Resolved Calabi-Yau Cones and Flows from L^{abc} Superconformal Field Theories

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ABSTRACT

We discuss D3-branes on cohomogeneity-three resolved Calabi-Yau cones over L^{abc} spaces, for which a 2-cycle or 4-cycle has been blown up. In terms of the dual quiver gauge theory, this corresponds to motion along the non-mesonic, or baryonic, directions in the moduli space of vacua. In particular, a dimension-two and/or dimension-six scalar operator gets a vacuum expectation value. These resolved cones support various harmonic $(2,1)$-forms which reduce the ranks of some of the gauge groups either by a Seiberg duality cascade or by Higgsing. We also discuss higher-dimensional resolved Calabi-Yau cones. In particular, we obtain square-integrable $(2,2)$-forms for eight-dimensional cohomogeneity-four Calabi-Yau metrics.

Contents

1   Introduction

The AdS/CFT correspondence relates type IIB string theory on $\text{AdS}_5 \times S^5$ to four-dimensional $\mathcal{N} = 4\ U(N)$ superconformal Yang-Mills theory [1, 2, 3]. More generally, type IIB string theory on $\text{AdS}_5 \times X^5$, where X^5 is an Einstein-Sasaki space such as $T^{1,1}$, Y^{pq} [4, 5] or L^{abc} [6, 7], corresponds to an $\mathcal{N} = 1$ superconformal quiver gauge theory. The dual gauge theories have been identified in [8] for $T^{1,1}$, in [9, 10] for Y^{pq} and in [11, 12, 13] for L^{abc}.

There is a prescription for mapping perturbations of the supergravity background to operators in the dual gauge theory [2, 3]. In particular, motion in the Kähler moduli space of the Calabi-Yau cone over the Einstein-Sasaki space corresponds to giving vacuum expectation values (vevs) to the fundamental fields, such that only non-mesonic operators get vevs. This is because the mesonic directions of the full moduli space correspond to the motion of the D3-branes in the Calabi-Yau space whereas the non-mesonic, or baryonic, directions are associated with either deformations of the geometry or turning on B-fields. This has been studied for a blown-up 2-cycle in the resolved conifold in [14], as well as for a blown-up 4-cycle in the resolved conifold [15], Y^{pq} cones [16], L^{abc} cones [17] and general Calabi-Yau cones [18]. All of these resolved Calabi-Yau cones with blown-up 4-cycles follow the general construction given in [19, 20].

In this paper, we shall apply the state/operator correspondence to a general class of resolved Calabi-Yau cones over L^{abc} with a blown-up 2-cycle or 4-cycle. These metrics can be obtained from the Euclideanization of the BPS limit of the six-dimensional Kerr-NUT-AdS solutions [21, 22].[1] In particular, blowing up a 2-cycle or 4-cycle corresponds to giving a vev to a real dimension-two and/or six scalar operator. Although cycles are being blown up, in all but two cases there remain singularities [24, 25]. However, there is a countably infinite subset of cases where there is an ALE singularity, on which perturbative string dynamics is well-defined. Some of these cases were studied in [18]. While adding a large number of D3-branes that are uniformly distributed, or "smeared", on the blown-up cycle ends up inducing a power-law singularity at short

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[1]This is the even-dimensional analog of the relation between the Einstein-Sasaki spaces constructed in [23] and odd-dimensional BPS Kerr-NUT-AdS solutions.

distance[2], the resulting backgrounds can nevertheless be reliably used to describe perturbations around the UV conformal fixed point of the quiver gauge theories. Close to the UV fixed point, blowing up a 2-cycle on the L^{abc} cone corresponds to giving a vev to an operator that is analogous to the case of the resolved conifold. Therefore, we shall refer to these spaces as resolved cones, though it should be understood that there are still orbifold-type singularities.

The supergravity background can also be perturbed by adding a harmonic 3-form which lives on the Calabi-Yau metrics. If this is a pure $(2,1)$-form then supersymmetry will be preserved. Furthermore, if this form carries nontrivial flux then it corresponds to D5-branes wrapped on a 2-cycle in the Calabi-Yau space. The introduction of these fractional D3-branes eliminates the conformal fixed point in the UV limit of the quiver gauge theory. The theory undergoes a Seiberg duality cascade and the ranks of some of the gauge groups are reduced with decreasing energy scale. The supergravity solutions corresponding to fractional branes have been constructed for the cones over T^{11} [27, 28], Y^{pq} [29] and L^{abc} spaces [30, 31]. Fractional branes have also been considered for Calabi-Yau spaces with blown-up cycles, such as the deformed conifold [32], resolved conifold [33] and regularized conifold [15], as well as the resolved Y^{pq} cones with blown-up 4-cycles [18]. We shall also consider continuous families of 3-forms that do not have nontrivial flux. In this case, there remains a conformal fixed point in the UV limit of the field theory. It has been proposed that the ranks of some of the gauge groups are reduced with decreasing energy scale via the Higgs mechanism [34].

Since the L^{abc} spaces have cohomogeneity two, the form fields constructed on the corresponding Calabi-Yau spaces will generally have nontrivial dependence on the radial direction as well as the two non-azimuthal coordinates of L^{abc}. In addition, these forms generally break the $U(1)_R \times U(1) \times U(1)$ global symmetry group of the theory down to a $U(1) \times U(1)$ symmetry group which, in particular, breaks the R-symmetry. However, this is done in such a way that the theory preserves $\mathcal{N} = 1$ supersymmetry.

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[2]This singularity is due to the smearing of the D3-brane charge on the blown-up cycle. A completely non-singular solution with D3-branes stacked at a single point on the resolved conifold has been constructed [26].

