VISCOUS REGULARIZATION OF THE EULER EQUATIONS AND ENTROPY PRINCIPLES*

JEAN-LUC GUERMOND† AND BOJAN POPOV‡

Abstract. This paper investigates a general class of viscous regularizations of the compressible Euler equations. A unique regularization is identified that is compatible with all the generalized entropies à la Harten et al. [10] and satisfies the minimum entropy principle. A connection with a recently proposed phenomenological model by Brenner [1] is made.

Key words. Conservation equations, hyperbolic systems, parabolic regularization, entropy, viscosity solutions.

AMS subject classifications. 76N15, 35L65, 65M12

1. Introduction.

Proving positivity of the density and internal energy and proving a minimum principle on the specific entropy of numerical approximations of the compressible Euler equations is a challenging task that has so far been achieved for very few numerical schemes on arbitrary meshes in two and higher space dimensions. The Godunov scheme (Godunov [7]) and some variants of the Lax[1] scheme (Lax [13]) are known to satisfy all these properties, (see Einfeldt et al. [2] for the Godunov scheme, Perthame and Shu [15, Appendix] for the explicit Lax algorithm, and Tang and Xu [20] for the implicit version of the Lax algorithm). The argumentation for the Godunov scheme relies on the fact that Riemann problems are solved exactly at each time step and averaging Riemann solutions preserves the above mentioned properties. None of the above arguments can be readily extended to central high-order schemes and more generally to schemes that are based on Galerkin approximations. One way to address this issue consists of using the standard parabolic regularization of the Euler equations to construct a scheme for which the vanishing viscosity is proportional to the mesh size. The problem with this approach is that the regularization acts on the conserved variables which are the density, momentum, and total energy. Since the momentum and total energy are not Galilean invariant, a change of reference frame by translation and/or rotation changes the regularization. A way out of this dilemma consists of considering the Navier-Stokes regularization as a starting point to construct a numerical method. However, one then encounters two serious difficulties. The first one is that the Navier-Stokes equations do not include any regularization in the continuity equation, which is inconsistent with most numerical discretizations. The second one is that whereas it is known that the Euler equations satisfy a minimum entropy principle on the specific entropy (see e.g., Tadmor [18]), it is also known that the Navier-Stokes equations violate this minimum principle if the thermal diffusivity is nonzero, see e.g., Serre [16, Thm 8.2.3]. These two observations make the Navier-Stokes regularization inconvenient for numerical purposes. One is then lead to

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[1]The Lax scheme is often called the Lax-Friedrichs scheme in the literature.

ponder on the following question: Is it possible to find a regularization of the Euler equations that is Galilean invariant, ensures positivity of the density and internal energy, satisfies a minimum entropy principle, and is compatible with a large class of entropies inequalities? The objective of this paper is to answer to this question.

The paper is organized as follows. The parabolic and the Navier-Stokes regularizations and their apparent shortcomings mentioned above are discussed in §2. A general family of regularizations is introduced and investigated in §3 and §4. The minimum entropy principle is investigated in §3 and the compatibility with entropy inequalities is studied in §4. The key result of this paper is Theorem 4.1: only one regularization technique satisfies the minimum entropy principle and is compatible with all the generalized entropies of Harten et al. [10]. This formulation is compared in §5 with a reformulation of the Navier-Stokes equations proposed by Brenner [1] that is based on heuristic arguments. A striking observation is that by distinguishing the so-called mass and volume velocities, it is possible to re-write the proposed regularization into a form similar to that of the Navier-Stokes equations with rotation invariant viscous fluxes. This way of looking at the regularization reconciles the parabolic and Navier-Stokes regularizations and shows that they are two faces of the same coin. The key results of the paper are summarized in §5.3. Standard identities and inequalities from thermodynamics that are used in this paper are collected in Appendix A.

2. Standard regularizations. We review in this section some well-known regularization techniques and discuss the pros and cons thereof.

2.1. Statement of the problem. Consider the compressible Euler equations in conservative form in $\mathbb{R}^d$,

$$\partial_t \rho + \nabla\cdot\boldsymbol{m} = 0, \tag{2.1}$$
$$\partial_t \boldsymbol{m} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{m}) + \nabla p = 0, \tag{2.2}$$
$$\partial_t E + \nabla\cdot(\boldsymbol{u}(E+p)) = 0, \tag{2.3}$$
$$\rho(\boldsymbol{x},0) = \rho_0, \qquad \boldsymbol{m}(\boldsymbol{x},0) = \boldsymbol{m}_0, \qquad E(\boldsymbol{x},0) = E_0, \tag{2.4}$$

where the dependent variables are the density, ρ, the momentum, $\boldsymbol{m}$ and the total energy, E. We adopt the usual convention that for any vectors $\boldsymbol{a}$, $\boldsymbol{b}$, with entries $\{a_i\}_{i=1,\dots,d}$, $\{b_i\}_{i=1,\dots,d}$, the following holds: $(\boldsymbol{a}\otimes\boldsymbol{b})_{ij} = a_i b_j$ and $\nabla\cdot\boldsymbol{a} = \partial_{x_j} a_j$, $(\nabla\boldsymbol{a})_{ij} = \partial_{x_i} a_j$. Moreover, for any order 2 tensors $\mathbb{g}$, $\mathbb{h}$, with entries $\{g_{ij}\}_{i,j=1,\dots,d}$, $\{h_{ij}\}_{i,j=1,\dots,d}$, we define $(\nabla\cdot\mathbb{g})_j = \partial_{x_i} g_{ij}$, $\boldsymbol{a}\cdot\nabla = a_i\partial_{x_i}$, $(\mathbb{g}\cdot\boldsymbol{a})_i = g_{ij}a_j$, $\mathbb{g}{:}\mathbb{h} = g_{ij}\mathbb{h}_{ij}$ where repeated indices are summed from 1 to d.

The pressure, p, is given by the equation of state which we assume to derive from a specific entropy, $s(\rho, e)$, through the thermodynamics identity:

$$T\,\mathrm{d}s := \mathrm{d}e + p\,\mathrm{d}\tau, \tag{2.5}$$

where $\tau := \rho^{-1}$, $e := \rho^{-1}E - \frac{1}{2}\boldsymbol{u}^2$ is the specific internal energy, $\boldsymbol{u} := \rho^{-1}\boldsymbol{m}$ is the velocity of the fluid particles. For instance it is common to take $s = \log(e^{\frac{1}{\gamma-1}}\rho^{-1})$ for an polytropic ideal gas. Using the notation $s_e := \frac{\partial s}{\partial e}$ and $s_\rho := \frac{\partial s}{\partial \rho}$, this definition implies that

$$s_e := T^{-1}, \qquad s_\rho := -pT^{-1}\rho^{-2}. \tag{2.6}$$

The equation of state takes the form $p := -\rho^2 s_\rho s_e^{-1}$, or

$$p s_e + \rho^2 s_\rho = 0. \tag{2.7}$$

The key structural assumption is that $-s$ is strictly convex with respect to $\tau := \rho^{-1}$ and e. Upon introducing $\sigma(\tau, e) := s(\rho, e)$, the convexity hypothesis is equivalent to assuming that $\sigma_{\tau\tau} \le 0$, $\sigma_{ee} \le 0$, and $\sigma_{\tau\tau}\sigma_{ee} - \sigma_{\tau e}^2 \le 0$ (see e.g., Godlewski and Raviart [6]). This in turn implies that

$$\partial_\rho(\rho^2 s_\rho) < 0, \qquad s_{ee} < 0, \qquad 0 < \partial_\rho(\rho^2 s_\rho)s_{ee} - \rho^2 s_{\rho e}^2, \tag{2.8}$$

or equivalently that the following matrix

$$\Sigma := \begin{pmatrix} \rho^{-1}\partial_\rho(\rho^2 s_\rho) & \rho s_{\rho e} \\ \rho s_{\rho e} & \rho s_{ee} \end{pmatrix}, \tag{2.9}$$

is negative definite. In the rest of the paper we assume that (2.8) holds and the temperature be positive

$$0 < s_e. \tag{2.10}$$

Remark 2.1. Note in passing that contrary to what is sometimes done in the literature, we do not assume that the pressure be positive, which requires $s_\rho < 0$ (see e.g., Godlewski and Raviart [6, p. 99], Harten et al. [10, (2.3)]). For instance, the assumptions (2.8) and (2.10) hold for stiffened gases, but the quantity s_ρ can change sign. It is shown in the Appendix (see Remark A.1) that the convexity assumption (2.8) and the positivity of the temperature (2.10) are sufficient to prove that the Euler system is hyperbolic. This fact was first established by Godunov [8] in one dimension. It was established again in Friedrichs and Lax [5] and Harten et al. [10].

The objective of the present paper is to introduce a viscous regularization of (2.1)–(2.4) that is compatible with thermodynamics and that can serve as a reasonable starting point for numerical approximation.

2.2. Monolithic parabolic regularization. A common regularization of (2.1) for theoretical and numerical purposes consists of the following monolithic parabolic regularization:

$$\partial_t \rho + \nabla\cdot\boldsymbol{m} = \epsilon\Delta\rho, \tag{2.11}$$

$$\partial_t \boldsymbol{m} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{m}) + \nabla p = \epsilon\Delta m, \tag{2.12}$$

$$\partial_t E + \nabla\cdot(\boldsymbol{u}(E+p)) = \epsilon\Delta E, \tag{2.13}$$

$$\rho(\boldsymbol{x},0) = \rho_0, \qquad \boldsymbol{m}(\boldsymbol{x},0) = \boldsymbol{m}_0, \qquad E(\boldsymbol{x},0) = E_0, \tag{2.14}$$

where ϵ is a small parameter. We call this regularization monolithic since no distinction is made between the conserved quantities, i.e., the operator $\epsilon\Delta$ is applied blindly to all the conserved quantities.

