncontestable (on equity grounds).[3]

[2]Our model is motivated by other-regarding preferences exhibited by agents who face distributive scenarios with anonymous partners as in most laboratory experiments and charitable giving. In particular, these agents do not have cardinal information about the welfare of the other agents. Thus, we concentrate on consumption externalities.

[3]"Envy" and "inequity" are commonly understood as consumption externalities that can be captured in our model. Referring to the allocations that pass our test as "fair," "envy-free," or "equitable" could mislead

For simplicity of the presentation, we assume, for most of the paper, that consumptions of money are required to be nonnegative. This is also a realistic restriction in some environments as the allocation of tasks and salary.[4] Our basic existence result, Theorem 1, states that if the preferences of at least $n-1$ agents satisfy a requirement that we refer to as the "compensation assumption," noncontestable allocations exist. Obviously, if the available budget cannot compensate the agents for not receiving one object or the other, noncontestable allocations may not exist. This can happen for two reasons. First, the agents do not care enough about their private consumption of money compared to the object they receive. Second, even if agents care about money sufficiently, there might not be enough money. These are exactly the situations that our compensation assumption excludes.

We investigate the implications of our basic existence result for two domains of other-regarding preferences of interest. First, we consider inequity-averse agents, i.e., agents who not only care about their own consumption, but also care about the equitability of the profile of consumptions. Such an agent subjectively assesses equity of an allocation. Her ideal is equality of consumption value. She loses welfare when, ceteris paribus, her perception of the value of the consumption of others deviates from the value of her own consumption. Our domain is parametrized by the agent's value functions, a coefficient that captures the agent's aversion to inequity against herself, and a coefficient that captures aversion to inequity against others as in Fehr and Schmidt (1999). In our domain, an agent's perception of the value of the allotment of others may depend on her own allotment. In particular, this can accommodate an agent's endowment effect, i.e., her propensity to assign a different value to a bundle of goods when it is assigned to her, compared to when it is assigned to others (Kahneman et al. 1990). It turns out that when the endowment effect is bounded for inequity-averse agents and the budget is large enough, there are noncontestable allocations (Theorem 2): this implies that under a mild restriction on inequity aversion, the agents' subjective assessment of equity does not compromise the existence of allocations satisfying our objective equity criterion. Second, we introduce a domain of preferences for which externalities are a linear function of the consumptions of money of the other agents. Here, noncontestable allocations exist, for a sufficiently large budget, whenever externality coefficients are uniformly bounded so that the direct effect of $1 allocated to an agent is greater than its aggregate effect on the other agents (Theorem 4).

Besides symmetry and equal opportunity, another important dimension of distributive justice is economic efficiency (Moulin 2003). It turns out that noncontestability may be incompatible with efficiency (Example 2 in the Appendix). For this incompatibility to hold, there must be an aggregate imbalance between private and social

the reader into thinking that these allocations minimize certain externality that the agents internalize in their preferences. Additionally, referring to these allocations as "swap-proof" may suggest a strategic role of all the agents involved in the swap. We adopt the more neutral terminology of "noncontestability," which suggests the application of an objective test of equity (see Kolm 1971, 1995, for a related discussion; see Rawls 1971, for a discussion on why distributive justice is not the absence of envy, understood as the pain at the well being of others).

[4] Section 5.2 extends our results to environments where consumptions of money can be negative.

interests, however. Indeed, our results reveal that for a wide range of externalities, noncontestable allocations are always efficient; thus, our results can be interpreted as first welfare theorems for equal income competitive allocations. For inequity-averse agents, noncontestable allocations are efficient for an open set of parameters that depend on the bounds on the marginal value of money for the agents. This set grows monotonically and continuously when the bounds get closer for each agent. In the limit, where the marginal value of money is constant and equal for all value functions for each agent, there are no restrictions on aversion to inequity against the agent herself, and for large n, there are few restrictions on the aversion to inequity against others (Theorem 3). Finally, the conditions under which we guarantee that noncontestable allocations exist for linear externality preferences also guarantee that each noncontestable allocation is efficient (Theorem 4).

Our existence results and welfare analysis provide a solid foundation for a theory of equitable allocation of objects and money with other-regarding preferences. Here it is also relevant to ask about the incentives issues associated with the implementation of noncontestable and efficient allocations (cf. Tadenuma and Thomson 1995); about the manipulability of rules selecting these allocations (cf. Andersson et al. 2014); about the refinement of this set based on further normative and practical requirements (cf. Tadenuma and Thomson 1995); about the existence of systematic ways to select noncontestable and efficient allocations that implement the notion of solidarity requiring that when the budget increases, all agents are better off (cf. Alkan et al. 1991, Velez 2015). These are all open questions, which are beyond the scope of this paper and are only meaningful now that we have established general existence of noncontestable and efficient allocations for a rich domain of admissible other-regarding preferences.

Our existence results and welfare analysis generalize previous literature on existence of envy-free allocations for assignment problems with no externalities (Svensson 1983, Maskin 1987, Alkan et al. 1991). In particular, our inequity-averse domain contains the domain of preferences with no externalities considered in these previous studies.[5] At a technical level, our results are related to the literature on land division (Dubins and Spanier 1961). Part of our contribution is to show how the combinatorial approach to land division introduced by Su (1999) is suitable to handle other-regarding preferences in our model (Section 3).

Our work joins the efforts to solve the technical and conceptual challenges that arise in economic models when one accounts for the possibility that agents may care about the consumption of others, e.g., Gale (1984) for general equilibrium in a market with indivisibilities and price externalities; Nogushi and Zame (2006) for general equilibrium with a continuum of agents; Dufwenberg et al. (2011) for general equilibrium in classical economies; Brânzei et al. (2013) for cake cutting procedures; and Azrieli and Shmaya

[5] The domain of profiles with no externalities for which we guarantee existence of noncontestable allocations in Theorem 1 properly contains the domain in which Svensson (1983, 1987), Maskin (1987), and Alkan et al. (1991) prove their existence results. First, we do not assume that an additional small amount of money necessarily makes the agent better off. Moreover, in Theorems 1, 2, and 6, we are able to waive the compensation assumption requirement for one agent.

(2014) for the allocation of indivisible goods that can be shared by several agents. We discuss in detail the relevant papers to us, i.e., Gale (1984), Dufwenberg et al. (2011), and Azrieli and Shmaya (2014), in Sections 3 and 5.

The remainder of the paper is organized as follows. Section 2 introduces our model. Section 3 presents and discusses our basic existence result. Section 4 presents our welfare analysis and the application of our existence results in domains of other-regarding preferences of interest. Section 5 discusses the relevance of our results for general equilibrium theory and the extension of our results when one allows negative consumption of money and a different number of agents and objects. The Appendix collects all proofs.

2. Model

We consider the problem of allocating a social endowment of n objects $A \equiv \{\alpha, \beta, \ldots\}$ and an amount $M \in \mathbb{R}$ of an infinitely divisible good, which we refer to as money, among a group of n agents $N \equiv \{1, 2, \ldots, n\}$. We assume that each agent should receive a bundle of one object and an amount of money. Agent i's generic consumption is (x_i, μ_i), where $x_i \in \mathbb{R}$ and $\mu_i \in A$. An allocation is a pair $z \equiv (x, \mu)$, where $x \equiv (x_i)_{i \in N} \in \mathbb{R}^n$ and $\mu \equiv (\mu_i)_{i \in N} \in A^n$, for which no two agents receive the same object, i.e., for $i \neq j$, $\mu_i \neq \mu_j$; z is feasible if $\sum_{i \in N} x_i = M$.[6] Alternatively, we write $z \equiv (z_i)_{i \in N}$, where $z_i \equiv (x_i, \mu_i)$ is agent i's allotment at z. We denote the consumption of money of the agent who receives object α at z by x_α. We refer to x_α as the consumption of money associated with object α at z. We denote the set of allocations by Z, the set of feasible allocations by Z_M, and the set of feasible allocations at which each agent's consumption of money is nonnegative by Z_M^+. We endow Z with its natural box topology.[7] It will be convenient for our purposes to reshuffle consumptions at a given allocation. That is, given $z \equiv (z_i)_{i \in N} \in Z$ and permutation $\pi : N \to N$, the allocation z_π is that at which agent i receives bundle $z_{\pi(i)}$.

Agents have complete, transitive, and continuous preferences on Z.[8] Given our continuity assumption, we introduce, without loss of generality, preference representations. Each agent is endowed with a continuous utility function on Z. Agent i's utility is $u_i : Z \to \mathbb{R}$ and the utility profile is $u \equiv (u_i)_{i \in N}$. We assume that externalities are anonymous, i.e., an agent's welfare is not affected by reshuffling the consumption bundles of the other agents. Formally, for $i \in N$ and $z \in Z$, if $\pi : N \to N$ is a permutation such that $\pi(i) = i$, then $u_i(z) = u_i(z_\pi)$. We denote the set of utility functions by $\mathcal{U}$ and the set of profiles by $\mathcal{U}^n$.

We investigate the existence and welfare properties of equitable allocations. Our welfare analysis is standard. An allocation $z \in Z_M$ is efficient for u if there is no other feasible allocation that each agent weakly prefers to z and at least one agent strictly prefers to z.

[6] We assume budget balance, a realistic assumption in some environments, which simplifies our presentation. All of our results generalize if we assume free disposal of money, which may be relevant in some environments with negative externalities in which a Pareto improvement may be attained by burning money.

[7] A sequence of allocations $\{z^k\}_{k \in \mathbb{N}}$ converges as $k \to \infty$ to $z \equiv (x_i, \mu_i)_{i \in N}$ if it converges component-wise, i.e., for each $i \in N$, $x_i^k \to x_i$ and there is some $K \in N$ such that for each $k \geq K$, $\mu_i^k = \mu_i$.

[8] A preference is continuous if its weak upper and weak lower contour sets are closed in Z.

Definition 1. The allocation $z \in Z$ is *noncontestable on equity grounds for u* (noncontestable for short) if no agent prefers the allocation obtained by swapping her consumption with any other agent.

Our equity test was originally proposed by Foley (1967) and since Varian (1974) has been specialized to environments without externalities. In this restricted environment it has been a central equity criterion (Thomson 2010). In what follows, we argue that the normative support that has been recognized for the test when there are no externalities is preserved when there are anonymous externalities.

First, the test is simple and avoids controversial interpersonal comparisons of utility. Second, it filters out the allocations that exhibit the clear-cut bias that one identifies when an agent is better off when she exchanges her place with another agent (Varian 1974). Thus, it only rejects allocations that are unambiguously biased. Third, the allocations passing the test can be thought of as being the result of an equitable process that gives equal opportunity to the agents (Kolm 1971). Indeed, for an allocation z, let B be the set of bundles that are allotted to the agents at z. Imagine that one asks each agent to select out of the set B her preferred consumption bundles given that the final consumptions in the economy will be those in B. This is a meaningful question for the agents in our environment. With anonymous externalities, fixing the final consumptions in the economy, the allocations that give an agent the same consumption bundle are welfare equivalent. If z is noncontestable, each agent selects a set of bundles containing her consumption at z. Thus, one can think of z as the outcome of a process in which each agent is provided with the same opportunity set, B, from which she selects her preferred choice.[9]

There are alternative extensions of the no-envy test to economies with externalities (see Thomson 2010). The most relevant to us is a proposal by Kolm (1995) that requires that no agent prefer the allocation at which all agents receive the consumption of some other agent to the allocation at which all agents receive her own consumption, i.e., $z \equiv (z_1, \dots, z_n)$ is equitable if for each pair of agents i and j, $(z_i, \dots, z_i)\, R_i\, (z_j, \dots, z_j)$. In our model, this requires rankings of allocations outside the space in which preferences are defined. Moreover, even if one extends preferences in an appropriate way, the test does not implement equal opportunity. Thus, we favor Foley's original formulation.

An alternative approach to identify equitable allocations is to consider an intuitively equitable institution and the outcomes it produces. The most prominent such institution is an equal income market. Varian (1976) argues in favor of these allocations because they provide equal opportunity in a common set that is robust to changes in tastes and population. In our environment, the set of equal income competitive allocations and the set of noncontestable allocations coincide (see Section 5). Thus, by studying the existence and welfare properties of noncontestable allocations, we are also studying the existence and welfare properties of equal income competitive allocations.

[9]We are abstracting from incentives issues, which can be analyzed only after allocations satisfying our normative requirements are known to exist.

Noncontestable allocations are desirable in environments beyond those with collective and symmetric ownership. Consider, for instance, a manager who has to allocate tasks and compensation among employees of the same rank. First, a noncontestable allocation promotes harmony in the workplace, for no employee finds the allocation is unambiguously biased toward another employee. Second, since they implement equal opportunity, they allow the manager to document nondiscrimination in the workplace. This reduces the likelihood of a law suit.