The various perturbations of the $AdS_5 \times L^{abc}$ supergravity background that will be discussed are shown in Figure 1. These perturbations, which can be superimposed with one another, correspond to continuous families of Renormalization Group (RG) flows from the UV superconformal fixed point of the quiver gauge theory.
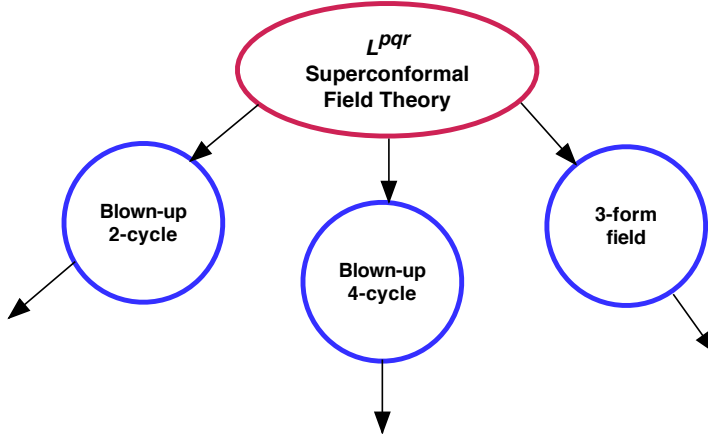

Figure 1: RG flows from the superconformal fixed point of the L^{abc} quiver gauge theory correspond to various deformations of the supergravity background.

The paper is organized as follows. In section 2, we discuss the geometry of the resolved Calabi-Yau cones over the L^{abc} spaces. A subset of these are the resolved cones over Y^{pq} and their various limits. We find various harmonic $(2, 1)$-forms on these metrics, some of which carry nontrivial flux and some of which do not. In section 3, we apply some of our results to the AdS/CFT correspondence. In particular, we relate the perturbations of the $AdS_5 \times L^{abc}$ background to various flows from the UV conformal fixed point of the dual quiver gauge theory. In section 4, we consider eight-dimensional resolved cones over L^{pqrs} and the various harmonic forms that live on them. In section 5, we carry out the corresponding analysis for the higher-dimensional resolved cones. Lastly, conclusions are presented in section 6.

2 Six-dimensional resolved Calabi-Yau cones

Although the L^{abc} spaces themselves are non-singular for appropriately chosen integers p, q, r [6, 7], the cones over these spaces have a power-law singularity at their apex. In the case of the cone over $T^{1,1}$, this singularity can be smoothed out in two different ways [35]. Firstly, one can blow up a 3-cycle, which corresponds to a complex

deformation. The resulting deformed conifold has been crucial for the construction of a well-behaved supergravity dual of the IR region of the gauge theory, providing a geometrical description of confinement [32].

One might hope that a similar resolution procedure could be performed on other L^{abc} cones. Although a first-order deformation of the complex structure of Y^{pq} cones has been found in [36], there exists an obstruction to finding the complex deformations beyond first order [37, 38]. There is also evidence from the field theory side that such deformations will break supersymmetry for the Y^{pq} cones [39, 40, 41, 42] as well as for a large class of L^{abc} cones [42]. Nevertheless, there are L^{abc} cones which allow for complex structure deformations [37, 38], which can be understood from the corresponding toric diagrams [12].[3] However, the explicit metrics for these deformed L^{abc} cones, let alone the solutions for D3-branes on these cones, are not known.
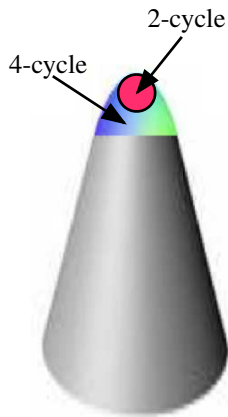

Figure 2: A 4-cycle within the base space of a cone over L^{abc} can be blown up. Within this 4-cycle lies a 2-cycle. The volumes of these two cycles correspond to two independent Kähler moduli.

The second way in which the $T^{1,1}$ cone can be rendered regular is by blowing up a 2-cycle [35]. Also, for the case of a cone over $T^{1,1}/\mathbb{Z}_2$, the singularity can be resolved by blowing up a 4-cycle. Both of these resolutions are examples of Kähler deformations which, as we shall see shortly, can also be performed on the L^{abc} cones $C(L^{abc})$. Moreover, the 2-cycle actually lives within the 4-cycle, as illustrated in Figure 2. This means that there are two Kähler moduli associated with the 4-cycle. For certain parameter choices, we can have the 4-cycle corresponds to the Einstein-Kähler base space of L^{abc}, whose metric can be obtained by taking a certain scaling limit of a

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[3]We thank Angel Uranga for discussions on this point.

Euclideanized form of the Plebanski-Demianski metric [30]. It is also possible to have the volume of the 4-cycle vanishes, whilst keeping a 2-cycle blown up.

It has been found that the cone over $Y^{2,1}$ can be rendered completely regular by blowing up an appropriate 4-cycle [24]. However this, together with the resolved cones over $T^{1,1}$ and $T^{1,1}/\mathbb{Z}_2$, constitute the only examples of non-singular resolved cones over the L^{abc} spaces [25]. Although we shall refer to these spaces as "resolved" L^{abc} cones, there are generally orbifold-type singularities remaining. In the limit of a vanishing 2-cycle, this can be seen simply because at short distance the geometry becomes a direct product of $\mathbb{R}^2$ and the four-dimensional Einstein-Kähler base space of L^{abc}, which is itself an orbifold. Nevertheless, the resolved cones over L^{abc} can be embedded in ten dimensions to give Ricci-flat backgrounds $\text{Mink}_4 \times C(L^{abc})$, on which perturbative string dynamics is well-defined. However, as we shall see in section 3, the back-reaction of D3-branes leads to a power-law singularity at short distance. This singularity is due to the fact that we are smearing the D3-branes on the blown-up cycle. For the case of the resolved conifold, it has been shown that if the D3-branes are stacked at a single point then the supergravity solution is completely regular [26].

