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Penrose Limits, PP-Waves and Deformed M2-branes

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ABSTRACT

Motivated by the recent discussions of the Penrose limit of $\mathrm{AdS}_5 \times S^5$, we examine a more general class of supersymmetric pp-wave solutions of the type IIB theory, with a larger number of non-vanishing structures in the self-dual 5-form. One of the pp-wave solutions can be obtained as a Penrose limit of a D3/D3 intersection. In addition to 16 standard supersymmetries these backgrounds always allow for supernumerary supersymmetries. The latter are in one-to-one correspondence with the linearly-realised world-sheet supersymmetries of the corresponding exactly-solvable type IIB string action. The pp-waves provide new examples where supersymmetries will survive in a T-duality transformation on the x^+ coordinate. The T-dual solutions can be lifted to give supersymmetric deformed M2-branes in $D = 11$. The deformed M2-brane is dual to a three-dimensional field theory whose renormalisation group flow runs from the conformal fixed point in the infra-red regime to a non-conformal theory as the energy increases. At a certain intermediate energy scale there is a phase transition associated with a naked singularity of the M2-brane. In the ultra-violet limit the theory is related by T-duality to an exactly-solvable massive IIB string theory.

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1 Introduction

Maximally supersymmetric type IIB pp-waves [1] turn out to arise as the Penrose limit [2] of $\mathrm{AdS}_5 \times S^5$ [3, 4]. The pp-wave provides a background that gives rise to an exactly-solvable string theory, with free massive fields in the light-cone gauge [5, 4]. This provides a way of studying the dual $\mathcal{N} = 4$ superconformal field theory in the Penrose limit [4]. Owing to the exact solvability of the system, there has been a burgeoning activity in the subject [6]–[14]. In this paper we shall consider other examples of supersymmetric pp-waves, with more general structures for the 5-form field strength. Our focus will be on examples where there are Killing spinors that are independent of the x^+ coordinate, thereby allowing us to obtain solutions that remain supersymmetric after T-dualisation. After lifting these to $D = 11$, we obtain supersymmetric deformed M2-branes. We also show that these pp-waves correspond to exactly-solvable massive string theories with linearly-realised world-sheet supersymmetries.

The Penrose limit of the $\mathrm{AdS}_5 \times S^5$ solution of the type IIB theory is given by

$$ds^2 = -4dx^+ \, dx^- - \tfrac{1}{16}\mu^2 \, z_i^2 \, dx^{+2} + dz_i^2 \,, \tag{1}$$

$$F_5 = \mu \, dx^+ \wedge \Phi_{(4)} \,, \tag{2}$$

where the 4-form $\Phi_{(4)}$, which is self-dual in the flat metric dz_i^2 on $\mathbb{E}^8$, is given by

$$\Phi_{(4)} = dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4 + dz^5 \wedge dz^6 \wedge dz^7 \wedge dz^8 \,. \tag{3}$$

This particular solution preserves all the supersymmetry [1].

This is a special case of a more general class of pp-wave solution, in which $\Phi_{(4)}$ in (2) is replaced by any constant self-dual 4-form on $\mathbb{E}^8$, and with the metric (1) generalised to

$$ds^2 = -4dx^+ \, dx^- + H \, dx^{+2} + dz_i^2 \,, \tag{4}$$

where H is a function on $\mathbb{E}^8$ satisfying the equation

$$\Box H = -\tfrac{1}{48} \, \mu^2 \, |\Phi_{(4)}|^2 \,. \tag{5}$$

Different choices of $\Phi_{(4)}$ give rise to different amounts of preserved supersymmetry; generically there will always be 16 Killing spinors, with additional ones for special choices $\Phi_{(4)}$, provided also that H is quadratic in z^i, $H = c_0 - \mu_i^2 \, z_i^2$, with appropriate choices for the μ_i. For example, there are an additional 16 Killing spinors if $\Phi_{(4)}$ has the form (3) (or a structure related to this one by symmetry) and H is quadratic with all the μ_i equal [1].[1]

We shall refer to the generic 16 Killing spinors that always occur in any pp-wave background as "standard Killing spinors," whilst the additional ones that occur only in special cases will be referred to as "supernumerary Killing spinors."

Writing (4) in the form

$$ds^2 = H\,(dx^+ - 2H^{-1}\,dx^-)^2 - 4\,H^{-1}\,dx^{-2} + dz_i^2\,, \tag{6}$$

we can perform a T-duality transformation on the $\partial/\partial x^+$ Killing direction, thereby obtaining a deformed string solution of the type IIA theory, where the function H has the 4-form $F_{(4)} = \mu\,\Phi_{(4)}$ as its source. Lifting this to $D = 11$, one then obtains a deformed M2-brane solution of M-theory, in which the 4-form carries an additional flux $\mu\,\Phi_{(4)}$:

$$\begin{aligned} ds_{11}^2 &= H^{-2/3}\,(-dt^2 + dx_1^2 + dx_2^2) + H^{1/3}\,dz_i^2\,, \\ F_{(4)} &= dt \wedge dx_1 \wedge dx_2 \wedge dH^{-1} + \mu\,\Phi_{(4)}\,. \end{aligned} \tag{7}$$

The coordinates t and x_1 are $2x^-$ and x^+ respectively, and x_2 is the eleventh coordinate. For convenience, we are assuming here that H is positive so that x^+ is spacelike and x^- is timelike. This can easily be achieved, for a range of the coordinates z^i, by taking the solution of (5) to be $H = c_0 - \mu_i^2\,z_i^2$, where c_0 is a positive constant. The inclusion of this constant is not in contradiction with what one can obtain from a Penrose limit, since it can be introduced merely by performing a coordinate transformation $x^- \longrightarrow x^- - \frac{1}{4}c_0\,x^+$. The inclusion of this constant has the merit of making the discussion of T-duality simpler, since the Killing coordinate can then be taken to be spacelike rather than timelike. We shall return to this point later, in section 6.

Although $\partial/\partial x^+$ is a Killing direction, allowing the above T-dualisation to be performed, the situation regarding supersymmetry is a little more complicated. For generic choices of $\Phi_{(4)}$, the associated 16 Killing spinors will all depend (periodically) on x^+, and consequently they will all be projected out, at the level of the low-energy effective field theory, in the circle reduction involved in the T-duality transformation. Of course at the level of the full string theory, where winding states are included too, the supersymmetry will survive the T-dualisation (provided the radius of the circle is appropriately chosen.) This is an example of the phenomenon of "supersymmetry without supersymmetry," which was discussed in

[1]Note that for the 16 Killing spinors that always occur, one can include solutions of the homogeneous equation, giving contributions of the form $Q/(z_i^2)^3$ in H. However for the additional Killing spinors that arise in special cases, H cannot include such a term.

[15, 16] (see also [17, 18, 19] for earlier work on supersymmetry under T-duality). The Penrose limit of $\mathrm{AdS}_5 \times S^5$, for which $\Phi_{(4)}$ is given by (3), is an example of a special case since there are supernumerary Killing spinors (16 in this example). We shall discuss these special cases below, after first giving a discussion applicable to the generic situation with only the 16 standard Killing spinors.

