ROM2F/2007/10
MIFP-07-16
hep-th/yymmddd

May 26, 2018



# Real Forms of Complex Higher Spin Field Equations and New Exact Solutions

Carlo Iazeolla[1,2], Ergin Sezgin[3] and Per Sundell[2]

[1] *Dipartimento di Fisica, Università di Roma "Tor Vergata"*
*INFN, Sezione di Roma "Tor Vergata"*
*Via della Ricerca Scientifica 1, 00133 Roma, Italy*

[2] *Scuola Normale Superiore and INFN*
*Piazza dei Cavalieri 7, 56100 Pisa, Italy*

[3] *George P. and Cynthia W. Mitchell Institute for Fundamental Physics*
*Texas A&M University*
*College Station, TX 77843-4242, USA*

## Abstract

We formulate four dimensional higher spin gauge theories in spacetimes with signature $(4-p,p)$ and nonvanishing cosmological constant. Among them are chiral models in Euclidean $(4,0)$ and Kleinian $(2,2)$ signature involving half-flat gauge fields. Apart from the maximally symmetric solutions, including de Sitter spacetime, we find: (a) $SO(4-p,p)$ invariant deformations, depending on a continuous and infinitely many discrete parameters, including a degenerate metric of rank one; (b) non-maximally symmetric solutions with vanishing Weyl tensors and higher spin gauge fields, that differ from the maximally symmetric solutions in the auxiliary field sector; and (c) solutions of the chiral models furnishing higher spin generalizations of Type D gravitational instantons, with an infinite tower of Weyl tensors proportional to totally symmetric products of two principal spinors. These are apparently the first exact 4D solutions with non-vanishing massless higher spin fields.

## Contents

# 1 Introduction

Given the impact Yang-Mills theory and Gravity have had on the development towards our present understanding of fundamental interactions, as formulated within Quantum Field Theory (QFT) and to some extent String Field Theory (SFT), it is natural to explore higher spin (HS) extensions of gauge symmetries (*i.e.* non-abelian gauge groups containing generators in representations of the Lorentz group with spins higher than one).

Presently, the only known full models of interacting higher-spin gauge fields are those based on the Vasiliev equations [1]. These equations are naturally formulated in terms of $SL(2;\mathbb{C})$ spinor oscillators in Lorentzian signature $(3,1)$. In this paper, we shall formulate them using spinor oscillators in Euclidean signature $(4,0)$ and Kleinian signature $(2,2)$ as well, and present nontrivial exact solutions with novel properties such as the excitation of all higher spin fields. Before we state our motivations for this work, let us first highlight some key elements of the HS theory.

To begin with, the Vasiliev equations that describe the HS theory involve two features that are relatively novel from the point-of-view of lower-spin QFT as well as the standard formulation of SFT. Firstly, they are written in a frame-like language, closely related to the constraint formulation of supergravity, known as free differential algebra (FDA), or unfolded dynamics. Here, *all* fields are differential forms, which live on an *a priori* unspecified base manifold. Moreover, for *each* differential form there is a, in general non-linear, differential constraint, written using the exterior derivative (and no contractions of curved indices using the metric). Thus, diffeomorphism invariance is manifest without the need to single out a metric or other component field.

Secondly, in order to accommodate an infinite number of physical as well as auxiliary fields, one works with master fields that, in addition to being differential forms, are functions of oscillator variables. The functions belong to, or, depending on taste, define a fiber over the base manifold consisting of representations of an underlying non-abelian higher-spin algebra. In particular, the master zero-form is directly related to the (massless) spectrum via the theorem of Flato and Fronsdal [2]. One may go further and associate the oscillators to a particle or other extended objects, perhaps related to discretization of tensionless strings and membranes in AdS [3], though these considerations are of course not crucial for setting up Vasiliev's formalism.

The simplest higher spin gauge theories of Vasiliev type, and indeed the first ones to appear in the literature [1], are based on higher-spin extensions of $SO(3,2)$ realized using oscillators that are $SL(2,\mathbb{C})$ doublets (coordinatizing the phase space of Dirac's $Sp(4)$ singleton). Here, the *master fields* are an adjoint *one-form* and a twisted-adjoint *zero-form*, sometimes referred to as the Weyl zero-form. The master field equations, which we again stress are manifestly background independent and diffeomorphism invariant, can then be written on a remarkably simple closed form. These equations can be treated in two almost opposite ways, namely by projecting to the fiber or by projecting to the base. In the latter case one can make contact with lower-spin field theory by taking the base manifold to be an ordinary spacetime and eliminate the auxiliary fields, treating *only* the Lorentz connection and the vierbein exactly. This well-defined, albeit tedious, approach yields manifestly reparametrization and locally Lorentz invariant physical field equations in a perturbative expansion in curvatures as well as higher-spin gauge fields.

The projection to the fiber, on the other hand, is a more tractable operation, since the Vasiliev equations can be solved locally on the base manifold using gauge functions [4]. This leaves equations on the non-commutative fiber, which are thus purely algebraic from the point-of-view of the base manifold. The simplest exact solution to these equations is the four-dimensional anti-de Sitter spacetime. In a recent paper, [5], we have given an exact $SO(3,1)$-invariant solution to these equations. The solution describes a locally time-dependent solution with a local space-like singularity that can be resolved by the method of patches. The solution is asymptotically AdS and periodic in time, so that one may think of it as an "instanton universe" inside AdS [6]. More recently, the gauge function method has been used to describe the BTZ black hole metric as a solution to full three dimensional HS gauge theory [7].

This raises the question how to Wick rotate solutions of the Lorentzian theory into solutions of a Euclidean theory. The main difficulty is to impose proper reality conditions given the doubling of the spinor oscillators due to the Euclidean signature. We resolve this by taking the master fields to be holomorphic functions of the left-handed and the right-handed spinor oscillators subject to pseudo-reality conditions.

In addition to the Euclidean signature, we shall consider the Kleinian signature as well. While in all signatures there is the possibility of a chiral asymmetry, in Euclidean and Kleinian signatures, the extreme case of parity violation involving half-flat gauge fields can also arise. We refer to the latter ones as *chiral models*. In HS gauge theory, the HS algebra valued gauge-field curvatures can be made, say, self-dual, but the model nonetheless contains the anti-self-dual gauge fields through the master zero-form which contains the corresponding Weyl tensor obeying the appropriate field equation. Although this is contrary to what happens in ordinary Euclidean gravity, where the field equations can contain only self-dual fields, it is not a surprise in higher spin theory since the underlying higher spin algebra, which is an extension of $SO(5)$, does not admit a chiral massless multiplet [1].

There are several reasons that make the investigation of HS theory in Euclidean and Kleinian signatures worthwhile. To begin with, just as the Euclidean version of gravity plays a significant role in the path integral formulation of quantum gravity, it is reasonable to expect that this may also be the case in the quantum formulation of HS theory, despite the fact that an action formulation is yet to be spelled out (see, [9] for recent progress). For reviews of Euclidean quantum gravity, see, for example, [10] and [11].

Another well known aspect of self-dual field theories is their capability to unify a wide class of integrable systems in two and three dimensions. It would be interesting to extend these mathematical structures to self-dual HS gauge theories to find new integrable systems.

The chiral HS theories in Kleinian signatures may also be of considerable interest in closed $N = 2$ string theory in which the self-dual gravity in $(2,2)$ dimensions arises as the effective target space theory [12]. However, there are some subtleties in treating the picture-changing operators in the BRST quantization which have raised the question of whether there are more

[1] We shall leave the group-theoretical analysis to [8], where we also give the spinor-oscillator formulations of the four-dimensional minimal bosonic models with $H_4$, $dS_4$ and $H_{3,2}$ vacua.

physical states [13], and in the case of open $N = 2$ theory an interpretation in terms of an infinite tower of massless higher spin states has been proposed [14]. It would be very interesting to establish whether these theories or their possible variants admit self-dual HS theory in the target space. While the $N = 2$ string theories may seem to be highly unrealistic, it should not be ruled out that they may be connected in subtle ways to all the other string theories which themselves are connected by a web of dualities in M theory.

In this paper, we shall take the necessary first steps to start the exploration of the Euclidean and Kleinian HS theories. We shall start by determining the real forms of the higher spin algebra based on infinite dimensional extension of $SO(5;\mathbb{C})$ and formulate the corresponding higher spin gauge theories in four-dimensional spacetime with signature $(4-p,p)$. Maximally symmetric four- dimensional constant curvature spacetimes, including de Sitter spacetime, defined by the embedding into five-plane with signature $(5-q,q)$ are readily exact solutions. Fluctuations about these spaces arrange themselves into all the irreducible representations of $SO(5-q,q)$ contained in the symmetric two-fold product of the fundamental singleton representation of this group, each occurring once. The details of this phenomenon will be provided in a separate paper [8].

We then devote the rest of the paper to finding a class of nontrivial exact solutions of these models, including the Euclidean and chiral cases. The key information about these solutions is encoded in the master zero-form which contains a real ordinary scalar field, and the *Weyl tensors* $\phi_{\alpha_1\cdots\alpha_{2s}}$ and $\phi_{\dot\alpha_1,\cdots\dot\alpha_{2s}}$ for spin $s = 2,4,6,...$ in the minimal bosonic model and $s = 1,2,3,4,...$ in a non-minimal bosonic model [15, 16]. Vasiliev's full higher spin field equations assume a form reminiscent of that of open string field theory, with master fields that are functions of spacetime as well as an internal noncommutative space of oscillators. Our new exact solutions are constructed by using the oscillators to build suitable projectors, with slightly different properties in the minimal and non-minimal models.

Our exact solutions fall into the following four classes:

<u>Type 0</u>:

These are *maximally symmetric solutions* (see Table 1) with

$$\begin{aligned}
\phi(x) &= 0\ , \qquad \Phi_{\alpha_1\cdots\alpha_{2s}} = 0\ , \qquad \Phi_{\dot\alpha_1\cdots\dot\alpha_{2s}} = 0\ , \\
e_\mu^a &= \frac{4\delta_\mu^a}{(1-\lambda^2x^2)^2}\ , \qquad W_\mu{}^{a_1\cdots a_{s-1}} = 0\ ,
\end{aligned} \tag{1.1}$$

describing the symmetric spaces $S^4, H_4, AdS_4, dS_4, H_{3,2} = SO(3,2)/SO(2,2)$, where $|\lambda|$ is the inverse radius of the symmetric space, $x^2 = x^a x^b \eta_{ab}$, and $\eta_{ab}$ is the tangent space metric. In the above the zero-forms have spin $s = 2,4,6,...$ in the minimal model and $s = 1,2,3,4,...$ in the non-minimal model, while for $W_\mu{}^{a_1\cdots a_{s-1}}$, $s = 4,6,...$ in the minimal model, and $s = 1,3,4,5,6,...$ in the non-minimal model.

<u>Type 1</u>:

These solutions, which arise in the minimal models (and therefore are evidently solutions also to the non-minimal models with vanishing odd spins), are $SO(p,4-p)$ *invariant deformations* of the maximally symmetric solutions with

$$\begin{aligned} \phi(x) &= \nu(1-\lambda^2x^2)\ , \qquad \Phi_{\alpha_1\cdots\alpha_{2s}}=0\ , \qquad \Phi_{\dot\alpha_1\cdots\dot\alpha_{2s}}=0\ , \quad (s=2,4,...) \\ e_\mu^a &= f_1\delta_\mu^a+\lambda^2 f_2 x_\mu x^a\ , \qquad W_\mu{}^{a_1\cdots a_{s-1}}=0 \quad (s=4,6,...)\ , \end{aligned} \tag{1.2}$$

where $\nu$ is a continuous parameter and $f_1, f_2$ (see (3.85)) are highly complicated functions of $x^2$, $\nu$, and a set of *discrete* parameters corresponding to whether certain projectors are switched on or off. The metric is Weyl-flat conformal to the maximally symmetric solution with a complicated conformal factor, and note that all the higher spin gauge fields vanish. Interestingly, a particular choice of the discrete parameters yield, in the $\nu\to 0$ limit, the *degenerate metric*:

$$g_{\mu\nu}=\frac{1}{(1-\lambda^2x^2)}\frac{x_\mu x_\nu}{\lambda^2x^2}\ . \tag{1.3}$$

Degenerate metrics are known to play a role topology change in spacetime (see, for example, [17], and references therein). Interestingly, here they arise in a natural way by simply taking a certain limit in the parameter space of our solution.

<u>Type 2</u>:

These are solutions of the non-minimal model that are *not* solutions to the minimal model. The spacetime component fields are identical to those of the maximally symmetric Type 0 solutions, but, unlike in the Type 0 solution, the spinorial master one-form is non-vanishing (see (3.102)). Even though all odd spin fields are vanishing, the solution exists only for the non-minimal model because the spinorial master field violates the kinematic conditions of the minimal model. In particular, this means that this type of solution cannot be a $\nu\to 0$ limit of the Type 1 solutions. Furthermore, the spinorial master field is parametrized by *discrete* parameters, again associated with projectors.

<u>Type 3</u>:

These are solutions of the *non-minimal chiral models* in Euclidean and Kleinian signatures, in which *all gauge fields are non-vanishing.* These solutions also depend on an infinite set of *discrete parameters* and for simple choices of these parameters we obtain two such solutions in both of which

$$\phi(x) \ = \ -1\ , \qquad \Phi_{\alpha_1\cdots\alpha_{2s}} \ = \ 0\ , \qquad W_\mu{}^{a_1\cdots a_{s-1}} \ \neq \ 0\ . \tag{1.4}$$

In one of the solutions the Weyl tensors and the vierbein take the form

$$\Phi_{\dot\alpha_1\cdots\dot\alpha_{2s}} \ = \ -2^{2s+1}(2s-1)!!\left(\frac{h^2-1}{\epsilon h^2}\right)^s U_{(\dot\alpha_1}\cdots U_{\dot\alpha_s}V_{\dot\alpha_{s+1}}\cdots V_{\dot\alpha_{2s})}\ , \tag{1.5}$$

$$e_\mu^a \ = \ \frac{-2}{h^2(1+2g)}\left[g_3\delta_\mu^a+g_4\lambda^2x_\mu x^a+g_5\lambda^2(Jx)_\mu(Jx)^a\right]\ , \tag{1.6}$$

where $h, g, g_3, g_4, g_5$ are functions of $x^2$ defined in (B.3), (3.126), and the almost complex structure $J_{ab}$ and spinors $(U, V)$ are defined in (B.8) and (B.11), and $\epsilon=\pm1$ as explained in Section

3.5. For the other solution we have

$$\Phi_{\dot\alpha_1\cdots\dot\alpha_{2s}} = -2^{2s+1}(2s-1)!!\left(\frac{1}{\epsilon h^2}\right)^s \bar\lambda_{(\dot\alpha_1}\cdots\bar\lambda_{\dot\alpha_s}\,\bar\mu_{\dot\alpha_{s+1}}\cdots\bar\mu_{\dot\alpha_{2s})}\ , \tag{1.7}$$

$$e_\mu^a = \frac{-2}{h^2(1+2\tilde g)}\left[\delta_\mu^a+\tilde g_4\lambda^2 x_\mu x^a+\tilde g_5\lambda^2(\tilde J x)_\mu(\tilde J x)^a\right]\ , \tag{1.8}$$

where the functions $\tilde g, \tilde g_4, \tilde g_5$ are defined in (3.134), and the almost complex structure $\tilde J_{ab}$ is defined in (B.10).

These are remarkable solutions in that they are, to our best knowledge, the first exact solution of higher spin gauge theory in which higher spin fields are non-vanishing. We also note that the Weyl tensors in these solutions corresponds to higher spin generalization of the Type D Weyl tensor that takes the form $\phi_{\dot\alpha\dot\beta\dot\gamma\dot\delta} \sim \lambda_{(\dot\alpha}\lambda_{\dot\beta}\mu_{\dot\gamma}\mu_{\dot\delta)}$ up to a scale factor [18]. Type D instanton solutions of Einstein's equation in Euclidean signature with and without cosmological constant have been discussed in [19]. Our solution provides their higher spin generalization.

After we describe the HS field equations in diverse signatures in Section 2, we shall present the detailed construction of our solutions in Section 3. We shall comment further on these solutions and open problems in the Conclusions.

# 2 The Bosonic 4D Models in Various Signatures

We shall first describe the field equations without imposing reality conditions on the master fields. These conditions will then be discussed separately leading to five different models in four-dimensional spacetimes with various signatures (see Table 1).