2.1 Resolved cones over Y^{pq}

Before turning to resolved cones over the general cohomogeneity-two L^{abc} spaces, it is instructive first to consider the subset involving the cohomogeneity-one Y^{pq} spaces. The resolved cone over Y^{pq} has the metric [21]

$$ds_6^2 = \frac{x+y}{4X}dx^2 + \frac{X}{x+y}\left(d\tau + \frac{y}{2\alpha}\,\sigma_3\right)^2 + \frac{x+y}{4Y}dy^2 + \frac{Y}{x+y}\left(d\tau - \frac{x}{2\alpha}\,\sigma_3\right)^2 + \frac{x\,y}{4\alpha}\,(\sigma_1^2 + \sigma_2^2)\,.$$
(2.1)

where

$$X = x(x+\alpha) - \frac{2\mu}{x}\,, \qquad Y = y(\alpha - y) + \frac{2\nu}{y}\,,$$
(2.2)

and that

$$\sigma_3 = d\psi + \cos\theta\,d\phi\,, \qquad \sigma_1^2 + \sigma_2^2 = d\theta^2 + \sin^2\theta\,d\phi^2\,.$$
(2.3)

It has been shown that the only completely regular examples are the resolved cones over $T^{1,1}$, $T^{1,1}/\mathbb{Z}_2$ and $Y^{2,1}$ [24, 25]. We shall now consider various limits of the metric (2.1).

Resolved conifold

In order to reduce to a resolved cone over T^{11} (or $T^{11}/\mathbb{Z}_2$), we need to select ν such that $Y(y)$ has a double root. This happens when $\nu = -\frac{2}{27}\alpha^3$. Making the coordinate redefinition

$$y = \tfrac{2}{3}\alpha + \epsilon \cos\tilde\theta\,, \qquad \nu = -\tfrac{2}{27}\alpha^3 + \tfrac{1}{2}\alpha\epsilon^2\,, \qquad \tau = -\frac{2}{9\epsilon}\tilde\phi\,, \qquad \sigma_3 \to \sigma_3 + \frac{2\alpha}{3\epsilon}d\tau\,, \quad (2.4)$$

and setting the parameter ϵ to zero, we find that the metric becomes

$$ds_6^2 = \frac{x + \tfrac{2}{3}\alpha}{4X}dx^2 + \frac{X}{9(x + \tfrac{2}{3}\alpha)}(\sigma_3 + \cos\tilde\theta\, d\tilde\phi)^2 + \tfrac{1}{6}(x + \tfrac{2}{3}\alpha)(d\tilde\theta^2 + \sin^2\tilde\theta\, d\tilde\phi^2) + \tfrac{1}{6}x\,(\sigma_1^2 + \sigma_2^2)\,.$$
(2.5)

If $\mu = 0$, there is a blown-up S^2 and the solution describes the resolved conifold [35]. If, on the other hand, $\alpha = 0$, then there is a blown-up $S^2 \times S^2$ and the solution describes the regularized conifold [15]. In fact, it has been shown that one can always blow up a 4-cycle on any cone over an Einstein-Sasaki space [19, 20]. We shall now take a look at the analogous limits for the resolved cones over the Y^{pq} spaces.

The $\alpha = 0$ limit

If we let $y \to \alpha y$, $\nu \to \alpha^3\nu$ and then take $\alpha \to 0$, we obtain the limit

$$ds^2 = \frac{x}{4X}dx^2 + \frac{X}{x}(d\tau + \tfrac{1}{2}y\sigma_3)^2 + x\left[\frac{dy^2}{4Y} + Y\,\sigma_3^2 + \tfrac{1}{4}y(\sigma_1^2 + \sigma_2^2)\right], \qquad (2.6)$$

where

$$X = x^2 - \frac{2\mu}{x}\,, \qquad Y = y(1 - y) + \frac{2\nu}{y}\,. \qquad (2.7)$$

There is a single Kähler modulus, which corresponds to a blown-up 4-cycle with a volume parameterized by μ. This is the analog of the resolved cone for general Y^{pq} spaces. However, unlike the $T^{1,1}/\mathbb{Z}_2$ case, this metric has an orbifold-type singularity at its apex, since the geometry reduces to the direct product of $\mathbb{R}^2$ and an Einstein-Kähler orbifold.

The $\mu = 0$ limit: blowing up 2-cycles

One can also consider the limit in which μ vanishes, in which case x runs from 0 to asymptotic ∞. Near $x = 0$, we can express the metric as

$$ds^2 = y\left(dr^2 + \tfrac{1}{4}r^2(\sigma_3 + \frac{2}{y}d\tau)^2 + \tfrac{1}{4}r^2(\sigma_1^2 + \sigma_2^2) + \frac{dy^2}{4Y}\right) + Y(d\tau - \tfrac{1}{2}r^2\sigma_3)^2\,, \qquad (2.8)$$

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where $x = r^2$. At $r = 0$ there is a collapsing 3-sphere, instead of a circle as in the previous limit. There is a single Kähler modulus corresponding to the volume of a blown-up 2-cycle, which is parameterized by α. However, unlike the analogous resolved conifold for which there is a smooth 2-sphere, in general this 2-cycle is a "tear-drop" with a conical singularity.