Deformed M2-branes of the form (7), with the transverse 8-metric dz_i^2 replaced by a Ricci-flat metric of special holonomy, and $\Phi_{(4)}$ an L^2-normalisable self-dual harmonic 4-form, have been discussed extensively [20, 21, 22, 23, 24, 25]. The condition for a Killing spinor ϵ of the undeformed solution to survive the deformation is that [26, 20, 22]

$$\Phi_{abcd}\,\Gamma^{bcd}\,\epsilon = 0 \tag{8}$$

in the transverse space. In our present context we instead take the 8-space to be flat, and $\Phi_{(4)}$ to be constant (and hence non-normalisable).

A solution to (5) that is sufficient for our purposes is

$$H = c_0 + \frac{Q}{r^6} - \mu_i^2\, z_i^2\,, \tag{9}$$

where $r^2 \equiv z_i^2$, Q is the M2-brane charge, and

$$\sum_i \mu_i^2 = \tfrac{1}{96}\mu^2\, |\Phi_{(4)}|^2\,. \tag{10}$$

We shall refer to H as being "quadratic in the z^i" if the charge Q vanishes, regardless of whether or not the constant c_0 vanishes. The criterion for a Killing spinor in the original undeformed solution (with $Q \neq 0$ but with $\mu = 0$, and hence $\frac{1}{2}$ supersymmetry) to remain a Killing spinor after the deformation ($\mu \neq 0$) is still (8), and so by appropriate choices for the non-vanishing components of $\Phi_{(4)}$ one can arrange for some fraction of the supersymmetry to be preserved.

Let us now turn to the discussion of the supernumerary supersymmetries that can arise for special $\Phi_{(4)}$ and μ_i distributions (with $Q = 0$). As we mentioned, an example of this is the Penrose limit of $\mathrm{AdS}_5 \times S^5$, where it was shown in [1] that for $\Phi_{(4)}$ given by (3), and $Q = 0$ and the μ_i all equal in (9), there are 16 supernumerary Killing spinors. In section 3, we shall show that these are all independent of x^+, implying that they will survive in a T-dualisation, giving a supersymmetric deformed M2-brane. However, by naively applying the criterion (8) to the expression for $\Phi_{(4)}$ given in (3), one would draw the conclusion that this particular M2-brane should have no supersymmetry, since the operator in (8) has no zero eigenvalues. As we shall discuss in section 7, in this particular case it is incorrect to check the supersymmetry by first finding the Killing spinors of the undeformed solution and then testing to see whether they survive in the criterion (8). Specifically, there are in fact 16 Killing spinors when $\mu \neq 0$ and $Q = 0$, but they are disjoint from the 16 that one has when $Q \neq 0$ and $\mu = 0$. Their existence depends on non-zero contributions from the $\Phi_{(4)}$ terms in the supersymmetry transformation rules cancelling against the other terms that are present even in the undeformed background. Thus although the general solution with both Q and μ non-vanishing has no supersymmetry in this example, one gets two disjoint sets of 16 Killing spinors in the two cases $Q = 0$ or $\mu = 0$.

In fact the Penrose limit of $\mathrm{AdS}_5 \times S^5$ is just one of many possible examples of special cases with extra x^+-independent supersymmetries, which, in the M2-brane picture, cannot be found by simply applying the criterion (8). In summary, the supersymmetry criterion (8) is correct for testing "standard Killing spinors," but not for testing "supernumerary Killing spinors," in the T-dualised M-theory picture. We shall discuss these issues further in section 4.

One of the purposes of this paper is to consider various possibilities for the constant self-dual 4-form $\Phi_{(4)}$ in (2), focusing on those cases where the corresponding T-dualised deformed solutions have surviving supersymmetries. In general it is not clear that the type IIB pp-wave solutions with $\Phi_{(4)}$ more general than (3) will have interpretations as Penrose limits. However, in one example we find that such an interpretation does arise. From the deformed M2-brane point of view, this example is inspired by considering the transverse space $\mathbb{E}^8$ as an orbifold limit of K3×K3, where the harmonic 4-form is a direct product of the Kähler forms of the two K3 factors;

$$\Phi_{(4)} = (dz^1 \wedge dz^2 + dz^3 \wedge dz^4) \wedge (dz^5 \wedge dz^6 + dz^7 \wedge dz^8)\,. \tag{11}$$

It is straightforward to verify from (8) that the associated deformed M2-brane will have $\frac{1}{4}$ supersymmetry (i.e. 8 Killing spinors), if $Q \neq 0$. When $Q = 0$ we find an additional 8 supernumerary Killing spinors, if four of the μ_i in (9) vanish, and the other four are equal and non-zero. We shall show in section 2 that from the type IIB point of view, this half-supersymmetric pp-wave solution can be obtained as a Penrose limit of the near-horizon limit of an intersection of two D3-branes. This near-horizon limit itself is $\mathrm{AdS}_3 \times S^3 \times T^4$ (or $\mathrm{AdS}_3 \times S^3 \times$K3).

In section 3 we discuss the supersymmetry of the general pp-wave solutions, focusing in particular on the question of whether there are Killing spinors that are independent of the x^+ coordinate, and thus will survive at the field theory level after a T-duality transformation. In section 4, we present a general class of $\Phi_{(4)}$ structures that will give rise to

x^+-independent Killing spinors. We obtain examples that give rise to deformed M2-branes that preserve a variety of fractions of supersymmetry. For example, if Q is non-zero (implying that only the 16 standard supersymmetries in the type IIB pp-wave arise), we can obtain supersymmetry fractions $n/16$ in the associated deformed M2-branes, with $1 \le n \le 6$. Even more possibilities arise when the special cases where the pp-waves have supernumerary supersymmetries are considered. In section 5 we consider the solvable massive string actions associated with these pp-wave solutions.

In section 6 we address the issue of the spacelike or timelike nature of the x^+ coordinate, and its effect on the nature of the T-dualised theories. In section 7 we study the supersymmetry of the deformed M2-branes that are T-dual to the pp-waves, paying particular attention to the cases where there are supernumerary Killing spinors. As an illustrative example, we examine in detail the example of the Penrose limit of $\mathrm{AdS}_5 \times S^5$, showing how it is supersymmetric despite violating the usual supersymmetry criterion (8).