## 2.1 The Complex Field Equations

To formulate the complex field equations we use independent $SL(2;\mathbb{C})_L$ doublet spinors $(y_\alpha, z_\alpha)$ and $SL(2;\mathbb{C})_R$ doublet spinors $(\bar y_{\dot\alpha}, \bar z_{\dot\alpha})$ generating an oscillator algebra with non-commutative and associative product $\star$ defined by

$$y_\alpha\star y_\beta = y_\alpha y_\beta + i\epsilon_{\alpha\beta}\ ,\qquad y_\alpha\star z_\beta = y_\alpha z_\beta - i\,\epsilon_{\alpha\beta}\ , \tag{2.1}$$

$$z_\alpha\star y_\beta = z_\alpha y_\beta + i\,\epsilon_{\alpha\beta}\ ,\qquad z_\alpha\star z_\beta = z_\alpha z_\beta - i\,\epsilon_{\alpha\beta}\ , \tag{2.2}$$

and

$$\bar y_{\dot\alpha}\star\bar y_{\dot\beta} = \bar y_{\dot\alpha}\bar y_{\dot\beta} + i\epsilon_{\dot\alpha\dot\beta}\ ,\qquad \bar z_{\dot\alpha}\star\bar y_{\dot\beta} = \bar z_{\dot\alpha}\bar y_{\dot\beta} - i\epsilon_{\dot\alpha\dot\beta}\ , \tag{2.3}$$

$$\bar y_{\dot\alpha}\star\bar z_{\dot\beta} = \bar y_{\dot\alpha}\bar z_{\dot\beta} + i\epsilon_{\dot\alpha\dot\beta}\ ,\qquad \bar z_{\dot\alpha}\star\bar z_{\dot\beta} = \bar z_{\dot\alpha}\bar z_{\dot\beta} - i\epsilon_{\dot\alpha\dot\beta}\ , \tag{2.4}$$

where the juxtaposition denotes the symmetrized, or Weyl-ordered, products. For example, $y_\alpha y_\beta = \frac{1}{2}(y_\alpha \star y_\beta + y_\beta \star y_\alpha)$. Equivalently, Weyl-ordered functions obey[2]

$$\begin{aligned} &\widehat{f}(y,\bar{y},z,\bar{z}) \ \star \ \widehat{g}(y,\bar{y},z,\bar{z}) \qquad (2.5) \\ = &\int \frac{d^4\xi d^4\eta}{(2\pi)^4}\ e^{i\eta^\alpha\xi_\alpha + i\bar{\eta}^{\dot\alpha}\bar{\xi}_{\dot\alpha}}\ \widehat{f}(y+\xi,\bar{y}+\bar{\xi},z+\xi,\bar{z}-\bar{\xi})\ \widehat{g}(y+\eta,\bar{y}+\bar{\eta},z-\eta,\bar{z}+\bar{\eta})\ , \end{aligned}$$

where the hats are used to denote functions of all oscillators, while functions of only $y_\alpha$ and $\bar{y}_{\dot\alpha}$ will be unhatted.

The complex master fields are the *adjoint* one-form $\widehat{A}$ and the *twisted-adjoint* zero-form $\widehat{\Phi}$ defined by

$$\widehat{A} \ = \ dx^\mu \widehat{A}_\mu(x;y,\bar{y},z,\bar{z}) + dz^\alpha \widehat{A}_\alpha(x;y,\bar{y},z,\bar{z}) + d\bar{z}^{\dot\alpha}\widehat{A}_{\dot\alpha}(x;y,\bar{y},z,\bar{z})\ , \qquad (2.6)$$

$$\widehat{\Phi} \ = \ \widehat{\Phi}(x;y,\bar{y},z,\bar{z})\ , \qquad (2.7)$$

where $x^\mu$ are coordinates on a commutative base manifold (which can, but need not, be fixed to be four-dimensional spacetime). One also defines the total exterior derivative

$$d \ = \ dx^\mu\partial_\mu + dz^\alpha\frac{\partial}{\partial z^\alpha} + d\bar{z}^{\dot\alpha}\frac{\partial}{\partial \bar{z}^{\dot\alpha}}\ , \qquad (2.8)$$

with the property $d(\widehat{f}\wedge\star\,\widehat{g}) = (d\widehat{f})\wedge\star\,\widehat{g} + (-1)^{\deg\widehat{f}}\widehat{f}\wedge\star\,d\widehat{g}$ for general differential forms. In what follows we shall suppress the $\wedge$. The master fields can be made subject to the following discrete symmetry conditions[3] [15, 16]

$$\text{Minimal model } (s=0,2,4,...) \ : \ \tau(\widehat{A}) \ = \ -\widehat{A}\ , \qquad \tau(\widehat{\Phi}) \ = \ \bar{\pi}(\widehat{\Phi})\ , \qquad (2.9)$$

$$\text{Non-minimal model } (s=0,1,2,3,...) \ : \ \pi\bar{\pi}(\widehat{A}) \ = \ \widehat{A}\ , \qquad \pi\bar{\pi}(\widehat{\Phi}) \ = \ \widehat{\Phi}\ , \qquad (2.10)$$

where $\tau$ is the $\star$-product algebra anti-automorphism defined by

$$\tau(\widehat{f}(y,\bar{y};z,\bar{z})) \ = \ \widehat{f}(iy,i\bar{y};-iz,-i\bar{z})\ , \qquad (2.11)$$

and $\pi$ and $\bar{\pi}$ are two involutive $\star$-product automorphisms defined by

$$\pi(\widehat{f}(y,\bar{y};z,\bar{z})) \ = \ \widehat{f}(-y,\bar{y};-z,\bar{z})\ , \qquad \bar{\pi}(\widehat{f}(y,\bar{y};z,\bar{z})) \ = \ \widehat{f}(y,-\bar{y};z,-\bar{z})\ . \qquad (2.12)$$

We note that

$$\tau(\widehat{f}\star\widehat{g}) \ = \ (-1)^{\deg(\widehat{f})\deg(\widehat{g})}\tau(\widehat{g})\star\tau(\widehat{f})\ , \qquad (2.13)$$

$$\pi(\widehat{f}\star\widehat{g}) \ = \ \pi(\widehat{f})\star\pi(\widehat{g})\ , \qquad (2.14)$$

$$\bar{\pi}(\widehat{f}\star\widehat{g}) \ = \ \bar{\pi}(\widehat{f})\star\bar{\pi}(\widehat{g})\ , \qquad (2.15)$$

[2] The integration measure is defined by $d^4\xi = d^2\xi^1 d^2\xi^2$, where $d^2z = idz\wedge d\bar{z} = 2dx\wedge dy$ for $z = x+iy$. With this normalization, $\mathbb{I}\star\widehat{f} = \widehat{f}$.

[3] The exterior derivative obeys $\tau d = d\tau$ and $\pi d = d\pi$, and the $\tau$ and $\pi$ maps do not act on the commutative coordinates.

and that $\tau^2 = \pi\bar{\pi}$. The automorphisms are inner and can be generated by conjugation with the functions $\kappa$ and $\bar{\kappa}$ given by

$$\kappa \;=\; \exp(iy^\alpha z_\alpha)\ , \qquad \bar{\kappa} \;=\; \exp(-i\bar{y}^{\dot{\alpha}}\bar{z}_{\dot{\alpha}})\ , \tag{2.16}$$

such that

$$\kappa \star \widehat{f}(y,z) \;=\; \kappa \widehat{f}(z,y)\ , \qquad \widehat{f}(y,z)\star\kappa \;=\; \kappa\widehat{f}(-z,-y)\ , \qquad \kappa\star\widehat{f}\star\kappa \;=\; \pi(\widehat{f})\ , \tag{2.17}$$

$$\bar{\kappa} \star \widehat{f}(\bar{y},\bar{z}) \;=\; \bar{\kappa} \widehat{f}(-\bar{z},-\bar{y})\ , \qquad \widehat{f}(\bar{y},\bar{z})\star\bar{\kappa} \;=\; \bar{\kappa}\widehat{f}(\bar{z},\bar{y})\ , \qquad \bar{\kappa}\star\widehat{f}\star\bar{\kappa} \;=\; \bar{\pi}(\widehat{f})\ . \tag{2.18}$$

The full complex field equations are

$$\widehat{F} \;=\; \frac{i}{4}\left[c_1 dz^\alpha \wedge dz_\alpha \widehat{\Phi}\star\kappa + c_2 d\bar{z}^{\dot{\alpha}}\wedge d\bar{z}_{\dot{\alpha}}\widehat{\Phi}\star\bar{\kappa}\right]\ , \tag{2.19}$$

$$\widehat{D}\widehat{\Phi} \;=\; 0\ , \tag{2.20}$$

where $c_1$ and $c_2$ are complex constants and the curvatures and gauge transformations are given by

$$\widehat{F} \;=\; d\widehat{A} + \widehat{A}\star\widehat{A}\ , \qquad \delta_{\widehat{\epsilon}}\widehat{A} \;=\; \widehat{D}\widehat{\epsilon} \tag{2.21}$$

$$\widehat{D}\widehat{\Phi} \;=\; d\widehat{\Phi} + [\widehat{A},\widehat{\Phi}]_\pi\ , \qquad \delta_{\widehat{\epsilon}}\widehat{\Phi} \;=\; -[\widehat{\epsilon},\widehat{\Phi}]_\pi\ , \tag{2.22}$$

with

$$[\widehat{f},\widehat{g}]_\pi \;=\; \widehat{f}\star\widehat{g} - (-1)^{\deg(\widehat{f})\deg(\widehat{g})}\widehat{g}\star\pi(\widehat{f})\ . \tag{2.23}$$

Since $\widehat{\Phi}$ is defined up to rescalings by complex numbers, the model only depends on one complex parameter, that we can take to be

$$c \;=\; \frac{c_2}{c_1}\ . \tag{2.24}$$

In components, the constraints read

$$\widehat{F}_{\mu\nu} \;=\; 0\ , \qquad \widehat{D}_\mu\widehat{\Phi} \;\equiv\; \partial_\mu\widehat{\Phi} + [\widehat{A}_\mu,\widehat{\Phi}]_\pi \;=\; 0\ , \tag{2.25}$$

$$\widehat{F}_{\mu\alpha} \;=\; 0\ , \qquad \widehat{F}_{\mu\dot{\alpha}} \;=\; 0\ , \tag{2.26}$$

$$\widehat{F}_{\alpha\beta} \;=\; -\tfrac{ic_1}{2}\epsilon_{\alpha\beta}\widehat{\Phi}\star\kappa\ , \qquad \widehat{F}_{\dot{\alpha}\dot{\beta}} \;=\; -\tfrac{ic_2}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\widehat{\Phi}\star\bar{\kappa}\ , \tag{2.27}$$

$$\widehat{F}_{\alpha\dot{\alpha}} \;=\; 0\ , \tag{2.28}$$

$$\widehat{D}_\alpha\widehat{\Phi} \;\equiv\; \partial_\alpha\widehat{\Phi} + \widehat{A}_\alpha\star\widehat{\Phi} + \widehat{\Phi}\star\pi(\widehat{A}_\alpha) \;=\; 0\ , \tag{2.29}$$

$$\widehat{D}_{\dot\alpha}\widehat{\Phi} \ \equiv\ \partial_{\dot\alpha}\widehat{\Phi}+\widehat{A}_{\dot\alpha}\star\widehat{\Phi}+\widehat{\Phi}\star\bar\pi(\widehat{A}_{\dot\alpha}) \ = \ 0\ , \tag{2.30}$$

where (2.30) can be derived using $\pi\bar\pi(\widehat{A}_{\dot\alpha})=-\widehat{A}_{\dot\alpha}$. Introducing [1]

$$\widehat{S}_\alpha \ = \ z_\alpha-2i\widehat{A}_\alpha\ ,\qquad \widehat{S}_{\dot\alpha}\ =\ \bar z_{\dot\alpha}-2i\widehat{A}_{\dot\alpha}\ , \tag{2.31}$$

the component form of the equations carrying at least one spinor index now take the form

$$\partial_\mu\widehat{S}_\alpha+[\widehat{A}_\mu,\widehat{S}_\alpha]_\star \ = \ 0\ ,\qquad \partial_\mu\widehat{S}_{\dot\alpha}+[\widehat{A}_\mu,\widehat{S}_{\dot\alpha}]_\star\ =\ 0\ , \tag{2.32}$$

$$[\widehat{S}_\alpha,\widehat{S}_\beta]_\star \ = \ -2i\epsilon_{\alpha\beta}(1-c_1\widehat{\Phi}\star\kappa)\ ,\qquad [\widehat{S}_{\dot\alpha},\widehat{S}_{\dot\beta}]_\star\ =\ -2i\epsilon_{\dot\alpha\dot\beta}(1-c_2\widehat{\Phi}\star\bar\kappa)\ , \tag{2.33}$$

$$[\widehat{S}_\alpha,\widehat{S}_{\dot\beta}]_\star \ = \ 0\ , \tag{2.34}$$

$$\widehat{S}_\alpha\star\widehat{\Phi}+\widehat{\Phi}\star\pi(\widehat{S}_\alpha) \ = \ 0\ , \tag{2.35}$$

$$\widehat{S}_{\dot\alpha}\star\widehat{\Phi}+\widehat{\Phi}\star\bar\pi(\widehat{S}_{\dot\alpha})\ =\ 0\ . \tag{2.36}$$

This form of the equations makes the following $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry manifest:

$$\widehat{S}_\alpha \ \to\pm\widehat{S}_\alpha\ ,\qquad \widehat{S}_{\dot\alpha}\ \to\ \pm\widehat{S}_{\dot\alpha}\ , \tag{2.37}$$

(where the two transformations can be performed independently) keeping $\widehat{A}_\mu$ and $\widehat{\Phi}$ fixed. We note that $\widehat{S}_\alpha\to-\widehat{S}_\alpha$ is equivalent to $\widehat{A}_\alpha\to-\widehat{A}_\alpha-iz_\alpha$, *idem* $\widehat{S}_{\dot\alpha}$ and $\widehat{A}_{\dot\alpha}$.

All component fields are of course complex at this level. Next we shall discuss various reality conditions on the (hatted) master fields that will lead to models with real physical fields living in spacetimes with different signatures.

## 2.2 Real Forms

In order to define the real forms of the field equations one has to impose reality conditions on both adjoint one-form and twisted-adjoint zero-form, corresponding to suitable real forms of the higher-spin algebra and signatures of spacetime. There are three distinct real forms of the complex higher-spin algebra itself. In two of these cases there are two distinct reality conditions that can be imposed on the zero-form, leading to five distinct models in total, as shown in Table 1. The reality conditions are

$$\widehat{A}^\dagger\ =\ -\sigma(\widehat{A})\ ,\qquad \widehat{\Phi}^\dagger\ =\ \sigma(\pi(\widehat{\Phi}))\ , \tag{2.38}$$

where the possible actions of the dagger [4] on the spinor oscillators and consequential selections of real forms of $SO(4;\mathbb{C}) \simeq SL(2;\mathbb{C}) \times SL(2;\mathbb{C})$ are given by

$$\begin{aligned} SU(2)_L \times SU(2)_R \quad : \quad & (y^\alpha)^\dagger = y^\dagger_\alpha \ , \quad (z^\alpha)^\dagger = z^\dagger_\alpha \ , \\ & (\bar{y}^{\dot\alpha})^\dagger = \bar{y}^\dagger_{\dot\alpha} \ , \quad (\bar{z}^{\dot\alpha})^\dagger = \bar{z}^\dagger_{\dot\alpha} \ , \end{aligned} \tag{2.39}$$

$$SL(2;\mathbb{C})_{\rm diag} \quad : \quad (y^\alpha)^\dagger = \bar{y}^{\dot\alpha} \ , \quad (z^\alpha)^\dagger = \bar{z}^{\dot\alpha} \ , \tag{2.40}$$

$$\begin{aligned} Sp(2;\mathbb{R})_L \times Sp(2;\mathbb{R})_R \quad : \quad & (y^\alpha)^\dagger = y^\alpha \ , \quad (z^\alpha)^\dagger = -z^\alpha \ , \\ & (\bar{y}^{\dot\alpha})^\dagger = \bar{y}^{\dot\alpha} \ , \quad (\bar{z}^{\dot\alpha})^\dagger = -\bar{z}^{\dot\alpha} \ , \end{aligned} \tag{2.41}$$

and the map $\sigma$ is given in Table 1, with the isomorphism $\rho$ given by

$$\rho(\widehat{f}(y^\dagger_\alpha, \bar{y}^\dagger_\alpha, z^\dagger_\alpha, \bar{z}^\dagger_\alpha)) = \widehat{f}(y_\alpha, \bar{y}_\alpha, -z_\alpha, -\bar{z}_\alpha) \tag{2.42}$$

in the case of $(4,0)$ signature. Note that $\sigma$ is an oscillator-algebra automorphism in signatures $(3,1)$ and $(2,2)$, while it is an isomorphism in signature $(4,0)$. Here, the $SU(2)$ doublets are pseudo real in the sense that from $(y_\alpha)^\dagger = -y^{\dagger\alpha}$ *idem* $(z_\alpha)^\dagger$, $(\bar{y}_{\dot\alpha})^\dagger$ and $(\bar{z}_{\dot\alpha})^\dagger$ it follows that $(y_\alpha, \bar{y}_{\dot\alpha}; z_\alpha, \bar{z}_{\dot\alpha})$ and $(y^\dagger_\alpha, \bar{y}^\dagger_{\dot\alpha}; z^\dagger_\alpha, \bar{z}^\dagger_{\dot\alpha})$ generate equivalent oscillator algebras with isomorphism $\rho$. The reality property of the exterior derivative takes the following form in different signatures:

$$\text{Signature } (3,1) \text{ and } (2,2) \quad : \quad d^\dagger = d \ , \tag{2.43}$$

$$\text{Signature } (4,0) \quad : \quad \rho \circ d^\dagger = d \circ \rho \ . \tag{2.44}$$

We note that the Euclidean case is consistent in the sense that

$$\rho(dz^\alpha)^\dagger = \rho d^\dagger (z^\alpha)^\dagger = d\rho(z^\dagger_\alpha) = -dz_\alpha \tag{2.45}$$

is compatible with representing $d\widehat{f}$ using $\partial\widehat{f}/\partial z^\alpha = \frac{i}{2}[z_\alpha, \widehat{f}]_\star$, which yields

$$\rho\left(\frac{i}{2} dz^\alpha [z_\alpha, \widehat{f}]_\star\right)^\dagger = \frac{i}{2} dz_\alpha \rho\left([\widehat{f}^\dagger, -z^{\dagger\alpha}]_\star\right) = \frac{i}{2} dz_\alpha [\rho\widehat{f}^\dagger, z^\alpha]_\star = \frac{i}{2} dz^\alpha [z_\alpha, \rho\widehat{f}^\dagger]_\star \ . \tag{2.46}$$

Demanding compatibility between the reality conditions (2.38) and the master field equations (2.19) and (2.20), and using

$$\rho\left((\kappa)^\dagger\right) = \kappa \ , \qquad \rho\left((idz^\alpha \wedge dz_\alpha)^\dagger\right) = -idz^\alpha \wedge dz_\alpha \ , \tag{2.47}$$

one finds the following reality conditions on the parameters

$$\text{Signature } (3,1) \quad : \quad c_1^* = c_2 \ , \tag{2.48}$$

$$\text{Signature } (4,0) \text{ and } (2,2) \quad : \quad c_1^* = c_1 \ , \quad c_2^* = c_2 \ . \tag{2.49}$$

[4] The dagger acts as usual complex conjugation on component fields; in this paper we shall denote the conjugate of a complex number $x$ by $x^*$, while reserving the bar for denoting quantities associated with the $R$-handed oscillators.