Even without externalities, it is evident that noncontestable allocations may not exist if the budget is not enough to compensate agents for not receiving one object or the other. For instance, suppose that at least two agents care only about the object they receive and prefer the same object. No matter how the budget is distributed, at least one of these agents will prefer to swap her consumption with the agent who receives her preferred object. Thus, to guarantee existence of noncontestable allocations, one has to assume that some agents care about money and that the budget is enough to compensate them for not receiving one object or the other.

Definition 2. The utility function $u_i \in \mathcal{U}$ satisfies the *compensation assumption* if for each $z \in Z_M^+$ at which agent i receives a bundle with zero money, she weakly prefers to swap her consumption with some other agent who receives a positive consumption of money.[10]

The compensation assumption is a statement about the budget in relation to the agent's preferences. It requires that if the agent receives no money at an allocation, no matter how the budget is distributed among the other agents, she finds it attractive to swap her allotment with at least one agent who receives some money. The assumption also implicitly bounds the agent's externalities. For other-regarding preferences the requirement may not be satisfied for arbitrary large M even if the utility of the agent is large when, ceteris paribus, her consumption of money is large. Thus, for it to be satisfied, it is also necessary that, ceteris paribus, the direct effect of a large consumption of money by the agent dominates the externality on the other bundles (see our discussion of linear preferences leading up to Theorem 4).

We say that there are noncontestable allocations for $u \in \mathcal{U}^n$ when the budget is large enough if the following property is satisfied: there is M_u such that if $M \geq M_u$, there are noncontestable allocations for u in Z_M^+.

3. Existence

Noncontestable allocations exist when agents care about money and there is enough money to distribute.

Theorem 1. *Let $u \in \mathcal{U}^n$. If the utility functions of at least $n-1$ agents satisfy the compensation assumption, there are noncontestable allocations for u in Z_M^+.*

[10] Formally, for each $z \in Z_M^+$ such that $x_i = 0$, there is a permutation π that coincides with the identity function except for agents $\{i, j\}$ and such that $u_i(z_\pi) \geq u_i(z)$ and $x_{\pi(i)} > 0$.

Theorem 1 proves possible our equity requirement in a model with a more realistic description of the agents than the self-regarding preferences model. Since the result is stated for an abstract domain, it is not immediate to realize what its implications are for preference domains of interest. To do so, we show in Section 4 that our results apply for inequity-averse preferences (Fehr and Schmidt 1999) and for a general domain with linear externalities that we propose. We identify conditions on preferences in these domains guaranteeing they satisfy the compensation assumption when the budget is large enough.

The proof of Theorem 1 is based on a combinatorial approach to equitable land division introduced by Stromquist (1980) and Woodall (1980) and refined by Su (1999). Part of our contribution is to show how these problems and our model with other-regarding preferences are connected. Land division is concerned with the equitable allocation of a measurable space among some agents whose preferences are represented by nonatomic measures. The first results in this literature provided existence of allocations at which each agent finds her assignment at least as good as that of the other agents (Dubins and Spanier 1961). These results were unsatisfactory because assigned lots were unrestricted measurable sets, leaving the possibility that an agent would receive a countable union of small pieces of land. To solve this, the problem was redefined as the division of an interval, say $[0, 1]$, into n subintervals among n agents. This provided an equitable division of any bounded set in a Euclidean space by means of $n-1$ "cuts." This problem is equivalent to finding l in Δ^{n-1}, the standard simplex, at which each agent prefers a different subinterval in $\{[0, l_1], [l_1, l_1 + l_2], \dots, [l_1 + \cdots + l_{n-1}, 1]\}$. This is relevant for our model with externalities. The utility of an agent from the kth interval in $l \in \Delta^{n-1}$ depends on $l_1, \dots, l_k$, for the location of the interval can only be determined when the length of the previous intervals is known. Our proof, which is based on the direct application of Sperner's lemma introduced by Su (1999) to land division and chore allocation, reveals that the simplicial subdivision methods developed to solve this problem are suitable to handle anonymous consumption externalities in our model. Theorem 1, even when it is restricted to externality-free economies, is not a consequence of the previous results in this literature, however. In particular, it guarantees existence of noncontestable allocations for economies in which the preferences of one agent are not required to satisfy the compensation assumption.[11] Using the covering method developed by Woodall (1980), one can prove our theorem under the additional assumption that each agent is indifferent to swap any two consumption bundles that have no money. This is very restrictive in our environment. In a contemporary paper, Azrieli and Shmaya (2014) generalize the simplicial covering methods in Stromquist (1980) and Woodall (1980) to accommodate

[11] Our theorem states that the requirement of the compensation assumption can be waived for one agent. Intuitively, if the budget is not enough to compensate one agent for not receiving a certain object, that agent can be allocated that object; as long as one can compensate all other agents, it is always possible to find a noncontestable allocation (note that this does not say that one can construct a noncontestable allocation without knowing the preferences of one agent). Our compensation assumption requirement cannot be dropped for more than one agent. That is, for a given M and preferences of $n-2$ agents, one can easily find preferences for two additional agents such that the budget necessary for the existence of noncontestable allocations is greater than M.

problems where an object may be shared by several agents. Again, our results are independent. Their covering methods require boundedness assumptions for all agents, so our theorem is not a consequence of theirs.

If externalities are nonanonymous, the conclusion of Theorem 1 may not hold. The issue is that nonanonymous externalities may limit the extent to which money can reduce the desire of an agent to swap her consumption with another agent.

Example 1 (Nonanonymous externalities). Let $A \equiv \{\alpha, \beta, \gamma\}$ and $N \equiv \{1, 2, 3\}$. Agent 3 cares only about money, i.e., $u_3(z) \equiv x_3$. Preferences of the other agents are given by

$$u_1(z) \equiv \begin{cases} x_1 - 2 & \text{if } \mu_2 = \gamma \\ x_1 & \text{otherwise,} \end{cases} \qquad u_2(z) \equiv \begin{cases} x_2 + 1 & \text{if } \mu_2 = \gamma \\ x_2 & \text{otherwise.} \end{cases}$$

These preferences are continuous and satisfy the compensation assumption when $M \geq 2$. Preferences of agents 2 and 3 are externality-free. In fact, they are quasi-linear. Agent 1's preferences are nonanonymous, for her welfare depends on the object received by agent 2. There is no noncontestable allocation for this economy. To see this, suppose that $z \equiv (x, \mu)$ is noncontestable. Clearly, $x_3 \geq \max\{x_1, x_2\}$. Now, since agent 1 does not prefer to swap her consumption with agent 3, $x_1 \geq x_3$. Since $x_3 \geq x_1$, we have that $x_1 = x_3 \geq x_2$. Thus, $\mu_2 = \gamma$, for otherwise agent 2 would prefer to swap her consumption with the agent who receives object γ. Consequently, $x_2 \geq x_1 - 1$, for otherwise agent 2 would prefer to swap her consumption with agent 1. This implies that agent 1 prefers to swap her consumption with agent 2. Thus, z cannot be noncontestable for u.[12] ◊

4. Welfare analysis and economic domains

Noncontestability may be incompatible with efficiency. The issue here is that with other-regarding preferences, the utility of each member of a group of agents can increase by changing the profile of consumption of money among them, while their aggregate consumption of money decreases. Thus, even in an economy in which ceteris paribus each agent prefers more money and for which efficient allocations exist, each noncontestable allocation may be Pareto dominated by extracting money from some agents and giving it to the rest (Example 2 in the Appendix; see also our discussion leading up to Theorems 3 and 4).

This analysis suggests that the incompatibility of noncontestability and efficiency is the result of an aggregate imbalance between private and social interests. Indeed, when there are no externalities, noncontestable allocations of objects and money are always efficient (Svensson 1983). Thus, one can expect that for a reasonable range of externalities there is general compatibility of these two properties. We show that this is so in two economic domains that capture natural other-regarding preferences.

[12] Observe that our argument for nonexistence of noncontestable allocations is independent of budget M and whether consumption of money is required to be nonnegative.

4.1 *Inequity aversion*

An inequity-averse agent not only cares about her own consumption, but also about the equitability of the profile of consumptions. The following preferences capture these other-regarding preferences when each agent has equality of values as her own standard of equity. For each $z \equiv (x, \mu) \in Z$,

$$u_i(z) \equiv v^i_{\mu_i}(z_i) - \frac{a_i}{n-1} \sum_{j \in N} \max\{v^i_{\mu_i}(z_j) - v^i_{\mu_i}(z_i), 0\} - \frac{c_i}{n-1} \sum_{j \in N} \max\{v^i_{\mu_i}(z_i) - v^i_{\mu_i}(z_j), 0\},$$

where a_i and c_i are nonnegative constants such that $c_i < 1$ and $c_i \leq a_i$, and for each i and each $\alpha \in A$, $v^i_\alpha : \mathbb{R} \times A \to \mathbb{R}$ is a continuous, unbounded function that is strictly monotone with respect to the first component and represents the value that agent i assigns to the consumption bundles when receiving object α.[13]

This domain of preferences, which we denote by $\mathcal{F}$, generalizes a proposal by Fehr and Schmidt (1999) for the allocation of money.[14] Essentially, the agent takes her consumption as a reference to evaluate deviations from equality. Her ideal would be equality of values. The agent loses some welfare when other agents receive different consumption value than her. Ceteris paribus, she loses more welfare when the deviation is against her than when the deviation is against others ($c_i \leq a_i$). The requirement $c_i < 1$ guarantees that, ceteris paribus, the agent prefers more money.

The value that an inequity-averse agent assigns to the consumption of another agent may depend on the object she receives. This is realistic in environments in which receiving a certain object may give the agent a particular vantage point that affects her assessment of others' consumption. In particular, this can reflect the "endowment effect," i.e., an agent's propensity to assign a different value to a bundle of goods when it is assigned to her compared to when it is assigned to others (Kahneman et al. 1990). It turns out that when this effect has a limit, noncontestability is compatible with the agents' subjective assessment of equity. We say that the endowment effect is bounded for agent i if there is $K > 0$ such that for each pair of objects α and β and each $x \in \mathbb{R}$, $|v^i_\alpha(x, \alpha) - v^i_\beta(x, \alpha)| \leq K$.

Theorem 2. *Let $u \in \mathcal{F}^n$ and suppose the endowment effect is bounded for at least $n-1$ agents. If the budget is large enough, there are noncontestable allocations for u.*

The domain of quasi-linear inequity-averse preferences illustrates well the implications of Theorem 2. We say that an inequity-averse preference, $u_i \in \mathcal{F}$, is quasi-linear when each v^i_α has the form $v^i_\alpha(x_\beta, \beta) \equiv \nu^i_\alpha(\beta) + x_\beta$; here, $\nu^i_\alpha(\beta)$ is the monetary value that agent i assigns to receiving object β when she receives object α. Thus, the difference in the value that agent i assigns to consumption bundle (x_α, α) when she receives objects α and β is just the difference in the monetary value assigned to α in both situations, i.e., $\nu_\alpha(\alpha) - \nu_\beta(\alpha)$. An immediate consequence of Theorem 2 is that noncontestable allocations exist when inequity-averse preferences are quasi-linear.

[13] Note that the functions v^i_α are anonymous, i.e., do not depend on the identities of the other agents, only on their consumption.

[14] Characterizations of Fehr–Schmidt preferences are discussed by Neilson (2006), Sandbu (2008), Rohde (2010).

Corollary 1. *Let $u \in \mathcal{F}^n$ be quasi-linear. If the budget is large enough, there are noncontestable allocations for u.*

Noncontestable allocations for inequity-averse preferences may not be efficient when the equity concerns of the agents overwhelm their private interests. For this to be so, it must be the case that for some allocations it is possible to achieve a Pareto improvement by reassigning consumption of money without reshuffling objects. To see this, let $u \in \mathcal{F}^n$, $z \equiv (x, \mu)$ be a noncontestable allocation for u, and $z' \in Z_M$. Let z_π be the allocation obtained from z by permuting consumption bundles so each agent receives the object she receives at z'. Since z is noncontestable for u, no agent prefers z_π to z. Thus, z' can Pareto dominate z only if it Pareto dominates z_π.

There are essentially two worst-case scenarios in which an allocation can be Pareto dominated for $u \in \mathcal{F}^n$ by reassigning money. Suppose that the consumption value of each agent $i \neq n$ is greater than her perceived consumption value for the consumption of agent n. If each of the first $n-1$ agents gives one unit of money to agent n, there can be a Pareto improvement if for each of these agents the reduction in inequality of value induced by the change offsets its direct effect. The following assumption precludes that this is so.