2 D3/D3 brane intersection and its Penrose limit

Let us begin by considering a standard D3/D3 brane intersection, for which the metric and self-dual 5-form are given by

$$\begin{aligned} ds_{10}^2 &= (H_1\, H_2)^{-1/2} \Big[-dt^2 + dx_1^2 + H_1\, (dz_1^2 + dz_2^2) + H_2\, (dz_3^2 + dz_4^2) \\ &\quad + H_1\, H_2\, (dr^2 + r^2\, d\Omega_3^2) \Big] \,, \\ F_{(5)} &= -dt \wedge dx_1 \wedge (dz_1 \wedge dz_2 + dz_3 \wedge dz_4) \wedge d(H_1^{-1} + H_2^{-1}) + \mathrm{dual} \,, \end{aligned} \tag{12}$$

where

$$H_1 = 1 + \frac{Q_1}{r^2} \,, \qquad H_2 = 1 + \frac{Q_2}{r^2} \,. \tag{13}$$

For simplicity, we shall take $Q_1 = Q_2 = \lambda^2$. In the near-horizon limit, we then find that the metric becomes

$$ds_{10}^2 = \lambda^2 \Big[d\Omega_3^2 + ds_3^2 \Big] + dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 \,. \tag{14}$$

After transforming from the Poincaré coordinates (r, t, x_1) to global coordinates (ρ, t, γ) in the AdS_3 metric ds_3^2, and writing

$$d\Omega_3^2 = d\theta^2 + \cos^2\theta\, d\psi^2 + \sin^2\theta\, d\phi^2 \,, \qquad ds_3^2 = d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\, d\gamma^2 \,, \tag{15}$$

the Penrose limit can be taken as

$$\rho \longrightarrow \frac{\rho}{\lambda} \,, \qquad \theta \longrightarrow \frac{\theta}{\lambda} \,, \qquad t = x^+ + \frac{x^-}{\lambda^2} \,, \qquad \psi = x^+ - \frac{x^-}{\lambda^2} \,, \tag{16}$$

with the constant λ sent to infinity. By this means, the solution becomes the pp-wave

$$\begin{aligned} ds_{10}^2 &= -4dx^+\,dx^- + H\,{dx^+}^2 + dz_i^2\,, \\ F_{(5)} &= \mu\,dx^+ \wedge (dz_1 \wedge dz_2 + dz_3 \wedge dz_4) \wedge (dz_5 \wedge dz_6 + dz_7 \wedge dz_8)\,, \end{aligned} \tag{17}$$

where $z_5 = \theta\,\cos\phi$, $z_6 = \theta\,\sin\phi$, $z_7 = \rho\,\cos\gamma$, $z_8 = \rho\,\sin\gamma$. (The constant μ is introduced here by making the replacements $x^+ \longrightarrow \mu\,x^+$, $x^- \longrightarrow x^-/\mu$.) The function H is given by

$$H = -\tfrac{1}{4}\mu^2 \sum_{i=5}^{8} z_i^2\,. \tag{18}$$

Comparing with (2), we see that the self-dual 4-form $\Phi_{(4)}$ is precisely of the form given in (11).

It is worth pointing out that in this example, and indeed in all analogous Penrose limits, we can generalise (16) in the following way. The expressions for t and ψ in (16) can be replaced by

$$t = (1 - \tfrac{1}{4}c_0\,\lambda^{-2})\,x^+ + \frac{x^-}{\lambda^2}\,, \qquad \psi = (1 - \tfrac{1}{4}c_0\,\lambda^{-2})\,x^+ - \frac{x^-}{\lambda^2}\,, \tag{19}$$

where c_0 is a constant. Now we obtain a metric of the same form as in (17), except that now H is given by

$$H = c_0 - \tfrac{1}{4}\mu^2 \sum_{i=5}^{8} z_i^2\,. \tag{20}$$

This means that there can be a regime where H is positive, implying that x^+ is then a spatial coordinate and x^- a timelike coordinate. Note that this same generalisation can be made in the standard Penrose limit of $\mathrm{AdS}_5 \times S^5$, allowing H in equation (4) to become $H = c_0 - \frac{1}{16}\mu^2\,z_i^2$. In fact the generalised Penrose limit that we are introducing here is nothing but a general coordinate transformation in which one sends $x^- \longrightarrow x^- - \frac{1}{4}c_0\,x^+$.

Note that although the summation range is restricted in (18), we could instead solve the type IIB equations of motion for configurations of the form (17) with a summation over the entire range $1 \le i \le 8$, although then the interpretation as a Penrose limit of the D3/D3 brane intersection would be lost. Furthermore, as we shall see in the next section, one would have only 16 rather than 24 Killing spinors.

In order to study the supersymmetry of these type IIB solutions, it will prove to be useful to give a more general discussion of supersymmetry for solutions of the form (4) and (2). This forms the topic of the next section.

3 Supersymmetry analysis for the pp-wave solutions

The discussion in this section follows the strategy described in [1], with appropriate generalisation and adjustment to our notation and conventions. The spinor covariant derivative in the metric (4) can be seen to take the form

$$\nabla_+ = \partial_+ + \tfrac{1}{4}\partial_i H\, \Gamma_-\, \Gamma_i\,, \qquad \nabla_- = \partial_-\,, \qquad \nabla_i = \partial_i\,. \tag{21}$$

With a suitable normalisation for the self-dual 5-form, the supercovariant derivative D_M is given by

$$D_M = \nabla_M + \mathrm{i}\, \Omega_M\,, \tag{22}$$

where

$$\Omega_M = \tfrac{1}{192} F_{M N_1\cdots N_4}\, \Gamma^{N_1\cdots N_4}\,. \tag{23}$$

With $F_{(5)}$ given by (2), one then has

$$\Omega_- = 0\,, \qquad \Omega_+ = \tfrac{1}{8}\mu\, W\,, \qquad \Omega_i = -\tfrac{1}{16}\mu\, \Gamma_-\, [\Gamma_i, W]\,, \tag{24}$$

where

$$W \equiv \tfrac{1}{24}\Phi_{ijk\ell}\, \Gamma_{ijk\ell}\,. \tag{25}$$

Note that in the type IIB theory there are two independent supersymmetry parameters ϵ, which are both of the same chirality. Thus each could potentially give up to 16 Killing spinors, implying a total maximum of 32.

Following the arguments presented in [1], one can now straightforwardly show that the Killing spinors, which satisfy $D_M\, \epsilon = 0$, are independent of x^-, and are given by

$$\epsilon = (1 - \mathrm{i}\, z^i\, \Omega_i)\, \chi\,, \tag{26}$$

where χ has only x^+ dependence, governed by

$$\partial_+\, \chi + \mathrm{i}\, \mu\, W\, \chi = 0\,. \tag{27}$$

Additionally, one has the requirement $\mu\, z^i\, [W, \Omega_i]\, \chi + \frac{1}{4}\partial_i H\, \Gamma_-\, \Gamma_i\, \chi = 0$, which therefore implies, using the fact that for the chiral spinors χ we have $W\, \Gamma_-\, \chi = 0$,

$$(\mu^2\, z^i\, W^2 + 32\partial_i H)\, \Gamma_i\, \Gamma_-\, \chi = 0\,. \tag{28}$$

It is this equation that determines the number of Killing spinors. Note that for any solution $H(z_i)$ one is guaranteed to have at least 16 Killing spinors $\chi = \Gamma_-\, \chi_0$, since every term in (28) contains a factor of Γ_-. It follows from (26) that these 16 Killing spinors are independent of the z^i coordinates. These are the standard Killing spinors that we spoke of earlier.