As a result, the parameter $c$ is a phase factor in Lorentzian signature and a real number in Euclidean and Kleinian signatures. The parameters can be restricted further by requiring invariance under the parity transformation

$$P(y_\alpha) \;=\; \bar{y}_{\dot\alpha}\ , \qquad P\circ d \;=\; d\circ P\ , \qquad P^2 \;=\; \mathrm{Id}\ . \tag{2.50}$$

Taking $\widehat{A}$ to be invariant and assigning intrinsic parity $\epsilon=\pm 1$ to $\widehat{\Phi}$,

$$P(\widehat{A}) \;=\; \widehat{A}\ , \qquad P(\widehat{\Phi}) \;=\; \epsilon\widehat{\Phi}\ , \tag{2.51}$$

one finds that the master equations are parity invariant provided that [20]

$$c \;=\; \epsilon \;=\; \begin{cases} 1 & \text{Type A model (scalar)} \\ -1 & \text{Type B model (pseudoscalar)} \end{cases} \tag{2.52}$$

In Lorentzian signature, there is no loss of generality in choosing $c_1=c_2=1$ in the Type A model and $c_1=-c_2=i$ in the Type B model, while in Euclidean and Kleinian signatures, one may always take $c_1=c_2=1$ in the Type A model and $c_1=-c_2=1$ in the Type B model. More generally, the parity transformation maps different models into each other as follows,

$$P(c_1) \;=\; \epsilon c_2\ , \qquad P(c_2) \;=\; \epsilon c_1\ , \qquad P(c) \;=\; \frac{1}{c}\ , \tag{2.53}$$

leaving invariant the Type A and B models. The *maximally parity violating* cases are

$$\text{Signature } (3,1) \;:\; c \;=\; \exp(i\pi/4)\ , \tag{2.54}$$

$$\text{Signature } (4,0) \text{ and } (2,2) \;:\; c \;=\; 0\ . \tag{2.55}$$

The case with $c=0$ shall be referred to as the *chiral model*, that we shall discuss in more detail below.

The HS equations in Lorentzian signature have the $\mathbb{Z}_2$ symmetry acting as $(\widehat{S}_\alpha,\widehat{S}_{\dot\alpha})\to(\epsilon\widehat{S}_\alpha,\epsilon\widehat{S}_{\dot\alpha})$, and $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry in $(4,0)$ and $(2,2)$ signatures acting as $(\widehat{S}_\alpha,\widehat{S}_{\dot\alpha})\to(\epsilon\widehat{S}_\alpha,\epsilon'\widehat{S}_{\dot\alpha})$, where $\epsilon=\pm1$ and $\epsilon'=\pm1$.

Finally, let us give the reality conditions at the level of the $SO(5;\mathbb{C})$ algebra and its minimal bosonic higher-spin extension. The adjoint representation of the complex minimal bosonic higher-spin Lie algebra is defined by [5]

$$\mathfrak{ho}(5;\mathbb{C}) \;=\; \{Q(y,\bar y)\ :\ \ \tau(Q) \;=\; -Q\}\ , \tag{2.56}$$

and the corresponding minimal twisted-adjoint representation by

$$T[\mathfrak{ho}(5;\mathbb{C})] \;=\; \{S(y,\bar y)\ :\ \ \ \tau(S) \;=\; \pi(S)\}\ . \tag{2.57}$$

[5] A more detailed description of the complex higher-spin algebra and its representations is given in [8].

The real forms are defined by

$$\mathfrak{ho}(5-q,q) \;=\; \Big\{Q(y,\bar{y})\in\mathfrak{ho}(5;\mathbb{C}) \;:\; Q^\dagger \;=\; -\sigma(Q)\Big\}\ , \tag{2.58}$$

$$T[\mathfrak{ho}(5-q,q)] \;=\; \Big\{S(y,\bar{y})\in T[\mathfrak{ho}(5-q,q) \;:\; S^\dagger \;=\; \sigma(\pi(S))\Big\}\ . \tag{2.59}$$

The finite-dimensional $SO(5;\mathbb{C})$ subalgebra is generated by $M_{AB}$, that we split into Lorentz rotations and translations $(M_{ab}, P_a)$ defined by

$$\pi(M_{ab}) \;=\; M_{ab}\ , \qquad \pi(P_a) \;=\; -P_a\ . \tag{2.60}$$

For these generators, which by convention arise in the expansion of the master fields together with a factor of $i$, the reality condition (2.38) implies

$$(M_{AB})^\dagger \;=\; \sigma(M_{AB})\ . \tag{2.61}$$

This condition is solved by

$$M_{ab} \;=\; -\frac{1}{8}\Big((\sigma_{ab})^{\alpha\beta}y_\alpha y_\beta+(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}}\Big)\ ,\quad P_a \;=\; \frac{\lambda}{4}(\sigma_a)^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}}\ , \tag{2.62}$$

where the van der Waerden symbols are defined in Appendix A and $\lambda^2$ is proportional to the cosmological constant, as shown in Table 1. The van der Waerden symbols encode the spacetime signature $\eta_{ab}$, and the commutation relations among the $M_{AB}$ then fix the signature of the ambient space to be

$$\eta_{AB} \;=\; (\eta_{ab};-\lambda^2)\ . \tag{2.63}$$

## 2.3 The Chiral Model

In the chiral model with $c=0$, the master field $\widehat{\Phi}$ can be eliminated using (2.27), and expressed as

$$\widehat{\Phi} \;=\; (1+\frac{i}{2}\widehat{S}^\alpha\star\widehat{S}_\alpha)\star\kappa\ , \tag{2.64}$$

where we have chosen $c_1=1$ and $\widehat{S}_\alpha$ is given by (2.31). The remaining independent master-field equations now read

$$\widehat{F}_{\mu\nu} \;=\; 0\ , \qquad \widehat{D}_\mu\widehat{S}_\alpha \;=\; 0\ , \qquad \widehat{D}_\mu\widehat{S}_{\dot{\alpha}} \;=\; 0\ , \tag{2.65}$$

$$[\widehat{S}_\alpha,\widehat{S}_{\dot{\alpha}}]_\star \;=\; 0\ , \qquad [\widehat{S}_{\dot{\alpha}},\widehat{S}_{\dot{\beta}}]_\star \;=\; -2i\epsilon_{\dot{\alpha}\dot{\beta}}\ , \tag{2.66}$$

$$\widehat{S}_\alpha\star\widehat{S}^\beta\star\widehat{S}_\beta+\widehat{S}^\beta\star\widehat{S}_\beta\star\widehat{S}_\alpha \;=\; 4i\widehat{S}_\alpha\ . \tag{2.67}$$

We note that (2.36) holds identically in virtue of $\widehat{S}_{\dot\alpha}\star\widehat{\Phi}+\widehat{\Phi}\star\bar\pi(\widehat{S}_{\dot\alpha}))=[\widehat{S}_{\dot\alpha},1+\frac{i}{2}\widehat{S}^{\alpha}\star\widehat{S}_{\alpha}]_\star\star\kappa=0$, where we used $\kappa\bar\kappa\star\widehat{S}_{\dot\alpha}\star\kappa\bar\kappa=-\widehat{S}_{\dot\alpha}$ and $[\widehat{S}_{\alpha},\widehat{S}_{\dot\alpha}]_\star=0$. The chiral model can be truncated further by imposing

$$\widehat{A}_{\dot\alpha} \;=\; 0\;, \qquad \frac{\partial}{\partial z^{\dot\alpha}}\widehat{A}_\mu \;=\; 0\;, \qquad \frac{\partial}{\partial z^{\dot\alpha}}\widehat{A}_\alpha \;=\; 0\;. \tag{2.68}$$

In general, the chiral model also has interesting solutions with non-vanishing $\widehat{A}_{\dot\alpha}$, since flat connections in non-commutative geometry can be non-trivial.

## 2.4 Comments on the Weak-Field Expansion and Spectrum

The procedure, described in great detail in [21], for obtaining the manifestly diffeomorphism and locally Lorentz invariant weak-field expansion of the physical field equations can be extended straightforwardly to arbitrary signature. The expansion is in terms of spin-$s$ physical fields with $s\neq 2$ as well as higher derivatives of all fields, while the vierbein and Lorentz connection are treated exactly.

In this approach one first solves (2.26)–(2.30) subject to the initial condition

$$\Phi \;=\; \widehat{\Phi}|_{Z=0}\;, \tag{2.69}$$

$$A_\mu \;=\; \widehat{A}_\mu\Big|_{Z=0} \;=\; e_\mu+\omega_\mu+W_\mu+K_\mu\;, \tag{2.70}$$

where

$$e_\mu \;=\; \frac{1}{2i}e_\mu{}^a P_a\;, \qquad \omega_\mu \;=\; \tfrac{1}{4i}\omega_\mu{}^{ab}M_{ab}\;; \tag{2.71}$$

$W_\mu$ contains the higher-spin gauge fields (and also the spin $s=1$ gauge field in the non-minimal model); and the field redefinition

$$K_\mu \;=\; \frac{1}{4i}\omega_\mu{}^{\alpha\beta}\,\widehat{S}_\alpha\star\widehat{S}_\beta\Big|_{Z=0}+\frac{1}{4i}\bar\omega_\mu{}^{\dot\alpha\dot\beta}\,\widehat{S}_{\dot\alpha}\star\widehat{S}_{\dot\beta}\Big|_{Z=0} \tag{2.72}$$

$$\;=\; i\omega_\mu{}^{\alpha\beta}\,(\widehat{A}_\alpha\star\widehat{A}_\beta-\frac{\partial}{\partial y^\alpha}\widehat{A}_\beta)\Big|_{Z=0}+i\bar\omega_\mu{}^{\dot\alpha\dot\beta}\,(\widehat{A}_{\dot\alpha}\star\widehat{A}_{\dot\beta}-\frac{\partial}{\partial \bar y^{\dot\alpha}}\widehat{A}_{\dot\beta})\Big|_{Z=0}\;. \tag{2.73}$$

One also imposes the gauge condition

$$\widehat{A}^{(0)}_\alpha \;=\; 0\;, \qquad \widehat{A}^{(0)}_{\dot\alpha} \;=\; 0\;, \tag{2.74}$$

where we have defined the internal flat connection

$$\widehat{A}^{(0)}_\alpha \;=\; \widehat{A}_\alpha|_{\Phi=0}\;, \qquad \widehat{A}^{(0)}_{\dot\alpha} \;=\; \widehat{A}_{\dot\alpha}|_{\Phi=0}\;. \tag{2.75}$$

One then substitutes the resulting $\widehat{\Phi}$ and $\widehat{A}_\mu$, which can be obtained explicitly in a perturbative expansion in $\Phi$, into (2.25) and sets $Z=0$, which yields a manifestly spin-2 covariant complex

HS gauge theory on the base manifold. Up to this point the local structure of the base-manifold, nor the detailed structure of the gauge fields, have played any role. To proceed, one may refer to an ordinary spacetime, take ${e_\mu}^a$ to be an (invertible) vierbein, and treat $W_\mu$ as a weak field. This allows one to eliminate a large number of auxiliary fields in $\Phi$ and $W_\mu$, leaving a model consisting of a physical scalar $\phi = \Phi|_{y=\bar{y}=0}$, the vierbein ${e_\mu}^a$, and an infinite tower of (doubly traceless) HS gauge fields $\phi_{a(s)}$ residing in $W_\mu$.

The gauge choice (2.74) is convenient since it implies $\frac{\partial}{\partial y^\alpha}\widehat{A}_\beta|_{Z=0} = 0$ that simplifies the expansion [21]. However, there are also other gauges where $\widehat{A}_\alpha|_{\Phi=0}$ is a flat but non-trivial internal connection, and indeed this will be the case for the Type 1 and Type 2 solutions that we shall present in Section 3.

In the leading order in the weak fields, the two-form and one-form constraints for the minimal model read

$$s=2 \quad : \quad \left\{ \begin{array}{lll} \mathcal{R}_{\alpha\beta,\gamma\delta} = c_2\Phi_{\alpha\beta\gamma\delta} \ , & & \mathcal{R}_{\dot{\alpha}\dot{\beta},\gamma\delta} = 0 \ , \\ \mathcal{R}_{\alpha\beta,\gamma\dot{\delta}} = 0 \ , & & \mathcal{R}_{\dot{\alpha}\dot{\beta},\gamma\dot{\delta}} = 0 \ , \\ \mathcal{R}_{\alpha\beta,\dot{\gamma}\dot{\delta}} = 0 \ , & & \mathcal{R}_{\dot{\alpha}\dot{\beta},\dot{\gamma}\dot{\delta}} = c_1\Phi_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} \ , \end{array} \right. \tag{2.76}$$

$$s=4,6,\ldots \quad : \quad \left\{ \begin{array}{ll} F^{(1)}_{\alpha\beta,\gamma_1\ldots\gamma_{2s-2}} = c_2\Phi_{\alpha\beta\gamma_1\ldots\gamma_{2s-2}} \ , & F^{(1)}_{\dot{\alpha}\dot{\beta},\dot{\gamma}_1\ldots\dot{\gamma}_k\gamma_{k+1}\ldots\gamma_{2s-2}} = 0 \ , \\ F^{(1)}_{\alpha\beta,\gamma_1\ldots\gamma_k\dot{\gamma}_{k+1}\ldots\dot{\gamma}_{2s-2}} = 0 \ , & F^{(1)}_{\dot{\alpha}\dot{\beta},\dot{\gamma}_1\ldots\dot{\gamma}_{2s-2}} = c_1\Phi_{\dot{\alpha}\dot{\beta}\dot{\gamma}_1\ldots\dot{\gamma}_{2s-2}} \ , \end{array} \right. \tag{2.77}$$

$$\text{0-forms} \quad : \quad \nabla_\alpha{}^{\dot{\alpha}}\Phi_{\beta_1\ldots\beta_m}{}^{\dot{\beta}_1\ldots\dot{\beta}_n} = i\lambda\left(\Phi_{\alpha\beta_1\ldots\beta_m}{}^{\dot{\alpha}\dot{\beta}_1\ldots\dot{\beta}_n} - mn\epsilon_{\alpha(\beta_1}\epsilon^{\dot{\alpha}(\dot{\beta}_1}\Phi_{\beta_2\ldots\beta_m)}{}^{\dot{\beta}_2\ldots\dot{\beta}_n)}\right) \ , \tag{2.78}$$

where for higher spins $s = 4, 6, \ldots$ and $k = 0, \ldots, 2s-3$, and for 0-forms $|m-n| = 0 \mod 4$. In all cases, the zero-form system contains a physical scalar with field equation

$$(\nabla^2 + 2\lambda^2)\phi = 0 \ . \tag{2.79}$$

In the Lorentzian case, where both $c_1$ and $c_2 = c_1^*$ are non-zero, the spin-2 sector consists of gravity with cosmological constant $-3\lambda^2$, and the spin-$s$ sectors with $s = 4, 6, \ldots$ consist of higher-spin tensor gauge fields with critical masses proportional to $\lambda^2$. The criticality in the masses, that implies composite masslessness[6] in the case of AdS, holds in the dS case as well, where thus the physical spectrum is given by the symmetric tensor product of two (non-unitary) $SO(4,1)$ singletons [8].

In the Euclidean and Kleinian cases, the parameters $c_1$ and $c_2$ are real and independent. In case $c_1c_2 \neq 0$, the Lorentzian analysis carries over, leading to a composite massless spectrum given by symmetric tensor products of suitable singletons [8]. However, unlike the Lorentzian case, the spin-$s$ sector of the twisted adjoint representation can be decomposed into left-handed and

[6] By definition masslessness refers to reduction in the infinite-dimensional weight space of the various real forms of $SO(5;\mathbb{C})$. This is well-known for $SO(3,2)$ and similar situations arise for other signatures as well. By compositeness we mean that the massless states are composites of singletons [2].

right-handed sub-sectors of real states, corresponding to $\{\Phi_{\alpha_1\ldots\alpha_m,\dot\alpha_1\ldots\dot\alpha_n}\}$ with $m-n=\pm 2s$ [8]. These sub-sectors mix under HS transformations.