It can be shown that the Lax-Friedrichs scheme and its parabolic analog introduced in Perthame and Shu [15] are approximations of (2.11). For instance, considering a nonlinear conservation equation $\partial_t \boldsymbol{U} + \nabla\cdot\boldsymbol{F}(\boldsymbol{U}) = 0$, where $\boldsymbol{U}$ is the dependent vector-valued variable in $\mathbb{R}^m$, the scheme introduced in Lax [13, p.163] in one space dimension consists of considering

$$\begin{aligned}\boldsymbol{U}_i^{n+1} &= \frac{1}{2}(\boldsymbol{U}_{i+1}^n + \boldsymbol{U}_{i-1}^n) - \frac{1}{2}\lambda(\boldsymbol{F}(\boldsymbol{U}_{i+1}^n) - \boldsymbol{F}(\boldsymbol{U}_{i-1}^n)) \\ &= \boldsymbol{U}_i^n - \frac{1}{2}\lambda(\boldsymbol{F}(\boldsymbol{U}_{i+1}^n) - \boldsymbol{F}(\boldsymbol{U}_{i-1}^n)) + \tau\frac{1}{2}h^2\tau^{-1}\frac{(\boldsymbol{U}_{i+1}^n - 2\boldsymbol{U}_i^n + \boldsymbol{U}_{i-1}^n)}{h^2},\end{aligned} \tag{2.15}$$

where h is the mesh size, τ is the time step, and $\lambda := \tau h^{-1}$. Assuming the flux $\boldsymbol{F}$ to be uniformly Lipschitz, to simplify, and upon introducing the maximum wave speed $\beta := \|\boldsymbol{F}'\|_{L^\infty(\mathbb{R}^m;\mathbb{R}^m\times\mathbb{R}^m)}$ and the CFL number cfl $:= \beta\tau h^{-1}$, (2.15) is the centered second-order approximation of the following parabolic regularization of the conservation equation $\partial_t \boldsymbol{U} + \nabla\cdot\boldsymbol{F}(\boldsymbol{U}) - \epsilon\Delta\boldsymbol{U} = 0$, with the artificial viscosity $\epsilon := \frac{1}{2}h\lambda^{-1} = \frac{1}{\text{cfl}}\frac{1}{2}\beta h$. In other words, the Lax-Friedrichs scheme is a centered second-order approximation of (2.11)-(2.14) with the numerical viscosity $\epsilon = \frac{1}{\text{cfl}}\frac{1}{2}h\||\boldsymbol{u}| + c\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}_+)}$, where c is the speed of sound. That the CFL number appears at the denominator of the artificial viscosity makes this scheme over-dissipative. It is often more appropriate to consider the following alternative

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n - \frac{1}{2}\lambda(\boldsymbol{F}(\boldsymbol{U}_{i+1}^n) - \boldsymbol{F}(\boldsymbol{U}_{i-1}^n)) + \frac{1}{2}\lambda|\beta|h^2\frac{(\boldsymbol{U}_{i+1}^n - 2\boldsymbol{U}_i^n + \boldsymbol{U}_{i-1}^n)}{h^2},$$

which is also a centered second-order approximation of the parabolic regularization $\partial_t \boldsymbol{U} + \nabla\cdot\boldsymbol{F}(\boldsymbol{U}) - \epsilon\Delta\boldsymbol{U} = 0$ with the viscosity $\frac{1}{2}\beta h$, which is more traditionally associated with up-winding. This algorithm is often abusively referred to as the Lax-Friedrichs scheme. Both the above numerical schemes have interesting positivity and entropy properties, see e.g., Lax [12], Tadmor [18, 19], Perthame and Shu [15].

Despite its appealing mathematical properties, the above regularization is often criticized by physicists since it seemingly violates the Galilean and rotation invariance. It also dissipates the density, the momentum and the total energy, which seemingly are again aberrations from the physical point of view. When looking at (2.11)-(2.14), it is indeed difficult to see how this set of equations can be reconciled with the Navier-Stokes equations which are usually viewed by physicists to be the acceptable regularization of the Euler equations.

2.3. Navier-Stokes regularization. As mentioned above, a common "physical" way to regularize the Euler system (2.1)-(2.4) consists of considering this system as the limit of the Navier-Stokes equations

$$\partial_t\rho + \nabla\cdot\boldsymbol{m} = 0, \tag{2.16}$$
$$\partial_t\boldsymbol{m} + \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{m}) + \nabla p - \nabla\cdot\mathbb{g} = 0, \tag{2.17}$$
$$\partial_t E + \nabla\cdot(\boldsymbol{u}(E+p)) - \nabla\cdot(\boldsymbol{h} + \mathbb{g}\cdot\boldsymbol{u}) = 0, \tag{2.18}$$
$$\rho(\boldsymbol{x},0) = \rho_0, \qquad \boldsymbol{m}(\boldsymbol{x},0) = \boldsymbol{m}_0, \qquad E(\boldsymbol{x},0) = E_0. \tag{2.19}$$

where $\mathbb{g}$ and $\boldsymbol{h}$ are the viscous and thermal fluxes. The most elementary model compatible with Galilean invariance consists of assuming that

$$\mathbb{g} = 2\mu\nabla^s\boldsymbol{u} + \lambda\nabla\cdot\boldsymbol{u}\mathbb{I}, \qquad \boldsymbol{h} = \kappa\nabla T. \tag{2.20}$$

where $\nabla^s\boldsymbol{u} := \mu(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T)$, $\mathbb{I}$ is the identity matrix in $\mathbb{R}^d$, and T is the temperature, $T := s_e^{-1}$. The viscosity μ and the thermal diffusivity κ are required to be non-negative by the Clausius-Duhem inequality, although these two parameters may depend on the state (ρ, e).

We claim that (2.16)-(2.20) is not appropriate for numerical purposes and we identify at least two obstructions. The first problem is that the minimum entropy principle cannot be satisfied for general initial data if the thermal dissipation is not zero. More precisely, assuming $\kappa \neq 0$, for any $r \in \mathbb{R}$, there exist initial data so that the set $\{s \geq r\}$ is not positively invariant. Let us recall a simple proof of this statement borrowed from Serre [16, Thm 8.2.3]. The specific entropy for the Navier-Stokes system satisfies

$$\partial_t s + \boldsymbol{u}\cdot\nabla s = \frac{1}{\rho T}\left(\mathbb{g}{:}\nabla^s \boldsymbol{u} + \nabla\cdot(\kappa\nabla T)\right). \tag{2.21}$$

Assume that $\boldsymbol{u}_0 := \boldsymbol{m}_0\rho_0^{-1}$ is constant. Assume also that the equation of state of the fluid is such that $p_e \neq 0$, then one can use T and s as independent state variables since $\rho^2\det\left(\frac{D(T,s)}{D(\rho,e)}\right) = \frac{\rho^2}{s_e^2}(s_\rho s_{ee} - s_e s_{\rho e}) = p_e \neq 0$ (see (A.6)). One can then choose s_0 with global minimum at 0 and T_0 so that $\Delta T_0(0) < 0$ and $\nabla T_0(0) = 0$. Without loss of generality, we assume that $\kappa > 0$ in a neighborhood of 0. Then $\partial_t s(0,0) = \kappa\rho_0^{-1}(0)\Delta T_0(0) < 0$, thereby proving that $\{s \geq r\}$ is not positively invariant for the regularized system (2.16)–(2.20).

Another argument often invoked against the presence of thermal dissipation is that it is incompatible with symmetrization of the Navier-Stokes system when using the generalized entropies of Harten for polytropic ideal gases. (The function $\rho f(s)$ is said to be a generalized entropy if $f'\gamma^{-1} - f'' > 0$, $f' > 0$ and $f \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$, see Harten [9].) It is proved in Hughes et al. [11] that the only generalized entropy that symmetrizes the Navier-Stokes system (2.16)–(2.20) is the trivial one ρs when $\kappa \neq 0$, see also Tadmor [19, (2.11) and Remark 2, page 460]. Note though that symmetrization of the viscous fluxes is not necessary to prove entropy dissipation. It is nevertheless true that the Navier-Stokes system with $\kappa \neq 0$ does not admit a generalized entropy inequality if $f''(s) \neq 0$, and this fact is a consequence of the following quadratic form not being non-negative: $f'(s)X^2 - f''(s)XY$, $(X,Y) \in \mathbb{R}^2$. Symmetry of the viscous flux is not a necessary condition for entropy dissipation, see e.g., Serre [17, §1.1].

The above two arguments seem to imply that one should take $\kappa = 0$ if one wants to use the Navier-Stokes system as a numerical device that regularizes the Euler equations, satisfies the minimum entropy principle, and satisfies entropy inequalities. In that case, one then faces a serious obstruction when solving for contact waves. For instance assuming that the initial data, ρ_0, $\boldsymbol{m}_0$, $\boldsymbol{E}_0$ are such that the exact velocity is constant in time and space, say $\boldsymbol{u} = \beta\boldsymbol{e}_x$, the problem (2.16)–(2.19) reduces to solving two linear transport equations

$$\partial_t\rho + \beta\partial_x\rho = 0, \quad \rho(\cdot,0) = \rho_0, \tag{2.22}$$
$$\partial_t E + \beta\partial_x E = 0, \quad E(\cdot,0) = E_0. \tag{2.23}$$

Note that $\boldsymbol{u}$ being constant implies that the pressure gradient is zero. The exact solution is $\rho(\boldsymbol{x},t) = \rho_0(\boldsymbol{x} - \beta t\boldsymbol{e}_x)$. To make things a little bit more interesting assume that ρ_0 is piecewise constant, say $\rho_0(x) = 1$ if $x < 0$ and $\rho_0(x) = 2$ if $x > 0$. In the absence of some sort of regularization, the above two linear transport equations are difficult to solve numerically. Except for the method of characteristics and Lagrangian based techniques, we are not aware of any numerical methods that can solve these equations without resorting to some kind of viscous regularization.