If H is quadratic (i.e. $Q = 0$ in (9)), so that $\partial_i H = -2\mu_i^2\, z^i$, then the possibility exists for specific μ_i and $\Phi_{(4)}$ that there may be further solutions to (28). These supernumerary Killing spinors are constructed using spinors χ that satisfy

$$\Gamma_+\,\chi = 0\,, \qquad (\mu^2\, z^i\, W^2 + 32\partial_i H)\,\Gamma_i\,\chi = 0\,. \tag{29}$$

Note that because the chiral spinors χ satisfy $\Gamma_+\,\chi = 0$, we have $W\,\chi = 0$ for any W given by (25) for a self-dual $\Phi_{(4)}$:

$$\Gamma_{11}\,\chi = \chi \quad \text{and} \quad \Gamma_+\,\chi = 0 \quad \Longrightarrow \quad W\,\chi = 0\,. \tag{30}$$

It then follows from (35) that all supernumerary Killing spinors are necessarily independent of x^+.

Before discussing our new pp-wave solutions it is useful to review in our notation the construction of the 16 supernumerary Killing spinors in the Penrose limit of $\mathrm{AdS}_5 \times S^5$, which were found in [1]. The 4-form $\Phi_{(4)}$ is given by (3), and hence from (25) we have

$$W = \Gamma_{1234} + \Gamma_{5678}\,. \tag{31}$$

It can be seen that we therefore have $W^2\,\Gamma_i\,\chi = 4\Gamma_i\,\chi$ and so the maximum of 32 Killing-spinor solutions to (28) is achieved (16 standard plus 16 supernumerary Killing spinors), provided that one has

$$H = c_0 - \tfrac{1}{16}\mu^2\, z_i^2\,. \tag{32}$$

The constant c_0 was not included in the discussion in [1], nor is it usually presented in the Penrose limit of $\mathrm{AdS}_5 \times S^5$, but its inclusion does not affect the conclusions about supersymmetry, since H appears in the spin connection only via its derivative. Furthermore, as we showed in section 2, it can easily be introduced in the Penrose limit, via a coordinate transformation $x^- \longrightarrow x^- - \frac{1}{4}c_0\, x^+$.

Turning now to our pp-wave solution in section 2, associated with the Penrose limit of $\mathrm{AdS}_3 \times S^3$, we shall have

$$W = \Gamma_{1234} + \Gamma_{5678} + \Gamma_{1278} + \Gamma_{3456}\,. \tag{33}$$

There are, as always, 16 standard Killing spinors, corresponding to $\chi = \Gamma_-\,\chi_0$. After some algebra, we find that there are in addition 8 supernumerary Killing spinors satisfying (29), with $W^2\,\Gamma_i\,\chi = 0$ for $i = 1, 2, 3, 4$ and $W^2\,\Gamma_i\,\chi = 16\Gamma_i\,\chi$ for $i = 5, 6, 7, 8$ (in a convenient labelling convention for the gamma matrices). Thus we obtain the supernumerary 8 Killing spinors if H is given by

$$H = c_0 - \tfrac{1}{4}\mu^2 \sum_{i=5}^{8} z_i^2 \,. \qquad (34)$$

Thus the solution preserves $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ of the supersymmetry. If any other distribution of μ_i (besides relabelling) is chosen (still satisfying the field equation (10)), all the 8 supernumerary Killing spinors will be lost.

Having found all the Killing spinors, we can now address the question of which are independent of x^+; this is determined by (27), whose solution is

$$\chi = e^{-\mathrm{i}\,\frac{1}{8}\mu\, x^+\, W}\,\eta \,. \qquad (35)$$

If a particular solution for χ has the property that $W\,\chi = 0$, then the associated Killing spinor will be independent of the coordinate x^+. In particular, from (30), this means that all supernumerary Killing spinors, coming from solutions of (29), will be independent of x^+. For the standard 16 Killing spinors, on the other hand, the requirement $W\,\chi = 0$ is a further condition, which may or may not have solutions, depending on the structure of W.

For our new case where W is given by (33), it is straightforward to see that 8 of the 16 standard Killing spinors are annihilated by the operator W, and thus these 8 do not depend on the coordinate x^+. In total, therefore, we have the 8 supernumerary Killing spinors and 8 of the standard Killing spinors that do not depend on x^+. These 16 Killing spinors will therefore survive under a T-duality transformation, and so after lifting to $D = 11$ we should obtain a deformed M2-brane that preserves $\frac{1}{2}$ the supersymmetry. The 8 that are standard Killing spinors, which, as we saw earlier, are also independent of z^i, are the ones that one would expect in $D = 11$, by simply applying the standard supersymmetry condition (8). The 8 coming from the supernumerary Killing spinors in the pp-wave are rather remarkable from an M-theory viewpoint; we shall discuss these in detail in section 7.

In the case of the Penrose limit of $\mathrm{AdS}_5 \times S^5$, W given in (31) does not annihilate any of the 16 standard Killing spinors, but it does, as we saw earlier, annihilate the supernumerary Killing spinors, and so these 16 are independent of x^+. This verifies a statement made in [4]. It contradicts a statement in [1], where it was stated that all 32 Killing spinors depend on the x^+ coordinate (called x^- in [1]). The explanation for this is that [1] did not take into account that W has 16 zero eigenvalues.[2]

[2] Specifically, the 16 Killing spinors annihilated by Γ_- are not annihilated by W, and so they depend on x^+ but not on z^i. The remaining 16 Killing spinors are independent of x^+ but they do depend on the z^i.

The fact that this Penrose limit of $AdS_5 \times S^5$ has 16 Killing spinors that are independent of x^+ again has a remarkable consequence in this case; that there should exist, in the T-dualised picture, a deformed M2-brane whose additional 4-form flux violates the usual supersymmetry condition (8) and yet in fact preserves $\frac{1}{2}$ of the supersymmetry. We shall return to this in section 7, where we shall construct this M2-brane solution explicitly, and demonstrate its supersymmetry.

4 x^+-independent Killing spinors in pp-waves

Having seen in the previous section that there can exist pp-waves with Killing spinors that do not depend on the coordinate x^+, it becomes of interest to classify the possible 5-form structures and the distributions of μ_i coefficients in (9), to see what fractions of x^+-independent Killing spinors can be achieved.

4.1 x^+-independent standard Killing spinors

We shall first study this question for the 16 standard Killing spinors, which are all annihilated by Γ_- (and thus are all independent of the z^i coordinates). These exist for any choice of H, provided only that the field equation (5) is satisfied. In terms of W, defined in (25) this is

$$\Box H = -\tfrac{1}{64}\,\mu^2\,\mathrm{tr} W^2\,. \tag{36}$$

It remains, therefore, to check how many of the 16 standard Killing spinors are annihilated by W since as we showed from (35), this is the condition for them to be independent of x^+. In turn, this subset of the 16 standard Killing spinors will survive at the field theory level in a T-duality transformation.