Calabi-Yau structure

The Calabi-Yau structure on the metric (2.1) is given by a Kähler form J and a holomorphic $(3,0)$-form $G_{(3)}$. These can be expressed in the complex vielbein basis

$$\epsilon^1 = e^1 + \mathrm{i}\, e^2\,, \qquad \epsilon^2 = e^3 + \mathrm{i}\, e^4\,, \qquad \epsilon^3 = e^5 + \mathrm{i}\, e^6\,, \tag{2.9}$$

where the vielbein is conveniently chosen to be

$$e^1 = \sqrt{\frac{x+y}{4X}}\, dx\,, \quad e^2 = \sqrt{\frac{X}{x+y}}\left(d\tau + \frac{y}{2\alpha}\sigma_3\right)\,, \quad e^3 = \sqrt{\frac{x+y}{4Y}}\, dy\,,$$

$$e^4 = \sqrt{\frac{Y}{x+y}}\left(d\tau - \frac{x}{2\alpha}\sigma_3\right)\,, \quad e^5 = \sqrt{\frac{xy}{4\alpha}}\,\sigma_1\,, \quad e^6 = \sqrt{\frac{xy}{4\alpha}}\,\sigma_2\,. \tag{2.10}$$

The Kähler 2-form is then given by

$$J = \tfrac{\mathrm{i}}{2}\, \epsilon^i \wedge \bar{\epsilon}^i\,, \tag{2.11}$$

and the complex self-dual harmonic $(3,0)$-form is given by

$$G_{(3)} = e^{-3\mathrm{i}\tau}\, \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \equiv W_{(3)} + \mathrm{i} * W_{(3)}\,. \tag{2.12}$$

Harmonic $(2,1)$-forms

We are interested in harmonic $(2,1)$-forms that live on the resolved Y^{pq} cones, since their presence preserves the minimal supersymmetry of the theory. We find there exist the following five such $(2,1)$-forms:

$$\Phi_1 = \frac{e^{-3\mathrm{i}\tau}}{x\,X}\,\bar{\epsilon}_1 \wedge \epsilon_2 \wedge \epsilon_3\,, \quad \Phi_2 = \frac{e^{-3\mathrm{i}\tau}}{y\,Y}\,\bar{\epsilon}_2 \wedge \epsilon_1 \wedge \epsilon_3\,, \quad \Phi_3 = \frac{e^{3\mathrm{i}\tau}}{x\,y\,X\,Y}\bar{\epsilon}_3 \wedge \epsilon_1 \wedge \epsilon_2\,,$$

$$\Phi_4 = \frac{1}{x\,y\,\sqrt{x+y}}\left(\frac{1}{x\sqrt{Y}}\epsilon_2 \wedge (\bar{\epsilon}_3 \wedge \epsilon_3 - \bar{\epsilon}_1 \wedge \epsilon_1) - \frac{1}{y\sqrt{X}}\epsilon_1 \wedge (\bar{\epsilon}_3 \wedge \epsilon_3 - \bar{\epsilon}_2 \wedge \epsilon_2)\right)\,,$$

$$\Phi_5 = \frac{1}{\sqrt{x+y}}\left(\frac{1}{x^2\sqrt{Y}}\epsilon_2 \wedge (\bar{\epsilon}_3 \wedge \epsilon_3 - \bar{\epsilon}_1 \wedge \epsilon_1) + \frac{1}{y^2\sqrt{X}}\epsilon_1 \wedge (\bar{\epsilon}_3 \wedge \epsilon_3 - \bar{\epsilon}_2 \wedge \epsilon_2)\right)\,. \tag{2.13}$$

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All of these forms have singularities at all distances x, for certain values of y, except for Φ_1, which has a singularity only at small distance. Φ_1 has a rapid fall off at large distance, such that it does not support nontrivial flux. On the other hand, in the large-x limit the last harmonic form behaves like

$$
\begin{aligned}
\Phi_5 &= \tfrac{1}{4}\sigma_1\wedge\sigma_2\wedge(\sigma_3+\frac{2}{y}\,d\tau)+\frac{1}{2y^2}\sigma_3\wedge d\tau\wedge dy+\frac{1}{4x}\Big(-2\sigma_1\wedge\sigma_2\wedge d\tau \\
&\quad +\mathrm{i}\left[(\frac{1}{y}\sigma_1\wedge\sigma_2-\frac{1}{y^2}\sigma_3\wedge dy)\wedge dx+\frac{y}{Y}\sigma_1\wedge\sigma_2\wedge dy\right]\Big)+\mathcal{O}(\frac{1}{x^2}). \quad (2.14)
\end{aligned}
$$

This indicates that this form does support nontrivial flux. In the $\alpha=0$ limit, in which we have first rescaled $y\to\alpha y$, Φ_4 and Φ_5 reduce to the same form. This form has a singularity that is confined to small distance.