In conclusion, if positivity of the density, the minimum entropy principle and a reasonable approximation of contact discontinuities is desired, the Navier-Stokes regularization does not seem to be appropriate to regularize (2.1)–(2.4), whether κ is zero or not.

3. General regularization. We investigate in this section the properties of a class of regularizations that we expect to be as general as possible. More precisely, let us consider the following general regularization for the Euler system:

$$\partial_t \rho + \nabla\cdot \boldsymbol{m} - \nabla\cdot \boldsymbol{f} = 0, \tag{3.1}$$
$$\partial_t \boldsymbol{m} + \nabla\cdot(\boldsymbol{u}\otimes \boldsymbol{m}) + \nabla p - \nabla\cdot \mathbb{g} = 0, \tag{3.2}$$
$$\partial_t E + \nabla\cdot(\boldsymbol{u}(E+p)) - \nabla\cdot(\boldsymbol{h} + \mathbb{g}\cdot\boldsymbol{u}) = 0, \tag{3.3}$$

where for the time being we let the fluxes $\boldsymbol{f}$, $\mathbb{g}$, and $\boldsymbol{h}$ to be as general as possible. A theory of viscous regularization for general nonlinear hyperbolic system has been developed in Serre [17] and Serre [16, Chap 6]. This theory identifies classes of entropy-dissipative viscous regularizations and establishes short term existence results. Our objective in this paper is more restrictive. We want to construct the fluxes $\boldsymbol{f}$, $\mathbb{g}$, and $\boldsymbol{h}$ so that (3.1)-(3.3) gives a positive density, gives a minimum principle on the specific entropy, and is compatible with a large class of entropies. (Note in passing that the positivity of the internal energy will be a consequence of the positivity of the density and the minimum entropy principle.) In the rest of the paper, we are going to work under the assumption that (3.1)-(3.3) has a smooth solution.

3.1. Positivity of the density. Let us now choose the flux $\boldsymbol{f}$ so that it regularizes the mass conservation equation. From the theory of second-order elliptic equation we conjecture that $a(\rho,e)\nabla\rho$ should be appropriate, where $a(\rho,e)$ is a smooth positive function of ρ and e. In particular, it is reasonable to expect that the following choice implies positivity of the density:

$$a(\rho,e) = \chi(\rho,e)\varphi'(\rho), \tag{3.4}$$

where χ is a smooth positive function of ρ and e and φ is a strictly increasing function. This definition gives $\boldsymbol{f} = \chi(\rho,e)\nabla\varphi(\rho)$. This regularization is at least compatible with the positive density principle as stated in the following.

LEMMA 3.1 (Positive Density Principle). *Let $\boldsymbol{f} = a(\rho,e)\nabla\rho$ in (3.1), with $a \in L^\infty(\mathbb{R}^2;\mathbb{R})$ and $\inf_{(\xi,\eta)\in\mathbb{R}^2} a(\xi,\eta) > 0$. Assume that $\boldsymbol{u}$ and $\nabla\cdot\boldsymbol{u} \in L^\infty(\mathbb{R}^d\times\mathbb{R}_+;\mathbb{R})$. Assume also that there are constant states at infinity ρ^∞, $\boldsymbol{u}^\infty$, so that the supports of $\rho(\cdot,\cdot)-\rho^\infty$ and $\boldsymbol{u}(\cdot,\cdot)-\boldsymbol{u}^\infty$ are compact in $\mathbb{R}^d\times(0,t)$, for any $t>0$. Assume finally that $\rho_0-\rho_\infty \in L^2(\mathbb{R}^d;\mathbb{R})$. Then the solution of (3.1) is such that*

$$\operatorname*{ess\,inf}_{\boldsymbol{x}\in\mathbb{R}^d} \rho(\boldsymbol{x},t) \ge 0, \qquad \forall t\ge 0. \tag{3.5}$$

Proof. Owing to the assumed regularity of $\boldsymbol{u}$ and ρ_0, the theory of parabolic equations implies that there is a unique solution to (3.1) such that $\rho-\rho_\infty \in L^\infty((0,\infty);L^2(\mathbb{R}^d))\cap L^2((0,\infty);H^1(\mathbb{R}^d))$ and $\partial_t\rho \in L^2((0,\infty);H^{-1}(\mathbb{R}^d))$, see e.g., Evans [3, p.356].

Let $\epsilon>0$ and let $h_\epsilon(x)$ be a smooth concave function that approximates $\min(x,0)$ uniformly over $\mathbb{R}$; say there is $c>0$ so that $\sup_{s\in\mathbb{R}} |h_\epsilon(s)-\min(s,0)|+|h_\epsilon(s)-sh'_\epsilon(s)| < c\epsilon$ and $h''\le 0$. Let $t>0$ be some fixed time. Let $B(0,R)$ be the ball centered at 0 of radius R such that the supports of $\rho(\cdot,\tau)-\rho^\infty$ and $\boldsymbol{u}(\cdot,\tau)-\boldsymbol{u}^\infty$ are in $B(0,R)$ for all $\tau\in[0,t]$. Let χ be a regularized characteristic function with the following properties: $\chi|_{B(0,R)} = 1$ and $\chi|_{\mathbb{R}^d\setminus B(0,R+1)} = 0$. Multiplying the weak form of (3.1) by the legitimate test function $\chi h'_\epsilon(\rho)$ we obtain

$$\int_{\mathbb{R}^d} \Big((\partial_t h_\epsilon(\rho) + \boldsymbol{u}\nabla h_\epsilon(\rho) + \rho h'_\epsilon(\rho)\nabla\cdot\boldsymbol{u})\chi(\boldsymbol{x}) + a\nabla\rho\nabla(\chi h'_\epsilon(\rho)) \Big)\,\mathrm{d}\boldsymbol{x} = 0.$$

Using that the properties of χ, we simplify the above equation as follows:

$$\int_{\mathbb{R}^d} \Big(\partial_t h_\epsilon(\rho) + \boldsymbol{u}\nabla h_\epsilon(\rho) + \rho h'_\epsilon(\rho)\nabla\cdot\boldsymbol{u} + h''_\epsilon(\rho)a|\nabla\rho|^2\Big)\,\mathrm{d}\boldsymbol{x} = 0$$
$$\int_{\mathbb{R}^d} \Big(\partial_t h_\epsilon(\rho) + \nabla\cdot(h_\epsilon(\rho)\boldsymbol{u}) + (\rho h'_\epsilon(\rho) - h_\epsilon(\rho))\nabla\cdot\boldsymbol{u} + h''_\epsilon(\rho)a|\nabla\rho|^2\Big)\,\mathrm{d}\boldsymbol{x} = 0$$

Now, we integrate over time and, owing to the assumptions regarding the behavior of $\boldsymbol{u}$, ρ and a, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} h_\epsilon(\rho(\boldsymbol{x},t))\,\mathrm{d}\boldsymbol{x} &\geq -\int_0^t\int_{\mathbb{R}^d} |\rho h'_\epsilon(\rho) - h_\epsilon(\rho)||\nabla\cdot\boldsymbol{u}|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_{\mathbb{R}^d} h_\epsilon(\rho_0(\boldsymbol{x}))\\ &\geq -c\epsilon + \int_{\mathbb{R}^d} h_\epsilon(\rho_0(\boldsymbol{x})).\end{aligned}$$

We can now pass to the limit on ϵ using the Lebesgue dominated convergence and we obtain $\int_{\mathbb{R}^d}\min(\rho(\boldsymbol{x},t),0) \geq 0$. The result follows readily. $\square$

3.2. Minimum entropy principle. We now investigate under which conditions on the fluxes $\boldsymbol{f}$, $\mathbb{g}$ and $\boldsymbol{h}$, a minimum principle on the specific entropy holds. In order to account for impact of the viscous part in the mass conservation, we change the notation of the various viscous fluxes as stated in the following lemma.

LEMMA 3.2. *Setting* $\mathbb{g} = \mathbb{G} + \boldsymbol{f}\otimes\boldsymbol{u}$, *and* $\boldsymbol{h} = \boldsymbol{l} - \frac{1}{2}\boldsymbol{u}^2\boldsymbol{f}$, *the specific entropy for the system* (3.1)–(3.3) *satisfies*

$$\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) + \nabla\cdot((es_e - \rho s_\rho)\boldsymbol{f} - s_e\boldsymbol{l}) - \boldsymbol{f}\cdot\nabla(es_e - \rho s_\rho) + \boldsymbol{l}\cdot\nabla s_e - s_e\mathbb{G}{:}\nabla\boldsymbol{u} = 0. \quad (3.6)$$

Proof. We re-write (3.1)–(3.3) in non-conservative form as follows:

$$\begin{aligned}&\partial_t\rho + \boldsymbol{u}\cdot\nabla\rho + \rho\nabla\cdot\boldsymbol{u} - \nabla\cdot\boldsymbol{f} = 0,\\ &\rho(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) + \boldsymbol{u}\nabla\cdot\boldsymbol{f} + \nabla p - \nabla\cdot\mathbb{g} = 0,\\ &\rho(\partial_t\mathcal{E} + \boldsymbol{u}\cdot\nabla\mathcal{E}) + \mathcal{E}\nabla\cdot\boldsymbol{f} + \nabla\cdot(\boldsymbol{u}p) - \nabla\cdot(\boldsymbol{h} + \mathbb{g}\cdot\boldsymbol{u}) = 0.\end{aligned}$$

where we have defined $\mathcal{E} = \rho^{-1}E$. Then we obtain the equation controlling the internal energy, $e = \mathcal{E} - \frac{1}{2}\boldsymbol{u}^2$, by multiplying the momentum equation by $\boldsymbol{u}$ and subtracting the result from the total energy equation:

$$\rho(\partial_t e + \boldsymbol{u}\cdot\nabla e) + (e - \tfrac{1}{2}\boldsymbol{u}^2)\nabla\cdot\boldsymbol{f} + p\nabla\cdot\boldsymbol{u} - \nabla\cdot\boldsymbol{h} - \mathbb{g}{:}\nabla\boldsymbol{u} = 0,$$