Assumption F1. *The utility function $u_i \in \mathcal{F}$ is such that there are $\overline{m}_i > \underline{m}_i$ such that for each pair of objects α and β, the derivative of $v^i_\beta(\cdot, \alpha)$ is bounded below by $\underline{m}_i$ and bounded above by $\overline{m}_i$, and*[15,16]

$$\frac{c_i}{n-1}((n-1)\overline{m}_i + \underline{m}_i) < \underline{m}_i.$$

Now, suppose that the consumption value of each agent $i \neq n$ is lower than her perceived consumption value for the allotment of each agent $j \in N \setminus \{i, n\}$, and is greater than her perceived consumption value for the allotment of agent n. If each agent $i \neq n$ gives \$1 to agent n, a Pareto improvement can be achieved not only because the inequality against agent n is reduced, but also because the inequality against agent i may be reduced. There are three ways in which one can guarantee that this situation cannot compromise efficiency of a noncontestable allocation. The first is to require that it does not happen at all.

Assumption F2. *The utility function $u_i \in \mathcal{F}$ is such that there are bounds on the marginal value of money $\underline{m}_i$ and $\overline{m}_i$ as in Assumption F1 and*

$$\frac{c_i}{n-1}((n-1)\overline{m}_i + \underline{m}_i) + (n-2)\frac{a_i}{n-1}(\overline{m}_i - \underline{m}_i) < \underline{m}_i.$$

[15]A similar observation was made by Dufwenberg et al. (2011) for marginal effects of income in indirect internal utility functions for inequity-averse agents in a general equilibrium setting.

[16]We are implicitly assuming that the functions $v^i_\beta(\cdot, \alpha)$ are differentiable. This is only for simplicity in the presentation. The assumption can be replaced by the existence of $\underline{m}_i$ and $\overline{m}_i$ such that for each $x \in \mathbb{R}$ and each $\delta > 0$, $\underline{m}_i \leq (v^i_\beta(x+\delta, \alpha) - v^i_\beta(x, \alpha))/\delta \leq \overline{m}_i$.

Second, it is enough to assume that there is no endowment effect. This implies that no agent perceives inequity against herself at each noncontestable allocation. Let $z \equiv (x, \mu)$ be noncontestable for u, $z' \in Z$, and let z_π be the allocation obtained from z by permuting consumption bundles so that each agent receives the object she receives at z'. Let i be the agent whose consumption of money is reduced the most from z_π to z'. Suppose that agent i gains in utility by the reduction in inequity against her by reassigning money from z_π to z'. If this is so and there is no endowment effect for agent i, she must prefer z to z_π, for otherwise she would not perceive inequity against her at z_π. It turns out that agent i's loss from z to z_π is never compensated by the gain from z_π to z'.

Assumption F3. *The utility function $u_i \in \mathcal{F}$ is such that for each pair of objects α and β, $v^i_\alpha(\cdot, \alpha) = v^i_\beta(\cdot, \alpha)$.*

Finally, observe that Assumption F2 precludes worst-case scenarios that would not arise if the agent's marginal value of money of her own consumption is always greater than that of her perceived marginal value of money for the consumption of the other agents.

Assumption F4. *The utility function $u_i \in \mathcal{F}$ is such that for each pair of different objects α and β, the derivative of $v^i_\alpha(\cdot, \alpha)$ is bounded below and this bound is an upper bound of the derivative of $v^i_\alpha(\cdot, \beta)$.*

Theorem 3. *Let $u \in \mathcal{F}^n$. Suppose that for each u_i, one of the following items is satisfied: (i) Assumptions F1 and F3; (ii) Assumption F2; (iii) Assumptions F1 and F4. Then each noncontestable allocation for u, $z \in Z_M$, is efficient for u.*

If our assumptions in Theorem 3 are not satisfied, the result may not hold. In the Appendix, we construct $u \in \mathcal{F}^n$ for which noncontestable inefficient allocations exist, and such that each u_i satisfies Assumption F1 and violates Assumptions F2–F4 (Example 3). This example materializes the worst-case scenario described before Assumption F2.

Noncontestability and efficiency are compatible when inequity-averse preferences satisfy the assumptions of Theorems 2 and 3. These theorems reveal that, provided the lower bound on marginal value of money is positive for each agent, this always holds for an open set of parameters a_i and c_i that depends on the bounds on the marginal value of money. This set grows monotonically and continuously when the bounds get closer. In the limit, where the marginal value of money is always constant and equal for all value functions of each agent, there are no restrictions on a_is and for large n, almost all c_is are acceptable, i.e., $c_i < (n-1)/n$. Quasi-linear inequity-averse preferences are the preferences that reach this maximal compatibility of noncontestability and efficiency.[17]

Corollary 2. *Let $u \in \mathcal{F}^n$ be quasi-linear and such that for each $i \in N$, $c_i < (n-1)/n$. Then each noncontestable allocation for u, $z \in Z_M$, is efficient for u.*

[17] Quasi-linear preferences satisfy Assumptions F1, F2, and F4, and may violate Assumption F3.

4.2 *Linear externalities*

We introduce a domain of preferences in which the welfare of an agent is a linear function of the consumption of the other. Consider two agents, say i and j, who receive objects α and β at some allocation, respectively. The key assumption here is that the effect that agent j's consumption of money, x_j, has on agent i's welfare is a constant proportion of x_j. We denote this proportion by $c_i(\alpha, \beta)$: note that this proportion may depend on the identity of the agent whose welfare is affected, the object that this agent receives, and the object that the other agent receives, but not on her identity. In other words, this proportion is independent of the identity of agent j. We assume quasi-linearity with respect to the agent's own consumption of money, i.e., for each $z \equiv (x, \mu) \in Z$,

$$u_i(z) \equiv v^i(\mu_i) + x_i + \sum_{j \neq i} c_i(\mu_i, \mu_j) x_j,$$

where $c_i(\mu_i, \mu_j) \in [0, 1)$ is the fraction of each dollar assigned to agent j at z that agent i can use for her own benefit when she receives object μ_i and $v^i : A \to \mathbb{R}$. We denote the domain of these preferences by $\mathcal{L}$.

Our linear externalities model is a natural representation when objects are interconnected. Think, for instance, of fields that have to be irrigated (money in our model represents any infinitely divisible homogeneous good). Some proportion of water allocated to field α will end up serving field β. This proportion may depend on the distance between the fields and can be asymmetric, for one field may be higher than the other. Our domain can even accommodate situations in which this proportion depends on the agent who is assigned field α. This is a realistic assumption when agents may have differential access to the technology that allows them to use the resources allocated to others. This flexibility is also desirable in other environments. For instance, consider a group of n senior partners in a consulting firm who are responsible for the completion of a project; there are n tasks, each partner can assume responsibility for only one task, and the firm has allocated a budget M to complete the project. Each partner will receive a task and a share of the budget in order to assemble a team, buy equipment and software, pay travel expenses, etc. Partners and their respective teams do not work in isolation, for tasks are not completely independent. Software that team α purchases may also be of use to team β. A travel intensive task may have less spillovers on the tasks that require more computational power, however. Thus, how easy or difficult it is to guarantee the completion of a task for a partner depends on her own competence and experience for the task, her assigned budget, the budget assigned to each other task, and the ability of the partner to use the possible spillovers from the other tasks.

The compensation assumption may not be satisfied for any budget $M > 0$ for some linear externalities profiles. To illustrate this, assume that agent i's valuations of all objects are equal and normalized to zero. Suppose further that when agent i receives object α, the effect that the consumption of money of each other agent has on her welfare is a proportion greater than or equal to $1/(n-1)$. That is, $c_i(\alpha, \cdot) \geq 1/(n-1)$. Then her utility when receiving object α and no money will be greater than or equal to M. If for each $\beta \neq \alpha$, $c_i(\beta, \cdot) = 0$, her utility from receiving an object different from α and a budget share less than the entire budget will be less than M. Thus, no matter how large M is, there will be allocations at which she would not swap receiving α and no money for any other bundle. Fortunately, when we rule out this example, we guarantee that not only noncontestable allocations exist, but also they are efficient. We say that $c > 0$ strictly bounds externality coefficients for agent i if for each pair α and β, $c_i(\alpha, \beta) < c$.

Theorem 4. *Let $u \in \mathcal{L}^n$. Suppose that $1/(n-1)$ strictly bounds externality coefficients for all agents. If the budget is large enough, there are noncontestable allocations for u. Moreover, each noncontestable allocation for u, $z \in Z_M$, is efficient for u.*

It is worth noting that $1/(n-1)$ is the greatest uniform bound on externality coefficients guaranteeing that the direct effect of \$1 allocated to an agent is greater than the aggregate indirect effect on the other agents.

5. Discussion

5.1 *Noncontestability and the market*

An intuitive equitable institution is the market when each agent is given an equal share of the aggregate income (here we cannot endow agents with equal parts of the aggregate endowment, for some goods are indivisible). This institution operates as follows in our environment. There is a vector of prices $p \in \mathbb{R}^A$, in terms of money, for the indivisible goods. At these prices, the aggregate income in the economy is $I_p \equiv M + \sum_{\alpha \in A} p_\alpha$. Each agent receives an equal share of I_p and maximizes her utility given the prices and income. More precisely, each agent chooses her best bundle among the ones she can exactly afford, given that the final consumptions in the economy will be the bundles she can exactly afford. Since externalities are anonymous, this is a well defined utility maximization. Finally, the allocation is feasible.

Definition 3. The pair $(p, z) \in \mathbb{R}^A \times Z$ is an *equal income competitive equilibrium* for $u \in \mathcal{U}^n$, if for each $i \in N$,

(i) $z_i \in B(p) \equiv \{(I_p/n - p_\alpha, \alpha) : \alpha \in A\}$

(ii) $z \in \arg\max\{u(z') : z' \in Z$ and for each $j \in N$, $z'_j \in B(p)\}$

(iii) $z \in Z_M$.

Equal income competitive allocations are noncontestable. More interestingly, each noncontestable allocation is decentralizable in an equal income market.[18] That is, for each such allocation, say z, there is a price vector p such that (p, z) is an equal income competitive equilibrium.[19]

Given the equivalence of noncontestable allocations and the equal income competitive allocations, it is evident that Theorem 1 and its applications, i.e., Theorems 2, 4,

[18]The connection between noncontestability and equal income competitive allocations when there are indivisibilities and no externalities was first noticed by Svensson (1983).

[19]Take p such that for each $\alpha \in A$, $p_\alpha = \max_{\delta \in A} x_\delta - x_\alpha$.

5, and 6 (below), provide conditions under which equal income competitive allocations exist. Conversely, one can ask whether previous literature providing existence of competitive equilibrium with externalities imply Theorem 1. To the extent of our knowledge, the only two relevant papers here are Gale (1984) and Dufwenberg et al. (2011). None of these works implies Theorem 1. Gale (1984) shows existence of competitive equilibria in an environment with indivisible goods were demand correspondences, the primitive of his model, may depend on the vector of nominal prices. To guarantee existence of competitive equilibria, Gale assumes that if the price of an object is large enough, the object is undesirable compared to a free object. If this assumption is satisfied, one cannot guarantee, for any budget, that prices are bounded by income in an equal income market. Dufwenberg et al. (2011) study conditions under which the market behavior of an agent with other-regarding preferences is indistinguishable from an agent with self-regarding preferences. Under these conditions standard existence results apply. In contrast, all of our results apply to economies in which externalities may affect the behavior of agents in the market. That is, if bundles (x_α, α) and (x_β, β) are both affordable to an agent at prices p and p', an agent with preferences in $\mathcal{U}$ may demand (x_α, α) at prices p and (x_β, β) at prices p', depending on the effect that the other consumptions have on the welfare of the agent.

Our results in Section 4 are also relevant for general equilibrium analysis. Example 2 in the Appendix implies that there can be situations in which no equal income competitive equilibrium is efficient. This impossibility is not the end of the road, however. Section 4 uncovers natural domains of other-regarding preferences in which the first welfare theorem for equal income competitive allocations holds, i.e., domains in which each equal income competitive outcome is efficient. These are the first results to uncover such conditions. The closest result in previous literature is Dufwenberg et al. (2011, Theorem 5), which provides conditions for a classical inequity-averse economy, akin to our Theorem 3, under which a competitive equilibrium cannot be Pareto dominated by an income redistribution at market prices. Our results are independent. Unlike receiving more income in a classical economy, receiving more money than the agent who previously received a certain object does not guarantee a welfare gain to an agent. This requires that we develop alternative bounds for an agent's welfare loss when there is a reallocation of resources.

5.2 *Negative consumption of money*

Our results are relevant for the existence of noncontestable allocations when consumptions of money can be negative.[20] Our welfare analysis already applies to this case. In some circumstances, agents may have initial individual budgets that allow them to compensate the other agents so as to get one object or the other. For instance, in the allocation of tasks and budget shares among senior partners in a consulting firm (Section 4.2), a partner can accept to give part of her division's budget to other divisions that will perform more demanding tasks. One needs also to consider negative consumptions

[20]This problem has been of independent interest in the literature, e.g., Alkan et al. (1991).

of money when the budget is negative, as in the allocations of the rooms and the division of the rent among roommates.