In case either $c_1$ or $c_2$, but not both, vanishes, that we shall refer to as the chiral models, the metric and the higher-spin gauge fields become half-flat. For definiteness, let us consider the case $c_2=0$. The components of the zero-form that drop out in the two-form constraint, *i.e.* $\Phi_{\alpha_1\ldots\alpha_{2s}}$, now become *independent* physical fields, obeying field equations following from (2.78).

| HSA | Signature $\eta_{ab}$ | Spinors | $\lambda^2$ | Reality $\sigma$ | Symmetric space | Hermitian isometries |
|---|---|---|---|---|---|---|
| $\mathfrak{ho}(5)$ | $(4,0)$ | $SU(2)_L\times SU(2)_R$ | $-1$ | $\rho$ | $S^4$ | $\mathfrak{so}(2)\otimes\mathfrak{so}(3)$ |
| $\mathfrak{ho}(4,1)$ | $(4,0)$ | $SU(2)_L\times SU(2)_R$ | $+1$ | $\rho\pi$ | $H_4$ | $\mathfrak{so}(3,1)$ |
| $\mathfrak{ho}(4,1)$ | $(3,1)$ | $SL(2,\mathbb{C})_{\rm diag}$ | $-1$ | $\pi$ | $dS_4$ | $\mathfrak{so}(3,1)'$ |
| $\mathfrak{ho}(3,2)$ | $(3,1)$ | $SL(2,\mathbb{C})_{\rm diag}$ | $+1$ | id | $AdS_4$ | $\mathfrak{so}(3,2)$ |
| $\mathfrak{ho}(3,2)$ | $(2,2)$ | $SL(2,\mathbb{R})_L\times SL(2,\mathbb{R})_R$ | $-1$ | id | $H_{3,2}$ | $\mathfrak{so}(3,2)$ |

Table 1: The minimal bosonic higher-spin algebras $\mathfrak{ho}(p',5-p')\supset\mathfrak{so}(5-p',p')$ in signature $(p,4-p)$ can be realized with spinor oscillators transforming as doublets under the groups listed in the third column. These realizations obey reality conditions $(M_{AB})^\dagger=\sigma(M_{AB})$, with hermitian subalgebras listed above [8]. The symmetric spaces with unit radius have cosmological constant $\Lambda=-3\lambda^2$.

## 3 Exact Solutions

In this section we shall give four types of exact solutions to the 4D HS models given in the previous section. The salient features of these are summarized in the Introduction. Here we stress that (a) the Type 0 solutions are maximally symmetric spaces; (b) the Type 1 solutions are $SO(4-p,p)$ invariant deformations of Type 0; (c) the Type 2 solutions, which exist necessarily in the non-minimal model, have vanishing spacetime component fields but non-vanishing spinorial master one-form; (d) the Type 3 solutions, which exist in the non-minimal chiral model only, have the remarkable feature that all higher spin gauge fields are non-vanishing in such a way that the Weyl zero-forms are covariantly constant, in a certain sense that will be explained below. Before we give these four types of solutions we shall describe briefly the method for solving the master field equations using gauge functions.

### 3.1 The Gauge Function Ansatz

In order to construct an interesting class of solutions we shall use the $Z$-space approach [22, 5] in which the constraints carrying at least one curved spacetime index, *viz.*

$$\widehat{F}_{\mu\nu} \;=\; 0\;,\qquad \widehat{D}_\mu\widehat{\Phi} \;=\; 0\;, \tag{3.1}$$

$$\widehat{F}_{\mu\alpha} \;=\; 0\;,\qquad \widehat{F}_{\mu\dot{\alpha}} \;=\; 0\;, \tag{3.2}$$

are integrated in simply connected spacetime regions given the spacetime zero-forms at a point $p$,

$$\widehat{\Phi}' \;=\; \widehat{\Phi}|_p\;,\qquad \widehat{S}'_\alpha \;=\; \widehat{S}_\alpha|_p\;,\qquad \widehat{S}'_{\dot{\alpha}} \;=\; \widehat{S}_{\dot{\alpha}}|_p\;, \tag{3.3}$$

and expressed explicitly as

$$\widehat{A}_\mu \;=\; \widehat{L}^{-1}\star\partial_\mu\widehat{L}\;,\qquad \widehat{\Phi} \;=\; \widehat{L}^{-1}\star\widehat{\Phi}'\star\pi(\widehat{L})\;, \tag{3.4}$$

$$\widehat{S}_\alpha \;=\; \widehat{L}^{-1}\star\widehat{S}'_\alpha\star\widehat{L}\;,\qquad \widehat{S}_{\dot{\alpha}} \;=\; \widehat{L}^{-1}\star\widehat{S}'_{\dot{\alpha}}\star\widehat{L}\;, \tag{3.5}$$

where $\widehat{L}=\widehat{L}(x,z,\bar{z};y,\bar{y})$ is a gauge function, and

$$\widehat{L}|_p \;=\; 1\;,\qquad \partial_\mu\widehat{\Phi}' \;=\; 0\;,\qquad \partial_\mu\widehat{S}'_\alpha \;=\; 0\;,\qquad \partial_\mu\widehat{S}'_{\dot{\alpha}} \;=\; 0\;. \tag{3.6}$$

The internal connections $\widehat{A}_\alpha$ and $\widehat{A}_{\dot{\alpha}}$ can be reconstructed from $\widehat{S}_\alpha$ and $\widehat{S}_{\dot{\alpha}}$ using (2.31). In particular note the relation

$$\widehat{A}_\alpha=\widehat{L}\star\partial_\alpha\widehat{L}+\widehat{L}^{-1}\star\widehat{A}'_\alpha\star\widehat{L}\;, \tag{3.7}$$

and it follows that

$$\widehat{S}'_\alpha=z_\alpha-2i\widehat{A}'_\alpha\;. \tag{3.8}$$

The remaining constraints in $Z$-space, *viz.*

$$[\widehat{S}'_\alpha,\widehat{S}'_\beta]_\star \;=\; -2i\epsilon_{\alpha\beta}(1-c_1\widehat{\Phi}'\star\kappa)\;,\qquad [\widehat{S}'_{\dot{\alpha}},\widehat{S}'_{\dot{\beta}}]_\star \;=\; -2i\epsilon_{\dot{\alpha}\dot{\beta}}(1-c_2\widehat{\Phi}'\star\bar{\kappa})\;, \tag{3.9}$$

$$[\widehat{S}'_\alpha,\widehat{S}'_{\dot{\beta}}]_\star \;=\; 0\;, \tag{3.10}$$

$$\widehat{S}'_\alpha\star\widehat{\Phi}'+\widehat{\Phi}'\star\pi(\widehat{S}'_\alpha) \;=\; 0\;, \tag{3.11}$$

$$\widehat{S}'_{\dot{\alpha}}\star\widehat{\Phi}'+\widehat{\Phi}'\star\bar{\pi}(\widehat{S}'_{\dot{\alpha}}) \;=\; 0\;, \tag{3.12}$$

are then to be solved with an initial condition

$$C'(y,\bar{y}) \;=\; \widehat{\Phi}'|_{Z=0}\;, \tag{3.13}$$

and some assumption about the topology of the internal flat connections

$$\widehat{S}'^{(0)}_\alpha \;=\; \widehat{S}'_\alpha|_{C'=0}\;,\qquad \widehat{S}'^{(0)}_{\dot{\alpha}} \;=\; \widehat{S}'_{\dot{\alpha}}|_{C'=0}\;. \tag{3.14}$$

In what follows, we shall restrict the class of solutions further by assuming that

$$\widehat{L} \;=\; L(x;y,\bar{y}) \;. \tag{3.15}$$

The gauge fields can then be obtained from (2.70), (2.73) and (3.5), *viz.*

$$e_\mu + \omega_\mu + W_\mu \;=\; L^{-1}\partial_\mu L - K_\mu \;, \tag{3.16}$$

where

$$K_\mu \;=\; \frac{1}{4i}\, L^{-1}\star\left(\omega_\mu{}^{\alpha\beta}\widehat{S}'_\alpha\star\widehat{S}'_\beta + \bar{\omega}_\mu{}^{\dot{\alpha}\dot{\beta}}\widehat{S}'_{\dot{\alpha}}\star\widehat{S}'_{\dot{\beta}}\right)\star L\Big|_{Z=0} \;. \tag{3.17}$$

Hence, the gauge fields, including the metric, can be obtained algebraically without having to solve any differential equations in spacetime.

## 3.2 Ordinary Maximally Symmetric Spaces (Type 0)

The complex master-field equations are solved by

$$\widehat{\Phi} \;=\; 0\;, \qquad \widehat{S}_\alpha \;=\; z_\alpha\;, \qquad \widehat{S}_{\dot{\alpha}} \;=\; \bar{z}_{\dot{\alpha}}\;, \qquad \widehat{A}_\mu \;=\; L^{-1}\star\partial_\mu L\;, \tag{3.18}$$

where the gauge function [22]

$$L(x;y,\bar{y}) \;=\; \frac{2h}{1+h}\exp\left[\frac{i\lambda x^{\alpha\dot{\alpha}}y_\alpha\bar{y}_{\dot{\alpha}}}{1+h}\right]\;, \tag{3.19}$$

gives

$$ds^2_{(0)} \;=\; \frac{4dx^2}{(1-\lambda^2x^2)^2}\;, \tag{3.20}$$

which we identify as the metric of the symmetric spaces listed in Table 1 for the different real forms of the model, in stereographic coordinates with inverse radius $|\lambda|$. This metric is invariant under the inversion

$$x^a \to -x^a/(\lambda^2x^2)\;, \tag{3.21}$$

and $H_4$ is covered by a single coordinate chart, while the remaining symmetric spaces require two charts, related by the inversion. If we let $\tilde{x}^a = -x^a/(\lambda^2x^2)$, the atlases are given by

$$S^4 \quad (\lambda^2=-1) \;:\; \{x^\mu : 0\le -\lambda^2x^2\le 1\}\cup\{\tilde{x}^\mu : 0\le -\lambda^2\tilde{x}^2\le 1\}\;, \tag{3.22}$$

$$H^4 \quad (\lambda^2=1) \;:\; \{x^\mu : 0\le \lambda^2x^2 < 1\}\;, \tag{3.23}$$

$$dS_4 \quad (\lambda^2=-1) \;:\; \{x^\mu : -1< -\lambda^2x^2\le 1\}\cup\{\tilde{x}^\mu : -1< -\lambda^2\tilde{x}^2\le 1\}\;, \tag{3.24}$$

$$AdS_4 \quad (\lambda^2=1) \;:\; \{x^\mu : -1\le \lambda^2x^2< 1\}\cup\{\tilde{x}^\mu : -1\le \lambda^2\tilde{x}^2< 1\}\;, \tag{3.25}$$

$$H_{3,2} \quad (\lambda^2=-1) \;:\; \{x^\mu : -1< -\lambda^2x^2\le 1\}\cup\{\tilde{x}^\mu : -1< -\lambda^2\tilde{x}^2\le 1\}\;, \tag{3.26}$$

where the overlap between the charts is given by $\{x^\mu : \lambda^2x^2=-1\}$ in the cases of $S^4$, $dS_4$, $AdS_4$ and $H_{3,2}$, and the boundary is $\{x^\mu : \lambda^2x^2=1\}$ in the case of $H_4$ and $\{x^\mu : \lambda^2x^2=1\}\cup\{\tilde{x}^\mu : \lambda^2\tilde{x}^2=1\}$ in the cases of $dS_4$, $AdS_4$ and $H_{3,2}$. The $H_{3,2}$ space can be described as the coset $SO(3,2)/SO(2,2)$.

## 3.3 $SO(4-p,p)$ Invariant Solutions to the Minimal Model (Type 1)

### 3.3.1 Internal Master Fields

A particular class of $SO(4;\mathbb{C})$-invariant solutions is given by the ansatz

$$\widehat{\Phi}' = \nu , \qquad \widehat{S}'_\alpha = z_\alpha \, S(u) , \qquad \widehat{S}'_{\dot\alpha} = \bar z_{\dot\alpha} \, \bar S(\bar u) \tag{3.27}$$

where

$$u = y^\alpha z_\alpha , \qquad \bar u = \bar y^{\dot\alpha} \bar z_{\dot\alpha} . \tag{3.28}$$

The above ansatz solves (3.10)-(3.12). There remains to solve (3.9), which now takes the form

$$[\widehat{S}'^\alpha, \widehat{S}'_\alpha]_\star = 4i(1-c_1\nu e^{iu}) , \qquad [\widehat{S}'^{\dot\alpha}, \widehat{S}'_{\dot\alpha}]_\star = 4i(1-c_2\nu e^{-i\bar u}) \tag{3.29}$$

Following [4], we use the integral representation

$$S(u) = \int_{-1}^{1} ds \; n(s) \; e^{\frac{i}{2}(1+s)u} , \tag{3.30}$$

$$\bar S(\bar u) = \int_{-1}^{1} ds \; \bar n(s) \; e^{-\frac{i}{2}(1+s)\bar u} . \tag{3.31}$$

which reduces (3.29) to

$$(n\circ n)(t) = \delta(t-1) - \frac{c_1\nu}{2}(1-t) , \tag{3.32}$$

$$(\bar n\circ \bar n)(t) = \delta(t-1) - \frac{c_2\nu}{2}(1-t) . \tag{3.33}$$

with $\circ$ defined by [4]

$$(f\circ g)(t) = \int_{-1}^{1} ds \int_{-1}^{1} ds' \delta(t-ss') \; f(s) \; g(s') . \tag{3.34}$$

Even and odd functions, denoted by $f^\pm(t)$, are orthogonal with respect to the $\circ$ product. Thus, one finds

$$(n^+\circ n^+)(t) = \iota_0^+(t) - \frac{c_1\nu}{2} , \qquad (n^-\circ n^-)(t) = \iota_0^-(t) + \frac{c_1\nu}{2}t , \tag{3.35}$$

$$(\bar n^+\circ \bar n^+)(t) = \iota_0^+(t) - \frac{c_2\nu}{2} , \qquad (\bar n^-\circ \bar n^-)(t) = \iota_0^-(t) + \frac{c_2\nu}{2}t , \tag{3.36}$$

where

$$\iota_0^\pm(t) = \frac{1}{2}\left[\delta(1-t) \pm \delta(1+t)\right] . \tag{3.37}$$

One proceeds [4], by writing

$$n^\pm(t) = m^\pm(t) + \sum_{k=0}^{\infty} \lambda_k p_k^\pm , \tag{3.38}$$

where $m^{\pm}$ are expanded in terms of $\iota_0^{(\pm)}(t)$ and the functions $(k \geq 1)$

$$\begin{aligned}
\iota_k^{\sigma}(t) &= [\text{sign}(t)]^{\frac{1}{2}(1-\sigma)} \int_{-1}^{1} ds_1 \cdots \int_{-1}^{1} ds_k \; \delta(t - s_1 \cdots s_k) \\
&= [\text{sign}(t)]^{\frac{1}{2}(1-\sigma)} \frac{\left(\log \frac{1}{t^2}\right)^{k-1}}{(k-1)!} \; ,
\end{aligned} \tag{3.39}$$

obeying the algebra $(k, l \geq 0)$

$$\iota_k^{\sigma} \circ \iota_l^{\sigma} = \iota_{k+l}^{\sigma} \; , \tag{3.40}$$

and $p_k^{\sigma}(t)$ $(k \geq 0)$ are the $\circ$-product projectors

$$p_k^{\sigma}(t) = \frac{(-1)^k}{k!} \delta^{(k)}(t) \; , \qquad \sigma = (-1)^k \; , \tag{3.41}$$

obeying

$$p_k^{\sigma} \circ f = L_k[f] p_k^{\sigma} \; , \qquad L_k[f] = \int_{-1}^{1} dt \; t^k f(t) \; . \tag{3.42}$$

In particular,

$$p_k^{\sigma} \circ p_l^{\sigma} = \delta_{kl} p_l^{\sigma} \; . \tag{3.43}$$

Substituting the expansion (3.38) into (3.35) and (3.36), one finds, in view of (3.40), (3.42) and (3.43), manageable algebraic equations. Transforming back one finds, after some algebra [5],

$$m(t) = \delta(1+t) + q(t) \; , \tag{3.44}$$

$$q(t) = -\frac{c_1 \nu}{4} \left( {}_1F_1 \left[ \frac{1}{2}; 2; \frac{c_1 \nu}{2} \log \frac{1}{t^2} \right] + t \, {}_1F_1 \left[ \frac{1}{2}; 2; -\frac{c_1 \nu}{2} \log \frac{1}{t^2} \right] \right) \; , \tag{3.45}$$

and

$$\lambda_k = -2\theta_k L_k[m] \; , \qquad \theta_k \in \{0, 1\} \; , \tag{3.46}$$

where

$$L_k[m] = (-1)^k + L_k[q] \; , \tag{3.47}$$

$$L_k[q] = -\frac{1+(-1)^k}{2} \left( 1 - \sqrt{1 - \frac{c_1 \nu}{1+k}} \right) - \frac{1-(-1)^k}{2} \left( 1 - \sqrt{1 + \frac{c_1 \nu}{2+k}} \right) \; . \tag{3.48}$$

The overall signs in $m^{\pm}$ have been fixed in (3.45) by requiring that

$$S(u) = 1 \;\; \text{for} \;\; \nu = 0 \;\; \text{and} \;\; \theta_k = 0 \; . \tag{3.49}$$

Treating $\bar{n}$ the same way, one finds

$$\bar{m}(t) = \delta(1+t) + \bar{q}(t) \ , \tag{3.50}$$

$$\bar{q}(t) = -\frac{c_2\nu}{4}\left({}_1F_1\left[\frac{1}{2};2;\frac{c_2\nu}{2}\log\frac{1}{t^2}\right] + t\,{}_1F_1\left[\frac{1}{2};2;-\frac{c_2\nu}{2}\log\frac{1}{t^2}\right]\right) \ , \tag{3.51}$$

$$\bar{\lambda}_k = -2\bar{\theta}_k L_k[\bar{m}] \ , \qquad \bar{\theta}_k \in \{0,1\} \ , \tag{3.52}$$

$$L_k[\bar{m}] = (-1)^k + L_k[\bar{q}] \ , \tag{3.53}$$

$$L_k[\bar{q}] = -\frac{1+(-1)^k}{2}\left(1-\sqrt{1-\frac{c_2\nu}{1+k}}\right) - \frac{1-(-1)^k}{2}\left(1-\sqrt{1+\frac{c_2\nu}{2+k}}\right) \ . \tag{3.54}$$