The key to obtain the equation that controls the entropy is to multiply the mass conservation by ρs_ρ, multiply the internal energy balance by s_e, and add the two resulting equations. This linear combination is motivated by the following observation $\partial_\alpha s = s_\rho\partial_\alpha\rho + s_e\partial_\alpha e$ which holds for any independent variable $\alpha\in\{t,\boldsymbol{x}\}$. We then obtain

$$\begin{aligned}\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) + s_e(e - \tfrac{1}{2}\boldsymbol{u}^2)\nabla\cdot\boldsymbol{f} + (ps_e + \rho^2 s_\rho)\nabla\cdot\boldsymbol{u}&\\ - s_e(\nabla\cdot\boldsymbol{h} + \mathbb{g}\cdot\nabla\boldsymbol{u}) - \rho s_\rho\nabla\cdot\boldsymbol{f} = 0&\end{aligned}$$

The definition of the pressure implies that the quantity $ps_e + \rho^2 s_\rho$ is zero, see (2.7). This simplification yields

$$\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) + (es_e - \rho s_\rho)\nabla\cdot\boldsymbol{f} - s_e(\mathbb{g}{:}\nabla\boldsymbol{u}) - s_e\tfrac{1}{2}\boldsymbol{u}^2\nabla\cdot\boldsymbol{f} - s_e\nabla\cdot\boldsymbol{h} = 0.$$

We now regroup the terms

$$\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) + (es_e - \rho s_\rho)\nabla\cdot\boldsymbol{f} - s_e\nabla\cdot(\boldsymbol{h} + \tfrac{1}{2}\boldsymbol{u}^2\boldsymbol{f}) - s_e(\mathbb{g}{:}\nabla\boldsymbol{u} - (\boldsymbol{f}\otimes\boldsymbol{u}){:}\nabla\boldsymbol{u}) = 0,$$

and conclude by using the definitions $\mathbb{g} = \mathbb{G}(\nabla^s\boldsymbol{u}) + \boldsymbol{f}\otimes\boldsymbol{u}$ and $\boldsymbol{h} = \boldsymbol{l} - \frac{1}{2}\boldsymbol{u}^2\boldsymbol{f}$. ◻

From now on we assume that the following structure holds for the viscous fluxes introduced in (3.1)–(3.3):

$$\mathbb{g} = \mathbb{G}(\nabla^s\boldsymbol{u}) + \boldsymbol{f}\otimes\boldsymbol{u}, \qquad \boldsymbol{h} = \boldsymbol{l} - \tfrac{1}{2}\boldsymbol{u}^2\boldsymbol{f}, \qquad \mathbb{G}(\nabla^s\boldsymbol{u}){:}\nabla\boldsymbol{u} \ge 0. \tag{3.7}$$

We also assume that $\boldsymbol{f}$ has the following form:

$$\boldsymbol{f} = a(\rho,e)\nabla\rho \qquad\qquad a(\rho,e) \ge 0, \tag{3.8}$$

and $\boldsymbol{l}$ is defined so that

$$\boldsymbol{l} = s_e^{-1}(es_e - \rho s_\rho)\boldsymbol{f} + d(\rho,e)\rho s_e^{-1}\nabla s, \qquad\qquad d(\rho,e) \ge 0. \tag{3.9}$$

Remark 3.1. The conditions $\mathbb{G}(\nabla^s\boldsymbol{u}){:}\nabla\boldsymbol{u} \ge 0$, $a(\rho,e) \ge 0$, and $d(\rho,e) \ge 0$ are essential to establish the minimum principle on the specific entropy and the entropy inequalities (see Theorem 3.4 and Theorem 4.1).

Remark 3.2. The structural assumption $\boldsymbol{l} = s_e^{-1}(es_e - \rho s_\rho)\boldsymbol{f} + d(\rho,e)\rho s_e^{-1}\nabla s$ is crucial. This condition is equivalent to assuming that the conservative term in (3.6) is of the following form: $(es_e - \rho s_\rho)\boldsymbol{f} - s_e\boldsymbol{l} = -\nabla\cdot(d\rho\nabla s)$. The definition of $\boldsymbol{l}$ makes sense since thermodynamics requires that $s_e = T^{-1} > 0$, (see (2.10)). Note that given (3.8) the following alternative forms hold $\boldsymbol{l} = (d-a)\rho s_\rho s_e^{-1}\nabla\rho + ae\nabla\rho + d\rho\nabla e$, or $\boldsymbol{l} = (a-d)(p\rho^{-1} + e)\nabla\rho + d\nabla(\rho e)$.

Let us define the quantity

$$J := -\boldsymbol{f}\cdot\nabla(es_e - \rho s_\rho) + \boldsymbol{l}\cdot\nabla s_e + a\nabla\rho\cdot\nabla s \tag{3.10}$$

which is a quadratic form with respect to $\nabla\rho$ and ∇e and whose coefficients depend on ρ, e, $a(\rho,e)$, $c(\rho,e)$, and $d(\rho,e)$.

Let $\mathbb{I}_d$ be the $d{\times}d$ identity matrix. For any symmetric $2{\times}2$ block matrix $\mathbb{N}$

$$\mathbb{N} = \begin{pmatrix} n_{11}\mathbb{I}_d & n_{12}\mathbb{I}_d \\ n_{12}\mathbb{I}_d & n_{22}\mathbb{I}_d \end{pmatrix} \quad \text{we denote} \quad \mathbb{N}_2 := \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix}.$$

Given row vectors $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^d$, the quadratic form $(\boldsymbol{X},\boldsymbol{Y})\cdot\mathbb{N}\cdot(\boldsymbol{X},\boldsymbol{Y})^T$, generated by the $2{\times}2$ block matrix $\mathbb{N}$, is negative semi-definite if and only if $\mathbb{N}_2$ is negative semi-definite, i.e., $n_{22} \le 0$ and $\det(\mathbb{N}_2) \le 0$.

Lemma 3.3. *Assume that* (3.8)-(3.9) *hold. The quadratic form J is negative semi-definite if and only if*

$$ad\det(\Sigma) - \tfrac{1}{4}(d-a)^2\rho^{-2}s_e^2p_e^2 \ge 0. \tag{3.11}$$

Moreover, let $\lambda \in \mathbb{R}$ such that $d(1+\lambda) = a$, then

$$J + \lambda d\frac{\rho}{s_e}\nabla s_e\cdot\nabla s \le 0. \tag{3.12}$$

The inequality (3.12) *becomes strict if $a > 0$ and $d > 0$.*

Proof. Using the definition of $\boldsymbol{l}$, we re-write $\boldsymbol{J}$ in the following form:

$$\begin{aligned}\boldsymbol{J} = -as_e\nabla\rho\nabla e - ae\nabla\rho\nabla s_e + as_\rho|\nabla\rho|^2 + a\rho\nabla\rho\nabla s_\rho + ae\nabla\rho\nabla s_e - a\rho s_\rho\nabla\rho\nabla s_e \\ + d\rho s_e^{-1}\nabla s_e(s_\rho\nabla\rho + s_e\nabla e) + a\nabla\rho(s_\rho\nabla\rho + s_e\nabla e).\end{aligned}$$

This expression can be further simplified as follows:

$$\begin{aligned}\boldsymbol{J} &= 2as_\rho|\nabla\rho|^2 + a\rho\nabla\rho(s_{\rho\rho}\nabla\rho + s_{\rho e}\nabla e) \\ &\quad + (d-a)\rho s_\rho s_e^{-1}\nabla\rho(s_{\rho e}\nabla\rho + s_{ee}\nabla e) + d\rho\nabla e(s_{\rho e}\nabla\rho + s_{ee}\nabla e) \\ &= (\nabla\rho,\nabla e)^T\mathbb{N}(\nabla\rho,\nabla e),\end{aligned}$$

where the matrix $\mathbb{N}$ is defined by

$$\mathbb{N} = \begin{pmatrix} n_{11}\mathbb{I}_d & n_{12}\mathbb{I}_d \\ n_{12}\mathbb{I}_d & n_{22}\mathbb{I}_d\end{pmatrix}; \qquad \begin{aligned} n_{11} &= (d-a)\rho s_\rho s_e^{-1}s_{\rho e} + a\rho^{-1}\partial_\rho(\rho^2 s_\rho), \\ 2n_{12} &= (d-a)\rho s_\rho s_e^{-1}s_{ee} + (d+a)\rho s_{\rho e}, \\ n_{22} &= d\rho s_{ee}.\end{aligned}$$

Let us define the 2×2 block matrix $\mathbb{Q}$ obtained by setting $a = 0$ and $d = 1$ in $\mathbb{N}$:

$$q_{11} = \rho s_\rho s_e^{-1}s_{\rho e}, \qquad q_{12} = \rho s_\rho s_e^{-1}s_{ee} + \rho s_{\rho e}, \qquad q_{22} = \rho s_{ee}.$$

Notice that this definition implies that the quadratic form induced by $\mathbb{Q}$ is

$$(\nabla\rho,\nabla e)\cdot\mathbb{Q}\cdot(\nabla\rho,\nabla e)^T = \frac{\rho}{s_e}\nabla s_e\cdot\nabla s.$$

Now let us consider the following 2×2 block matrix $\mathbb{M} = \mathbb{N} + \lambda d\mathbb{Q}$ where $\lambda\in\mathbb{R}$. Let us set $d' = d(1+\lambda)$ and observe that

$$\begin{aligned} m_{11} &= (d'-a)\rho s_\rho s_e^{-1}s_{\rho e} + a\rho^{-1}\partial_\rho(\rho^2 s_\rho), \\ 2m_{12} &= (d'-a)\rho s_\rho s_e^{-1}s_{ee} + (d'+a)\rho s_{\rho e}, \\ m_{22} &= d'\rho s_{ee}.\end{aligned}$$

Observe finally that $J + \lambda d\frac{\rho}{s_e}\nabla s_e\cdot\nabla s = (\nabla\rho,\nabla e)\cdot\mathbb{M}\cdot(\nabla\rho,\nabla e)^T$.