Theorem 1 implicitly provides conditions for the existence of noncontestable allocations when consumptions of money can be negative. The direct implication of this theorem is that for a fixed preference profile, if the budget is large enough, then there are noncontestable allocations at which each agent gets a nonnegative consumption of money. However, our choice of zero as minimal consumption of money here can be easily modified. Instead we can require consumptions of money to be at least some lower bound b and modify the compensation assumption accordingly.

Definition 4. The utility function $u_i \in \mathcal{U}$ satisfies the *compensation assumption at (consumption reference)* b if for each $z \in Z_M$ at which each agent's consumption of money is at least b and $x_i = b$, agent i weakly prefers to swap her consumption with some other agent who receives a consumption of money greater than b.

For a fixed budget and a fixed preference profile u, if the compensation assumption is eventually satisfied for b low enough, there are noncontestable allocations for u.[21] (We omit the proof of the following result, which is a straightforward modification of our proof of Theorem 1.)

Theorem 5. *Let $u \in \mathcal{U}^n$. If the utility functions of at least $n-1$ agents satisfy the compensation assumption at b, then there is $(x, \mu) \in Z_M$ that is noncontestable for u and such that for each $i \in N$, $x_i \geq b$.*

The same conditions that guarantee existence of noncontestable allocations when the budget is large enough in Theorems 2 and 4 guarantee that these allocations exist if consumptions of money can be negative.

Theorem 6. *Let $M \in \mathbb{R}$. There are noncontestable allocations in Z_M for each $u \in \mathcal{U}^n$ satisfying any of the following statements:*

(i) The profile $u \in \mathcal{F}^n$ and the endowment effect is bounded for at least $n-1$ agents.

(ii) The profile $u \in \mathcal{L}^n$ and $1/(n-1)$ strictly bounds externality coefficients for at least $n-1$ agents.

5.3 *Different numbers of agents and objects*

One can easily extend Theorem 1 to an environment were the numbers of agents and objects may differ. Our welfare analysis is not easily extended when there are more objects than agents, however. If there are no consumption externalities and $|A| > n$, one

[21] Another variant of Theorem 1 can be easily obtained by considering different minimal consumptions of money for each object, as in the allocation of tasks and salary with task-dependent minimal salary constraints.

can prove the existence of noncontestable efficient allocations by introducing $|A| - n$ agents who only care about money. If z is a noncontestable allocation for the extended economy, the restriction of z to N is efficient (Alkan et al. 1991). Moreover, one can guarantee that for an arbitrary budget M, there will be an extended economy for which the aggregate consumption of the agents in N is M (Velez 2015). A modified version of Example 1 shows that this approach fails in our environment with externalities. Consider the economy with objects $\{\alpha, \beta, \gamma\}$ and agents $\{1, 2\}$ as in the example. Note that their preferences are anonymous (here we interpret agent 1's preference as being $u_1(z) = x_1$ if no other agent receives γ and $u_1(z) = x_1 - 2$ if another agent receives γ). However, the economy in which we introduce a fictitious agent is our example for which there are no noncontestable allocations.

It is an open question to obtain conditions under which there are noncontestable and efficient allocations when there are more objects than agents.

Appendix

Proof of Theorem 1. Let $u \in \mathcal{U}^n$ be such that the utilities of agents $1, \ldots, n-1$ satisfy the compensation assumption. We assume without loss of generality that $M = 1$. We will construct a sequence of allocations that converges and its limit is a noncontestable allocation. A nonnegative distribution of money is a vector $x \equiv (x_\alpha)_{\alpha \in A}$ such that $x \geq 0$ and $\sum_{\alpha \in A} x_\alpha = 1$. Note that a distribution of money is a vector indexed by objects. Moreover, the space of distributions of money is isomorphic to the standard simplex Δ^{n-1} (one can think of the set of objects as being $\{\alpha_1, \ldots, \alpha_n\}$ and $x \in \Delta^{n-1}$ being the distribution of money in which for each $i = 1, \ldots, n$, $x_{\alpha_i} = x_i$). Thus, for simplicity we will refer to the simplex of distributions of money Δ^{n-1}. We will say that allocation $z \equiv (y, \mu)$ has distribution of money $x \equiv (x_\alpha)_{\alpha \in A}$ whenever for each $i \in N$, her consumption of money at z, i.e., y_i, is equal to x_{μ_i}. Observe that all the allocations with a given distribution of money x and at which agent i receives object α are obtained by permuting the consumptions of money among $N \setminus \{i\}$. Thus, since we assume anonymity of externalities, the set of allocations with distribution x at which agent i receives a given object is an indifference class for agent i.

To construct the elements of our sequence of allocations, we will make use of the so-called barycentric triangulations of Δ^{n-1} (see Vick 1994, for details). A triangulation of Δ^{n-1} is a collection of simplices with the same dimension as Δ^{n-1} that cover Δ^{n-1}, with the property that if two simplices intersect, it has to be on a common face (a face of a simplex is the convex hull of a subset of its extremes, the simplex itself being the largest face; a facet is a face generated by $n-1$ vertices). The first barycentric triangulation of Δ^{n-1}, which we denote by T_1, is defined inductively over n as follows. When $n = 1$, i.e., the simplex is a single point, T is the trivial covering $\{\Delta^{n-1}\}$. Now, suppose that the barycentric triangulations of Δ^k have been defined for $k = 1, \ldots, n-2$. Then the barycentric triangulation of Δ^{n-1} is the collection of sets conv.hull(Δ, b), where Δ runs over all simplices in the first barycentric triangulations of the facets of Δ^{n-1} and b is the barycenter of Δ^{n-1}. Figure 1(a) shows the first barycentric triangulation when $n = 3$. For a triangulation T, let $V(T)$ be the set of all the vertices of simplices in T.

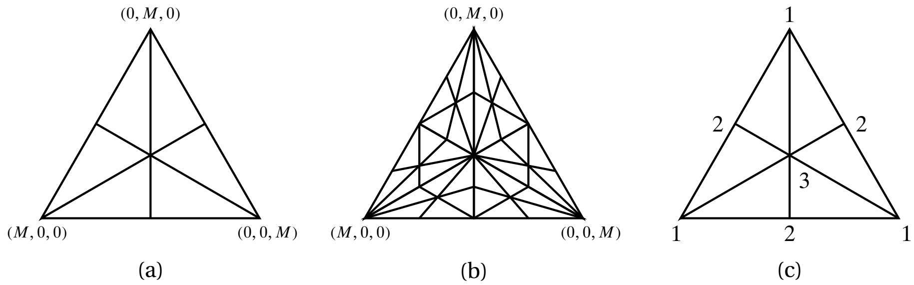

FIGURE 1. (a) The first barycentric triangulation of Δ^2. (b) The second barycentric triangulation of Δ^2. (c) The agent-labeling of the vertices of the second barycentric triangulation of Δ^2. The barycenters of the faces generated by one vertex of Δ^2 are labeled 1; the barycenters of the faces generated by two vertices of Δ^2 are labeled 2; the barycenters of the faces generated by three vertices of Δ^2 are labeled 3.

Notice that the vertices of T_1 are the barycenters of all the faces of Δ^{n-1}. Thus, one can partition these vertices in those that are the barycenters of the faces generated by one vertex of Δ^{n-1}, the barycenters of the faces generated by two vertices of Δ^{n-1}, and so on.

The subsequent barycentric triangulations of Δ^{n-1} are obtained recursively as follows. Suppose that $T_1, \dots, T_k$ have been defined. Then T_{k+1} is the collection of all the covering subsimplices of the first barycentric triangulation of all simplices in T_k (this is, T_k contains all first barycentric triangulations of the triangles in T_{k-1}). Figure 1(b) shows the second barycentric triangulation, i.e., T_2, when $n = 3$. If we denote $T_0 \equiv \{\Delta^{n-1}\}$, for each $k > 1$, $V(T_k)$ is the set of all barycenters of the faces of all subsimplices in T_{k-1}.

An agent-labeling of a triangulation T of Δ^{n-1} is a function that assigns an agent to each vertex of T, i.e., $L : V(T) \to \{1, \dots, n\}$. We define an agent-labeling L_k for the kth barycentric triangulation of Δ^{n-1} as follows: assign to agent 1 the vertices that are the barycenters of the faces generated by one vertex of some $\Delta \in T_{k-1}$; assign to agent 2 the vertices that are the barycenters of the faces generated by two vertices of some $\Delta \in T_{k-1}$; and so on. Since any two subsimplices in T_{k-1} can intersect only in a common face and representation inside a simplex is unique, L_k is well defined. Notice that agent n gets assigned only interior vertices. Moreover, each subsimplex in T_k is fully labeled, i.e., each of its extremes is assigned to a different agent.

We construct the kth element of our sequence of allocations. Recall that for a fixed $x \in \Delta^{n-1}$, the allocations with distribution of money x at which an agent receives a given object are an indifference class for the agent. For $x \in V(T_k)$ ask agent $L_k(x)$, "What are your preferred indifference classes from those associated with the objects when the distribution of money is x?" The answers to this question can be recorded as the objects that are received by the agent in the maximal indifference classes. We define an object-labeling, i.e., a function $O_k : V(T_k) \to A$, that selects for each $x \in V(T_k)$, one of the answers of agent $L_k(x)$, breaking ties in favor of objects with positive amount of money at x. Since the utility of agents $1, \dots, n-1$ satisfy the compensation assumption, and agent n only L_k-labels interior points of Δ^{n-1}, we are able to select as an answer an object, say α, that receives a positive amount of money at x, i.e., $x_\alpha > 0$. Thus, the object-labeling O_k is a Sperner's labeling, i.e., the O_k-label of each vertex of Δ^{n-1} is different and for each $x \in V(T_k)$, $O_k(x)$ is one of the labels of the vertices of Δ^{n-1} that generate x. By a well known result in combinatorics (Sperner's lemma), for each Sperner's labeling of T_k, there is an odd number of fully labeled simplices in T_k (see Border 1999, McLennan and Tourky 2008, Su 1999, for the statement and elemental proofs of Sperner's lemma). Thus, there is at least a subsimplex in T_k that is fully O_k-labeled. We define the kth element of the sequence of allocations as follows. Take $\Delta \in T_k$ that is fully O_k-labeled and let $x \in \Delta$. From the bundles $\{(x_\alpha, \alpha)\}_{\alpha \in A}$, assign to agent $i \in N$ the bundle that contains her answer to the question for the vertex of Δ whose L_k-label is i. Since each agent was assigned one of the extremes of Δ and since Δ is fully O_k-labeled, this defines a feasible allocation, z^k.

Since there is a finite number of objects, we can select a subsequence of $\{z^k\}_{k\in\mathbb{N}}$ in which each agent receives the same object.[22] Since Δ^{n-1} is a compact set, we can select a subsequence whose distributions converge to a limit x^*. Thus, this subsequence of allocations converges to an allocation z^* with distribution of money x^* and in which each agent receives the object she receives in each element of the sequence. Since the mesh sizes of the barycentric triangulations vanish as $k \to \infty$ (Vick 1994), for each agent and each neighborhood of the limit allocation, there is an allocation at which the agent receives the object in the limit allocation and at which the agent would not prefer to swap her consumption with any other agent.[23] By continuity of preferences, no agent prefers to swap her consumption with any other agent and the limit allocation is noncontestable. □

Example 2 ($u \in \mathcal{U}^n$ satisfying the compensation assumption for each agent, such that the set of efficient allocations for u is nonempty, but no noncontestable allocation for u is efficient for u). We assume that the budget is zero and drop the nonnegativity requirement in consumptions of money; u can be easily shifted to accommodate nonnegative consumptions. We have $A \equiv \{\alpha, \beta, \gamma\}$ and $N \equiv \{1, 2, 3\}$. For each $i \in N$ and each $z \equiv (x, \mu) \in Z_M$,

$$u_i(z) \equiv \begin{cases} x_i + f(x_\beta) & \text{if } \mu_i = \alpha \\ f(x_\alpha) + x_i & \text{if } \mu_i = \beta \\ x_i & \text{if } \mu_i = \gamma, \end{cases} \qquad \text{where } f(a) \equiv \begin{cases} 0 & \text{if } a \in]-\infty, -2] \\ a+2 & \text{if } a \in]-2, -1] \\ -a & \text{if } a \in]-1, 0] \\ 0 & \text{if } a \in]0, +\infty[. \end{cases}$$

[22]One can construct this sequence by refining $\{z^k\}_{k\in\mathbb{N}}$ agent by agent. More precisely, since there is a finite number of objects, there has to be an object that agent 1 receives in infinitely many elements of the sequence. Thus, one can pass to a subsequence in which agent 1 receives the same object. From here one can refine it further so agent 2 receives the same object, and so on.