2.2 Resolved cones over L^{abc}

We now turn to the resolved cones over the general cohomogeneity-two L^{abc} spaces. The metric is given by [22]

$$
\begin{aligned}
ds^2 &= \tfrac{1}{4}(u^2dx^2+v^2dy^2+w^2dz^2)+\frac{1}{u^2}(d\tau+(y+z)d\phi+yz\,d\psi)^2 \\
&\quad +\frac{1}{v^2}(d\tau+(x+z)d\phi+xz\,d\psi)^2+\frac{1}{w^2}(d\tau+(x+y)d\phi+xy\,d\psi)^2, \quad (2.15)
\end{aligned}
$$

where the functions u,v,w are given by

$$
\begin{aligned}
u^2 &= \frac{(y-x)(z-x)}{X}, \qquad v^2 = \frac{(x-y)(z-y)}{Y}, \qquad w^2 = \frac{(x-z)(y-z)}{Z}, \\
X &= x(\alpha-x)(\beta-x)-2M, \qquad Y = y(\alpha-y)(\beta-y)-2L_1, \\
Z &= z(\alpha-z)(\beta-z)-2L_2.
\end{aligned} \tag{2.16}
$$

Notice that the coordinates x, y and z appear in the metric on a symmetrical footing. We shall choose x to be the radial direction, and y and z to be the non-azimuthal coordinates on the L^{abc} level sets. This reduces to the Y^{pq} subset when $a=p-q$, $b=p+q$ and $c=d=p$.

Calabi-Yau structure

The complex vielbein can be written as

$$
\epsilon^1 = e^1+\mathrm{i}\,e^2, \qquad \epsilon^2 = e^3+\mathrm{i}\,e^4, \qquad \epsilon^3 = e^5+\mathrm{i}\,e^6, \tag{2.17}
$$

in the vielbein basis

$$e^1 = \tfrac{1}{2}u\,dx\,, \qquad e^2 = \frac{1}{u}(d\tau+(y+z)d\phi+yz\,d\psi)\,,$$

$$e^3 = \tfrac{1}{2}v\,dy\,, \qquad e^4 = \frac{1}{v}(d\tau+(x+z)d\phi+xz\,d\psi)\,,$$

$$e^5 = \tfrac{1}{2}w\,dz\,, \qquad e^6 = \frac{1}{w}(d\tau+(x+y)d\phi+xy\,d\psi)\,. \tag{2.18}$$

Then the Kähler 2-form and complex self-dual harmonic $(3,0)$-form are given by

$$J = \frac{\mathrm{i}}{2}\bar\epsilon_i\wedge\epsilon_i\,, \qquad G_{(3)} = e^{\mathrm{i}\,\nu}\,\epsilon_1\wedge\epsilon_2\wedge\epsilon_3\,, \tag{2.19}$$

where

$$\nu = 3\tau+2(\alpha+\beta)\phi+\alpha\beta\psi\,. \tag{2.20}$$

Harmonic $(2,1)$-forms

There is a harmonic $(2,1)$-form given by

$$\Psi_1 = \frac{e^{\mathrm{i}\,\nu}}{X}\,\bar\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\,. \tag{2.21}$$

Using this, one can then construct a general class of harmonic $(2,1)$-forms

$$\Phi_1 = f(\gamma)\,\Psi_1\,, \tag{2.22}$$

for any function f so long as $d\gamma\wedge\Psi_1 = 0$. This orthogonality condition is obeyed by

$$\gamma = \frac{YZ}{X}\,e^{\mathrm{i}\,2\nu}\,, \tag{2.23}$$

as can be seen by calculating its exterior derivative:

$$d\gamma = \frac{2\gamma}{(x-y)(y-z)(z-x)}\left(u(y-z)X'\,\bar\epsilon^1-v(z-x)Y'\,\epsilon^2-w(x-y)Z'\,\epsilon^3\right). \tag{2.24}$$

We can consider the special case for which

$$\Phi_1 = \frac{(YZ)^\delta}{X^{\delta+1}}e^{\mathrm{i}(2\delta+1)\nu}\,\bar\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\,, \tag{2.25}$$

where δ is a continuous parameter. Due to the ν dependence, this field only preserves $U(1)^2$ of the $U(1)^3$ isometry of the six-dimensional space. Although the full $U(1)^3$ is preserved for $\delta = -1/2$, the form field would blow up at the degeneracies of X,

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Y and Z, which would lead to a singular surface in the ten-dimensional geometry. In order for the singularity to be confined to $X = 0$, so that we have a reasonable gravity description near the UV region of the dual field theory, we require that $\delta \geq 0$.

We find there exist the following $(2, 1)$-forms:

$$
\begin{aligned}
\Phi_1 &= f\left(\frac{YZ}{X}e^{i\,2\nu}\right)\frac{e^{i\nu}}{X}\bar{\epsilon}^1\wedge\epsilon^2\wedge\epsilon^3\,, \\
\Phi_2 &= f\left(\frac{XZ}{Y}e^{i\,2\nu}\right)\frac{e^{i\nu}}{Y}\epsilon^1\wedge\bar{\epsilon}^2\wedge\epsilon^3\,, \\
\Phi_3 &= f\left(\frac{XY}{Z}e^{i\,2\nu}\right)\frac{e^{i\nu}}{Z}\epsilon^1\wedge\epsilon^2\wedge\bar{\epsilon}^3\,, \\
\Phi_4 &= a_1 A\,\epsilon^1\wedge(\bar{\epsilon}^2\wedge\epsilon^2-\bar{\epsilon}^3\wedge\epsilon^3)+a_2 B\,\epsilon^2\wedge(\bar{\epsilon}^3\wedge\epsilon^3-\bar{\epsilon}^1\wedge\epsilon^1) \\
&\quad + a_3 C\,\epsilon^3\wedge(\bar{\epsilon}^1\wedge\epsilon^1-\bar{\epsilon}^2\wedge\epsilon^2)\,, \\
\Phi_5 &= b_1 Ax\,\epsilon^1\wedge(\bar{\epsilon}^2\wedge\epsilon^2-\bar{\epsilon}^3\wedge\epsilon^3)+b_2 By\,\epsilon^2\wedge(\bar{\epsilon}^3\wedge\epsilon^3-\bar{\epsilon}^1\wedge\epsilon^1) \\
&\quad + b_3 Cz\,\epsilon^3\wedge(\bar{\epsilon}^1\wedge\epsilon^1-\bar{\epsilon}^2\wedge\epsilon^2)\,,
\end{aligned}
\tag{2.26}
$$