As we discussed earlier, the T-dualised solutions can be lifted to $D = 11$, where they become deformed M2-branes with an extra self-dual 4-form flux in the 8-dimensional flat transverse space. One might expect that the smallest degree of unbroken supersymmetry that could be achieved for such deformed M2-branes would be in a case where the 4-form flux was related to that seen in the harmonic 4-form of a transverse space of Spin(7) holonomy. Motivated by this, it is therefore natural here to consider a self-dual 4-form $\Phi_{(4)} = L_{(4)} + {*L_{(4)}}$ where $L_{(4)}$ has 7 structures, which can be taken to be

$$L_{(4)} = m_1\, dz^{1234} + m_2\, dz^{1256} + m_3\, dz^{1357} + m_4\, dz^{1467} + m_5\, dz^{2367} + m_6\, dz^{2457} + m_7\, dz^{3456}\,, \tag{37}$$

where $dz^{ijk\ell} \equiv dz^i \wedge dz^j \wedge dz^k \wedge dz^\ell$, and the m_i are constants. From this, one constructs the matrix W given in (25), which can be written as

$$W = \sum_{\alpha=1}^{7} m_\alpha \, W_\alpha \,, \tag{38}$$

and then one determines what fraction of the 16 standard Killing spinors (which satisfy $\chi = \Gamma_- \chi_0$) are annihilated by W. In fact the choice of seven structures in (37) can be characterised by the fact that they give the maximal set of W_α that all commute.

It is a simple exercise to obtain the eigenvalues of the matrix W, since the terms W_α all commute. Projected into the 16-dimensional subspace of chiral spinors χ satisfying $\Gamma_- \chi = 0$ (i.e. 2 copies of an 8-dimensional subspace of the 32×32 matrix W, corresponding to the $\mathcal{N} = 2$ supersymmetry in the type IIB theory), we find that they are given by λ_i, where

$$\begin{aligned}
\lambda_1 &= 2(m_1 + m_2 - m_3 + m_4 - m_5 - m_6 - m_7)\,, \\
\lambda_2 &= -2(m_1 - m_2 + m_3 + m_4 + m_5 - m_6 - m_7)\,, \\
\lambda_3 &= -2(m_1 - m_2 - m_3 - m_4 - m_5 + m_6 - m_7)\,, \\
\lambda_4 &= 2(m_1 + m_2 + m_3 - m_4 + m_5 + m_6 - m_7)\,, \\
\lambda_5 &= -2(m_1 + m_2 + m_3 - m_4 - m_5 - m_6 + m_7)\,, \\
\lambda_6 &= 2(m_1 - m_2 - m_3 - m_4 + m_5 - m_6 + m_7)\,, \\
\lambda_7 &= 2(m_1 - m_2 + m_3 + m_4 - m_5 + m_6 + m_7)\,, \\
\lambda_8 &= -2(m_1 + m_2 - m_3 + m_4 + m_5 + m_6 + m_7)\,,
\end{aligned} \tag{39}$$

If the constants m_α are chosen so that any number $n \le 6$ of the λ_i vanish, there will then be $2n$ standard Killing spinors in the type IIB solution that are independent of x^+. If 7 or 8 of the λ_i vanish, then all the m_α vanish and the solution becomes trivial.

The standard Killing spinors that we have been considering in this subsection are all annihilated by Γ_-, and so from (26) they are all independent of z^i. Thus the subsets that are annihilated by W are independent of all the coordinates.

4.2 x^+-independent supernumerary Killing spinors

As we showed in section 3, all the supernumerary Killing spinors, satisfying (29), are independent of x^+, and so it is merely necessary to count them. For these Killing spinors, the charge Q in (9) must vanish, so that H is quadratic in the z^i. Furthermore, we find that we must have

$$\mu_i^2 = \tfrac{1}{64}\mu^2\, \lambda_i^2\,, \tag{40}$$

and that the number of these supernumerary Killing spinors is equal to the degeneracy k of the least degenerate of the squared eigenvalues λ_i^2, multiplied by a factor of 2 because of the $\mathcal{N}=2$ supersymmetry of the type IIB theory.

To prove these facts, we begin by noting that there exists a similarity transformation matrix S such that $S\,W\,S^t$ is diagonal. The 32×32 matrix itself has 16 zero eigenvalues and 2 of each of the λ_i given in (39). We can therefore change to a new basis for the gamma matrices, $\Gamma_A \longrightarrow S\,\Gamma_A\,S^t$, in which W is diagonal. After projection into the subspace of chiral spinors that are also annihilated by Γ_+, we have just the eight diagonal entries λ_i.

In this diagonal basis, we find that for each i and j in the range 1 to 8, the matrix $-\Gamma_{ij}\,W\,\Gamma_{ij}$ is again diagonal, with entries that are a permutation of those in W. Furthermore, for each W_α in (38) we have $\Gamma_{ij}\,W_\alpha\,\Gamma_{ij}=\pm W_\alpha$, with a sign that depends on i and j. Suppose now that $\Gamma_1\,\chi$ is an eigenvector of W with eigenvalue λ. It follows, by acting with Γ_{1j}, that $\Gamma_j\,\chi$ will be an eigenvector of the corresponding permuted matrix, say $\widetilde{W}$, with the same eigenvalue. However, the permuted matrix can be transformed back into W itself by an appropriate set of sign reversals for the m_α. It follows that $\Gamma_j\,\chi$ is therefore an eigenvector of W with one of the other eigenvalues of W. Thus we have established that the set of eigenvectors of W can be expressed as $\Gamma_i\,\chi$, once one has established that any one of these is one of the eigenvectors, and so with a suitable labelling order for the gamma matrices we have

$$W\,\Gamma_i\,\chi = \lambda_i\,\Gamma_i\,\chi\,,\qquad 1\le i\le 8\,. \tag{41}$$

Substituting into (29), with H given by (9) and $Q=0$, we therefore obtain the result (40).

By this means we can obtain 8 solutions to the conditions (29). However, in general each solution will require that the set of constants μ_i in H will be permuted differently from each other solution. Thus for a fixed choice for the μ_i, there will in general only be one solution to (29), implying, after the doubling because of $\mathcal{N}=2$ supersymmetry in type IIB, two supernumerary Killing spinors. If, however, there are degeneracies among the λ_i, there can accordingly be more than one solution to (29) with the *same* set of μ_i in H. It is clear that the largest number of solutions that one could have for a given choice of μ_i is therefore equal to the smallest degeneracy factor, k, among the eigenvalues λ_i. It turns out that this largest number is in fact attained, and so we get $2k$ supernumerary Killing spinors in the pp-wave solution.

Consider, for example, the case of the Penrose limit of $\mathrm{AdS}_5\times S^5$, where we have $m_1=1$ and all other $m_\alpha=0$, implying that all $\lambda_i^2=4$. The smallest degeneracy is therefore $k=8$, and we recover the 16 supernumerary Killing spinors previously found [1] in this example.

For the pp-wave associated with the Penrose limit of $\mathrm{AdS}_3 \times S^3$, constructed in section 2, we have $m_1 = m_7 = 1$, with all other m_α vanishing. This implies $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda_4^2 = 0$, and $\lambda_5^2 = \lambda_6^2 = \lambda_7^2 = \lambda_8^2 = 16$, and hence $k = 4$. This reproduces the 8 supernumerary Killing spinors that we found for this example in (3).

In general, the possible values of least degeneracy that can occur are $k = 1$, 2, 4 or 8, implying 2, 4, 8 or 16 supernumerary Killing spinors. It is interesting to note that there must therefore always be at least two supernumerary Killing spinors for any of the configurations for $\Phi_{(4)}$ contained within (37), provided, of course, that the μ_i are chosen according to (40) (and Q in (9) is set to zero).