Thus, the internal solution is given by

$$\widehat{\Phi}' = \nu \ , \tag{3.55}$$

together with $\widehat{S}'_\alpha$ and $\widehat{S}'_{\dot\alpha}$ as given in (3.8) with

$$\widehat{A}'_\alpha = \widehat{A}'^{(reg)}_\alpha + \widehat{A}'^{(proj)}_\alpha \ , \qquad \widehat{A}'_{\dot\alpha} = \widehat{A}'^{(reg)}_{\dot\alpha} + \widehat{A}'^{(proj)}_{\dot\alpha} \ , \tag{3.56}$$

$$\widehat{A}'^{(reg)}_\alpha = \frac{i}{2}z_\alpha\int_{-1}^{1} dt\ q(t)\,e^{\frac{i}{2}(1+t)u} \ , \qquad \widehat{A}'^{(reg)}_{\dot\alpha} = \frac{i}{2}\bar{z}_{\dot\alpha}\int_{-1}^{1} dt\ \bar{q}(t)\,e^{-\frac{i}{2}(1+t)\bar{u}} \ , \tag{3.57}$$

$$\widehat{A}'^{(proj)}_\alpha = -iz_\alpha\sum_{k=0}^{\infty}\theta_k(-1)^k L_k[m]P_k(u) \ , \qquad \widehat{A}'^{(proj)}_{\dot\alpha} = -i\bar{z}_{\dot\alpha}\sum_{k=0}^{\infty}\bar{\theta}_k(-1)^k L_k[\bar{m}]\bar{P}_k(\bar{u}) \tag{3.58}$$

where

$$P_k(u) = \int_{-1}^{1} ds\ e^{\frac{i}{2}(1-s)u}p_k(s) = \frac{1}{k!}\left(\frac{-iu}{2}\right)^k e^{\frac{iu}{2}} \ , \tag{3.59}$$

$$\bar{P}_k(\bar{u}) = \int_{-1}^{1} ds\ e^{-\frac{i}{2}(1-s)\bar{u}}p_k(s) = \frac{1}{k!}\left(\frac{i\bar{u}}{2}\right)^k e^{-\frac{i\bar{u}}{2}} \tag{3.60}$$

are projectors in the $\star$-product algebra given by functions of $u$ and $\bar{u}$, *viz.*

$$P_k\star F = L_k[f]P_k \ , \qquad P_k\star P_l = \delta_{kl}P_k \ , \tag{3.61}$$

$$\bar{P}_k\star \bar{F} = L_k[\bar{f}]\bar{P}_k \ , \qquad \bar{P}_k\star \bar{P}_l = \delta_{kl}\bar{P}_k \ , \tag{3.62}$$

for $F(u) = \int_{-1}^{1} ds e^{\frac{i}{2}(1-s)u}f(s)$ and $\bar{F}(\bar{u}) = \int_{-1}^{1} ds e^{-\frac{i}{2}(1-s)\bar{u}}\bar{f}(s)$ with $L_k[f]$ and $L_k[\bar{f}]$ given in (3.42). The projectors also obey $(u-2ik)\star P_k = 0$ and $y^\alpha\star P_k\star z_\alpha = i(k+1)(P_{k-1}+P_{k+1})$ with $P_{-1}\equiv 0$. We note the opposite signs in front of $s$ in the exponents of (3.30), (3.31) and (3.59), (3.60), resulting in the $(-1)^k$ in the projector part (3.58) of the internal connection, which we can thus write as

$$\widehat{A}'^{(proj)}_\alpha = -iz_\alpha\sum_{k=0}^{\infty}\left[\theta_k P_k - \left(1-\sqrt{1-\frac{c_1\nu}{1+2k}}\right)\theta_{2k}P_{2k} + \left(1-\sqrt{1+\frac{c_1\nu}{3+2k}}\right)\theta_{2k+1}P_{2k+1}\right] \ , \tag{3.63}$$

$$\widehat{A}'^{(proj)}_{\dot\alpha} = -i\bar{z}_{\dot\alpha}\sum_{k=0}^{\infty}\left[\bar{\theta}_k\bar{P}_k - \left(1-\sqrt{1-\frac{c_2\nu}{1+2k}}\right)\bar{\theta}_{2k}\bar{P}_{2k} + \left(1-\sqrt{1+\frac{c_2\nu}{3+2k}}\right)\bar{\theta}_{2k+1}\bar{P}_{2k+1}\right] \ , \tag{3.64}$$

which are analytic functions of $\nu$ in a finite region around the origin. For example, for $c_1 = c_2 = 1$, they are real analytic for $-3 < \mathrm{Re}\nu < 1$, where also the particular solution can be shown to be real analytic [5]. The reality conditions on the $\theta_k$ and $\bar{\theta}_k$ parameters are as follows:

$$(4,0) \text{ and } (2,2) \text{ signature} \; : \; \theta_k \;, \quad \bar{\theta}_k \qquad \text{independent} \;, \tag{3.65}$$

$$(3,1) \text{ signature} \; : \; \theta_k \; = \; \bar{\theta}_k \;. \tag{3.66}$$

Taking $\nu = 0$ there remains only the projector part, leading to the following "vacuum" solutions

$$\widehat{\Phi}' \;=\; 0 \;, \tag{3.67}$$

$$\widehat{A}'_\alpha \;=\; -iz_\alpha \sum_{k=0}^{\infty} \theta_k \frac{1}{k!} \left(\frac{-iu}{2}\right)^k e^{\frac{iu}{2}} \;, \qquad \widehat{A}'_{\dot\alpha} \;=\; -i\bar{z}_{\dot\alpha} \sum_{k=0}^{\infty} \bar{\theta}_k \frac{1}{k!} \left(\frac{i\bar{u}}{2}\right)^k e^{-\frac{i\bar{u}}{2}} \;. \tag{3.68}$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (2.37) acts by

$$\theta_k \;\to\; 1 - \theta_k \;, \qquad \bar{\theta}_k \;\to\; 1 - \bar{\theta}_k \;. \tag{3.69}$$

The maximally symmetric spaces discussed in Section 3.2 are recovered by setting $\theta_k = \theta$ and $\bar{\theta}_k = \bar{\theta}$ for all $k$. In Euclidean and Kleinian signatures, $\theta$ and $\bar{\theta}$ are independent, leading to four solutions related by $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation. In Lorentzian signature, $\theta = \bar{\theta}$ leading to two solutions related by $\mathbb{Z}_2$ symmetry.

### 3.3.2 Spacetime Component Fields

The calculation of the component fields follow the same steps as in [5]. The spin $s \geq 1$ Weyl tensors vanish, while the scalar field is given by

$$\phi(x) \;=\; \nu h^2(x^2) \;=\; \nu(1 - \lambda^2 x^2) \;. \tag{3.70}$$

In order to compute the gauge fields, we first need to compute the quantity $K_\mu$ given in (3.17). This calculation is formally the same as the one spelled out in the case of $\theta_k = \bar{\theta}_k = 0$ in [5], and result is

$$K_\mu \;=\; \frac{Q}{4i} \omega_\mu^{\alpha\beta} v_\alpha v_\beta + \frac{\bar{Q}}{4i} \bar{\omega}_\mu^{\dot\alpha\dot\beta} \bar{v}_{\dot\alpha} \bar{v}_{\dot\beta} \;, \tag{3.71}$$

where

$$Q \;=\; -\frac{(1-a^2)^2}{4} \int_{-1}^{1} ds \int_{-1}^{1} ds' \frac{(1+s)(1+s')n(s)n(s')}{(1-ss'a^2)^4} \;, \tag{3.72}$$

$$\bar{Q} \;=\; -\frac{(1-a^2)^2}{4} \int_{-1}^{1} ds \int_{-1}^{1} ds' \frac{(1+s)(1+s')\bar{n}(s)\bar{n}(s')}{(1-ss'a^2)^4} \;. \tag{3.73}$$

and

$$v_\alpha = (1+a^2)y_\alpha + 2(a\bar{y})_\alpha \ , \qquad \bar{v}_{\dot\alpha} = (1+a^2)\bar{y}_{\dot\alpha} + 2(\bar{a}y)_{\dot\alpha} \ , \tag{3.74}$$

with $\bar{a}_{\dot\alpha\alpha} = a_{\alpha\dot\alpha}$ defined in (B.3). We can simplify $Q$ using $n(t) = \delta(1+t) + q(t) + \sum_k \lambda_k p_k(t)$, with $p_k(t)$ given by (3.41) and $\lambda_k$ by (3.46) and (3.47). After some algebra we find

$$Q(\nu;\{\theta_k\}) = Q^{(reg)}(\nu) + Q^{(proj)}(\nu;\{\theta_k\}) \ , \tag{3.75}$$

$$Q^{(reg)} = -\frac{(1-a^2)^2}{4}\int_{-1}^{1} ds \int_{-1}^{1} ds' \frac{(1+s)(1+s')q(s)q(s')}{(1-ss'a^2)^4} \ , \tag{3.76}$$

$$Q^{(proj)} = (1-a^2)^2 \sum_{k=0}^{\infty} \frac{4_k a^{2k}}{k!}(\theta_k - \theta_{k+1})^2 \Big((-1)^k + L_k(q)\Big)\Big((-1)^{k+1} + L_{k+1}(q)\Big) , \tag{3.77}$$

where we note that $Q$ depends on $\theta_k$ only via $\theta_k - \theta_{k+1}$. The same expression with $q \to \bar{q}$ and $\theta_k \to \bar\theta_k$ holds for $\bar{Q}$. The regular part, which was computed in [5], is given by

$$Q^{(reg)} = Q_+^{(reg)} + Q_-^{(reg)} \ , \tag{3.78}$$

$$Q_+^{(reg)} = -\frac{(1-a^2)^2}{4}\sum_{p=0}^{\infty}\binom{-4}{2p} a^{4p}\left(\sqrt{1-\frac{c_1\nu}{2p+1}} - \sqrt{1+\frac{c_1\nu}{2p+3}}\right)^2 \tag{3.79}$$

$$Q_-^{(reg)} = \frac{(1-a^2)^2}{4}\sum_{p=0}^{\infty}\binom{-4}{2p+1} a^{4p+2}\left(\sqrt{1-\frac{c_1\nu}{2p+3}} - \sqrt{1+\frac{c_1\nu}{2p+3}}\right)^2 \ , \tag{3.80}$$

while a similar expression, obtained by replacing $c_1 \to c_2$, holds for $\bar{Q}$.

Since $K_\mu$ is bilinear in the $y_\alpha$ and $\bar{y}_{\dot\alpha}$ oscillators, it immediately follows that all higher spin fields vanish. Moreover, after some algebra, we find that the vierbein and $\mathfrak{so}(4;\mathbb{C})$ connection are given by

$$e^a = f_1(x^2)dx^a + f_2(x^2)x^a dx^b x_b \ , \tag{3.81}$$

$$\omega_{\alpha\beta} = f(x^2)\omega^{(0)}_{\alpha\beta} \ , \qquad \bar\omega_{\dot\alpha\dot\beta} = \bar{f}(x^2)\bar\omega^{(0)}_{\dot\alpha\dot\beta} \ , \tag{3.82}$$

where

$$f = \frac{1+(1-a^2)^2\bar{Q}}{[1+(1+a^2)^2Q]\left[1+(1+a^2)^2\bar{Q}\right] - 16a^4 Q\bar{Q}} \ , \tag{3.83}$$

$$\bar{f} = \frac{1+(1-a^2)^2 Q}{[1+(1+a^2)^2Q]\left[1+(1+a^2)^2\bar{Q}\right] - 16a^4 Q\bar{Q}} \ , \tag{3.84}$$

and

$$f_1 + \lambda^2 x^2 f_2 = \frac{2}{h^2} \ , \qquad f_2 = \frac{2(1+a^2)^4}{(1-a^2)^2}(fQ + \bar{f}\bar{Q}) \ . \tag{3.85}$$

By a change of coordinates, the metric can be written locally, in a given coordinate chart, as a foliation

$$ds^2 \;=\; \epsilon d\tau^2 + R^2 d\Omega_3^2 \;, \qquad R^2(\tau) \;=\; \eta^2 |\sinh^2(\sqrt{\epsilon}\tau)| \;, \tag{3.86}$$

where $x^2 = \epsilon \tan^2 \frac{\tau}{2}$ with $\epsilon = \pm 1$, and $d\Omega_3^2$ is a three-dimensional metric of constant curvature with suitable signature, and [6]

$$\eta \;=\; \frac{f_1 h^2}{2} \;. \tag{3.87}$$

One has the following simplifications in specific models:

$$\text{Type A model:} \qquad Q \;=\; \bar{Q} \;, \qquad \eta \;=\; \frac{1+(1-a^2)^2 Q}{1+(1+6a^2+a^4)Q} \;, \tag{3.88}$$

$$\text{Chiral model:} \qquad \bar{Q} \;=\; 0 \;, \qquad \eta \;=\; \frac{1+(1-a^2)^2 Q}{1+(1+a^2)^2 Q} \;. \tag{3.89}$$

The metric may have conical singularities, namely zeroes $R(\tau_0) = 0$ for which $\partial_\tau R|_{\tau_0} \neq 1$ (we note that $\eta|_{\tau=0} = 1$, so that $\tau = 0$ is not a conical singularity). The scale factor depend heavily on $\nu$ as well as the choice of the infinitely many discrete parameters $\theta_k$ and $\bar{\theta}_k$. This makes the analysis unyielding, and we shall therefore limit ourselves to the case of vanishing discrete parameters and $\nu \ll 1$. In Lorentzian signature, the resulting analysis was performed in [6], and it generalizes straightforwardly to Euclidean and Kleinian signatures. To this end, one expands $Q^{(reg)}$ in $\nu$ around $\nu = 0$, and finds

$$Q^{(reg)} \;=\; \frac{c_1 \nu}{6} \log(1+a^2) + \mathcal{O}(\nu^2) \;. \tag{3.90}$$

Focusing on a single chart, as listed in (3.22)-(3.26), since $a^2$ is then bounded from below by $(1-\sqrt{2})(1+\sqrt{2})^{-1}$, we see that, if $|\nu| \ll 1$, then $|Q| \ll 1$, and consequently the factor $\eta$ defined in (3.87) remains finite. Thus, for small enough $\nu$, there are no conical singularities within the coordinate charts. However, they may appear for some finite critical $\nu$.

While the $Q$ functions are highly complicated for $\nu \neq 0$, they simplify drastically at $\nu = 0$, where we find

$$Q \;=\; -(1-a^2)^2 \sum_{k=0}^{\infty} \frac{4_k a^{2k}}{k!} (\theta_k - \theta_{k+1})^2 \;. \tag{3.91}$$

An analogous expression can be found for $\bar{Q}$. Setting $(\theta_k - \theta_{k+1})^2 = 1$, yields

$$Q \;=\; -\frac{1}{(1-a^2)^2} \;. \tag{3.92}$$

If $Q = \bar{Q} = -(1-a^2)^{-2}$, which is necessarily the case in the Lorentzian models, then the equation system for $\omega_{\alpha\beta}$ and $\bar{\omega}_{\dot{\alpha}\dot{\beta}}$ becomes degenerate, and one finds

$$\omega_{\alpha\beta} \;=\; -\frac{(1-a^2)^2}{8a^2} \omega^{(0)}_{\alpha\beta} \;=\; \frac{(\sigma^{ab})_{\alpha\beta} dx_a x_b}{2x^2} \;, \tag{3.93}$$

$$\bar{\omega}_{\dot{\alpha}\dot{\beta}} \;=\; -\frac{(1-a^2)^2}{8a^2} \bar{\omega}^{(0)}_{\dot{\alpha}\dot{\beta}} \;=\; \frac{(\bar{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} dx_a x_b}{2x^2} \;, \tag{3.94}$$

leading to the degenerate vierbein

$$e_{\alpha\dot{\alpha}} \;=\; -\frac{\lambda x_{\alpha\dot{\alpha}} x^a dx_a}{x^2 h^2} \;, \qquad (3.95)$$

and metric

$$ds^2 \;=\; \frac{4(x^a dx_a)^2}{\lambda^2 x^2 h^2} \;. \qquad (3.96)$$

## 3.4 Solutions of Non-minimal Model (Type 2)

### 3.4.1 Internal Master Fields

The non-minimal model admits the following solutions

$$\widehat{\Phi}' \;=\; 0 \;, \qquad \widehat{S}'_\alpha \;=\; z_\alpha \star \Gamma(y,\bar{y}) \;, \qquad \widehat{S}'_{\dot{\alpha}} \;=\; \bar{z}_{\dot{\alpha}} \star \bar{\Gamma}(y,\bar{y}) \;, \qquad (3.97)$$

provided that

$$\Gamma \star \Gamma \;=\; \bar{\Gamma} \star \bar{\Gamma} \;=\; 1 \;, \qquad [\Gamma, \bar{\Gamma}]_\star \;=\; 0 \;, \qquad \pi\bar{\pi}(\Gamma) \;=\; \Gamma \;, \qquad \pi\bar{\pi}(\bar{\Gamma}) \;=\; \bar{\Gamma} \;. \qquad (3.98)$$

The elements $\Gamma$ and $\bar{\Gamma}$ can be written as

$$\Gamma \;=\; 1 - 2P \;, \qquad \bar{\Gamma} \;=\; 1 - 2\bar{P} \;, \qquad (3.99)$$

where $P(y,\bar{y})$ and $\bar{P}(y,\bar{y})$ are projectors obeying

$$P \star P \;=\; P \;, \qquad \bar{P} \star \bar{P} \;=\; \bar{P} \;, \quad [P, \bar{P}]_\star \;=\; 0 \;, \qquad \pi\bar{\pi}(P) \;=\; P \;, \qquad \pi\bar{\pi}(\bar{P}) \;=\; \bar{P} \;. \qquad (3.100)$$

A set of such projectors is described in Appendix C, where we also explain why the projectors can be subject to the $\tau$-conditions of the non-minimal model, given in (2.10), but not those of the minimal model, given in (2.9), unless one develops some further formalism for handling certain divergent $\star$-products.