[23]Let $r > 0$ and consider the open ball in Δ^{n-1} centered at x^* with radius $\frac{1}{2}r$. For k large enough, x^k, i.e., the distribution of money in z^k, is $\frac{1}{2}r$-close to x^* because as $k \to \infty$, $x^k \to x^*$. Now, for k large enough, the mesh size of T_k, i.e., the maximum diameter of a subsimplex, is less than $\frac{1}{2}r$. Let k be large enough so these two conditions are satisfied and let Δ_k be the subsimplex that defines z^k. Consider an agent, say $i \in N$. This agent labels one of the vertices of Δ_k in labeling L_k. Let $v^{i,k}$ be this vertex. Suppose that the object-labeling of this vertex is α. That means that agent i receives object α at both z^k and z. Moreover, among all allocations with distribution of money $v^{i,k}$, agent i's preferred allocations are those at which she receives α. Since the diameter of Δ_k is at most $\frac{1}{2}r$, then $v^{i,k}$ is r close to x^*.

One can easily see that u_i satisfies the compensation assumption for $M \geq 0$. Let $z \equiv (x, \mu) \in Z_M$ be the allocation $z_1 \equiv (-4, \alpha)$, $z_2 \equiv (-4, \beta)$, and $z_3 \equiv (8, \gamma)$. We claim that z is efficient for u. Let $z' = (x', \mu') \in Z_M$ be such that $z' \, R \, z$. Then $x'_3 \geq 8$. Feasibility implies that another agent gets at most -4. Suppose that $x'_2 \leq -4$. We claim that $x'_2 = -4$. Clearly if $\mu_2 = \gamma$, $x_2 = -4$. Suppose that $\mu_2 = \alpha$. The only way that agent 2 can get a utility boost from the consumption of the agent who gets β is that $x_\beta \in [-2, 0]$. But if $x_\beta \in [-2, 0]$, $x_2 \leq -6$ and $u_2(z') < -4$. Thus, $x_2 = -4$. The problem is symmetric when $\mu_2 = \beta$. The same argument shows that $x_1 = -4$. Then $x' = x$. Thus, $z \, R \, z'$. Now, at each noncontestable allocation for u, consumptions of money have to be zero. If there is a consumption that is positive, the consumption of the agent who receives γ has to be positive. The utility of the agent who receives the minimal consumption of money is at most zero. Thus, that agent will prefer to swap her consumption with the agent who receives γ. If consumptions of money are all zero, there is a Pareto improvement by taking $\varepsilon \in (0, 1)$ from each agent who receives α and β, and giving 2ε to the agent who receives γ. ◊

Example 3 ($u \in \mathcal{F}^n$ and $z \equiv (x, \mu) \in Z_M$ that is noncontestable for u such that u satisfies Assumption F1, u violates Assumption F2 with the minimal slack possible, u also violates Assumptions F3 and F4, the set of efficient allocations for u is nonempty, and z is not efficient for u). Our example has $n \geq 3$ agents and the budget is $n-1$. For simplicity we label objects $A \equiv \{1, 2, \dots, n\}$. Let z be the allocation at which agent i receives object i, agents $1, \dots, n-1$ receive \$1 each, and agent n receives \$0. We construct preferences such that the money derivative of $v^i_\beta(\cdot, \alpha)$ is bounded below by 1 and above by K, where $K \in (1, 2)$. We only make requirements for each i on the functions $v^i_i(\cdot, j)$ with $j \in A$, i.e., the value function for agent i when she receives object i. One can define $v^i_j(\cdot, \cdot)$ for $j \neq i$ such that the allocation is noncontestable for u. This requires that we violate Assumption F3.[24] For each agent $i = 1, \dots, n-1$, $c_i \equiv 0$ and $a_i \equiv (n-1)/((n-2)(K-1))$. Thus, Assumption F2 will be violated with the minimal slack for these agents. For agent n, $c_n = a_n \equiv 0$. We specify preferences for agent 1 such that if agent 1 gives up $1/n$ in consumption of money, each agent from $\{2, \dots, n-1\}$ does the same, and agent n receives the $(n-1)/n$ collected from agents $1, \dots, n-1$, agent 1 is not worse off when there is no reshuffling of objects. By symmetry, one can specify preferences for agents $2, \dots, n-1$ such that each of them is not worse off with the change. Since money is desirable for agent n, she is better off. Thus, our allocation is not efficient for u. Our preference for agent 1 is such that for each $j \neq 1$, $v^1_1(1, 1) = v^1_1(1 - (1 - 1/K)/n, j) \equiv 0$, $v^1_1(1 - 1/n, 1) = v^1_1(1 - 1/n, j) \equiv -1/n$, and $v^1_1(1, j) \equiv K(1 - 1/K)/n$. One can complete the definition of this functions such that the derivative of $v^1_1(\cdot, 1)$ is 1 and the derivative of each $v^1_1(\cdot, j)$ with $j \neq 1$ is K (since our example holds for arbitrary $K > 1$, one can say that u violates Assumption F4 for an arbitrarily small margin).[25] Now, agent 1 is not

[24] As part of the proof of Theorem 3 we prove that if Assumption F3 is satisfied, z is noncontestable for u only if for each i and j, $v^i_i(z_i) \geq v^i_i(z_j)$.

[25] The only restrictions that our assumptions impose are $[v^1_1(1, 1) - v^1_1(1 - 1/n, 1)]/[1/n] = 1$ for each $j \neq i$, $[v^1_1(1, j) - v^1_1(1 - (1 - 1/K)/n, j)]/[(1 - 1/K)/n] = K$, and $[v^1_1(1 - (1 - 1/K)/n, j) - v^1_1(1 - 1/n, j)]/[1 - (1 - 1/K)/n - (1 - 1/n)] = K$.

worse off with the redistribution of money whenever

$$v_1^1(1-1/n,1) \geq v_1^1(1,1) - \frac{a_1}{n-1}\Bigg(\sum_{j=2,\dots,n-1} v_1^1(1,j) - v_1^1(1,1)\Bigg).$$

This inequality holds with equality when $a_1 \equiv (n-1)/((n-2)(K-1))$.

Finally, note that since consumption externalities are negative for each $u \in \mathcal{F}$, there is a level of utility $\overline{u}$ that cannot be surpassed by each agent simultaneously at an allocation in $z \in Z_M$. By continuity, the set $\arg\max_{z\in Z_M} \min_{i\in N} u_i(z)$ is nonempty; each allocation that, for some order, lexicographically maximizes the utility levels of the agents in this set is efficient for u. $\diamondsuit$

Proof of Theorem 2. Let $K > 0$ and $u_i \in \mathcal{F}$ be such that for each pair α and β, and each $x \in \mathbb{R}$, $|v_\alpha^i(x,\alpha) - v_\beta^i(x,\alpha)| \leq K$. We will prove a general result that will also help us prove Theorem 6. Let $b \in \mathbb{R}$ be such that $M > nb$ and suppose that for each pair $\{\alpha,\beta\} \subseteq A$, $v_\beta^i(M/n,\beta) - v_\alpha^i(b,\alpha) > K$. Let $z \equiv (x,\mu) \in Z_M$, and suppose that $x_i = b$ and for each $j \neq i$, $x_j \geq b$. Since $\sum_{j\in N} x_j = M$, there is $j \in N$ such that $x_j \geq M/n > b$. Since value functions are increasing in money, then for each pair $\{\alpha,\beta\} \subseteq A$, $v_\beta^i(x_\beta,\beta) - v_\alpha^i(b,\alpha) > K$. Let $\hat{z}$ be the allocation obtained by swapping the consumption bundles of agents i and j at z. To simplify notation we denote $\beta = \mu_j$ and $\alpha = \mu_i$. Let $B \equiv \{\gamma \in A : v_\beta^i(x_\beta,\beta) > v_\beta^i(x_\gamma,\gamma)\}$ and $C \equiv \{\gamma \in A : v_\alpha^i(x_\alpha,\alpha) > v_\alpha^i(x_\gamma,\gamma)\}$. We obtain conditions guaranteeing $u_i(\hat{z}) > u_i(z)$, i.e.,

$$\begin{aligned} v_\beta^i(x_\beta,\beta) - \frac{c_i}{n-1}\sum_{\gamma\in B} v_\beta^i(x_\beta,\beta) - v_\beta^i(x_\gamma,\gamma) - \frac{a_i}{n-1}\sum_{\gamma\in A\setminus B} v_\beta^i(x_\gamma,\gamma) - v_\beta^i(x_\beta,\beta) \\ > v_\alpha^i(b,\alpha) - \frac{c_i}{n-1}\sum_{\gamma\in C} v_\alpha^i(b,\alpha) - v_\alpha^i(x_\gamma,\gamma) - \frac{a_i}{n-1}\sum_{\gamma\in A\setminus C} v_\alpha^i(x_\gamma,\gamma) - v_\alpha^i(b,\alpha). \end{aligned}$$

Let $\gamma \in A \setminus B$. Thus, $v_\beta^i(x_\gamma,\gamma) - v_\beta^i(x_\beta,\beta) \geq 0$. Moreover, $v_\alpha^i(x_\gamma,\gamma) - v_\alpha^i(b,\alpha) \geq v_\beta^i(x_\gamma,\gamma) - v_\beta^i(x_\beta,\beta)$, for otherwise $v_\beta^i(x_\beta,\beta) - v_\alpha^i(b,\alpha) < v_\beta^i(x_\gamma,\gamma) - v_\alpha^i(x_\gamma,\gamma) \leq K$. Thus, $\gamma \in A \setminus C$. Thus, $A \setminus B \subseteq A \setminus C$ and $u_i(\hat{z}) > u_i(z)$ whenever

$$\begin{aligned} v_\beta^i(x_\beta,\beta) - v_\alpha^i(b,\alpha) - \frac{c_i}{n-1}\sum_{\gamma\in B\cap C} v_\beta^i(x_\beta,\beta) - v_\beta^i(x_\gamma,\gamma) - v_\alpha^i(b,\alpha) + v_\alpha^i(x_\gamma,\gamma) \\ - \frac{c_i}{n-1}\sum_{\gamma\in B\setminus C} v_\beta^i(x_\beta,\beta) - v_\beta^i(x_\gamma,\gamma) \\ - \frac{a_i}{n-1}\sum_{\gamma\in A\setminus B} v_\beta^i(x_\gamma,\gamma) - v_\beta^i(x_\beta,\beta) - v_\alpha^i(x_\gamma,\gamma) + v_\alpha^i(b,\alpha) > 0. \end{aligned}$$

Since for each $\gamma \in A$, $v_\beta^i(x_\beta,\beta) - v_\alpha^i(b,\alpha) > K \geq v_\beta^i(x_\gamma,\gamma) - v_\alpha^i(x_\gamma,\gamma)$, this inequality is satisfied whenever

$$\begin{aligned} v_\beta^i(x_\beta,\beta) - v_\alpha^i(b,\alpha) - \frac{c_i}{n-1}\sum_{\gamma\in B\cap C} v_\beta^i(x_\beta,\beta) - v_\alpha^i(b,\alpha) - v_\beta^i(x_\gamma,\gamma) + v_\alpha^i(x_\gamma,\gamma) \\ - \frac{c_i}{n-1}\sum_{\gamma\in B\setminus C} v_\beta^i(x_\beta,\beta) - v_\beta^i(x_\gamma,\gamma) > 0. \end{aligned}$$

For each $\gamma \in A \setminus C$, $v^i_\alpha(b,\alpha) - K \leq v^i_\alpha(x_\gamma,\gamma) - K \leq v^i_\beta(x_\gamma,\gamma)$. Thus, the inequality above holds whenever

$$v^i_\beta(x_\beta,\beta) - v^i_\alpha(b,\alpha) - \frac{c_i}{n-1}\sum_{\gamma\in B\cap C} v^i_\beta(x_\beta,\beta) - v^i_\alpha(b,\alpha) - v^i_\beta(x_\gamma,\gamma) + v^i_\alpha(x_\gamma,\gamma)$$
$$- \frac{c_i}{n-1}\sum_{\gamma\in B\setminus C} v^i_\beta(x_\beta,\beta) - v^i_\beta(b,\alpha) - \frac{c_i}{n-1}\sum_{\gamma\in B\setminus C} K > 0.$$

Since $|B| \leq n-1$, this inequality holds whenever

$$(1-c_i)(v^i_\beta(x_\beta,\beta) - v^i_\alpha(b,\alpha)) + \frac{c_i}{n-1}\sum_{\gamma\in B\cap C} v^i_\beta(x_\gamma,\gamma) - v^i_\alpha(x_\gamma,\gamma) - \frac{c_i}{n-1}\sum_{\gamma\in B\setminus C} K > 0.$$

Since for each $\gamma \in A$, $v^i_\beta(x_\gamma,\gamma) - v^i_\alpha(x_\gamma,\gamma) \geq -K$, this inequality holds whenever $v^i_\beta(x_\beta,\beta) - v^i_\alpha(b,\alpha) > c_iK/(1-c_i)$.