where

$$
\begin{aligned}
A^{-1} &= (y-z)^2\sqrt{(y-x)(z-x)X}\,, \qquad B^{-1} = (x-z)^2\sqrt{(x-y)(z-y)Y}\,, \\
C^{-1} &= (x-y)^2\sqrt{(x-z)(y-z)Z}\,,
\end{aligned}
\tag{2.27}
$$

and a_i and b_i are constants which satisfy the conditions $a_1+a_2+a_3 = 0$ and $b_1 + b_2 + b_3 = 0$. Notice that the first three forms in (2.27) are related to each other by interchanging the x, y and z coordinates, while the last two forms remain invariant. This reflects the fact that the x, y and z coordinates appear in a completely symmetric manner in the metric of the resolved cone over L^{abc}. Φ_1 has a singularity that is confined to small distance, as do Φ_4 and Φ_5 if one performs the rescaling $y \to \alpha y$, $z \to \alpha z$ and then takes the limit $\alpha \to 0$. Φ_4 and Φ_5 have nontrivial flux, while Φ_1 does not.

In the cohomogeneity-two limit, the resolved L^{abc} cones reduce to the resolved Y^{pq} cones. In this limit, Φ_4 and Φ_5 reduce to the corresponding forms given in (2.13), while the first three forms generalize those in (2.13) to include an arbitrary function f. In particular, taking $f = 1$ reproduces the Φ_1 and Φ_2 in (2.13), whilst taking f to be the inverse of its argument reproduces Φ_3.

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3 D3-branes and the AdS/CFT correspondence

A supersymmetric D3-brane solution of the type IIB theory with six-dimensional Calabi-Yau transverse space is given by

$$
\begin{aligned}
ds &= H^{-\frac{1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H^{\frac{1}{2}} ds_6^2 \,, \\
F_5 &= G_{(5)} + *G_{(5)} \,, \qquad G_{(5)} = dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dH^{-1} \,, \\
F_{(3)} &= F_{(3)}^{\mathrm{RR}} + \mathrm{i}\, F_{(3)}^{\mathrm{NS}} = m\, \omega_{(3)} \,,
\end{aligned}
\tag{3.1}
$$

with

$$
\Box_6 H = m^2 |\omega_{(3)}|^2 \,.
\tag{3.2}
$$

Here the $\Box_6$ is a Laplacian of the Calabi-Yau metric ds_6^2 and $\omega_{(3)}$ is a harmonic $(2,1)$-form in ds_6^2. We shall refer to this as a modified D3-brane solution, owing to the inclusion of the additional 3-form. If this 3-form carries nontrivial flux, then it corresponds to fractional a D3-brane.

We shall take the six-dimensional metric ds_6^2 of the transverse space to be the resolved cone over L^{abc}. We first consider the case of vanishing m. It was shown in [43, 44, 45] that the Klein-Gordon equation for the general AdS-Kerr-NUT solutions constructed in [22] is separable. Since our metrics arise as the Euclideanization of the supersymmetric limit of AdS-Kerr-NUT solutions, the corresponding equation for H is hence also separable. To see this, we consider a real superposition of the ansatz

$$
H = H_1(x)\, H_2(y)\, H_3(z)\, e^{2\mathrm{i}(a_0 \psi - a_1 \phi + a_2 \tau)} \,.
\tag{3.3}
$$

In general, this ansatz breaks the $U(1)^3$ global symmetry.

The Laplace equation is then given by

$$
\begin{aligned}
0 = \;& \frac{1}{(y-x)(z-x)}\left(\frac{(X\, H_1')'}{H_1} - \frac{(a_0 + a_1 x + a_2 x^2)^2}{X} \right) \\
&+ \frac{1}{(x-y)(z-y)}\left(\frac{(Y\, H_2')'}{H_2} - \frac{(a_0 + a_1 y + a_2 y^2)^2}{Y} \right) \\
&+ \frac{1}{(x-z)(y-z)}\left(\frac{(Z\, H_3')'}{H_3} - \frac{(a_0 + a_1 z + a_2 z^2)^2}{Z} \right) \,,
\end{aligned}
\tag{3.4}
$$

where a prime denotes a derivative with respect to the separated variable associated with the function H_i. This equation can be expressed as three separate equations in

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x, y and z:

$$(X\,H_1')' - \left(\frac{(a_0 + a_1 x + a_2 x^2)^2}{X} + b_0 + b_1 x\right) H_1 = 0\,,$$

$$(Y\,H_2')' - \left(\frac{(a_0 + a_1 y + a_2 y^2)^2}{Y} + b_0 + b_1 y\right) H_2 = 0\,,$$

$$(Z\,H_3')' - \left(\frac{(a_0 + a_1 z + a_2 z^2)^2}{Z} + b_0 + b_1 z\right) H_3 = 0\,, \qquad (3.5)$$

where b_0 and b_1 are separation constants. These equations do not have explicit closed-form solutions for general a_i and b_i. We shall consider the simplest solution obtained by setting all of the a_i and b_i to zero and letting H depend on x only. The solution is given by

$$H = c_0 - \frac{c_1\,\log(x - x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{c_1\,\log(x - x_2)}{(x_2 - x_1)(x_2 - x_3)} + \frac{c_1\,\log(x - x_3)}{(x_3 - x_1)(x_3 - x_2)}\,. \qquad (3.6)$$

where x_1, x_2 and x_3 are the three roots of X, satisfying

$$x_1 + x_2 + x_3 = \alpha + \beta\,, \qquad x_1 x_2 + x_1 x_3 + x_2 x_3 = \alpha\,\beta\,, \qquad x_1 x_2 x_3 = 2M\,. \qquad (3.7)$$

Consider the radial coordinate x with $x_1 \leq x \leq \infty$. Then the function H has a logarithmic divergence at x_1.