4.3 Supersymmetry of the deformed M2-branes

Having determined the numbers of x^+-independent Killing spinors for these pp-wave solutions of the type IIB theory, we can now directly examine the associated deformed M2-brane solutions in $D = 11$ supergravity.

In general for a deformed M2-brane with H given by (9), with $Q \neq 0$, the criterion for unbroken supersymmetry is given by substituting the 4-form $\Phi_{(4)}$ in the transverse space into (8). After examining all the cases, we find that the numbers of Killing spinors obtained by setting $n \le 6$ of the λ_i in (39) to zero is indeed $2n$, as is expected from T-duality. It is interesting to note that the supersymmetry fractions that are thus achieved, namely $\{\frac{1}{16}, \frac{1}{8}, \frac{3}{16}, \frac{1}{4}, \frac{5}{16}, \frac{3}{8}\}$, include some unusual values. The first four are seen also in regular deformed M2-brane solutions using Spin(7), Kähler_8, hyper-Kähler and $\mathrm{K3}\times T^4$ manifolds respectively for the transverse 8-space. The final two examples, with $\frac{5}{16}$ and $\frac{3}{8}$ supersymmetry, do not have any known corresponding regular counterparts.

When $Q = 0$, and the μ_i are given by (40), the existence of the $2k$ supernumerary Killing spinors, which are x^+-independent, in the type IIB pp-wave implies that there should be $2k$ further Killing spinors in the T-dualised deformed M2-brane. As we shall show in section 7, these Killing spinors do indeed exist in the deformed M2-branes, but they arise in a somewhat subtle way, since the usual supersymmetry criterion (8) is not satisfied. This is related to their non-standard z^i dependence, which can be seen in the type IIB picture from equation (26).

5 String actions on the pp-waves

The light-cone gauge string actions for the pp-wave solutions we have obtained in this paper are given by

$$\mathcal{S} = \frac{1}{2\pi\alpha'} \int d\tau \int_0^{2\pi\alpha' p^+} d\sigma\, \mathcal{L}\,, \tag{42}$$

where

$$\mathcal{L} = \sum_{i=1}^{8} (\tfrac{1}{2}\dot{z}_i^2 - \tfrac{1}{2}z_i'^2 - \tfrac{1}{2}\mu_i^2\, z_i^2) + \mathrm{i}\,\bar\Psi(\partial\!\!\!/ + \tfrac{1}{8}\mu\, W)\,\Gamma_+\,\Psi\,, \tag{43}$$

where W is given by (25).[3] This is an exactly-solvable massive free string theory, whose character is determined by W and the pattern of non-vanishing μ_i. The number of zero eigenvalues of W associated with eigenspinors that are not annihilated by Γ_- determines the number of massless fermions. In fact the fermion masses are given by $\frac{1}{8}\mu\,\lambda_i$, where the λ_i are the eigenvalues of W, given for our examples in (39). We have presented the action in the case where the additive constant c_0 in the function H is set to zero. If it is non-zero, one gets an additional additive constant in $\mathcal{L}$; this does not affect the exact solvability.

In the case associated with the Penrose limit of $\mathrm{AdS}_5 \times S^5$, all the relevant eigenvalues of W are non-zero, and thus one has 8 massive fermions (with equal mass). All the bosonic masses are equal too, since H is isotropic in z^i. This solution has the supernumerary 16 Killing spinors that are independent of x^+ (the time coordinate in this context), implying that the linearly-realised supersymmetry commutes with the Hamiltonian and hence that the fermionic and bosonic masses are equal here [4].

In our further examples of supersymmetric pp-waves with $\Phi_{(4)}$ given by (37), the matrix W can have n zero eigenvalues when $1 \le n \le 6$ of the λ_i in (39) vanish. This implies that n fermions are massless while $(8-n)$ are massive. For a general distribution of μ_i satisfying (10), the masses of the bosons and fermions are not equal. This is because there are no supernumerary Killing spinors in general, and hence there are no linearly-realised supersymmetries that would imply a bose/fermi mass equality. However, if instead we choose the μ_i to be given by (40), the masses of the bosons will match with the masses of the fermions, suggesting the existence of linearly-realised supersymmetries. Indeed, as we showed, supernumerary Killing spinors arise in this case, which give rise to these linearly-realised supersymmetries. The number of these supernumerary Killing spinors is governed by the smallest degeneracy k among the eigenvalues λ_i of W, which are the masses of the

[3]This result can in general be obtained by light-cone gauge fixing in the type IIB Green-Schwarz action derived in [27]. The results for the cases of Penrose limits for $\mathrm{AdS}_5 \times S^5$ and $\mathrm{AdS}_3 \times S^3$ were obtained in [5, 4].

fermions. The larger the degeneracy, the larger the number of equal-mass fields. This is consistent with the associated larger number of linearly-realised supersymmetries.

The example of the Penrose limit of the D3/D3 system, discussed in section 2, is a special case of (37) where $\Phi_{(4)}$ is given by (11). The light-cone Lagrangian is given by

$$\mathcal{L} = \sum_{i=1}^{8} (\tfrac{1}{2}\dot{z}_i^2 - \tfrac{1}{2}z_i'^2) - \tfrac{1}{4}\mu^2 \sum_{i=5}^{8} z_i^2 + \mathrm{i}\,\Psi(\partial\!\!\!/ + \tfrac{1}{8}\mu\, W)\,\Gamma_+\,\Psi\,, \tag{44}$$

with W given by (33). In this case there are four massive fermions and bosons (with equal masses), and four massless fermions and bosons. The equality of the fermion and boson masses is a consequence of having the supernumerary Killing spinors that we discussed earlier. This string action is similar to the result one obtains from the D1/D5 system [4, 10]. However, the fermion structure is different because in the D1/D5 system there is a non-vanishing R-R 3-form, whilst in our case it is the 5-form that is non-vanishing. Note that we can replace the 4-space whose coordinates are z^i for $1 \le i \le 4$ by a T^4/Z_N orbifold. The twisted states arising from this orbifolding can be analysed in a conventional way, since the associated target-space coordinates correspond to massless free fields.

6 Spacelike vs. timelike T-duality, and phase transitions

In presenting the pp-wave solution (4) or (6), we have chosen to treat x^+ as a spacelike coordinate, by taking H to be positive. This means that we can perform a spacelike T-duality transformation on the coordinate x^+, leading to deformed M2-brane solutions in M-theory. As we discussed in section 1, the positivity of H can be justified since solutions to (5) can be taken to be (9), implying positivity in some region of spacetime. Moreover, even though the term involving Q would not arise in a Penrose limit the constant c_0 does, and so it is always possible to find a region of spacetime where x^+ is spacelike.