For a fixed b, let M^* be such that $v^i_\beta((1/n)M^*,\beta) - v^i_\alpha(b,\alpha) > K/(1-c_i)$. Thus, for each $M \geq M^*$, u_i satisfies the compensation assumption at b. The theorem then follows from Theorem 1. □

Proof of Theorem 3. Let $u \in \mathcal{F}^n$. Let $z \equiv (x,\mu) \in Z_M$ be a noncontestable allocation for u. We prove that z is efficient for u for each set of assumptions in the theorem. We claim that if $z' \equiv (x',\mu') \in Z_M$ is such that for each $i \in N$, $u_i(z') \geq u_i(z)$, it must be the case that for each $\alpha \in A$, $x'_\alpha \geq x_\alpha$. This implies that z' is obtained from z by reshuffling consumption bundles, and thus for each $i \in N$, $u_i(z) \geq u_i(z')$. Suppose by means of contradiction that there is $\alpha \in A$ such that $x'_\alpha < x_\alpha$. Suppose without loss of generality that $\alpha \in \arg\max_{\gamma\in A} x_\gamma - x'_\gamma$. Let $j \in N$ be such that $\mu'_j = \alpha$.

Suppose first that u_j satisfies Assamptions F1 and F4. Since z is noncontestable for u, then $u_j(z) \geq u_j(\hat{z})$, where $\hat{z}$ is obtained from z by swapping the consumption bundles of agent j and the agent who receives α at z. We prove that $u_j(\hat{z}) > u_j(z')$, which implies $u_j(\hat{z}) > u_j(z)$, a contradiction. In what follows, we will only analyze statements of preference for agent j when receiving object α. To simplify notation, we drop the superindices and subindices associated with this agent and this object in $v^j_\alpha(\cdot,\cdot)$, $\underline{m}_j$, and $\overline{m}_j$. Let $\Delta u \equiv u_j(\hat{z}) - u_j(z')$ and for each $\gamma \in A$, let $\Delta v(\gamma) \equiv v(x_\gamma,\gamma) - v(x'_\gamma,\gamma)$. For each $y \in \mathbb{R}^A$ and each $\gamma \in A$, let

$$\Phi(y,\gamma) \equiv -a_j \max\{v(y_\gamma,\gamma) - v(y_\alpha,\alpha), 0\} - c_j \max\{v(y_\alpha,\alpha) - v(y_\gamma,\gamma), 0\}.$$

Then

$$\Delta u = \Delta v(\alpha) + \frac{1}{n-1}\sum_{\gamma\in A} \Phi(x,\gamma) - \Phi(x',\gamma).$$

For each $\gamma \in A$, let $\Delta_\gamma \equiv \Phi(x,\gamma) - \Phi(x',\gamma)$. Let $A_1 \equiv \{\gamma \in A : v(x_\gamma,\gamma) \leq v(x_\alpha,\alpha)\}$ and $\gamma \in A_1$. Note that $A_1 \neq \emptyset$ for $\alpha \in A_1$. We claim that $\Delta_\gamma \geq -c_j(\Delta v(\alpha) - \Delta v(\gamma))$. There are two cases.

Case 1: $v(x'_\gamma, \gamma) \le v(x'_\alpha, \alpha)$. Then $\Delta_\gamma = -c_j(v(x_\alpha, \alpha) - v(x_\gamma, \gamma)) + c_j(v(x'_\alpha, \alpha) - v(x'_\gamma, \gamma))$.

Case 2: $v(x'_\gamma, \gamma) > v(x'_\alpha, \alpha)$. Then $\Delta_\gamma = -c_j(v(x_\alpha, \alpha) - v(x_\gamma, \gamma)) + a_j(v(x'_\gamma, \gamma) - v(x'_\alpha, \alpha)) \ge -c_j(v(x_\alpha, \alpha) - v(x_\gamma, \gamma))$. Now $v(x_\alpha, \alpha) - v(x_\gamma, \gamma) = v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) + v(x'_\alpha, \alpha) - v(x_\gamma, \gamma)$. By the case hypothesis, $v(x'_\gamma, \gamma) > v(x'_\alpha, \alpha)$. Thus, $v(x_\alpha, \alpha) - v(x_\gamma, \gamma) < v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) + v(x'_\gamma, \gamma) - v(x_\gamma, \gamma)$. Thus, $\Delta_\gamma > -c_j(v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) + v(x'_\gamma, \gamma) - v(x_\gamma, \gamma))$.

We claim that for each $\gamma \in A \setminus A_1$, $\Delta_\gamma \ge 0$. Let $\gamma \in A \setminus A_1$. Then $v(x_\gamma, \gamma) > v(x_\alpha, \alpha)$. There are two cases.

Case 1: $x'_\gamma \ge x_\gamma$. Then $v(x'_\gamma, \gamma) \ge v(x_\gamma, \gamma)$. Since $\gamma \in A \setminus A_1$, $v(x_\gamma, \gamma) > v(x_\alpha, \alpha)$. Thus, $v(x'_\gamma, \gamma) > v(x'_\alpha, \alpha)$. Thus, $\Delta_\gamma = -a_j(v(x_\gamma, \gamma) - v(x_\alpha, \alpha)) + a_j(v(x'_\gamma, \gamma) - v(x'_\alpha, \alpha)) = a_j([v(x_\alpha, \alpha) - v(x'_\alpha, \alpha)] + [v(x'_\gamma, \gamma) - v(x_\gamma, \gamma)]) \ge 0$.

Case 2: $x'_\gamma < x_\gamma$. By Assamption F4, the derivative of $v(\cdot, \alpha)$ is always greater than or equal to that of $v(\cdot, \gamma)$. Since $x_\alpha - x'_\alpha \ge x_\gamma - x'_\gamma$, $v(x_\gamma, \gamma) - v(x'_\gamma, \gamma) \le v(x_\alpha, \alpha) - v(x'_\alpha, \alpha)$. Thus, $v(x_\gamma, \gamma) - v(x_\alpha, \alpha) \le v(x'_\gamma, \gamma) - v(x'_\alpha, \alpha)$. Since $\gamma \in A \setminus A_1$, then $0 < v(x_\gamma, \gamma) - v(x_\alpha, \alpha)$. Thus, $v(x'_\alpha, \alpha) < v(x'_\gamma, \gamma)$. Thus, $\Delta_\gamma = -a_j(v(x_\gamma, \gamma) - v(x_\alpha, \alpha)) + a_j(v(x'_\gamma, \gamma) - v(x'_\alpha, \alpha)) = a_j([v(x_\alpha, \alpha) - v(x'_\alpha, \alpha)] - [v(x_\gamma, \gamma) - v(x'_\gamma, \gamma)])$. Recall that $v(x_\gamma, \gamma) - v(x'_\gamma, \gamma) \le v(x_\alpha, \alpha) - v(x'_\alpha, \alpha)$. Thus, $\Delta_\gamma \ge 0$.

Since for each $\gamma \in A_1$, $\Delta_\gamma \ge -c_j(\Delta v(\alpha) - \Delta v(\gamma))$ and for each $\gamma \in A \setminus A_1$, $\Delta_\gamma \ge 0$, then

$$\Delta u \ge \Delta v(\alpha) - \frac{c_j}{n-1} \sum_{\gamma \in A_1} \Delta v(\alpha) + \frac{c_j}{n-1} \sum_{\gamma \in A_1} \Delta v(\gamma). \tag{1}$$

Let $B \equiv \{\gamma \in A_1 : x_\gamma \le x'_\gamma\}$. Recall that by Assamption F1, the derivative of $v(\cdot, \gamma)$ is bounded above by $\overline{m}$ and below by $\underline{m}$. Thus, if $\gamma \in B$, then $x'_\gamma \ge x_\gamma$ and $\Delta v(\gamma) \ge \overline{m}(x_\gamma - x'_\gamma)$; if $\gamma \in A_1 \setminus B$, then $x'_\gamma < x_\gamma$ and $\Delta v(\gamma) \ge \underline{m}(x_\gamma - x'_\gamma)$. Thus, $\sum_{\gamma \in A_1} \Delta v(\gamma) \ge (\overline{m} - \underline{m}) \sum_{\gamma \in B} (x_\gamma - x'_\gamma) + \underline{m} \sum_{\gamma \in A_1} (x_\gamma - x'_\gamma)$.

We claim that for each $C \subseteq A$, $\sum_{\gamma \in C} x_\gamma - x'_\gamma \ge -(1/\underline{m}) \sum_{\gamma \in A \setminus C} \Delta v(\alpha)$. Since $z \in Z^+_M$, then $\sum_{\gamma \in A} x'_\gamma \le \sum_{\gamma \in A} x_\gamma$.[26] Thus, $\sum_{\gamma \in C} x_\gamma - x'_\gamma \ge \sum_{\gamma \in A \setminus C} x'_\gamma - x_\gamma \ge \sum_{\gamma \in A \setminus C} x'_\alpha - x_\alpha$, where the last inequality holds because for each $\gamma \in A$, $x_\alpha - x'_\alpha \ge x_\gamma - x'_\gamma$. By Assamption F1, $\Delta v(\alpha) \ge \underline{m}(x_\alpha - x'_\alpha)$. Thus, $x'_\alpha - x_\alpha \ge -(1/\underline{m}) \Delta v(\alpha)$ and $\sum_{\gamma \in C} x_\gamma - x'_\gamma \ge -(1/\underline{m}) \sum_{\gamma \in A \setminus C} \Delta v(\alpha)$.

Thus, $\sum_{\gamma \in A_1} \Delta v(\gamma) \ge -((\overline{m} - \underline{m})/\underline{m}) \sum_{\gamma \in A \setminus B} \Delta v(\alpha) - \sum_{\gamma \in A \setminus A_1} \Delta v(\alpha)$. Combining this inequality with inequality (1), we have that

$$\Delta u \ge \Delta v(\alpha) \left(1 - \frac{c_j}{n-1} \left(|A_1| + |A \setminus A_1| + \left(\frac{\overline{m} - \underline{m}}{\underline{m}}\right) 1_{B \neq \emptyset} |A \setminus B|\right)\right),$$

where $1_{B \neq \emptyset}$ is the indicator function that equal to 1 when $B \neq \emptyset$ and 0 otherwise. Thus,

$$\Delta u \ge \Delta v(\alpha) \left(1 - \frac{c_j}{n-1} \left(n + \left(\frac{\overline{m} - \underline{m}}{\underline{m}}\right)(n-1)\right)\right) = \Delta v(\alpha) \left(1 - c_j \left(\frac{1}{n-1} + \frac{\overline{m}}{\underline{m}}\right)\right).$$

By Assamption F1, $\Delta u > 0$.

[26] Observe that our argument applies when there is free disposal of money for a modified bound in which the set D at the end of our estimation can be taken to be such that $|D| = n - 1$.

Now, suppose that u_j satisfies Assumption F2 (Assumption F2 implies Assumption F1). Note that the only place that Assumption F4 was used above was to prove that for each $\gamma \in A \setminus A_1$ such that $x'_\gamma < x_\gamma$, $\Delta_\gamma \geq 0$. We prove now that for each such γ, $\Delta_\gamma \geq -a_j(\overline{m}/\underline{m} - 1)\Delta v(\alpha)$. Since $\gamma \in A \setminus A_1$, then

$$\Delta_\gamma = -a_j(v(x_\gamma, \gamma) - v(x_\alpha, \alpha)) + c_j \max\{v(x'_\alpha, \alpha) - v(x'_\gamma, \gamma), 0\} + a_j \max\{v(x'_\gamma, \gamma) - v(x'_\alpha, \alpha), 0\}.$$

Since $x_\gamma > x'_\gamma$, $v(x_\gamma, \gamma) - v(x'_\gamma, \gamma) \leq \overline{m}(x_\gamma - x'_\gamma) \leq \overline{m}(x_\alpha - x'_\alpha) \leq (\overline{m}/\underline{m})(v(x_\alpha, \alpha) - v(x'_\alpha, \alpha))$. Then

$$v(x_\gamma, \gamma) - v(x'_\gamma, \gamma) \leq \frac{\overline{m}}{\underline{m}} \Delta v(\alpha). \tag{2}$$

There are two cases.