### 3.4.2 Spacetime Component Fields

Turning to the computation of the space components of the master fields, since $z_\alpha$ star-commutes with $L$, it immediately follows from (3.16), (2.73) and (3.97) that

$$K_\mu \;=\; 0 \;. \qquad (3.101)$$

From (3.16) this in turn implies that all HS gauge fields and the spin-1 gauge field vanish, while the metric is that of maximally symmetric spacetime. To that extent, the Type 1 solution looks like the Type 0 solution, but it does differ in an important way, namely, here the internal connection, *i.e.* the spinor component $\widehat{A}_\alpha$ of the master 1-form, is non-vanishing. Indeed, (3.97), (3.98) and (3.8) give the result

$$\widehat{A}_\alpha \;=\; -iz_\alpha \star V(x;y,\bar{y}) \;, \qquad \widehat{A}_{\dot{\alpha}} \;=\; -i\bar{z}_{\dot{\alpha}} \star \bar{V}(x;y,\bar{y}) \;, \qquad (3.102)$$

where the quantities $V$ and $\bar{V}$, which shall be frequently encountered in what follows, are defined by

$$V \ = \ L^{-1} \star P \star L \ , \qquad \bar{V} \ = \ L^{-1} \star \bar{P} \star L \ . \tag{3.103}$$

Their explicit evaluation is given in Appendix D, with the result (D.21).

Whilst the internal connection does not turn on any spacetime component fields, it does, however, affect the interactions as it does not obey the physical gauge condition normally used in the weak-field expansion [21], namely that the internal connection should vanish when the zero-form vanishes. In this sense, the internal connection may be viewed as a non-trivial flat connection in the non-commutative space.

## 3.5 Solutions of Non-minimal Chiral Model (Type 3)

### 3.5.1 Internal Master Fields

In the case of the non-minimal chiral model, defined in Section 2.3, it is possible to use projectors $P(y,\bar{y})$ to build solutions with non-vanishing Weyl zero-form and higher spin fields. They are

$$\widehat{\Phi}' \ = \ (1-P)\star\kappa \ , \qquad \widehat{S}'_\alpha \ = \ z_\alpha \star P \ , \qquad \widehat{S}'_{\dot\alpha} \ = \ \bar{z}_{\dot\alpha}\star\bar{\Gamma} \ , \tag{3.104}$$

where

$$P\star P \ = \ P \ , \qquad \bar{\Gamma}\star\bar{\Gamma} \ = \ 1 \ , \qquad [P,\bar{\Gamma}]_\star \ = \ 0 \ , \qquad \pi\bar{\pi}(P) \ = \ P \ , \qquad \pi\bar{\pi}(\bar{\Gamma}) \ = \ \bar{\Gamma} \ . \tag{3.105}$$

These elements of the $\star$-product algebra can be constructed as in Section 3.4 and Appendix C.

For the purpose of exhibiting explicitly the spacetime component fields, we shall choose to work with the simplest possible projectors, namely

$$P_+(y) \ = \ 2e^{-2\epsilon uv} \ = \ 2e^{\epsilon yby} \ , \tag{3.106}$$

$$P_-(\bar{y}) \ = \ 2e^{-2\epsilon\bar{u}\bar{v}} \ = \ 2e^{\epsilon\bar{y}\bar{b}\bar{y}} \ , \tag{3.107}$$

where $\epsilon=\pm1$, and $u$, $v$, $\bar{u}$, $\bar{v}$, $b_{\alpha\beta}$ and $\bar{b}_{\dot\alpha\dot\beta}$ are defined in Appendices B and C.

### 3.5.2 Spacetime Component Fields

The master gauge field and zero-form is given by

$$e_\mu+\omega_\mu+W_\mu \ = \ e_\mu^{(0)}+\omega_\mu^{(0)}+\frac{\omega_\mu^{\alpha\beta}}{4i}\frac{\partial^2 V}{\partial y^\alpha\partial y^\beta} \ , \tag{3.108}$$

and

$$\Phi \ = \ \left[L^{-1}\star(1-P)\star\kappa\star\pi(L)\right]|_{Z=0} \ = \ 1-V|_{y_\alpha=0} \ , \tag{3.109}$$

where $K_\mu$ is defined by (2.73); we have used (2.17); and $V$ is given by (D.21). Remarkably, since there is no $y$-dependence in the Weyl zero-form $\Phi$, it is covariantly constant in the sense that $\Phi_{\alpha(m)\dot{\alpha}(n)}$ vanishes unless $m = 0$. Moreover, using (D.21), it is straightforward to compute the constant value of the physical scalar field, with the result

$$\phi(x) \;=\; 1 - 4 \sum_{n_1, n_2 \in \mathbb{Z}+\frac{1}{2}} (-1)^{n_1+n_2-\frac{\epsilon_1+\epsilon_2}{2}} \theta_{n_1,n_2} \;. \tag{3.110}$$

Summing over all $n_2$, and using (C.17) with $x = 0$, *i.e.* $\sum_{k=0}^{\infty}(-1)^k = \frac{1}{2}$, one finds that for the reduced projector (C.19), the scalar field is given by

$$\phi(x) \;=\; 1 - 2 \sum_{n \in \mathbb{Z}+\frac{1}{2}} (-1)^{n-\frac{\epsilon}{2}} \theta_n \;. \tag{3.111}$$

Finally setting all $\theta$-parameters equal to 1, one ends up with $P = 1$, *i.e.* in the Type 0 case, where indeed $\phi(x) = 0$.

In the special cases of (3.106) and (3.107), one finds

$$V_+ \;=\; L^{-1} \star P_+ \star L \;=\; 2 \exp\left(-\epsilon \frac{[2\bar{y}\bar{a} - (1+a^2)y]\, b\, [2a\bar{y} + (1+a^2)y]}{(1-a^2)^2}\right) \;, \tag{3.112}$$

$$V_- \;=\; L^{-1} \star P_- \star L \;=\; 2 \exp\left(-\epsilon \frac{[2ya - (1+a^2)\bar{y}]\, \bar{b}\, [2\bar{a}y + (1+a^2)\bar{y}]}{(1-a^2)^2}\right) \tag{3.113}$$

where $a_{\alpha\dot{\alpha}}$ and $b_{\alpha\beta}$ are defined in Appendix B. The physical scalar is now given in both cases by

$$\phi(x) \;=\; -1 \;, \tag{3.114}$$

and the self-dual Weyl tensors in both cases by ($s = 1, 2, 3, ....$)

$$\Phi_{\alpha(2s)} \;=\; 0 \;, \tag{3.115}$$

while the anti-self-dual Weyl tensors take the form

$$\Phi^{+}_{\dot{\alpha}_1 \cdots \dot{\alpha}_{2s}} \;=\; -2^{2s+1}(2s-1)!! \left(\frac{h^2-1}{\epsilon h^2}\right)^s U_{(\dot{\alpha}_1} \cdots U_{\dot{\alpha}_s} V_{\dot{\alpha}_{s+1}} \cdots V_{\dot{\alpha}_{2s})} \;, \tag{3.116}$$

$$\Phi^{-}_{\dot{\alpha}_1 \cdots \dot{\alpha}_{2s}} \;=\; -2^{2s+1}(2s-1)!! \left(\frac{1}{\epsilon h^2}\right)^s \bar{\lambda}_{(\dot{\alpha}_1} \cdots \bar{\lambda}_{\dot{\alpha}_s} \bar{\mu}_{\dot{\alpha}_{s+1}} \cdots \bar{\mu}_{\dot{\alpha}_{2s})} \;, \tag{3.117}$$

with spinors $(U, V)$ defined in (B.11).

In the case of $\lambda^2 = 1$ in Euclidean signature, we only need to use one coordinate chart, in which $0 \leq h^2 \leq 1$. The Weyl tensors blow up in the limit $h^2 \to 0$, preventing the solution from approaching $H_4$ in this limit. In this sense the above solution is a non-perturbative solution without weak-field limit in any region of spacetime. Indeed, in the perturbative weak-field expansion around the $H_4$ solution, the scalar field has non-vanishing mass, preventing the linearized scalar field from being a non-vanishing constant.

In the case of $\lambda^2 = -1$ in Euclidean signature, the base manifold consists of two charts, covered by the coordinates in (3.22). Thus, in each chart we have $1 \leq h^2 < 2$, and so the local representatives (3.116) and (3.117) of the Weyl tensors are well-defined throughout the base manifold.

Finally, in the case of $\lambda^2 = -1$ in Kleinian signature, one also needs two charts (since we are working with stereographic coordinates), with $0 \leq h^2 \leq 2$, and hence the Weyl tensors blow up in the limit $h^2 \to 0$ preventing the solution from approaching $H_{3,2}$ in this limit.

From the Weyl tensors, which are not in themselves HS gauge invariant quantities, one can construct an infinite set of invariant (and thus closed) zero-forms [5], namely

$$\mathcal{C}^-_{2p} \;=\; \int \frac{d^4y d^4z}{(2\pi)^4}[(\widehat{\Phi}\star\pi(\widehat{\Phi})]^{\star p}\star\kappa\bar{\kappa}\ . \tag{3.118}$$

Remarkably, on our solution they all assume the same value, given by the constant value of the scalar field, *viz.*

$$\mathcal{C}^-_{2p} \;=\; (1-V)^{\star 2p}|_{y=\bar{y}=0} \;=\; 1-4\sum_{n_1,n_2\in\mathbb{Z}+\frac{1}{2}}(-1)^{n_1+n_2-\frac{\epsilon_1+\epsilon_2}{2}}\theta_{n_1,n_2}\ . \tag{3.119}$$

The calculation of the metric in the two models proceeds in a parallel fashion as follows:

**The $P_+$ Solution:**

From (3.108) and (3.112) a straightforward computation yields the result

$$e_{\mu\dot{\alpha}\alpha} \;=\; e^{(0)}_{\mu\dot{\alpha}\alpha}+12(1+h)h^{-4}\,b_{(\alpha\beta}(ba)_{\gamma)\dot{\alpha}}\,\omega_\mu^{\beta\gamma}\ , \tag{3.120}$$

$$\omega_{\mu\alpha\beta} \;=\; \omega^{(0)}_{\mu\alpha\beta}+12h^{-4}b_{(\alpha\beta}b_{\gamma\delta)}\,\omega_\mu^{\gamma\delta}\ , \tag{3.121}$$

$$\bar{\omega}_{\mu\dot{\alpha}\dot{\beta}} \;=\; \bar{\omega}^{(0)}_{\mu\dot{\alpha}\dot{\beta}}+4(1+h)^2h^{-4}\left[-(\bar{a}ba)_{\dot{\alpha}\dot{\beta}}b_{\gamma\delta}+2(\bar{a}b)_{\dot{\alpha}\gamma}(\bar{a}b)_{\dot{\beta}\delta}\right]\omega_\mu^{\gamma\delta}\ . \tag{3.122}$$

First we solve for the spin connection from (3.121) by inverting the hyper-matrix that multiplies $\omega^{(0)}$, obtaining the result

$$\omega_{\mu\alpha\beta} = g_1\left[\omega^{(0)}_{\mu\alpha\beta}-8g(b\omega^{(0)}_\mu b)_{\alpha\beta}\right]+g_2 b_{\alpha\beta}b^{\gamma\delta}\omega^{(0)}_{\mu\gamma\delta}\ , \tag{3.123}$$

where

$$g_1 = \frac{1}{1-4g^2}\ ,\qquad g_2 = \frac{4g}{(1-2g)(1-4g)}\ ,\qquad g = h^{-4}\ . \tag{3.124}$$

Substituting this result in (3.120) then gives the vierbein

$$e_\mu^a = \frac{-2}{h^2(1+2g)}\left[g_3\delta_\mu^a+g_4\lambda^2 x_\mu x^a+g_5\lambda^2(Jx)_\mu(Jx)^a\right]\ , \tag{3.125}$$

where

$$g_3 = 1 + 2h^{-2} \ , \qquad g_4 = 2g \qquad g_5 = \frac{6g}{1-4g} \ , \tag{3.126}$$

and the spin connections are given in (3.123) and (3.122). Thus, the metric $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ takes the form

$$g_{\mu\nu} = \frac{4}{h^4(1+2g)^2} \left[ g_3^2 \eta_{\mu\nu} + g_4(\lambda^2 x^2 g_4 + 2g_3) x_\mu x_\nu + g_5(\lambda^2 x^2 g_5 + 2g_3)(Jx)_\mu (Jx)_\nu \right] \ . \tag{3.127}$$

The vierbein thus has potential singularities at $h^2 = 0$ and $h^2 = 2$. The limit $h^2 \to 0$ is a boundary in the case of $\lambda^2 = 1$ in Euclidean signature and $\lambda^2 = -1$ in Kleinian signature. At these boundaries $e_\mu{}^a \sim h^{-2} x_\mu x^a$, *i.e.* a scale factor times a degenerate vierbein. In the limit $h^2 \to 2$ one approaches the boundary of a coordinate chart in the case of $\lambda^2 = 1$ in Euclidean signature and $\lambda^2 = -1$ in Kleinian signature. Also in this limit, the vierbein becomes degenerate, *viz.* $e_\mu{}^a \sim h^{-2}(Jx)_\mu (Jx)^a$. **The $P_-$ Solution:**

A parallel computation that uses (3.108) and (3.113) yields the result

$$e_{\mu\dot\alpha\alpha} = e^{(0)}_{\mu\dot\alpha\alpha} + 12\lambda^2 x^2 (1+h) h^{-4}\, \tilde b_{(\alpha\beta} (\tilde b a)_{\gamma)\dot\alpha}\, \omega_\mu^{\beta\gamma} \ , \tag{3.128}$$

$$\omega_{\mu\alpha\beta} = \omega^{(0)}_{\mu\alpha\beta} + 12(\lambda^2 x^2)^2 h^{-4} \tilde b_{(\alpha\beta} \tilde b_{\gamma\delta)}\, \omega_\mu^{\gamma\delta} \ , \tag{3.129}$$

$$\bar\omega_{\mu\dot\alpha\dot\beta} = \bar\omega^{(0)}_{\mu\dot\alpha\dot\beta} + 4(1+h)^2 h^{-4} \left[ -(\bar a \tilde b a)_{\dot\alpha\dot\beta} \tilde b_{\gamma\delta} + 2(\tilde b a)_{\gamma\dot\alpha} (\tilde b a)_{\delta\dot\beta} \right] \omega_\mu^{\gamma\delta} \ , \tag{3.130}$$

where $\tilde b_{\alpha\beta}$ is defined in (B.9). As before, solving for the spin connection from (3.129) by inverting the hyper-matrix that multiplies $\omega^{(0)}$, we obtain

$$\omega_{\mu\alpha\beta} = \tilde g_1 \left[ \omega^{(0)}_{\mu\alpha\beta} - 8\tilde g (\tilde b \omega^{(0)}_\mu \tilde b)_{\alpha\beta} \right] + \tilde g_2 \tilde b_{\alpha\beta} \tilde b^{\gamma\delta} \omega^{(0)}_{\mu\gamma\delta} \ , \tag{3.131}$$

where

$$\tilde g_1 = \frac{1}{1-4\tilde g} \ , \qquad \tilde g_2 = \frac{4\tilde g}{(1-2\tilde g)(1-4\tilde g)} \ , \qquad \tilde g = (\lambda^2 x^2)^2 h^{-4} \ . \tag{3.132}$$

Substituting this result in (3.128) then gives the vierbein

$$e_\mu^a = \frac{-2}{h^2\,(1+2\tilde g)} \left[ \delta_\mu^a + \tilde g_4 \lambda^2 x_\mu x^a + \tilde g_5 \lambda^2 (\tilde J x)_\mu (\tilde J x)^a \right] \ , \tag{3.133}$$

where $\tilde J_{ab}$ is defined in (B.10)

$$\tilde g_4 = 2\lambda^2 x^2 h^{-4} \qquad \tilde g_5 = \frac{6\lambda^2 x^2 h^{-4}}{1-4\tilde g} \ , \tag{3.134}$$

and the spin connections are given in (3.131) and (3.130). Thus, the metric $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ takes the form

$$g_{\mu\nu} = \frac{4}{h^4\,[1+2\tilde g]^2} \left[ \eta_{\mu\nu} + \tilde g_4(\lambda^2 x^2 \tilde g_4 + 2) x_\mu x_\nu + \tilde g_5 (\lambda^2 x^2 \tilde g_5 + 2)(\tilde J x)_\mu (\tilde J x)_\nu \right] \ . \tag{3.135}$$

The vierbein has potential singularities at $h^2 = 0$, $h^2 = 2$ and $h^2 = \frac{2}{3}$. The singularities at $h^2 = 0$ and $h^2 = 2$ are related to degenerate vierbeins exactly as for the $P^+$ solution. The singularity at $h^2 = \frac{2}{3}$, which arises in the case of $\lambda^2 = 1$ in Euclidean and Kleinian signature, also gives a degenerate vierbein. This is an intriguing situation since the degeneration occurs inside the coordinate charts.