To have a negative semi-definite form we need $m_{22} = d'\rho s_{ee} \le 0$, which means $0 \le d'$ since $s_{ee} < 0$ owing to the convexity assumption (2.8). We also need $\det(\mathbb{M}_2)$ to be non-negative,

$$\begin{aligned}\det(\mathbb{M}_2) &= ((d'-a)\rho s_\rho s_e^{-1}s_{\rho e} + a\rho^{-1}\partial_\rho(\rho^2 s_\rho))d'\rho s_{ee} \\ &\quad - \tfrac{1}{4}((d'-a)\rho s_\rho s_e^{-1}s_{ee} + (d'+a)\rho s_{\rho e})^2 \\ &= ad'\left(\partial_\rho(\rho^2 s_\rho)s_{ee} - \rho^2 s_{\rho e}^2\right) - \tfrac{1}{4}(d'-a)^2\rho^2 s_e^{-2}(s_e s_{\rho e} - s_\rho s_{ee})^2.\end{aligned}$$

Now if we set λ so that $d' = d(1+\lambda) = a$, then $\det(\mathbb{M}_2)$ is non-negative and $d' = a \ge 0$. Note in passing that upon setting $\lambda = 0$, this computation shows that $J \le 0$ if and only if (3.11) holds. ∎

Remark 3.3. Note that we could avoid invoking the convexity of the entropy in the above argument by taking $a = 0$ and $\lambda = -1$. This would however defeat the purpose of our enterprise whose primary goal is to find a nonzero viscous regularization of the mass conservation equation that ensures positivity of the density and is entropy compatible.

Remark 3.4. Note that $J < 0$ when $a = d$.

THEOREM 3.4 (Minimum Entropy Principle). *Assume that ρ_0 and e_0 are constant outside some compact set. Assume also that* (3.7)-(3.8)-(3.9) *hold. Assume that the solution to* (3.1)–(3.3) *is smooth, then the minimum entropy principle holds,*

$$\inf_{\boldsymbol{x}\in\mathbb{R}^d} s(\boldsymbol{x},t) \geq \inf_{\boldsymbol{x}\in\mathbb{R}^d} s_0(\boldsymbol{x}), \qquad \forall t \geq 0.$$

Proof. We re-write (3.6) as follows:

$$\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) + \nabla\cdot((es_e - \rho s_\rho)\boldsymbol{f} - s_e\boldsymbol{l}) - \boldsymbol{f}\cdot\nabla(es_e - \rho s_\rho) + \boldsymbol{l}\cdot\nabla s_e - s_e\mathbb{G}{:}\nabla\boldsymbol{u} = 0.$$

Upon using (3.9) we obtain

$$\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) - \nabla\cdot(d\rho\nabla s) - \boldsymbol{f}\cdot\nabla(es_e - \rho s_\rho) + \boldsymbol{l}\cdot\nabla s_e - s_e\mathbb{G}{:}\nabla\boldsymbol{u} = 0.$$

Let $N := -\boldsymbol{f}\cdot\nabla(es_e - \rho s_\rho) + \boldsymbol{l}\cdot\nabla s_e$. Using definition (3.10) we have $N = J - a\nabla\rho\cdot\nabla s$ and

$$\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) - \nabla\cdot(d\rho\nabla s) - a\nabla\rho\cdot\nabla s = -J + s_e\mathbb{G}{:}\nabla\boldsymbol{u} \geq 0. \tag{3.13}$$

Owing to Lemma 3.3 there is $\lambda \in \mathbb{R}$ so that $J + \lambda d\frac{\rho}{s_e}\nabla s_e\cdot\nabla s \leq 0$. Finally we have proved that

$$\begin{aligned}\rho(\partial_t s + \boldsymbol{u}\cdot\nabla s) - \nabla\cdot(d\rho\nabla s) - (a\nabla\rho + \lambda d\frac{\rho}{s_e}\nabla s_e)\cdot\nabla s&\\ = -J - \lambda d\frac{\rho}{s_e}\nabla s_e\cdot\nabla s + s_e\mathbb{G}{:}\nabla\boldsymbol{u} \geq 0.&\end{aligned} \tag{3.14}$$

By assumption all the fields are smooth and s is constant outside some compact set (i.e., ρ and e are constant outside some time-dependent compact set since the initial data are constant outside a compact set and the speed of propagation is finite). For each time t, s reaches its minimum; let $x_{\min}(t)$ be one point where the minimum of s is reached, then $\nabla s(x_{\min}(t),t) = 0$ and $\Delta s(x_{\min}(t),t) \geq 0$. The equation (3.14) implies that

$$\rho\partial_t s((x_{\min}(t),t)) - d\rho\Delta s(x_{\min}(t),t) \geq 0,$$

which in turn implies that $\rho\partial_t s((x_{\min}(t),t)) \geq 0$, and we conclude that the minimum entropy principle holds. ∎

Remark 3.5. Note that the condition (3.11) is not required to hold for the minimum principle to hold.

4. Entropy inequalities. We investigate in this section whether the regularization of the Euler equations (3.1)–(3.3) is compatible with some or all generalized entropy inequalities.

4.1. Generalized entropies. Let us consider all the generalized entropy identified in Harten et al. [10]. A function $\rho f(s)$ is called a generalized entropy if f is twice differentiable and

$$f'(s) > 0, \qquad f'(s)c_p^{-1} - f''(s) > 0, \qquad \forall(\rho,e) \in \mathbb{R}^2_+, \tag{4.1}$$

where $c_p(\rho,e) = T\partial_T s(p,T)$ is the specific heat at constant pressure. It is shown in Harten et al. [10] that $-\rho f(s)$ is strictly convex if and only if (4.1) holds, i.e., (4.1) characterizes the maximal set of admissible entropies for the compressible Euler equations that are of the form $\rho f(s)$.

THEOREM 4.1 (Entropy Inequalities). *Assume that* (3.7)-(3.8)-(3.9) *hold. Any weak solution to the regularized system* (3.1)-(3.3) *satisfies the entropy inequality*

$$\partial_t(\rho f(s)) + \nabla\cdot\big(\boldsymbol{u}\rho f(s) - d\rho\nabla f(s) - af(s)\nabla\rho\big) \geq 0. \tag{4.2}$$

for all generalized entropies $\rho f(s)$ if and only if $a = d$.

Proof. Let us multiply (3.13) by $f'(s)$,

$$\begin{aligned}\rho(\partial_t f(s) + \boldsymbol{u}\cdot\nabla f(s)) - \nabla\cdot(d\rho\nabla f(s)) + d\rho f''(s)|\nabla s|^2 - af'(s)\nabla\rho\cdot\nabla s \\ + Jf'(s) = f'(s)s_e\mathbb{G}{:}\nabla\boldsymbol{u}.\end{aligned}$$

We now multiply the mass conservation equation (3.1) by $f(s)$ and we add the result to the above equation:

$$\begin{aligned}\partial_t(\rho f(s)) + \nabla\cdot(\boldsymbol{u}\rho f(s)) - \nabla\cdot(d\rho\nabla f(s) + af(s)\nabla\rho) \\ + d\rho f''(s)|\nabla(s)|^2 + Jf'(s) = f'(s)s_e\mathbb{G}{:}\nabla\boldsymbol{u}\end{aligned}$$

We now investigate the sign of the quantity $d\rho f''(s)|\nabla s|^2 + Jf'(s)$.

Owing to (4.1), we have

$$d\rho f''(s)|\nabla s|^2 + Jf'(s) < (d\rho c_p^{-1}|\nabla s|^2 + J)f'(s). \tag{4.3}$$

We now need to determine the sign of the quadratic form in the right hand side of the above inequality:

$$\begin{aligned}d\rho c_p^{-1}|\nabla s|^2 + J &= d\rho c_P^{-1}|s_\rho\nabla\rho + s_e\nabla e|^2 + J \\ &= d\rho c_P^{-1}(s_\rho^2|\nabla\rho|^2 + 2s_\rho s_e\nabla\rho\cdot\nabla e + s_e^2|\nabla e|^2) + J = d\rho(\nabla\rho,\nabla e)\cdot\mathbb{S}\cdot(\nabla\rho,\nabla e)^T,\end{aligned}$$

where the coefficients of the 2×2 block matrix $\mathbb{S}$ are defined as follows:

$$\begin{aligned}ds_{11} &= dc_P^{-1}s_\rho^2 + \big((d-a)s_\rho s_e^{-1}s_{\rho e} + a\rho^{-2}\partial_\rho(\rho^2 s_\rho)\big) \\ 2ds_{12} &= 2dc_P^{-1}s_\rho s_e + \big((d-a)s_\rho s_e^{-1}s_{ee} + (d+a)s_{\rho e}\big) \\ ds_{22} &= d(c_P^{-1}s_e^2 + s_{ee}),\end{aligned}$$

and can be re-written into the following form

$$\begin{aligned}ds_{11} &= d(c_P^{-1}s_\rho^2 + \rho^{-2}\partial_\rho(\rho^2 s_\rho)) + (d-a)s_e^{-1}\left(s_\rho s_{\rho e} - s_e\rho^{-2}\partial_\rho(\rho^2 s_\rho)\right) \\ 2ds_{12} &= 2d(c_P^{-1}s_\rho s_e + s_{\rho e}) + (d-a)s_e^{-1}\left(s_\rho s_{ee} - s_e s_{\rho e}\right) \\ ds_{22} &= d(c_P^{-1}s_e^2 + s_{ee}).\end{aligned}$$