We now consider solutions for which the 3-form $\omega_{(3)}$ is turned on. A simple solution can be obtained by rescaling $y \to \alpha y$, $z \to \beta z$ and then taking the limit $\alpha = \beta = 0$. The general construction of [19, 20] is recovered for the case of this class of resolved L^{abc} cones [17]. We can then take $\omega_{(3)}$ to be the harmonic $(2,1)$-form Ψ_1 given by (2.21). Then, for a certain choice of integration constants, the resulting H is given by

$$H = \frac{x}{18M(x^3 - 2M)}\,, \qquad (3.8)$$

which diverges at $x^3 = 2M$.

The divergence of the H function in both (3.6) and (3.8) corresponds to a naked singularity in the short-distance region of the geometry. This singularity of the D3-brane solution arises even in the case of the resolved cone over $Y^{2,1}$, which itself is completely regular [24]. This singularity is due to the fact that the D3-branes have been smeared over the blown-up 4-cycle. A shell of uniformly distributed branes tends to be singular at its surface. For the case of the resolved conifold, in which there is a blown-up 2-cycle, a completely regular solution has been found for which

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the D3-branes are stacked at a single point [26]. This involves solving the equations (3.5) for the case of $T^{1,1}$ for which there is a delta function source. The solution is expressed as an expansion in terms of the angular harmonics. It would be interesting to explore than analogous construction for the resolved L^{abc} cones. All of these other examples, with the sole exception of $Y^{2,1}$, will still have orbifold singularities.

Another possible way in which regular solutions can be obtained is to blow up a 3-cycle instead of a 4-cycle. Then an appropriate 3-form would prevent the 3-cycle from collapsing, as in the case of the deformed conifold [32]. As already discussed in the previous section, while there exists an obstruction to complex deformations of Y^{pq} cones there are other subsets of the L^{abc} cones which do allow for complex structure deformations [37, 38, 12]. However, the explicit metrics for these deformed L^{abc} cones are not known.

Although the solution describing D3-branes on a resolved L^{abc} cone becomes singular at short distance, we can still use this background at large distance to study various flows of the quiver gauge theory in the region of the UV conformal fixed point. At large x, (3.2) becomes

$$\frac{4}{x^2}\partial_x\left(X\partial_x H\right) = m^2|\omega_{(3)}|^2,\tag{3.9}$$

where X is given by (2.16). Note that this equation applies for arbitrary α and β, since for large x we can consistently neglect the dependence of H on the non-azimuthal "angular" coordinates y and z. Again considering the case of the self-dual harmonic $(2,1)$-form Ψ_1 given by (2.21), the resulting asymptotic expansion of H is

$$H = \frac{Q}{x^2}\left(1+\frac{c_2}{x}+\frac{c_4}{x^2}+\frac{c_6}{x^3}+\cdots\right),\tag{3.10}$$

where

$$
\begin{aligned}
c_2 &= \frac{2}{3}(\alpha+\beta),\\
c_4 &= \frac{1}{2}(\alpha^2+\alpha\beta+\beta^2),\\
c_6 &= \frac{1}{30}\left(\frac{m^2}{Q}+12(\alpha^2+\beta^2)(\alpha+\beta)+2M\right).
\end{aligned}\tag{3.11}
$$

We have set an additive constant to zero so that the geometry is asymptotically $\text{AdS}_5 \times L^{abc}$. This can be seen from the leading $x^{-2} \sim r^{-4}$ term in H (since x has

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dimension two, we can take $x \sim r^2$ for large x). The transformation properties and dimensions of the operators being turned on in the dual field theory can be read off from the linearized form of the supergravity solution (3.1). The metric perturbations due to H have the same form as those within the metric ds_6^2 itself. Therefore, from the asymptotic expansion of H given in (3.10), we can read off that there are scalar operators of dimension two, four and six with expectation values that go as c_2, c_4 and c_6, respectively. This is consistent with the perturbations of the 2-form and 4-form potentials. We shall now discuss the gauge theory interpretation of the blown-up 2-cycles, as well as the 3-form, in more detail.

Blown-up 2-cycle

First, we consider the case with vanishing M, for which the six-dimensional space is the L^{abc} analog of the resolved conifold, in the sense that there is a blown-up 2-cycle. The volume of the 2-cycle is characterized by the parameters α and β. This is a global deformation, in that it changes the position of the branes at infinity [18].