## 4 Conclusions

Starting from HS gauge theories in four dimensions based on infinite dimensional extensions of $SO(5;\mathbb{C})$, we have determined their real forms in spacetimes with Euclidean $(4,0)$ and Kleinian $(2,2)$ signature, in addition to the usual Lorentzian $(3,1)$ signature. We have then found three new types of solutions in addition to the maximally symmetric ones. Type 1 solutions, which are invariant under an infinite dimensional extension of $SO(4-p,p)$, give us a nontrivial deformation of the maximally symmetric solutions, and depend on a continuous real parameter as well as infinite set of discrete parameters. Interestingly, a particular choice of the discrete parameters, in the limit of vanishing continuous parameter, gives rise to a degenerate, indeed rank one, metric. Given that degenerate metrics are known to play an important role in topology change in quantum gravity [17], it is remarkable that such metrics emerge naturally in HS gauge theory.

Type 2 solutions, which provide another kind of deformation of the maximally symmetric solutions, have a non-vanishing spinorial master one-form. Type 3 solutions are particularly remarkable because all the higher spin fields are non-vanishing, and the corresponding Weyl tensors furnish a higher spin generalization of Type D gravitational instantons. It would be interesting to apply the framework we have used in this paper to finding pp-wave, black hole and domain wall solutions with non-vanishing HS fields.

We stress that our models in Euclidean and Kleinian signatures are formulated using the 4D spinor-oscillator formulation. It would be interesting to compare these models to the vector-oscillator formulation [23]. The latter exists in any dimension and signature, and relies on the gauging of an internal $Sp(2)$ gauge symmetry. At the full level, the vector-oscillator master field equations, in any dimension and signature, are formulated using a single $Sp(2)$-doublet $Z$-oscillator, leaving, apparently, no room for parity violating interactions. The precise relation between the spinor and vector-oscillator formulations in D=4 therefore deserve further study.

In the context of supersymmetric field theories, including supergravity, the non-Lorentzian signature typically presents obstacle since the spinor properties are sensitive to the spacetime signature. Here, however, we have considered bosonic HS gauge theories in which the spinor oscillators play an auxiliary role, and we have formulated the non-Lorentzian signature theories with suitable definition of the spinors without having to face such obstacles. Remarkably, non-supersymmetric 4D theories in Kleinian signature describing self-dual gravity arise in worldsheet $N = 2$ supersymmetric string theories, known as $N = 2$ strings. For reasons mentioned in the introduction, it is an interesting open problem to find a niche for Kleinian HS gauge theory in a variant of an $N = 2$ string.

There are several other open problems that deserve investigation. To begin with, we have not determined the symmetries of Type 2 and Type 3 solutions. While it may be useful in its own right to determine whether our Type 3 solutions support a complex, possibly Kähler, structure up to a conformal scaling, such results may be limited in shedding light to the geometry associated with infinitely many gauge fields present in HS gauge theory. The correct interpretation of the singularities or degeneracies in the metrics we have found also require sufficient knowledge of the HS geometry. Furthermore, a proper formulation of the HS geometry would also provide a framework for constructing invariants that could distinguish the gauge inequivalent classes of exact solutions.

It would also be interesting to study the fluctuations about our exact solutions, and explore their potential application in quantum gravity and cosmology. Similarities between the frameworks for studying instanton and soliton solutions of the noncommutative field theories (see, for example, [24]), and in particular open string field theory, are also worth investigating.

## Acknowledgements

We are grateful to J. Engquist and A. Sagnotti for discussions. P.S. thanks the Physics Department at Texas A& M University, and E.S. thanks the Scuola Normale Superiore, where part of this work was done, for hospitality. The research of C.I. was supported in part by INFN; by the MIUR-PRIN contract 2003-023852; by the EU contracts MRTN-CT-2004-503369 and MRTN-CT-2004-512194; by the INTAS contract 03-51-6346; and by the NATO grant PST.CLG.978785. The research of E.S. was supported in part by NSF Grant PHY-0555575. The research of P.S. was supported in part by a visiting professorship issued by Scuola Normale Superiore; by INFN; by the MIUR-PRIN contract 2003-023852; by the EU contracts MRTN-CT-2004-503369 and MRTN-CT-2004-512194; by the INTAS contract 03-51-6346; and by the NATO grant PST.CLG.978785.

## A General Conventions and Notation

We use the conventions of [25] in which the real form of the $SO(5;\mathbb{C})$ generators obey

$$[M_{AB},M_{CD}] \;=\; i\eta_{BC}M_{AD}+3\text{ more}\ ,\qquad (M_{AB})^\dagger \;=\; \sigma(M_{AB})\ , \tag{A.1}$$

where $\eta_{AB}=(\eta_{ab};-\lambda^2)$. The commutation relations above decompose as

$$[M_{ab},M_{cd}]_\star \;=\; 4i\eta_{[c|[b}M_{a]|d]}\ ,\qquad [M_{ab},P_c]_\star \;=\; 2i\eta_{c[b}P_{a]}\ ,\qquad [P_a,P_b]_\star \;=\; i\lambda^2 M_{ab}\ . \tag{A.2}$$

The corresponding oscillator realization is taken to be

$$M_{ab} \;=\; -\frac{1}{8}\Big[\,(\sigma_{ab})^{\alpha\beta}y_\alpha y_\beta+(\bar\sigma_{ab})^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\bar y_{\dot\beta}\,\Big]\ ,\qquad P_a \;=\; \frac{\lambda}{4}(\sigma_a)^{\alpha\dot\beta}y_\alpha\bar y_{\dot\beta}\ . \tag{A.3}$$

Our spinor conventions are

$$\epsilon^{\alpha\beta}\epsilon_{\gamma\delta} \;=\; 2\delta^{\alpha\beta}_{\gamma\delta}\ ,\qquad \epsilon^{\alpha\beta}\epsilon_{\alpha\gamma} \;=\; \delta^\beta_\gamma\ , \tag{A.4}$$

and

$$(\epsilon_{\alpha\beta})^\dagger \;=\; \begin{cases} \epsilon^{\alpha\beta} & \text{for } SU(2)\\ \epsilon_{\dot\alpha\dot\beta} & \text{for } SL(2,\mathbb{C})\\ \epsilon_{\alpha\beta} & \text{for } Sp(2)\end{cases} \tag{A.5}$$

Oscillator indices are raised and lowered according to the following conventions, $A^\alpha=\epsilon^{\alpha\beta}A_\beta$, $A_\alpha=A^\beta\epsilon_{\beta\alpha}$. The reality conditions on oscillators have been summarized in (2.39), (2.40) and (2.41). The van der Waerden symbols obey

$$(\sigma^a)_\alpha{}^{\dot\alpha}(\bar\sigma^b)_{\dot\alpha}{}^\beta \;=\; \eta^{ab}\delta^\beta_\alpha\;+\;(\sigma^{ab})_\alpha{}^\beta\ ,\qquad (\bar\sigma^a)_{\dot\alpha}{}^\alpha(\sigma^b)_\alpha{}^{\dot\beta} \;=\; \eta^{ab}\delta^{\dot\beta}_{\dot\alpha}\;+\;(\bar\sigma^{ab})_{\dot\alpha}{}^{\dot\beta}\ , \tag{A.6}$$

$$\tfrac12\epsilon_{abcd}(\sigma^{cd})_{\alpha\beta} \;=\; \epsilon(\sigma_{ab})_{\alpha\beta}\ ,\qquad \tfrac12\epsilon_{abcd}(\bar\sigma^{cd})_{\dot\alpha\dot\beta} \;=\; -\epsilon(\bar\sigma_{ab})_{\dot\alpha\dot\beta}\ , \tag{A.7}$$

where $\epsilon=\sqrt{\det\eta_{ab}}$, and the following reality conditions

$$((\sigma^a)_{\alpha\dot\beta})^\dagger \;=\; \begin{cases} -(\bar\sigma^a)^{\dot\beta\alpha} \;=\; -(\sigma^a)^{\alpha\dot\beta} & \text{for } SU(2)\\ (\bar\sigma^a)_{\dot\alpha\beta} \;=\; (\sigma^a)_{\beta\dot\alpha} & \text{for } SL(2,\mathbb{C})\\ (\bar\sigma^a)_{\dot\beta\alpha} \;=\; (\sigma^a)_{\alpha\dot\beta} & \text{for } Sp(2)\end{cases} \tag{A.8}$$

and

$$((\sigma^{ab})_{\alpha\beta})^\dagger \;=\; \begin{cases} (\sigma^{ab})^{\alpha\beta} & \text{for } SU(2)\\ (\bar\sigma^{ab})_{\dot\alpha\dot\beta} & \text{for } SL(2,\mathbb{C})\\ (\sigma^{ab})_{\alpha\beta} & \text{for } Sp(2)\end{cases}\ . \tag{A.9}$$

Convenient representations are:

$$SU(2)\ :\qquad \sigma^a=(i,\sigma^i)\ ,\qquad \bar\sigma^a=(-i,\sigma^i)\ ,\qquad \epsilon=i\sigma^2\ ; \tag{A.10}$$

$$SL(2,\mathbb{C})\ :\qquad \sigma^a=(-i\sigma^2,-i\sigma^i\sigma^2)\ ,\qquad \bar\sigma^a=(-i\sigma^2,i\sigma^2\sigma^i)\ ,\qquad \epsilon=i\sigma^2\ ; \tag{A.11}$$

$$Sp(2)\ :\qquad \sigma^a=(1,\tilde\sigma^i)\ ,\qquad \bar\sigma^a=(-1,\tilde\sigma^i)\ ,\qquad \epsilon=i\sigma^2\ , \tag{A.12}$$

where in the last case $\tilde{\sigma}^i = (\sigma^1, i\sigma^2, \sigma^3)$. Combining (2.71) with (A.3), the real form of the $\mathfrak{so}(5;\mathbb{C})$-valued connection $\Omega$ can be expressed as

$$\Omega \;=\; \frac{1}{4i}dx^\mu \left[\omega_\mu^{\alpha\beta}\; y_\alpha y_\beta + \bar{\omega}_\mu{}^{\dot{\alpha}\dot{\beta}}\; \bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}} + 2e_\mu^{\alpha\dot{\beta}}\; y_\alpha \bar{y}_{\dot{\beta}}\right] \;, \tag{A.13}$$

where

$$\omega^{\alpha\beta} \;=\; -\tfrac{1}{4}(\sigma_{ab})^{\alpha\beta}\;\omega^{ab}\;, \quad \bar{\omega}^{\dot{\alpha}\dot{\beta}} \;=\; -\tfrac{1}{4}(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\;\omega^{ab}\;, \quad e^{\alpha\dot{\alpha}} \;=\; \tfrac{\lambda}{2}(\sigma_a)^{\alpha\dot{\alpha}}\;e^a\;. \tag{A.14}$$

Likewise, for the curvature $\mathcal{R} = d\Omega + \Omega \wedge \star\Omega$ one finds

$$\mathcal{R}_{\alpha\beta} \;=\; d\omega_{\alpha\beta} + \omega_{\alpha\gamma}\wedge\omega_\beta{}^\gamma + e_{\alpha\dot{\delta}}\wedge e_\beta{}^{\dot{\delta}}\;, \tag{A.15}$$

$$\bar{\mathcal{R}}_{\dot{\alpha}\dot{\beta}} \;=\; d\bar{\omega}_{\dot{\alpha}\dot{\beta}} + \bar{\omega}_{\dot{\alpha}\dot{\gamma}}\wedge\bar{\omega}_{\dot{\beta}}{}^{\dot{\gamma}} + e_{\delta\dot{\alpha}}\wedge e^\delta{}_{\dot{\beta}}\;, \tag{A.16}$$

$$\mathcal{R}_{\alpha\dot{\beta}} \;=\; de_{\alpha\dot{\beta}} + \omega_{\alpha\gamma}\wedge e^\gamma{}_{\dot{\beta}} + \bar{\omega}_{\dot{\beta}\dot{\delta}}\wedge e_\alpha{}^{\dot{\delta}}\;, \tag{A.17}$$

and

$$\mathcal{R}^{ab} \;=\; d\omega^{ab} + \omega^a{}_c\wedge\omega^{cb} + \lambda^2 e^a\wedge e^b\;, \qquad \mathcal{R}^a \;=\; de^a + \omega^a{}_b\wedge e^b\;. \tag{A.18}$$

# B Further Notation Used for the Solutions

The gauge function $L(x;y,\bar{y})$ defined in (3.19) can be written as

$$L \;=\; \frac{2h}{1+h}\exp(-iya\bar{y})\;, \tag{B.1}$$

where

$$a_{\alpha\dot{\alpha}} \;=\; \frac{\lambda x_{\alpha\dot{\alpha}}}{1+h}\;, \qquad x_{\alpha\dot{\alpha}} = (\sigma^a)_{\alpha\dot{\alpha}}x_a\;, \tag{B.2}$$

$$x^2 \;=\; \eta_{ab}x^a x^b\;, \qquad h = \sqrt{1-\lambda^2 x^2}\;. \tag{B.3}$$

Useful relations that follow from these definitions are

$$a^2 = \frac{1-h}{1+h}\;, \qquad h = \frac{1-a^2}{1+a^2}\;. \tag{B.4}$$

The Maurer-Cartan form based on $L$ defined in (3.19) yields the the vierbein and Lorentz connection

$$e_{(0)}{}^{\alpha\dot{\alpha}} \;=\; -\frac{\lambda(\sigma^a)^{\alpha\dot{\alpha}}dx_a}{h^2}\;, \qquad \omega_{(0)}{}^{\alpha\beta} \;=\; -\frac{\lambda^2(\sigma^{ab})^{\alpha\beta}dx_a x_b}{h^2}\;, \tag{B.5}$$

with Riemann tensor given by

$$R_{(0)\mu\nu,\rho\sigma} \;=\; -\lambda^2\left(g_{(0)\mu\rho}g_{(0)\nu\sigma} - g_{(0)\nu\rho}g_{(0)\mu\sigma}\right)\;. \tag{B.6}$$

A further useful definition is

$$b_{\alpha\beta} \;=\; 2\lambda_{(\alpha}\mu_{\beta)}\ , \qquad \lambda^\alpha\mu_\alpha \;=\; \tfrac{i}{2}\ . \tag{B.7}$$

It obeys the relation $(b^2)_\alpha{}^\beta \;=\; -\frac{1}{4}\delta_\alpha^\beta$ and it defines an almost complex structure via the relations (see, for example, [26])

$$b_{\alpha\beta} = \frac{1}{8}(\sigma^{ab})_{\alpha\beta}\, J_{ab}\ , \qquad J_{ab} = (\sigma_{ab})^{\alpha\beta}\, b_{\alpha\beta}\ , \qquad J_a{}^c J_c{}^b = -\delta_a^b\ . \tag{B.8}$$

Similarly, using the definition

$$\tilde{b}_{\alpha\beta} = a^{-2}(a\bar{b}\bar{a})_{\alpha\beta}\ , \tag{B.9}$$

we have the relations

$$\tilde{b}_{\alpha\beta} = \frac{1}{8}(\sigma^{ab})_{\alpha\beta}\, \tilde{J}_{ab}\ , \qquad \tilde{J}_{ab} = (\sigma_{ab})^{\alpha\beta}\, \tilde{b}_{\alpha\beta}\ , \qquad \tilde{J}_a{}^c \tilde{J}_c{}^b = -\delta_a^b\ . \tag{B.10}$$

Finally, we have the following definition for spinors used in describing a Type 3 solution:

$$U_{\dot{\alpha}} = \frac{x^a}{\sqrt{x^2}}\,(\bar{\sigma}_a\lambda)_{\dot{\alpha}}\ , \qquad V_{\dot{\alpha}} = \frac{x^a}{\sqrt{x^2}}\,(\bar{\sigma}_a\mu)_{\dot{\alpha}}\ . \tag{B.11}$$

# C Weyl-ordered Projectors

Weyl-ordered projectors $P(y,\bar{y})$ can be constructed by recombining $(y,\bar{y})$ into a pair of Heisenberg oscillators $(a_i, b^j)$ $(i,j=1,2)$ obeying

$$[a_i, b^j]_\star \;=\; \delta_i^j\ . \tag{C.1}$$

For example, one can take

$$a_1 \;=\; u \;=\; \lambda^\alpha y_\alpha\ , \qquad b^1 \;=\; v \;=\; \mu^\alpha y_\alpha\ , \tag{C.2}$$

$$a_2 \;=\; \bar{u} \;=\; \bar{\lambda}^{\dot{\alpha}}\bar{y}_{\dot{\alpha}}\ , \qquad b^2 \;=\; \bar{v} \;=\; \bar{\mu}^{\dot{\alpha}}\bar{y}_{\dot{\alpha}}\ , \tag{C.3}$$

where the constant spinors are normalized as

$$\lambda^\alpha\mu_\alpha \;=\; \tfrac{i}{2}\ , \qquad \bar{\lambda}^{\dot{\alpha}}\bar{\mu}_{\dot{\alpha}} \;=\; \tfrac{i}{2}\ . \tag{C.4}$$