Then upon setting $x = 1 - \frac{a}{d}$ we infer that

$$s_{11} = h_{11} + x\rho^{-2}s_e p_\rho, \qquad 2s_{12} = 2h_{12} + x\rho^{-2}s_e p_e, \qquad s_{22} = h_{22} \tag{4.4}$$

where the 2×2 matrix $\mathbb{H}_2$ is defined by

$$\mathbb{H}_2 = \begin{pmatrix} s_\rho^2 c_P^{-1} + \rho^{-2}\partial_\rho(\rho^2 s_\rho) & s_\rho s_e c_P^{-1} + s_{\rho e} \\ s_\rho s_e c_P^{-1} + s_{\rho e} & s_e^2 c_P^{-1} + s_{ee}\end{pmatrix}$$

is shown to be negative in Lemma A.3. In particular we have $s_{22} = h_{22} = s_e^2 c_P^{-1} + s_{ee} < 0$ owing to the inequality $c_p T_e > 1$ established in (A.12). As a result, the matrix $\mathbb{S}$ is negative semi-definite if and only if the determinant of $\mathbb{S}_2$ is non-negative,

$$\begin{aligned}\det(\mathbb{S}_2) &= h_{11}h_{22} + xh_{22}\rho^{-2}s_e p_\rho - (h_{12} + \tfrac{1}{2}x\rho^{-2}s_e p_e)^2\\ &= \det(\mathbb{H}_2) + x\rho^{-2}s_e(h_{22}p_\rho - h_{12}p_e) - \tfrac{1}{4}x^2\rho^{-4}s_e^2 p_e^2.\end{aligned}$$

According to Lemma A.3 we have $\det(\mathbb{H}_2)$ and $h_{22}p_\rho - h_{12}p_e = 0$. This proves that

$$\det(\mathbb{S}_2) = -\tfrac{1}{4}x^2\rho^{-4}s_e^2 p_e^2.$$

In conclusion, $\mathbb{S}$ is negative semi-definite if and only if $x = 0$, ie $a = d$.

The above argument shows that $d\rho f''(s)|\nabla s|^2 + Jf'(s) < 0$ if $a = d$. This proves that all the generalized entropy inequalities are satisfied if $a = d$.

If $a \neq d$ we consider generalized entropies such that $f''(s) = (1-\epsilon)f'(s)c_p(s,\rho)$, $\epsilon \in (0,1)$ (it is always possible to solve this ODE for any fixed value of ρ). For this subclass of generalized entropies, we have

$$d\rho f''(s)|\nabla s|^2 + Jf'(s) = ((1-\epsilon)d\rho c_p^{-1}|\nabla s|^2 + J)f'(s). \tag{4.5}$$

From the proof of Theorem(4.1), we know that the quadratic form $d\rho c_p^{-1}|\nabla s|^2 + J = d\rho(\nabla\rho, \nabla e){\cdot}\mathbb{S}(\rho, e){\cdot}(\nabla\rho, \nabla e)^T$ is negative semi-definite if and and only $a = d$. Let $(\rho^*, e^*) \in \mathbb{R}_+^2$ be a pair of positive numbers so that $a(\rho^*, e^*) \neq d(\rho^*, e^*)$. Since the quadratic form generated by $\mathbb{S}(\rho^*, e^*)$ is not negative semi-definite, there exists a pair of row vectors $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^d$ so that $(\boldsymbol{X}, \boldsymbol{Y}){\cdot}\mathbb{S}(\rho^*, e^*){\cdot}(\boldsymbol{X}, \boldsymbol{Y})^T > 0$. It is always possible to choose ϵ small enough so that

$$(\boldsymbol{X}, \boldsymbol{Y}){\cdot}\mathbb{S}(\rho^*, e^*){\cdot}(\boldsymbol{X}, \boldsymbol{Y}) - \epsilon d^*\rho^*(c_p^*)^{-1}|s_\rho^*\boldsymbol{X} + s_e^*\boldsymbol{Y}|^2 f'(s^*) > 0.$$

Now we define an initial state so that in the neighborhood of the origin we have the following data: $\boldsymbol{m}_0 = 0$, $\rho_0(\boldsymbol{x}) = \rho^* + \boldsymbol{x}{\cdot}\boldsymbol{X}$, $e_0(\boldsymbol{x}) = e^* + \boldsymbol{x}{\cdot}\boldsymbol{Y}$. Notice that with this choice $\nabla\boldsymbol{u}_0 = 0$, $\nabla\rho_0 = \boldsymbol{X}$ and $\nabla e_0 = \boldsymbol{Y}$; therefore $d\rho_0 f''(s_0)|\nabla(s_0)|^2 + J(\rho_0, e_0)f'(s_0) - f'(s_0)s_e(\rho_0, e_0)\mathbb{G}{:}\nabla\boldsymbol{u}_0 > 0$, which proves that the entropy inequality is violated at the origin close to the initial time. In conclusion $a = d$ is a necessary condition for all the generalized entropy inequalities to be satisfied. ∎

Remark 4.1. Upon re-defining the velocity $\widetilde{\boldsymbol{u}} = \boldsymbol{u} + (d-a)\nabla\log\rho$, the entropy inequality (4.2) can be re-written into the following form

$$\partial_t(\rho f(s)) + \nabla{\cdot}(\widetilde{\boldsymbol{u}}\rho f(s)) - \nabla{\cdot}(d\rho\nabla\rho f(s)) \geq 0. \tag{4.6}$$

Remark 4.2. Theorem 4.1 proves that the family of regularization such that $a = d$ is the most robust in the sense that it is the most dissipative. This result suggests that the choice $a = d$ may be a very good candidate to construct a robust first-order numerical method for solving the compressible Euler equations.

Corollary 4.2. *Let α be a real number, $\alpha < 1$, and assume that* (3.7)-(3.8)-(3.9) *hold. Any weak solution to the regularized system* (3.1)-(3.3) *satisfies the entropy inequality* (4.2) *for all the generalized entropies $\rho f(s)$ such that $f' > 0$ and $\alpha c_p^{-1} f' \geq f''$ if $2\Gamma - 2\Delta^{\frac{1}{2}} < 1 - \frac{a}{d} < 2\Gamma + 2\Delta^{\frac{1}{2}}$ where $\Gamma = (1-\alpha)det(\Sigma)\rho^2 s_e^{-2} p_e^{-2}$ and $\Delta = \Gamma(1+\Gamma)$.*

Proof. We proceed as in the proof of Theorem 4.1 where we replace $\mathbb{H}$ by $\mathbb{H}^\alpha$ where c_p^{-1} is substituted by αc_p^{-1}. Upon replacing c_p^{-1} by αc_p^{-1} in the proof of Lemma A.3, we infer that $\det(\mathbb{H}_2^\alpha) = (1-\alpha)\rho^{-2}\det(\Sigma)$ and $s_e(h_{22}^\alpha p_\rho - h_{21}^\alpha p_e) = (1-\alpha)\rho^{-2}\det(\Sigma)$. Then by defining $\mathbb{S}^\alpha$ as in (4.4), where $\mathbb{H}$ is substituted by $\mathbb{H}^\alpha$, we obtain

$$\begin{aligned}\det(\mathbb{S}_2^\alpha) &= (1-\alpha)\rho^{-2}\det(\Sigma) + x\rho^{-2}(1-\alpha)\det(\Sigma) - \tfrac{1}{4}x^2\rho^{-4}s_e^2p_e^2 \\ &= \rho^{-2}((1-\alpha)\det(\Sigma)(1+x) - \tfrac{1}{4}x^2\rho^{-2}s_e^2p_e^2),\end{aligned}$$

where we defined $x = 1 - \frac{a}{d}$. Then upon setting $\Gamma = (1-\alpha)\det(\Sigma)\rho^2 s_e^{-2}p_e^{-2}$ and $\Delta = \Gamma(1+\Gamma)$, we conclude that the matrix $\mathbb{S}^\alpha$ is negative definite if

$$2\Gamma - 2\Delta^{\frac{1}{2}} < 1 - \frac{a}{d} < 2\Gamma + 2\Delta^{\frac{1}{2}},$$

which ends the proof. ▯

Corollary 4.3. *Any weak solution to the regularized system* (3.1)-(3.3) *satisfies the entropy inequality* (4.2) *for the physical entropy ρs (i.e., $f(s) = s$) if* $2\Gamma - 2\Delta^{\frac{1}{2}} < 1 - \frac{a}{d} < 2\Gamma + 2\Delta^{\frac{1}{2}}$ *where $\Gamma = det(\Sigma)\rho^2 s_e^{-2}p_e^{-2}$ and $\Delta = \Gamma(1+\Gamma)$.*

Proof. Take $\alpha = 0$ in Corollary 4.2 or use (3.11). ▯

4.2. Ideal gas. Let us illustrate the above theory in the case of ideal gases, i.e., $s = \log(e^{\frac{1}{\gamma-1}}\rho^{-1})$ with $\gamma > 1$. We have $c^2 = \gamma(\gamma-1)e$, $c_p = \gamma(\gamma-1)^{-1}$, $\det(\Sigma) = (\gamma-1)^{-1}e^{-2}$, $\boldsymbol{f} = a\nabla\rho$, and $\boldsymbol{l} = \gamma de(\frac{a}{d} - 1 + \frac{1}{\gamma})\nabla\rho + d\rho\nabla e$. The range for the ratio ad^{-1} for Corollary 4.3 to hold is

$$\frac{2}{\gamma-1}(1-\sqrt{\gamma}) < 1 - \frac{a}{d} < \frac{2}{\gamma-1}(1+\sqrt{\gamma}). \tag{4.7}$$

In particular the choice $1 - \frac{a}{d} = \frac{1}{\gamma}$ is clearly in the admissible range for the physical entropy inequality. This particular choice is such that $\boldsymbol{l} = d\rho\nabla e$ and $\boldsymbol{f} = d\frac{\gamma-1}{\gamma}\nabla\rho$, i.e., $\boldsymbol{l}$ does involve any mass dissipation.