Case 1: $v(x'_\gamma, \gamma) < v(x'_\alpha, \alpha)$. Then $\Delta_\gamma = -a_j(v(x_\gamma, \gamma) - v(x_\alpha, \alpha)) + c_j(v(x'_\alpha, \alpha) - v(x'_\gamma, \gamma))$. By the case hypothesis, $v(x'_\alpha, \alpha) > v(x'_\gamma, \gamma)$. Thus, $\Delta_\gamma \geq -a_j(v(x_\gamma, \gamma) - v(x_\alpha, \alpha))$. Now $v(x_\gamma, \gamma) - v(x_\alpha, \alpha) = v(x_\gamma, \gamma) - v(x'_\gamma, \gamma) + v(x'_\gamma, \gamma) - v(x'_\alpha, \alpha) + v(x'_\alpha, \alpha) - v(x_\alpha, \alpha)$. By the case hypothesis, $0 > v(x'_\gamma, \gamma) - v(x'_\alpha, \alpha)$. Thus,

$$v(x_\gamma, \gamma) - v(x_\alpha, \alpha) \leq v(x_\gamma, \gamma) - v(x'_\gamma, \gamma) - (v(x_\alpha, \alpha) - v(x'_\alpha, \alpha)) \leq \left(\frac{\overline{m}}{\underline{m}} - 1\right)\Delta v(\alpha),$$

where the last inequality follows from (2).

Case 2: $v(x'_\gamma, \gamma) \geq v(x'_\alpha, \alpha)$. Then $\Delta_\gamma = -a_j(v(x_\gamma, \gamma) - v(x_\alpha, \alpha)) + a_j(v(x'_\gamma, \gamma) - v(x'_\alpha, \alpha))$. Thus, $\Delta_\gamma = -a_j(v(x_\gamma, \gamma) - v(x'_\gamma, \gamma)) + a_j(v(x_\alpha, \alpha) - v(x'_\alpha, \alpha))$. Our claim then follows from (2).

Let $D \equiv \{\gamma \in A \setminus A_1 : x'_\gamma < x_\gamma\}$. Then

$$\Delta u \geq \Delta v(\alpha) - \frac{c_j}{n-1} \sum_{\gamma \in A_1} \Delta v(\alpha) + \frac{c_j}{n-1} \sum_{\gamma \in A_1} \Delta v(\gamma) - \frac{a_j}{n-1} \sum_{\gamma \in D} \left(\frac{\overline{m}}{\underline{m}} - 1\right)\Delta v(\alpha).$$

The remainder of the argument when Assumptions F1 and F4 hold is valid. Thus,

$$\Delta u \geq \Delta v(\alpha)\left(1 - \frac{c_j}{n-1}\left(\frac{1}{n-1} + \frac{\overline{m}}{\underline{m}}\right) - \frac{a_j}{n-1}\left(\frac{\overline{m}}{\underline{m}} - 1\right)|D|\right).$$

The cardinality of D is at most $n-2$, for otherwise $\sum_{\gamma \in A} x'_\gamma < \sum_{\gamma \in A} x_\gamma$.[27] Thus,

$$\Delta u \geq \Delta v(\alpha)\left(1 - c_j\left(\frac{1}{n-1} + \frac{\overline{m}}{\underline{m}}\right) - a_j \frac{n-2}{n-1}\left(\frac{\overline{m}}{\underline{m}} - 1\right)\right).$$

By Assumption F2, $\Delta u > 0$.

[27] If we assume free disposal of money, Assumption F2 must be modified to reflect the possibility that $|D| = n - 1$.

Finally, suppose that u_j satisfies Assumptions F1 and F3. Since Assumption F3 holds, we continue dropping the subindex α in $v^j_\alpha(\cdot,\cdot)$. Since we will only make statements about the rankings of agent j, we will also continue dropping the subindex and superindex j from the notation.

Let $s \equiv (y, \sigma) \in Z$ and let s' be the allocation obtained from s by swapping the consumption bundles of agents j and i. We claim that $u_j(s) \geq u_j(s')$ if and only if $v(s_j) \geq v(s_i)$. By definition of u_j, if $v(s_j) = v(s_i)$, then $u_j(s) = u_j(s')$. We prove that if $v(s_j) > v(s_i)$, then $u_j(s) > u_j(s')$, which completes the proof of our claim. Denote $\sigma_j \equiv \delta$ and $\sigma_i \equiv \eta$. Let $A_1 \equiv \{\gamma \in A : v(y_\gamma, \gamma) \geq v(y_\delta, \delta)\}$ and $B_1 \equiv \{\gamma \in A : v(y_\gamma, \gamma) \geq v(y_\eta, \eta)\}$. Then $u_j(s) > u_j(s')$ whenever

$$\begin{aligned} &v(y_\delta, \delta) - \frac{c_j}{n-1} \sum_{\gamma \in A \setminus A_1} v(y_\delta, \delta) - v(y_\gamma, \gamma) - \frac{a_j}{n-1} \sum_{\gamma \in A_1} v(y_\gamma, \gamma) - v(y_\delta, \delta) \\ &\quad > v(y_\eta, \eta) - \frac{c_j}{n-1} \sum_{\gamma \in A \setminus B_1} v(y_\eta, \eta) - v(y_\gamma, \gamma) - \frac{a_j}{n-1} \sum_{\gamma \in B_1} v(y_\gamma, \gamma) - v(y_\eta, \eta). \end{aligned}$$

Since $v(y_\delta, \delta) > v(y_\eta, \eta)$, then $A_1 \subseteq B_1$ and $A \setminus B_1 \subseteq A \setminus A_1$. Thus, the inequality above is satisfied whenever

$$\begin{aligned} &v(y_\delta, \delta) - v(y_\eta, \eta) - \frac{c_j}{n-1} \sum_{\gamma \in A \setminus B_1} v(y_\delta, \delta) - v(y_\eta, \eta) \\ &\qquad - \frac{c_j}{n-1} \sum_{\gamma \in (A \setminus A_1) \setminus (A \setminus B_1)} v(y_\delta, \delta) - v(y_\gamma, \gamma) > 0. \end{aligned}$$

Since for each $\gamma \in (A \setminus A_1) \setminus (A \setminus B_1) \subseteq B_1$, $v(y_\eta, \eta) \leq v(y_\gamma, \gamma)$, the inequality above is satisfied whenever, $v(y_\delta, \delta) - v(y_\eta, \eta) - (c_j/(n-1)) \sum_{\gamma \in A \setminus A_1} v(y_\delta, \delta) - v(y_\eta, \eta) > 0$. This inequality holds because $\delta \in A_1$ and $c_j < 1$.

Recall that $\alpha \in \arg\max_{\gamma \in A} x_\gamma - x'_\gamma$ and $\mu'_j = \alpha$. Let $\beta \equiv \mu_j$. We conclude by proving that $u_j(z) > u_j(z')$, a contradiction. By our claim above and since z is noncontestable for u, we have that for each $i \in N$, $v(z_j) \geq v(z_i)$. Let $E \equiv A \setminus \{\alpha, \beta\}$ and $C \equiv \{\gamma \in A : v(x'_\alpha, \alpha) \geq v(x'_\gamma, \gamma)\}$. Thus, $u_j(z) > u_j(z')$ whenever

$$\begin{aligned} &v(x_\beta, \beta) - \frac{c_j}{n-1} \sum_{\gamma \in E} (v(x_\beta, \beta) - v(x_\gamma, \gamma)) - \frac{c_j}{n-1} (v(x_\beta, \beta) - v(x_\alpha, \alpha)) \\ &\quad > v(x'_\alpha, \alpha) - \frac{c_j}{n-1} \sum_{\gamma \in C} (v(x'_\alpha, \alpha) - v(x'_\gamma, \gamma)) - \frac{a_j}{n-1} \sum_{\gamma \in A \setminus C} (v(x'_\gamma, \gamma) - v(x'_\alpha, \alpha)). \end{aligned}$$

Thus, $u_j(z) > u_j(z')$ whenever

$$\begin{aligned} &\left(1 - \frac{c_j}{n-1}\right) (v(x_\beta, \beta) - v(x_\alpha, \alpha)) + v(x_\alpha, \alpha) - v(x'_\alpha, \alpha) \\ &\qquad - \frac{c_j}{n-1} \sum_{\gamma \in E \cap C} (v(x_\beta, \beta) - v(x_\gamma, \gamma)) - (v(x'_\alpha, \alpha) - v(x'_\gamma, \gamma)) \\ &\qquad - \frac{c_j}{n-1} \sum_{\gamma \in E \setminus C} (v(x_\beta, \beta) - v(x_\gamma, \gamma)) > 0. \end{aligned}$$

Equivalently,

$$\begin{aligned}
&\left(1-\frac{c_j}{n-1}\right)(v(x_\beta,\beta)-v(x_\alpha,\alpha))+v(x_\alpha,\alpha)-v(x'_\alpha,\alpha)\\
&\quad-\frac{c_j}{n-1}\sum_{\gamma\in E\cap C}(v(x_\beta,\beta)-v(x_\alpha,\alpha)+v(x_\alpha,\alpha)-v(x'_\alpha,\alpha)+v(x'_\gamma,\gamma)-v(x_\gamma,\gamma))\\
&\quad-\frac{c_j}{n-1}\sum_{\gamma\in E\setminus C}(v(x_\beta,\beta)-v(x_\alpha,\alpha)+v(x_\alpha,\alpha)-v(x'_\alpha,\alpha)+v(x'_\gamma,\gamma)-v(x_\gamma,\gamma))\\
&\quad-\frac{c_j}{n-1}\sum_{\gamma\in E\setminus C}(v(x'_\alpha,\alpha)-v(x'_\gamma,\gamma))>0.
\end{aligned}$$

Since z is noncontestable for u, then $v(x_\beta,\beta)\geq v(x_\alpha,\alpha)$. Since $|E|=n-1$ only if $\beta=\alpha$ and $c_j<1$, we have that the inequality above holds whenever

$$\begin{aligned}
v(x_\alpha,\alpha)-v(x'_\alpha,\alpha)-\frac{c_j}{n-1}&\sum_{\gamma\in E\cap C}(v(x_\alpha,\alpha)-v(x'_\alpha,\alpha)+v(x'_\gamma,\gamma)-v(x_\gamma,\gamma))\\
&-\frac{c_j}{n-1}\sum_{\gamma\in E\setminus C}(v(x_\alpha,\alpha)-v(x'_\alpha,\alpha)+v(x'_\gamma,\gamma)-v(x_\gamma,\gamma))\\
&-\frac{c_j}{n-1}\sum_{\gamma\in E\setminus C}(v(x'_\alpha,\alpha)-v(x'_\gamma,\gamma))>0.
\end{aligned}$$

Since for each $\gamma\notin C$, $v(x'_\alpha,\alpha)<v(x'_\gamma,\gamma)$, then the inequality above holds whenever, $v(x_\alpha,\alpha)-v(x'_\alpha,\alpha)-(c_j/(n-1))\sum_{\gamma\in E}(v(x_\alpha,\alpha)-v(x'_\alpha,\alpha)+v(x'_\gamma,\gamma)-v(x_\gamma,\gamma))>0$. By the same argument after (1) we have that this inequality holds whenever

$$(v(x_\alpha,\alpha)-v(x'_\alpha,\alpha))\left(1-\frac{c_j}{n-1}\left(|E|+|A\setminus E|+\left(\frac{\overline{m}-\underline{m}}{\underline{m}}\right)1_{B'\neq\emptyset}|A\setminus B'|\right)\right)>0,$$

where $B'\equiv\{\gamma\in E: x'_\gamma\geq x_\gamma\}$. This inequality holds by Assumption F1. □

Proof of Theorem 4. Let $u\in\mathcal{L}^n$ be such that for each $i\in N$ and each pair of different objects α and β, $c_i(\alpha,\beta)<1/(n-1)$. Let $i\in N$. Recall that u_i has the form, for each $z\equiv(x,\mu)\in Z$,

$$u(z)\equiv v(\mu_i)+x_i+\sum_{j\neq i}c_i(\mu_i,\mu_j)x_j.$$

To simplify notation we write c_α^β instead of $c_i(\alpha,\beta)$. Suppose without loss of generality that there are different objects α and β such that $c_\alpha^\beta>0$, for otherwise the preferences exhibit no externalities, and the result is easily verified. Let $k\equiv\min_{\alpha\neq\beta,c_\alpha^\beta>0}1/c_\alpha^\beta$ and $\Gamma\equiv\max_{\{\delta,\gamma\}\subseteq A}|v(\delta)-v(\gamma)|$. By our hypothesis, $k>n-1$.