The projectors, obeying the appropriate reality conditions, take the form

$$P \;=\; \sum_{n_1,n_2\in\ \mathbb{Z}+\frac{1}{2}} \theta_{n_1,n_2} P_{n_1,n_2}\ , \qquad \bar{P} \;=\; \sum_{n_1,n_2\in\ \mathbb{Z}+\frac{1}{2}} \bar{\theta}_{n_1,n_2} P_{n_1,n_2}\ , \tag{C.5}$$

where $\theta_{n_1,n_2}\in\{0,1\}$ and $\bar{\theta}_{n_1,n_2}\in\{0,1\}$, with

$$(3,1)\text{ signature} \;:\; \theta_{n_1,n_2} \;=\; \bar{\theta}_{n_1,n_2}\ , \tag{C.6}$$

$$(4,0)\text{ and }(2,2)\text{ signatures} \;:\; \theta_{n_1,n_2}\ ,\bar{\theta}_{n_1,n_2} \quad \text{independent}\ , \tag{C.7}$$

and

$$P_{n_1,n_2} = 4(-1)^{n_1+n_2-\frac{\epsilon_1+\epsilon_2}{2}} e^{-2\sum_i \epsilon_i w_i} L_{n_1-\frac{\epsilon_1}{2}}(4\epsilon_1 w_1) L_{n_2-\frac{\epsilon_2}{2}}(4\epsilon_2 w_2) \ , \tag{C.8}$$

$$w_i = b^i a_i = b^i \star a_i + \tfrac{1}{2} = a_i \star b^i - \tfrac{1}{2} \qquad \text{(no sum)} \ , \tag{C.9}$$

with $\epsilon_i = n_i/|n_i|$ and $L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}(e^{-x} x^n)$ are the Laguerre polynomials. The projector property follows from

$$P_{m_1,m_2} \star P_{n_1,n_2} = \delta_{m_1 n_1} \delta_{m_2 n_2} P_{n_1,n_2} \ , \tag{C.10}$$

$$(w_i - n_i) \star P_{n_1,n_2} = 0 \ , \tag{C.11}$$

$$\tau(P_{n_1,n_2}) = P_{-n_1,-n_2} \ . \tag{C.12}$$

Here, $w_i - \frac{1}{2}$ is the Weyl-ordered form of the number operator, and

$$2(-1)^{n_i-\frac{\epsilon_i}{2}} e^{-2\epsilon_i w_i} L_{n_i-\frac{\epsilon_i}{2}}(4\epsilon_i w_i) = \begin{cases} |n_i\rangle\langle n_i| & \text{for } n_i > 0 \\ (-1)^{-n_i-\frac{1}{2}} |n_i\rangle\langle n_i| & \text{for } n_i < 0 \end{cases} \tag{C.13}$$

where $|n_i\rangle = \frac{(b^i)^{n_i-\frac{1}{2}}}{\sqrt{(n_i-\frac{1}{2})!}}|0\rangle$ with $n_i > 0$ belongs to the standard Fock space, built by acting with $b^i$ on the ground state $|0\rangle$ obeying $a_i|0\rangle = 0$, while $|n_i\rangle = \frac{(a_i)^{-n_i-\frac{1}{2}}}{\sqrt{(-n_i-\frac{1}{2})!}}|\tilde{0}\rangle$ for $n_i < 0$ are anti-Fock space states, built by acting with $a_i$ on the anti-ground state $|\tilde{0}\rangle = 0$ obeying $b^i|\tilde{0}\rangle = 0$. Formally, the inner product between a Fock space state and an anti-Fock space state vanishes. However, the corresponding Weyl-ordered projectors have divergent $\star$-products, as can be seen from the lemma

$$e^{suv} \star e^{tuv} = \frac{1}{1+\frac{st}{4}} \exp\left(\frac{s+t}{1+\frac{st}{4}}\, uv\right) \ . \tag{C.14}$$

Thus, lacking, at present, a suitable regularization scheme that does not violate associativity and other basic properties of the $\star$-product algebra, we shall restrict our attention to projectors that are constructed in either the Fock space or the anti-Fock space, *i.e.*

$$\theta_{n_1,n_2} = 1 \quad \text{only if } (n_1,n_2) \in Q \ , \tag{C.15}$$

where $Q$ is anyone of the four quadrants in the $(n_1, n_2)$ plane. From (C.12), it follows that these projectors are not invariant under the $\tau$ map, and therefore the master fields Type 2 and Type 3 solutions will be those of the non-minimal model, where the $\tau$ conditions are relaxed to $\pi\bar{\pi}$ conditions, which are certainly satisfied.

We also note that in order to solve the higher-spin equations it is essential that

$$[P, \bar{P}]_\star = \sum_{n_1,n_2} (\theta_{n_1,n_2}\bar{\theta}_{n_1,n_2} - \bar{\theta}_{n_1,n_2}\theta_{n_1,n_2}) P_{n_1,n_2} = 0 \ , \tag{C.16}$$

which holds for independent $\theta_{n_1,n_2}$ and $\bar{\theta}_{n_1,n_2}$ parameters (in the Euclidean and Kleinian signatures). Moreover, one can work with a reduced set of oscillators, say $a_1 = u$ and $b^1 = v$, by summing over all values of $n_2$ using

$$\sum_{k=0}^{\infty} t^k L_k(x) \quad = \quad (1-t)^{-1} \exp(-xt(1-t)^{-1}) \ . \tag{C.17}$$

Setting $n = n_1$ and $\epsilon = \epsilon_1$, this leads to

$$P \quad = \quad \sum_{n \in \ \mathbb{Z}+\frac{1}{2}} \theta_n P_n \ , \qquad \bar{P} \quad = \quad \sum_{n \in \ \mathbb{Z}+\frac{1}{2}} \bar{\theta}_n P_n \tag{C.18}$$

$$P_n \quad = \quad 2(-1)^{n-\frac{\epsilon}{2}} e^{-2\epsilon uv} L_n(4\epsilon uv) \ , \tag{C.19}$$

with suitable reality conditions on the $\theta_n$ parameters. Finally, using (C.17) once more, one finds that setting all $\theta$-parameters equal to 1 gives $P = 1$.

# D Calculation of $V = L^{-1} \star P \star L$

In this Appendix we compute $V = L^{-1} \star P \star L$ where $L$ is the gauge function given in (B.1) and $P$ is a projector of the form given in (C.5). Let us begin by considering the case of $P = P_{\frac{1}{2}} = 2e^{-2uv}$, *i.e.*

$$V \quad = \quad \frac{8h^2}{(1+h)^2} e^{iya\bar{y}} \star e^{yby} \star e^{-iya\bar{y}} \ , \tag{D.1}$$

where $ya\bar{y} = y^\alpha a_\alpha{}^{\dot{\alpha}} \bar{y}_{\dot{\alpha}}$ and $yby = y^\alpha b_\alpha{}^\beta y_\beta$, with $a_{\alpha\dot{\alpha}}$ and $b_{\alpha\beta}$ given by (B.2) and (B.7). The first $\star$-product can be performed treating the integration variables $(\xi_\alpha, \eta_\alpha)$ and $(\bar{\xi}_{\dot{\alpha}}, \bar{\eta}_{\dot{\alpha}})$ as separate real variables. Using the formulae (B.1) provided in [5], we find

$$V \quad = \quad \frac{8h^2}{(1+h)^2} e^{iya\bar{y}+(y-\bar{y}a)b(y+a\bar{y})} \star e^{-iya\bar{y}} \ . \tag{D.2}$$

The remaining $\star$-product leads to the Gaussian integral

$$V \quad = \quad \frac{8h^2}{(1+h)^2} \int \frac{d^4\xi d^4\eta}{(2\pi)^4} e^{\frac{1}{2}\Xi^I M_I{}^J \Xi_J + \Xi^I N_I + (y-\bar{y}a)b(y+a\bar{y})} \ , \tag{D.3}$$

where $\Xi^I = (\xi^\alpha, \bar{\xi}^{\dot{\alpha}}; \eta^\alpha, \bar{\eta}^{\dot{\alpha}})$ and $\Xi_I = (\xi_\alpha, \bar{\xi}_{\dot{\alpha}}; \eta_\alpha, \bar{\eta}_{\dot{\alpha}}) = \Xi^J \Omega_{JI}$, with block-diagonal symplectic metric $\Omega = \epsilon \oplus \bar{\epsilon} \oplus \epsilon \oplus \bar{\epsilon}$, and

$$M = \begin{bmatrix} A & -i \\ i & B \end{bmatrix} , \tag{D.4}$$

$$A = \begin{bmatrix} 2b & ia + 2ba \\ ia - 2ba & -2\bar{a}ba \end{bmatrix} , \qquad B = \begin{bmatrix} 0 & -ia \\ -i\bar{a} & 0 \end{bmatrix} , \tag{D.5}$$

$$N = \begin{bmatrix} i(1-2ib)a\bar{y} + 2by \\ -2\bar{a}ba\bar{y} + i\bar{a}(1+2ib)y \\ -ia\bar{y} \\ -i\bar{a}y \end{bmatrix} . \tag{D.6}$$

The Gaussian integration gives

$$V = \frac{8h^2}{(1+h)^2\sqrt{\det M}}\, e^{\frac{1}{2}N^I (M^{-1})_I{}^J N_J + (y - \bar{y}a)b(y + a\bar{y})} . \tag{D.7}$$

From $\det M = \det(1 + AB)$, and noting that the matrices defined as

$$C \equiv \frac{BA - a^2}{2i} = \begin{bmatrix} a^2 b & a^2 ba \\ -\bar{a}b & -\bar{a}ba \end{bmatrix} , \qquad \tilde{C} \equiv \frac{AB - a^2}{2i} = \begin{bmatrix} -a^2 b & -ba \\ \bar{a}ba^2 & \bar{a}ba \end{bmatrix} , \tag{D.8}$$

are nilpotent, i.e. $C^2 = \tilde{C}^2 = 0$, one finds

$$\det M = (1 - a^2)^4 , \tag{D.9}$$

and, using $1 - a^2 = 2h/(1+h)$, the pre-factor in $V$ is thus given by

$$\frac{8h^2}{(1+h)^2\sqrt{\det M}} = 2 . \tag{D.10}$$

Next, using geometric series expansions, one finds

$$M^{-1} = \frac{i}{(1-a^2)} \begin{bmatrix} i(1-a^2)B + 2B\tilde{C} & -(1-a^2) - 2iC \\ 1 - a^2 + 2i\tilde{C} & i(1-a^2)A + 2AC \end{bmatrix} , \tag{D.11}$$

and

$$\tfrac{1}{2} N^I (M^{-1})_I{}^J N_J = \frac{4a^2 yby + 2(1 + 4a^2 - a^4) yba\bar{y} - (3 - a^2)(1 + a^2)\bar{y}\bar{a}ba\bar{y}}{(1-a^2)^2} . \tag{D.12}$$

Adding the classical term in the exponent in (D.3) yields the final result

$$V = 2\exp\left( -\frac{[2\bar{y}\bar{a} - (1+a^2)y]\, b\, [2a\bar{y} + (1+a^2)y]}{(1-a^2)^2} \right) . \tag{D.13}$$

The projector property $V \star V = V$ follows manifestly from

$$V = 2\exp(-2\tilde{u}\tilde{v}) , \qquad [\tilde{u}, \tilde{v}]_\star = 1 , \tag{D.14}$$

where

$$\tilde{u} \;=\; \lambda^\alpha \eta_\alpha \;, \qquad \tilde{v} \;=\; \mu^\alpha \eta_\alpha \;, \tag{D.15}$$

with

$$\eta_\alpha \;=\; \frac{[(1+a^2)y + 2a\bar{y}]_\alpha}{1-a^2} \;, \qquad [\eta_\alpha, \eta_\beta]_\star \;=\; 2i\epsilon_{\alpha\beta} \;. \tag{D.16}$$

Thus, the net effect of rotating the projector $P_{\frac{1}{2}}(u,v)$ given in (C.19) is to replace the oscillators $u$ and $v$ by their rotated dittos $\tilde{u}$ and $\tilde{v}$. We claim, without proof, that this generalizes to any $n$, *viz.*

$$L^{-1} \star P_n(u,v) \star L \;=\; P_n(\tilde{u},\tilde{v}) \;. \tag{D.17}$$

Similarly, for $P_n(\bar{u},\bar{v})$ we have

$$L^{-1} \star P_n(\bar{u},\bar{v}) \star L \;=\; P_n(\tilde{\bar{u}},\tilde{\bar{v}}) \;, \tag{D.18}$$

where

$$\tilde{\bar{u}} \;=\; \bar{\lambda}^{\dot{\alpha}} \bar{\eta}_{\dot{\alpha}} \;, \qquad \tilde{\bar{v}} \;=\; \bar{\mu}^{\dot{\alpha}} \bar{\eta}_{\dot{\alpha}} \;, \tag{D.19}$$

with

$$\bar{\eta}_{\dot{\alpha}} \;=\; \frac{[(1+a^2)\bar{y} + 2\bar{a}y]_{\dot{\alpha}}}{1-a^2} \;, \qquad [\bar{\eta}_{\dot{\alpha}}, \bar{\eta}_{\dot{\beta}}]_\star \;=\; 2i\epsilon_{\dot{\alpha}\dot{\beta}} \;. \tag{D.20}$$

Finally, using $[\eta_\alpha, \bar{\eta}_{\dot{\alpha}}]_\star = 0$, we deduce that

$$V \;=\; L^{-1} \star P \star L \;=\; \sum_{n_1,n_2} \theta_{n_1,n_2} P_{n_1,n_2}(\tilde{u},\tilde{v};\tilde{\bar{u}},\tilde{\bar{v}}) \;. \tag{D.21}$$

## References


[1] M.A. Vasiliev, *Consistent equations for interacting gauge fields of all spins in* $3+1$ *dimensions*, Phys. Lett. **B243** (1990) 378.

[2] M. Flato and C. Fronsdal, "One Massless Particle Equals Two Dirac Singletons: Elementary Particles In A Curved Space. 6," Lett. Math. Phys. **2** (1978) 421.

[3] J. Engquist, P. Sundell and L. Tamassia, "Singleton strings," arXiv:hep-th/0701081.

[4] S. F. Prokushkin and M. A. Vasiliev, "Higher-spin gauge interactions for massive matter fields in 3D AdS space-time," Nucl. Phys. B **545** (1999) 385 [arXiv:hep-th/9806236].

[5] E. Sezgin and P. Sundell, "An exact solution of 4D higher-spin gauge theory," Nucl. Phys. B **762**, 1 (2007) [arXiv:hep-th/0508158].

[6] E. Sezgin and P. Sundell, "On an exact cosmological solution of higher spin gauge theory," arXiv:hep-th/0511296.

[7] V. E. Didenko, A. S. Matveev and M. A. Vasiliev, "BTZ black hole as solution of 3d higher spin gauge theory," arXiv:hep-th/0612161.

[8] C. Iazeolla and P. Sundell, in preparation.

[9] J. Engquist and O. Hohm, "Higher-spin Chern-Simons theories in odd dimensions," arXiv:0705.3714 [hep-th].

[10] G. W. Gibbons and S.W. Hawking, (eds.), *Euclidean Quantum Gravity*, (World Scientific, 993).

[11] G. W. Gibbons, *Euclidean quantum gravity: The view from 2002*, in *The future of Theoretical Physics and Cosmology*, p.351 (Cambridge University Press, 2002).

[12] H. Ooguri and C. Vafa, "Selfduality and N=2 string magic," Mod. Phys. Lett. A **5** (1990) 1389.

[13] K. Junemann and O. Lechtenfeld, "Chiral BRST cohomology of N = 2 strings at arbitrary ghost and picture number," Commun. Math. Phys. **203** (1999) 53 [arXiv:hep-th/9712182].

[14] C. Devchand and O. Lechtenfeld, "Extended self-dual Yang-Mills from the N = 2 string," Nucl. Phys. B **516** (1998) 255 [arXiv:hep-th/9712043].

[15] M. A. Vasiliev, "Higher-spin gauge theories in four, three and two dimensions," Int. J. Mod. Phys. D **5** (1996) 763 [arXiv:hep-th/9611024].

[16] J. Engquist, E. Sezgin and P. Sundell, "On N = 1,2,4 higher spin gauge theories in four dimensions," Class. Quant. Grav. **19** (2002) 6175 [arXiv:hep-th/0207101].

[17] G. T. Horowitz, "Topology change in classical and quantum gravity," Class. Quant. Grav. **8** (1991) 587.

[18] E. J. Flaherty, "Hermitian and Kahlerian Geometry in Relativity", p. 175 (Springer-Verlag, 1976).

[19] A. S. Lapedes and M. J. Perry, "Type D Gravitational Instantons," Phys. Rev. D **24** (1981) 1478.

[20] E. Sezgin and P. Sundell, "Holography in 4D (super) higher spin theories and a test via cubic scalar couplings," JHEP **0507** (2005) 044 [arXiv:hep-th/0305040].

[21] E. Sezgin and P. Sundell, "Analysis of higher spin field equations in four dimensions," JHEP **0207** (2002) 055 [arXiv:hep-th/0205132].

[22] K. I. Bolotin and M. A. Vasiliev, "Star-product and massless free field dynamics in AdS(4)", Phys. Lett. B **479** (2000) 421 [arXiv:hep-th/0001031].

[23] M. A. Vasiliev, "Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)," Phys. Lett. B **567** (2003) 139 [arXiv:hep-th/0304049].

[24] F. A. Schaposnik, "Noncommutative solitons and instantons," Braz. J. Phys. **34** (2004) 1349 [arXiv:hep-th/0310202].

[25] E. Sezgin and P. Sundell, "Higher spin N = 8 supergravity," JHEP **9811** (1998) 016 [arXiv:hep-th/9805125].

[26] C. N. Pope, "Kahler Manifolds And Quantum Gravity," J. Phys. A **15** (1982) 2455.