5. Discussion. We show in this section that the regularization proposed above is a bridge between the Navier-Stokes and parabolic regularizations of the Euler equations that reconciles the two point of views.

5.1. Parabolic regularization. The first natural question that comes to mind is how different is the general regularization (3.1)-(3.3) from other known regularizations. In particular how does it differ from the parabolic regularization (2.11)-(2.14)? The answer is given by the following, somewhat a priori frustrating result:

Proposition 5.1. *The parabolic regularization* (2.11)-(2.13) *is identical to* (3.1)-(3.3) *with* (3.7)-(3.9) *where $a = d = \epsilon$, $\mathbb{G} = \epsilon\rho\nabla\boldsymbol{u}$.*

Proof. The equality $a = \epsilon$ comes from the identification $\boldsymbol{f} = \epsilon\nabla\rho$ in the mass conservation equation in (2.11) and (3.1). The identity $\epsilon\nabla\boldsymbol{m} = \epsilon\nabla\rho\otimes\boldsymbol{u} + \epsilon\rho\nabla\boldsymbol{u}$ implies that upon setting $\mathbb{g} = \boldsymbol{f}\otimes\boldsymbol{u} + \mathbb{G}$ with $\mathbb{G} = \epsilon\rho\nabla\boldsymbol{u}$, the momentum conservation equations in (2.12) and (3.2) are identical. Upon observing that

$$\mathbb{g}\cdot\boldsymbol{u} = \boldsymbol{u}^2\boldsymbol{f} + \mathbb{G}\cdot\boldsymbol{u} = \epsilon\boldsymbol{u}^2\nabla\rho + \frac{1}{2}\epsilon\rho\nabla\boldsymbol{u}^2 = \epsilon\nabla\frac{1}{2}\rho\boldsymbol{u}^2 + \frac{1}{2}\boldsymbol{u}^2\boldsymbol{f},$$

we obtain that

$$\epsilon\nabla\boldsymbol{E} = \epsilon\nabla(\rho e) + \nabla\frac{1}{2}\epsilon\rho\boldsymbol{u}^2 = \epsilon\nabla(\rho e) - \frac{1}{2}\boldsymbol{u}^2\boldsymbol{f} + \mathbb{g}\cdot\boldsymbol{u}.$$

The energy equations in (2.13) and (3.3) are identical if one sets $\boldsymbol{h} = \boldsymbol{l} - \frac{1}{2}\boldsymbol{u}^2\boldsymbol{f}$, with $\boldsymbol{l} = \epsilon\nabla(\rho e)$, meaning $d = \epsilon$. ▯

Remark 5.1. Even when $a = d$, one important interest of the class of regularization (3.1)-(3.3), when compared to the monolithic parabolic regularization, is that it decouples the regularization on the velocity from that on the density and internal energy. In particular the regularization on the velocity can be made rotation invariant by making the tensor $\mathbb{G}$ a function of the symmetric gradient $\nabla^s \boldsymbol{u}$. This decoupling was not a priori evident (at least to us) when looking at the monolithic parabolic regularization (2.11)-(2.13).

5.2. Connection with phenomenological models. When introducing the structural assumptions (3.7)-(3.9) in the balance equations (3.1)-(3.3) we obtain the following system:

$$\partial_t \rho + \nabla\cdot \boldsymbol{m} - \nabla\cdot \boldsymbol{f} = 0, \tag{5.1}$$

$$\partial_t \boldsymbol{m} + \nabla\cdot(\boldsymbol{u}\otimes \boldsymbol{m}) + \nabla p - \nabla\cdot(\mathbb{G}(\nabla^s \boldsymbol{u}) + \boldsymbol{f}\otimes \boldsymbol{u}) = 0, \tag{5.2}$$

$$\partial_t E + \nabla\cdot(\boldsymbol{u}(E+p)) - \nabla\cdot(\boldsymbol{l} + \tfrac{1}{2}\boldsymbol{u}^2 \boldsymbol{f} + \mathbb{G}(\nabla^s \boldsymbol{u})\cdot \boldsymbol{u}) = 0, \tag{5.3}$$

When looking at (5.1)-(5.3) it is not immediately clear how this system can be reconciled either with the Navier-Stokes regularization or with any phenomenological modeling of dissipation.

It is remarkable that this exercise can actually been done by introducing the quantity $\boldsymbol{u}_m = \boldsymbol{u} - \rho^{-1}\boldsymbol{f}$. The conservation equations then becomes

$$\partial_t \rho + \nabla\cdot(\boldsymbol{u}_m \rho) = 0, \tag{5.4}$$

$$\partial_t \boldsymbol{m} + \nabla\cdot(\boldsymbol{u}_m \otimes \boldsymbol{m}) + \nabla p - \nabla\cdot(\mathbb{G}(\nabla^s \boldsymbol{u})) = 0, \tag{5.5}$$

$$\partial_t E + \nabla\cdot(\boldsymbol{u}_m E) - \nabla\cdot(\boldsymbol{l} - e\boldsymbol{f}) + \nabla\cdot\big((p\mathbb{I} - \mathbb{G}(\nabla^s \boldsymbol{u}))\cdot \boldsymbol{u}\big) = 0, \tag{5.6}$$

with again $\boldsymbol{m} = \rho \boldsymbol{u}$ and $E = \rho e + \frac{1}{2}\rho \boldsymbol{u}^2$. It is surprising that this system involves two velocities. It is also somewhat surprising to observe that the above system resembles the Navier-Stokes regularization. In particular if one sets $a = d$, the term $\boldsymbol{l} - e\boldsymbol{f}$ becomes $d\rho\nabla e$, which upon assuming $\mathrm{d}e = c_v\,\mathrm{d}T$, reduces to $d(\rho,e)\rho c_v \nabla T$, i.e., one obtains Fourier's law: $\boldsymbol{l} - e\boldsymbol{f} = d(\rho,e)\rho c_v \nabla T$.

During the preparation of this paper, it has been brought to our attention that the regularization model that we propose above somewhat resembles, at least formally, a model of fluid dynamics of Brenner [1] (see e.g., equations (1) to (5) in Brenner [1]). The author has derived the above system of conservation equations (up to some non-essential disagreement on the term $\boldsymbol{l} - e\boldsymbol{f}$) by invoking theoretical arguments from Öttinger [14] and phenomenological considerations. The mathematical properties of this system have been investigated thoroughly by Feireisl and Vasseur [4]. Brenner has been defending the idea that it makes phenomenological sense to distinguish the so-called mass velocity, $\boldsymbol{u}_m$, from the so-called volume velocity, $\boldsymbol{u}$, since 2004 (or so). We do not want to enter this debate, but this idea seems to be supported by our mathematical derivation (5.4)-(5.6) that did not invoke any had oc phenomenological assumption. Recall that our primal motivation in this project is to find a regularization of the compressible Euler equations that can serve as a good numerical device, and by being good we mean that the model must give positive density, positive internal energy, a minimum entropy principle and be compatible with a large class of entropy inequalities.

5.3. Conclusions. Let us finally rephrase our findings. In its most general form, the regularized system (5.4)-(5.6) can be re-written as follows:

$$\partial_t\rho + \nabla\cdot(\boldsymbol{u}_m\rho) = 0, \tag{5.7}$$
$$\partial_t\boldsymbol{m} + \nabla\cdot(\boldsymbol{u}_m\otimes\boldsymbol{m}) + \nabla p - \nabla\cdot(G(\nabla^s\boldsymbol{u})) = 0, \tag{5.8}$$
$$\partial_t E + \nabla\cdot(\boldsymbol{u}_m E) - \nabla\cdot\boldsymbol{q} + \nabla\cdot\big((p\mathbb{I} - G(\nabla^s\boldsymbol{u}))\cdot\boldsymbol{u}\big) = 0 \tag{5.9}$$
$$\boldsymbol{u}_m = \boldsymbol{u} - a(\rho,e)\nabla\log\rho \tag{5.10}$$
$$\boldsymbol{q} = (a-d)p\nabla\log\rho + d\rho\nabla e, \qquad a(\rho,e)\geq 0,\ d(\rho,e)\geq 0. \tag{5.11}$$

It is established in Lemma 3.1 that the definition of $\boldsymbol{f} = a(\rho,e)\nabla\rho$ is compatible with the positive density principle. The particular form of $\boldsymbol{q}$ in (5.11) results from the definition of $\boldsymbol{l}$, see (3.9), which is required for the minimum entropy principle to hold, as established in Theorem 3.4. It is finally proved in Theorem 4.1 that the most robust regularization, i.e., that which is compatible with all the generalized entropy à la Harten et al. [10], corresponds to the choice $a = d$. Various relaxations of the constraint $a = d$ are described in Corollary 4.2 and in Corollary 4.3. As observed in §5.1, the parabolic regularization can be put into the form (5.7)-(5.11) with the particular choice $\mathbb{G} = a\nabla\boldsymbol{u}$, which is not rotation invariant and uses the same viscosity coefficient for all fields.

Appendix A. Primer in thermodynamics. We collect in this appendix standard results from thermodynamics that are used in the paper.