On the other hand, the pp-wave solution that is of particular interest in the context of Penrose limits corresponds to setting $Q = 0$. For sufficiently large values of z_i^2 the coordinate x^+ becomes timelike, and indeed in the discussion of solvable string actions one chooses the light-cone gauge where x^+ is set equal to the worldsheet time coordinate [5, 4]. Whilst T-duality generalises to a timelike $U(1)$ isometry in the heterotic string theory [28], it breaks down for the type II string theories because of the Ramond-Ramond sector [29]. In fact it has been proposed that the type IIA and IIB theories are timelike T-dual to new theories called type IIB* and type IIA* respectively, where all the R-R field strengths have kinetic terms of reversed sign [30]. Thus an alternative viewpoint is to consider a timelike T-duality on the x^+ direction for the pp-wave solution, in the region where $H < 0$, and obtain now a regular 2-brane in the M* theory in $(2, 9)$ spacetime signature that was introduced in [31].

Whichever view one takes, it will not affect the conclusions about the supersymmetry of the T-dualisations of the pp-wave solutions that we have obtained here. From a practical point of view, it is simpler to analyse the supersymmetry in the region where H is positive.

If we consider the deformed M2-branes with H given by (9) where Q as well as μ is non-zero, the solution close to $r = 0$ is the near-horizon limit $\mathrm{AdS}_4 \times S^7$, which is dual to a three-dimensional superconformal field theory. On the other hand at large r, the solution approaches a deformed 2-brane of M*-theory that is T-dual to a pp-wave in the type IIB string. Thus the naked singularity of the M2-brane that arises as H passes through 0 can be argued to be a low-energy artefact of eleven-dimensional supergravity. It is in fact the point where a phase transition would take place, as one passes from the M-theory to M*-theory description. From the type IIB point of view, there is no singularity in the metric as H passes through zero (see (4)), although x^+ becomes a null coordinate at this transition point, which is why the T-duality transformed solution becomes singular there. On the other hand the type IIB pp-wave *is* singular at $r = 0$ (if Q is non-zero), whilst this is perfectly regular in the M-theory picture where it corresponds to the $\mathrm{AdS}_4 \times S^7$ near-horizon limit.

In the $r \longrightarrow 0$ limit the metric becomes $\mathrm{AdS}_4 \times S^7$ and hence the supersymmetry is fully restored. At large r, the solution can be T-dualised to the pp-wave in type IIB, which preserves at least half the supersymmetry. Thus the deformed M2-brane is dual to a three-dimensional field theory whose renormalisation group flow runs from the conformal fixed point in the infra-red regime (at small r) to a non-conformal theory as the energy increases. At a certain intermediate energy scale there is a phase transition associated with the naked singularity of the M2-brane. In the ultra-violet limit the theory is related by T-duality to an exactly-solvable massive IIB string theory.

7 New supersymmetric deformed M2-branes

We observed in section 3 that whenever a pp-wave in the type IIB theory has supernumerary Killing spinors, coming from (29), these will be independent of x^+ and thus they will survive in a T-dualisation and lifting to $D = 11$ supergravity. However, a naive application of the supersymmetry criterion (8) to check whether these Killing spinors are present in the deformed M2-brane solution will appear to lead to a contradiction, since (8) will be violated.

In this section we shall discuss the resolution of this puzzle, by showing how there are indeed Killing spinors in the deformed M2-brane solution corresponding to the supernumerary Killing spinors in the type IIB pp-wave. They are unusual in that they satisfy the $D = 11$ Killing spinor equations by virtue of a cancellation between contributions from the extra flux $\Phi_{(4)}$ and contributions from the spin-connection and standard M2-brane-charge terms in the supercovariant derivative. Since they therefore cannot be viewed as Killing spinors that existed already in the undeformed M2-brane, whose "survival" under the deformation is then being tested, the assumptions that were made in the derivation of the standard supersymmetry criterion (8) are not valid.

We shall illustrate this point by discussing in detail the example of the Penrose limit of $\mathrm{AdS}_5 \times S^5$, which gives the pp-wave with the maximal number 16 of supernumerary Killing spinors [1]. The calculations for the other pp-waves that we have been considering in this paper are very similar.

Substituting the pp-wave background (7) with a general self-dual 4-form $\Phi_{(4)}$ into the gravitino transformation rule $\delta\psi_M = \mathcal{D}_M\,\epsilon$, with

$$\mathcal{D}_M = \nabla_M - \frac{1}{288}\Big(F_{N_1\cdots N_4}\,\Gamma_M{}^{N_1\cdots N_4} - 8F_{MN_1\cdots N_3}\,\Gamma^{N_1\cdots N_3}\Big)\,, \tag{45}$$

we find that

$$\mathcal{D}_\mu = \partial_\mu - \tfrac{1}{6}H^{-3/2}\,\partial_i H\,\Gamma_\mu\,\Gamma_i\,(1-\Gamma) - \tfrac{1}{12}\mu\,H^{-1}\,\Gamma_\mu\,W\,, \tag{46}$$

$$\mathcal{D}_i = \partial_i + \tfrac{1}{12}H^{-1}\,\partial_j H\,\Gamma_{ij}\,(1-\Gamma) + \tfrac{1}{6}H^{-1}\,\partial_i H\,\Gamma + \tfrac{1}{24}\mu\,H^{-1/2}\,(\Gamma_i\,W - 3W\,\Gamma_i)\,, \tag{47}$$

where W is given by (25) and $\Gamma \equiv \frac{1}{6}\epsilon_{\mu\nu\rho}\,\Gamma^{\mu\nu\rho}$. (Note that all indices on Γ matrices are tangent-frame indices.)

For the specific case of the deformed M2-brane corresponding to the T-dualisation of the Penrose limit of $\mathrm{AdS}_5 \times S^5$ the self-dual 4-form $\Phi_{(4)}$ given by (3), and W for this example, given by (31), has the following properties:

$$W^2 = 2(1+\Gamma)\,, \quad W\,\Gamma = \Gamma\,W = W\,. \tag{48}$$

One can now verify, using these, that the following spinors ϵ satisfy the Killing-spinor condition $\mathcal{D}_M\,\epsilon = 0$:

$$\epsilon = H^{-1/6}\,(\partial_i H\,W\,\Gamma_i - H^{1/2})\,\eta \tag{49}$$

where η is a constant spinor satisfying $(1+\Gamma)\,\eta = 0$ (and hence $W\,\eta = 0$). In addition, the function H must satisfy

$$\partial_i\,\partial_j\,H = -\tfrac{1}{8}\mu^2\,\delta_{ij}\,, \tag{50}$$

or, in other words,

$$H = c_0 - \tfrac{1}{16}\mu^2\, z_i^2\,. \qquad (51)$$

Thus we have verified that there are 16 Killing spinors for this deformed M2-brane solution, which is precisely the one related by T-duality to the pp-wave that is the Penrose limit of $\mathrm{AdS}_5 \times S^5$.

It should be emphasised that the supersymmetry in this solution is achieved by "trading off" the contributions from the extra 4-form flux (the W terms in (46) and (47)) against the $\partial_i H$ terms that usually cancel by themselves in the supersymmetry transformation rules for a deformed M2-brane. It is for this reason that the usual supersymmetry criterion (8) is inapplicable in this special case.