We will prove a more general result that will also help us prove Theorem 6. Let $M\in\mathbb{R}$ and $b\leq 0$ be such that $b<M$. Let $z\equiv(x,\mu)\in Z_M$ be such that $x_i=b$ and for each $j\neq i$, $x_j\geq b$. Let $z'\equiv(x',\mu')$ be the allocation obtained by swapping agent i's consumption

bundle with that of one agent who gets the highest consumption of money at z. We obtain conditions guaranteeing that $u_i(z') \geq u_i(z)$. Let $B \equiv \{\alpha \in A : x_\alpha \leq 0\}$ and $C \equiv \{\alpha \in A : x_\alpha > 0\}$. Let $\alpha \equiv \mu_i$ and $\beta \equiv \mu_i'$. Since $\alpha \in B$, then $B \neq \emptyset$. Since $M > b$, then $\beta \in C \neq \emptyset$. Denote $c_\alpha^\alpha = c_\beta^\beta = 0$. Then $u_i(z') \geq u_i(z)$ whenever

$$v(\alpha) - v(\beta) + b - x_\beta + \sum_{\gamma \in B} (c_\alpha^\gamma - c_\beta^\gamma) x_\gamma + \sum_{\gamma \in C} (c_\alpha^\gamma - c_\beta^\gamma) x_\gamma \leq 0.$$

This inequality is satisfied whenever $\Gamma + b - x_\beta - (1/k) \sum_{\gamma \in B} x_\gamma + (1/k) \sum_{\gamma \in C} x_\gamma \leq 0$. Since for each $\gamma \in B$, $x_\gamma \geq b$ and for each $\gamma \in C$, $x_\gamma \leq x_\beta$, the inequality above holds whenever, $\Gamma + b - x_\beta - |B|b/k + |C|x_\beta/k \leq 0$. Reorganizing terms, $\Gamma + (|C| - |B|)b/k \leq (k - |C|)(x_\beta - b)/k$. Now $x_\beta - b \geq (M - nb)/(n-1)$, for otherwise $M = b + \sum_{\delta \neq \alpha} x_\delta < b + (n-1)b + M - nb = M$. Since $k > |C|$, then $u_i(z') \geq u_i(z)$ whenever, $\Gamma + (|C| - |B|)b/k \leq (k - |C|)(M - nb)/((n-1)k)$. Equivalently,

$$\Gamma + \left(\frac{(n-1)(k - |B|) + (k - |C|)}{(n-1)k} \right) b \leq \left(\frac{k - |C|}{(n-1)k} \right) M.$$

Since $k > |C|$, this inequality is satisfied whenever

$$\frac{(n-1)k}{k - (n-1)} \Gamma + \min_{0 \leq |C| \leq n-1, 0 \leq |B| \leq n-1, |C|+|B|=n} \left(\frac{(n-1)(k - |B|) + (k - |C|)}{k - |C|} \right) b \leq M. \quad (3)$$

For a fixed $b \leq 0$ the left term of the inequality above defines M_b^* such that for each $M > \max\{b, M_b^*\}$, u_i satisfies the compensation assumption at b. Thus, the first part of the theorem follows from Theorem 1.

Finally, we prove that if $z \in Z_M$ is noncontestable for u, then z is efficient for u. Let $M \in \mathbb{R}$ and let $z \equiv (x, \mu) \in Z_M$ be a noncontestable allocation for u. We claim that if $z' \equiv (x', \mu') \in Z$ is such that $\sum_{i \in N} x_i' \leq M$ and for each $i \in N$, $u_i(z') \geq u_i(z)$, it must be the case that for each $\alpha \in A$, $x_\alpha' \geq x_\alpha$.[28] This implies that z' is obtained from z by reshuffling consumption bundles, and thus for each $i \in N$, $u_i(z) \geq u_i(z')$. Suppose by contradiction that there is $\alpha \in A$ such that $x_\alpha' < x_\alpha$. Let $\alpha \in \arg\max\{x_\beta - x_\beta' : \beta \in A\}$. Let $\varepsilon \equiv x_\alpha - x_\alpha' > 0$. Let $i \in N$ be such that $\mu_i' = \alpha$. Since z is noncontestable for u, then $u_i(z) \geq u_i(\hat{z})$, where $\hat{z}$ is obtained from z by swapping the consumption bundle of agent i and the agent who receives α. We claim that $u_i(\hat{z}) > u_i(z')$. Writing c_α^β for $c_i(\alpha, \beta)$, this is

$$x_\alpha + \sum_{\beta \in A \setminus \{\alpha\}} c_\alpha^\beta x_\beta > x_\alpha' + \sum_{\beta \in A \setminus \{\alpha\}} c_\alpha^\beta x_\beta'.$$

Let $B \equiv \{\beta \in A \setminus \{\alpha\} : x_\beta < x_\beta'\}$ and $C \equiv \{\beta \in A \setminus \{\alpha\} : x_\beta \geq x_\beta'\}$. Since $\sum_{\beta \in C} c_\alpha^\beta (x_\beta - x_\beta') \geq 0$, then this inequality is satisfied whenever $(x_\alpha - x_\alpha') + \sum_{\beta \in B} c_\alpha^\beta (x_\beta - x_\beta') > 0$. Suppose without loss of generality that $B \neq \emptyset$, for otherwise the result is trivial. Let $\Upsilon \equiv \sum_{\beta \in B} c_\alpha^\beta (x_\beta' - x_\beta)$. We need to prove that $\Upsilon < x_\alpha - x_\alpha' = \varepsilon$. Now $\Upsilon \leq \max_{\beta \in A \setminus \{\alpha\}} c_\alpha^\beta \sum_{\beta \in B} (x_\beta' - x_\beta) < (1/(n-1)) \sum_{\beta \in B} (x_\beta' - x_\beta)$, where the strict

[28] Our argument also applies when there is free disposal of money.

inequality follows from $B \neq \emptyset$. Since $\sum_{\beta \in A} x'_\beta \leq \sum_{\beta \in A} x_\beta$, then $\sum_{\beta \in B}(x'_\beta - x_\beta) \leq \sum_{\beta \in A \setminus B}(x_\beta - x'_\beta) \leq |A \setminus B|\varepsilon$. Thus, $\Upsilon < (n-1)\varepsilon/(n-1)$. Thus, $u_i(\hat{z}) > u_i(z') \geq u_i(z) \geq u_i(\hat{z})$. This is a contradiction. □

Proof of Theorem 6. Let $M \in \mathbb{R}$ and $u_i \in \mathcal{F}$ be such that there is $K > 0$ such that for each pair of objects α and β and each $x \in \mathbb{R}$, $|v^i_\alpha(x, \alpha) - v^i_\beta(x, \alpha)| \leq K$. For a fixed M, let b^* be such that $M > nb^*$ and for each pair $\{\alpha, \beta\} \subseteq A$, $v^i_\beta(M/n, \beta) - v^i_\alpha(b^*, \alpha) > K/(1 - c_i)$. By the same argument as in Theorem 2, u_i satisfies the compensation assumption at each $b \leq b^*$.

Let $u \in \mathcal{L}^n$ and suppose that for each $i \in N$ and each pair of different objects α and β, $c_i(\alpha, \beta) < 1/(n-1)$. Observe that the coefficient that multiplies b in the left term of inequality (3) is positive. Thus, if we fix M, there is $b^* < \min\{0, M\}$ such that u_i satisfies the compensation assumption at each $b \leq b^*$.

The theorem then follows from Theorem 5. □

References

Alkan, Ahmet, Gabrielle Demange, and David Gale (1991), "Fair allocation of indivisible goods and criteria of justice." *Econometrica*, 59, 1023–1039. [382, 384, 395, 397]

Andersson, Tommy, Lars Ehlers, and Lars-Gunnar Svensson (2014), "Budget-balance, fairness and minimal manipulability." *Theoretical Economics*, 9, 753–777. [384]

Azrieli, Yaron and Eran Shmaya (2014), "Rental harmony with roommates." *Journal of Economic Theory*, 153, 128–137. [384, 385, 388]

Border, Kim C. (1999), *Fixed Point Theorems With Applications to Economics and Game Theory*. Cambridge University Press, Cambridge. [399]

Brânzei, Simina, Ariel D. Procaccia, and Jie Zhang (2013), "Externalities in cake cutting." In *Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence*, 55–61. [384]

Dubins, Lester E. and Edwin H. Spanier (1961), "How to cut a cake fairly." *American Mathematical Monthly*, 68, 1–17. [384, 388]

Dufwenberg, Martin, Paul Heidhues, Georg Kirchsteiger, Frank Riedel, and Joel Sobel (2011), "Other-regarding preferences in general equilibrium." *Review of Economic Studies*, 78, 613–639. [384, 385, 391, 395]

Fehr, Ernst and Klaus M. Schmidt (1999), "A theory of fairness, competition, and cooperation." *Quarterly Journal of Economics*, 114, 817–868. [382, 383, 388, 390]

Fehr, Ernst and Klaus M. Schmidt (2001), "Theories of fairness and reciprocity—Evidence and economic applications." CEPR Discussion Paper 2703. [382]

Foley, Duncan K. (1967), "Resource allocation and the public sector." *Yale Economic Essays*, 7, 45–98. [381, 386]

Gale, David (1984), "Equilibrium in a discrete exchange economy with money." *International Journal of Game Theory*, 13, 61–64. [384, 385, 395]

Kahneman, Daniel, Jack L. Knetsch, and Richard H. Thaler (1990), "Experimental tests of the endowment effect and the Coase theorem." *Journal of Political Economy*, 98, 1325–1348. [383, 390]

Kolm, Serge-Christophe (1971), *Justice et Equité*. CEPREMAP, Paris. An English translation is: *Justice and Equity*, MIT Press, Cambridge, Massachusetts, 1997. [381, 382, 383, 386]

Kolm, Serge-Christophe (1995), "The economics of social sentiments: The case of envy." *Japanese Economic Review*, 46, 63–87. [383, 386]

Maskin, Eric S. (1987), "On the fair allocation of indivisible goods." In *Arrow and the Foundations of the Theory of Economic Policy* (George R. Feiwel, ed.), 341–349, Macmillan, London. [382, 384]

McLennan, Andrew and Rabee Tourky (2008), "Using volume to prove Sperner's lemma." *Economic Theory*, 35, 593–597. [399]

Moulin, Hervé (2003), *Fair Division and Collective Welfare*. MIT Press, Cambridge, Massachusetts. [383]

Neilson, William S. (2006), "Axiomatic reference-dependence in behavior toward others and toward risk." *Economic Theory*, 28, 681–692. [390]

Nogushi, Mitsunori and William R. Zame (2006), "Competitive markets with externalities." *Theoretical Economics*, 1, 143–166. [384]

Rawls, John (1971), *A Theory of Justice*. Belknap Press, Cambridge, Massachusetts. [383]

Rohde, Kristen I. M. (2010), "A preference foundation for Fehr and Schmidt's model of inequity aversion." *Social Choice and Welfare*, 34, 537–547. [390]

Sandbu, Martin E. (2008), "Axiomatic foundations for fairness-motivated preferences." *Social Choice and Welfare*, 31, 589–619. [390]

Stromquist, Walter (1980), "How to cut a cake fairly." *American Mathematical Monthly*, 87, 640–644. [388]

Su, Francis E. (1999), "Rental harmony: Sperner's lemma in fair division." *American Mathematical Monthly*, 106, 930–942. [384, 388, 399]

Svensson, Lars-Gunnar (1983), "Large indivisibles: An analysis with respect to price equilibrium and fairness." *Econometrica*, 51, 939–954. [382, 384, 389, 394]

Svensson, Lars-Gunnar (1987), "Erratum to 'Large indivisibles: An analysis with respect to price equilibrium and fairness'." *Econometrica*, 55, 489. [384]

Tadenuma, Koichi and William Thomson (1995), "Games of fair division." *Games and Economic Behavior*, 9, 191–204. [384]

Thomson, William (2010), "Fair allocation rules." In *Handbook of Social Choice and Welfare*, Vol. 2 (Kenneth Arrow, Amartya Sen, and Kotaro Suzumura, eds.), North-Holland, Amsterdam. [381, 386]

Tinbergen, Jan (1946), *Redelijke Inkomensverdeling*. De Gulden Pers, Haarlem. [381]

Varian, Hal R. (1974), "Equity, envy, and efficiency." *Journal of Economic Theory*, 9, 63–91. [381, 382, 386]

Varian, Hal R. (1976), "Two problems in the theory of fairness." *Journal of Public Economics*, 5, 249–260. [382, 386]

Velez, Rodrigo A. (2015), "Sharing an increase of the rent fairly." Report, Texas A&M University. [384, 397]

Vick, James W. (1994), *Homology Theory: An Introduction to Algebraic Topology*, second edition. Springer-Verlag, New York. [397, 399]

Woodall, Douglas R. (1980), "Dividing a cake fairly." *Journal of Mathematical Analysis and Applications*, 78, 233–247. [388]

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