A.1. Chain rule. Let $\Phi : \mathbb{R}^2 \ni (\alpha,\beta) \longmapsto \Phi(\alpha,\beta) = (\phi(\alpha,\beta),\psi(\alpha,\beta)) \in \mathbb{R}^2$ be a $\mathcal{C}^1$-diffeomorphism. The following holds:

$$\frac{1}{\partial_\alpha\phi\partial_\beta\psi - \partial_\beta\phi\partial_\alpha\psi}\begin{pmatrix}\partial_\beta\psi & -\partial_\beta\phi\\ -\partial_\alpha\psi & \partial_\alpha\phi\end{pmatrix} = \begin{pmatrix}\partial_\phi\alpha & \partial_\psi\alpha\\ \partial_\phi\beta & \partial_\psi\beta\end{pmatrix}. \tag{A.1}$$

In particular if $\phi(\alpha,\beta) = \alpha$ we have

$$\partial_\alpha\beta(\alpha,\psi) = -\frac{\partial_\alpha\psi(\alpha,\beta)}{\partial_\beta\psi(\alpha,\beta)}, \qquad \partial_\psi\beta(\alpha,\psi) = \frac{1}{\partial_\beta\psi(\alpha,\beta)} \tag{A.2}$$

A.2. Speed of sound. The square of the speed of sound is defined to be

$$c^2 := \partial_\rho p(\rho,s), \tag{A.3}$$

i.e., c^2 is the partial derivative of the pressure as a function of the density and the specific entropy. Using the chain rule, this definition is equivalent to

$$c^2 = \partial_\rho p(\rho,s) = \partial_\rho p(\rho,e) + \partial_e p(\rho,e)\partial_\rho e(\rho,s), \tag{A.4}$$

and using (A.2) with $\alpha=\rho$, $\beta=e$, $\psi=s$, one obtains

$$c^2 = p_\rho - \frac{s_\rho}{s_e}p_e(\rho,e). \tag{A.5}$$

Using the following representations of p_e and p_ρ:

$$p_e = \rho^2 s_e^{-2}(s_\rho s_{ee} - s_e s_{\rho e}), \qquad p_\rho = s_e^{-2}(\rho^2 s_\rho s_{\rho e} - s_e\partial(\rho^2 s_\rho)), \tag{A.6}$$

the expression (A.5) also gives

$$c^2 = \rho^2 s_e^{-3}(2s_e s_\rho s_{\rho e} - s_e^2\rho^{-2}\partial(\rho^2 s_\rho) - s_\rho^2 s_{ee}). \tag{A.7}$$

A.3. Convexity of the entropy, det(Σ). Let us define the following matrix

$$\Sigma := \rho \begin{pmatrix} \rho^{-2}\partial_\rho(\rho^2 s_\rho) & s_{\rho e} \\ s_{\rho e} & s_{ee} \end{pmatrix}, \tag{A.8}$$

which, up to the ρ factor, is the Hessian of the entropy with respect to the variables (ρ^{-1}, e). The convexity assumption on the entropy implies that s_{ee} and $\rho^{-1}\partial_\rho(\rho^2 s_\rho)$ are negative. We have the following characterization of the determinant of Σ.

$$\det(\Sigma) = s_e^3(p_\rho T_e - p_e T_\rho). \tag{A.9}$$

To prove this statement, we observe that the following holds owing to (A.6):

$$\begin{aligned} s_e^2 T_e &= -s_{ee}, & s_e^2 T_\rho &= -s_{\rho e}, \\ s_e^2 p_e &= \rho^2(s_\rho s_{ee} - s_e s_{\rho e}) & s_e^2 p_\rho &= \rho^2\left(s_\rho s_{\rho e} - s_e \rho^{-2}\partial_\rho(\rho^2 s_\rho)\right). \end{aligned}$$

The result is now evident.

A.4. Specific heat at constant pressure . The specific heat at constant pressure is defined to be $c_p(\rho, e) = T\partial_T s(T, p)$.

Lemma A.1. *The quantities* $\det(\Sigma)$, c^2 *and* c_p *are related by*

$$c_p \det(\Sigma) = s_e^3 c^2. \tag{A.10}$$

Proof. Using the chain rule, we can re-write the above definition as follows:

$$c_p(\rho, e) = s_e^{-1}(s_\rho \rho_T(p, T) + s_e e_T(p, T)).$$

The change of variable formula (A.1) with the convention $(\alpha = \rho, \beta = e)$ and $(\phi = p, \psi = T)$ gives

$$\rho_T(p, T) = \frac{-p_e}{p_\rho T_e - p_e T_\rho}, \qquad e_T(p, T) = \frac{p_\rho}{p_\rho T_e - p_e T_\rho}.$$

We then have the following expression for c_p

$$c_p = s_e^{-1}\frac{(p_\rho s_e - p_e s_\rho)}{p_\rho T_e - p_e T_\rho}. \tag{A.11}$$

Then using the expression of c^2 in (A.5) and the relation (A.9), we arrive at the desired expression. ∎

Lemma A.2. *The following holds:*

$$c_p T_e > 1. \tag{A.12}$$

Proof. The definition of c_p implies that we need to estimate $Ts_T(p, T)T_e(\rho, e)$. The chain rule implies

$$1 = Ts_e(\rho, e) = Ts_p(p, T)p_e(\rho, e) + Ts_T(p, T)T_e(\rho, e).$$

The result will be established if we can prove that $s_p(p, T)p_e(\rho, e) < 0$. We now calculate $s_p(p, T)$. The chain rule implies again that

$$(s_p(p, T))^{-1} = p_s(s, T) = p_\rho(\rho, e)\rho_s(s, T) + p_e(\rho, e)e_s(s, T).$$

Then using (A.1) with the convention ($\alpha = \rho, \beta = e$) and ($\phi = s, \psi = T$) gives

$$\rho_s(s,T) = \frac{T_e}{s_\rho T_e - s_e T_\rho}, \qquad e_s(s,T) = \frac{-T_\rho}{s_\rho T_e - s_e T_\rho}.$$

This in turn implies that

$$(s_p(p,T))^{-1} = p_s(s,T) = \frac{p_\rho T_e - p_e T_\rho}{s_\rho T_e - s_e T_\rho} = -\frac{s_e^{-3}\det(\Sigma)}{\rho^{-2} p_e},$$

since $s_\rho T_e - s_e T_\rho = s_e^{-2}(-s_\rho s_{ee} + s_e s_{\rho e}) = -\rho^{-2} p_e$, where we used (A.6). In conclusion $s_p(p,T)p_e(\rho,e) = -s_e^3 p_e^2 \rho^{-2}\det(\Sigma)^{-1} < 0$, owing to (2.8) and (2.10), which concludes the proof. ☐

Remark A.1. Note in passing that the convexity assumption (2.8) implies that $T_e > 0$, which owing to (A.12) implies that $c_p > 0$. This in turn implies that $c^2 > 0$ owing to (A.10), i.e., the Euler system (2.1)-(2.4) is hyperbolic under the convexity assumption (2.8) and the positivity assumption on the temperature (2.10). Positivity of the pressure is not needed to establish this fact.

A.5. Matrix $\mathbb{H}_2$. Investigations on entropy inequalities involve the quadratic form induced by the matrix $\mathbb{H}_2$

$$\mathbb{H}_2 = \begin{pmatrix} s_\rho^2 c_P^{-1} + \rho^{-2}\partial_\rho(\rho^2 s_\rho) & s_\rho s_e c_P^{-1} + s_{\rho e} \\ s_\rho s_e c_P^{-1} + s_{\rho e} & s_e^2 c_P^{-1} + s_{ee} \end{pmatrix}$$

Some key properties of this matrix are collected in the following lemma.

Lemma A.3. *The following hold:*

(i) $\det(\mathbb{H}_2) = 0$*.*
(ii) $\mathbb{H}_2$ *is negative semi-definite.*
(iii) $h_{22} p_\rho - h_{12} p_e = 0$*.*

Proof. (i) Using the expressions (A.7) and (A.9) for the speed of sound, c^2, and $\det(\Sigma)$, and the relation (A.10), the determinant of $\mathbb{H}_2$ is re-written as follows:

$$\begin{aligned}\det(\mathbb{H}_2) &= (s_\rho^2 c_P^{-1} + \rho^{-2}\partial_\rho(\rho^2 s_\rho))(s_e^2 c_P^{-1} + s_{ee}) - (s_\rho s_e c_P^{-1} + s_{\rho e})^2 \\ &= \rho^{-2}\det(\Sigma) + c_P^{-1}(s_e^2 \rho^{-2}\partial_\rho(\rho^2 s_\rho) + s_\rho^2 s_{ee} - 2 s_\rho s_e s_{\rho e}) \\ &= \rho^{-2}\det(\Sigma) - c_P^{-1} c^2 \rho^{-2} s_e^3 = 0.\end{aligned}$$

This is essentially the result established in Harten et al. [10, p. 2126].

(ii) Owing to the inequality $1 < c_p T_e$ established in (A.12), we infer that $h_{22} = s_e^2 c_P^{-1} + s_{ee} < 0$, which together with (i) proves statement (ii).

(iii) Let us compute $s_e^{-2}(h_{22} p_\rho - h_{12} p_e)$,

$$\begin{aligned}s_e^{-2}(h_{22} p_\rho - h_{12} p_e) &= (c_p^{-1} - T_e) p_\rho - (s_\rho s_e^{-1} c_p^{-1} - T_\rho) p_e \\ &= p_e T_\rho - p_\rho T_e + c_p^{-1} s_e^{-1}(s_e p_\rho - s_\rho p_e).\end{aligned}$$

This proves that $s_e^{-2}(h_{22} p_\rho - h_{12} p_e) = 0$ owing to (A.11). ☐

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