It is straightforward to repeat the above analysis for all the other examples of pp-waves with supernumerary Killing spinors that we have constructed in this paper. We shall not present the details here, since the manipulations are very similar. Furthermore, since the x^+-independence of the supernumerary Killing spinors implies that they must survive as Killing spinors in the T-dualised picture, our illustrative example above suffices to establish the principle of how this can happen, despite the fact that the standard supersymmetry criterion (8) is not satisfied.

We can also use the results in this section to study the supersymmetries of the deformed M2-branes related by T-duality to the various pp-waves obtained in section (4) that have x^+-independent standard Killing spinors, with the 4-form $\Phi_{(4)}$ given by (37). In these cases there is no longer any "trading off" between the W terms and the $\partial_i H$ terms in (46) and (47), and hence any Killing spinors must be annihilated by these two types of term separately. This means that the supersymmetry criterion (8) applies in these examples, and so the Killing spinors are simply given by $\epsilon = H^{-1/6}\,\eta$, where η is any constant spinor satisfying $(1-\Gamma)\,\eta = 0$ and (8) (which can be expressed as $[\Gamma_i, W]\,\eta = 0$). One can now verify that there are $2n$ such Killing spinors if any n of the equations (39) are satisfied, with $1 \le n \le 6$. This is exactly in accordance with our supersymmetry discussion for the pp-waves in section 4, where we counted the subset of the 16 standard Killing spinors that were independent of x^+.

It is interesting to compare our results for these supersymmetric deformed M2-branes with the results in [32], where a self-dual 4-form with a structure contained within (37) was considered. In [32] the 4-form $L_{(4)}$ was taken to be the G_2-invariant structure constants in the multiplication table of the imaginary octonion units, and the solution was found to be non-supersymmetric. This corresponds to all the seven constants m_α having unit magnitude, $|m_\alpha| = 1$, and it is then evident from (39) that none of the eigenvalues λ_i will vanish, and thus none of the 16 standard Killing spinors will be independent of x^+. However, we have also seen that for any of the pp-waves constructed using (37), there will be at least two supernumerary Killing spinors, provided that one takes $Q = 0$ in (9), and chooses the constants μ_i to satisfy (40). For the particular example for $\Phi_{(4)}$ considered in [32], we find that the λ_i in (39) are given by

$$\lambda_i = (14, -2, -2, -2, -2, -2, -2)\,, \qquad (52)$$

and so the smallest degeneracy is $k = 1$, implying exactly two supernumerary Killing spinors. It follows, therefore, that with the choice for $\Phi_{(4)}$ made in [32], the "octonionic M2-brane" will have two supersymmetries if H is taken to have the non-isotropic form

$$H = c_0 - \tfrac{1}{16}\mu^2\,(49 z_1^2 + z_2^2 + z_3^2 + \cdots + z_8^2)\,. \qquad (53)$$

It is also worth remarking that we can obtain further examples of supersymmetric deformed M2-branes by taking $\Phi_{(4)}$ to be harmonic and given by $\Phi_{(4)} = r^{-8}\,(L_{(4)} + {*L_{(4)}})$, with $L_{(4)}$ again given by (37). The fractions of preserved supersymmetry are again governed by (39). For the solutions with $\Phi_{(4)} = r^{-8}\,(L_{(4)} + {*L_{(4)}})$ the 4-form is square integrable at large distance but divergent at small distance. The resulting solution is accordingly well-behaved at large distance, but has a naked singularity at small distance. The singularity can be resolved by replacing the flat transverse space by a Ricci-flat space with special holonomy that admits an L^2-normalisable harmonic 4-form [21, 23, 24, 25]. By dimensional reduction and T-duality we can then obtain pp-waves in type IIB supergravity where the flat 8-metric is replaced by the space of special holonomy. However, although these pp-waves are non-singular, there is no reason to expect that these backgrounds would correspond to exactly solvable string theories.

8 Conclusions

In the paper we have obtained a large class of pp-waves in type IIB supergravity theories, in which a constant five-form characterised by seven parameters is turned on. These solutions in general lead to exactly solvable string backgrounds. Our principal focus was on the analysis of the number of surviving supersymmetries. In addition to 16 "standard supersymmetries," these backgrounds always allow for the possibility of further "supernumerary supersymmetries." The conditions for the appearance of supernumerary supersymmetries precisely determines that the masses of the world-sheet bosonic fields must match those of the fermionic fields, thus ensuring linearly-realised world-sheet supersymmetry of the corresponding string action.

The analysis of the above class of pp-wave solutions demonstrates a one-to-one correspondence between the supernumerary supersymmetry and the world-sheet supersymmetry. The Penrose limit of $AdS_5 \times S^5$ is a particular example in this class, with all eight world-sheet superfields having equal and non-vanishing masses. We also found another example of a Penrose limit, namely the Penrose limit of $AdS_3 \times S^3$ space, which corresponds to the near horizon limit of a D3/D3 intersection. For this example, the supernumerary supersymmetry determines that four world-sheet superfields have zero masses, and the other four have equal and non-zero masses.

The pp-waves can all be T-dualised and then lifted to eleven-dimensions, where they become "deformed M2-branes" with an additional 4-form flux. Any Killing spinor in the pp-wave that is independent of the x^+ T-dualising coordinate will necessarily continue to be a Killing spinor in the $D = 11$ supergravity picture. The 16 standard Killing spinors in the pp-wave are always independent of z^i, but in general depend on x^+ unless they are annihilated by W. By contrast, any supernumerary Killing are automatically independent of x^+, but they depend in a non-trivial way on the z^i coordinates. After the T-dualisation, any of the 16 standard Killing spinors that are x^+-independent give rise to Killing spinors of the usual sort in the deformed M2-brane, which satisfy the standard supersymmetry criterion (8). By contrast, all the supernumerary Killing spinors give rise to Killing spinors in the deformed M2-brane for which the standard supersymmetry criterion (8) is not satisfied. They are annihilated by the $D = 11$ supercovariant derivative because of a cancellation between terms involving the extra 4-form flux and terms from the spin connection and the usual 4-form contribution.

We also observed that by starting from any deformed M2-brane solution in $D = 11$, with dz_i^2 replaced by an 8-metric ds_8^2 of special holonomy and $\Phi_{(4)}$ a self-dual harmonic 4-form in ds_8^2, we can obtain a supersymmetric pp-wave solution in the type IIB theory, by means of a dimensional reduction to type IIA and then a T-duality transformation. The resulting pp-wave will have 16 "standard" Killing spinors, regardless of whether or not the deformed M2-brane is supersymmetric. There will not be any supernumerary Killing spinors in these generalised pp-waves. If the deformed M2-brane has supersymmetries, then the corresponding subset of the 16 standard Killing spinors in the T-dual picture will be independent of x^+. The deformed M2-brane solutions could be of the type principally considered in [20, 21, 22, 23, 24, 25], where the self-dual harmonic 4-form $\Phi_{(4)}$ is square-integrable, or else one could take $\Phi_{(4)}$ to be a non-normalisable self-dual harmonic 4-form, such as the covariantly-constant associative 4-form in a space of Spin(7) holonomy, or $J \wedge J$ in a Ricci-flat Kähler 8-metric.

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