The parameters α and β specify the expectation values of dimension n non-mesonic scalar operators in the dual gauge theory. For the case $\beta = -\alpha$, c_2 and c_6 vanish, while c_4 can only vanish for $\alpha = \beta = 0$. To identify the specific dimension-two operator whose expectation value goes as c_2, it is helpful to consider the description of the resolved cone over L^{abc} in terms of four complex numbers z_i which satisfy the constraint

$$\sum_{i=1}^{4} Q_i \, |z_i|^2 = t \,, \tag{3.12}$$

where one then takes the quotient by a $U(1)$ action [9]. The parameter t is the area of the blown-up CP^1 and corresponds to the coefficient of the Fayet-Iliopoulos term in the Lagrangian of the field theory. The z_i correspond to the lowest components of chiral superfields. This can be described as a gauged linear sigma model with a $U(1)$ gauge group and 4 fields with charges Q_i. Then the above constraint corresponds to setting the D-terms of the gauged linear sigma model to zero to give the vacuum. For the L^{abc} spaces, the Q_i are given by $Q_i = (a, -c, b, -d)$ where $d = a+b-c$ [11]. The requirement $\sum_{i=1}^{4} Q_i = 0$ guarantees that the 1-loop β-function vanishes, so that the sigma model is Calabi-Yau.

Since t acts as a natural order-parameter in the gauge theory, from (3.12) it is reasonable to suppose that blowing up the 2-cycle corresponds to giving an expectation value that goes as c_2 to the dimension-two scalar operator[4]

$$\mathcal{K} = a\, A_\alpha \bar{A}^{\dot{\alpha}} - c\, B_{\dot{\alpha}} \bar{B}^{\dot{\alpha}} + b\, C_\alpha \bar{C}^\alpha - d\, D_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}\,. \tag{3.13}$$

This operator lies within the $U(1)$ baryonic current multiplet. Since this conserved current has no anomalous dimension, the dimension of $\mathcal{K}$ is protected. $\mathcal{K}$ reduces to the operator discussed in [18] for the case of a resolved cone over $T^{11}/\mathbb{Z}_2$, for which $a = b = c = d = 1$.

Blown-up 4-cycle

For nonvanishing M in the function X, one generically blows up a 4-cycle. Unlike the case of a blown-up 2-cycle, this is a local deformation since it does not change the position of the branes at infinity [18]. In the limit of vanishing α and β, one recovers the general construction obtained in [19, 20] that has been recently discussed in [16, 17, 18]. Also note that c_6 vanishes for the appropriate values of M, α and β.

It has been shown that the number of formal Fayet-Iliopoulos parameters can be matched with the possible deformations, which is suggestive that the dimension-six operator that is turned on is associated with the gauge groups in the quiver. Although the specific operator has not been identified, it has been proposed that they are of the schematic form [18]

$$\mathcal{O}_i = \sum_g c_{i,g} \mathcal{W}_g \bar{\mathcal{W}}_g\,, \tag{3.14}$$

where the gauge groups in the quiver have been summed over, $\mathcal{W}_g$ is an operator associated with the field strength for the gauge group g, and $c_{i,g}$ are constants. The dimension-six operator might also have contributions from the bifundamental fields of the form

$$a_1 A_\alpha \bar{A}^\alpha B_{\dot{\alpha}} \bar{B}^{\dot{\alpha}} C_\beta \bar{C}^\beta + a_2 A_\alpha \bar{A}^\alpha B_{\dot{\alpha}} \bar{B}^{\dot{\alpha}} D_{\dot{\beta}} \bar{D}^{\dot{\beta}} + a_3 A_\alpha \bar{A}^\alpha C_\beta \bar{C}^\beta D_{\dot{\beta}} \bar{D}^{\dot{\beta}} + a_4 B_{\dot{\alpha}} \bar{B}^{\dot{\alpha}} C_\beta \bar{C}^\beta D_{\dot{\beta}} \bar{D}^{\dot{\beta}},$$
$$\tag{3.15}$$

where the a_i are constants. It is proposed that a particular combination of all of these terms in (3.14) and (3.15) correspond to the blown-up 4-cycle[5]. One possibility

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[4]We thank Amihay Hanany and Igor Klebanov for correspondence on this point.

[5]We thank Sergio Benvenuti for correspondence on this point.

is that the contributions from the bifundamental fields in (3.15) are present only when α and β are nonvanishing.

Turning on the 3-form

Turning on a 3-form results in the ranks of some of the gauge groups of the dual quiver gauge theory being reduced with decreasing energy scale. For the case in which the 3-form has nontrivial flux, the theory undergoes a Seiberg duality cascade [27, 28, 32]. On the other hand, the 3-form Ψ_1 given by (2.21) does not have nontrivial flux. For a case such as this, it has been proposed that the reduction in ranks of gauge groups is due to Higgsing [34]. In particular, from (3.10), we see that the parameter m associated with the 3-form also contributes to the expectation value c_6 of a dimension-six scalar operator. An additional effect of this 3-form is that the $U(1)$ R-symmetry is broken. The theory still preserves $\mathcal{N} = 1$ supersymmetry.

4 Eight-dimensional resolved Calabi-Yau cones

4.1 Cohomogeneity-two metrics

We now turn to eight-dimensional Calabi-Yau spaces, which can be used to construct M2-brane solutions of eleven-dimensional supergravity. Before considering the general cohomogeneity-four resolved cones over L^{pqrs}, we shall first look at the cohomogeneity-two metrics, which can be built over an $S^2 \times S^2$ base space. These metrics are given by [21, 22]

$$
\begin{aligned}
ds_8^2 &= \tfrac{1}{4}u^2 dx^2 + \tfrac{1}{4}v^2 dy^2 + \frac{1}{u^2}\left[d\tau + \frac{y}{3\alpha}(\sigma_3 + \tilde\sigma_3)\right]^2 \\
&\quad + \frac{1}{v^2}\left[d\tau - \frac{x}{3\alpha}(\sigma_{(3)} + \tilde\sigma_3)\right]^2 + c^2(\sigma_1^2 + \sigma_2^2 + \tilde\sigma_1^2 + \tilde\sigma_2^2) \\
u^2 &= \frac{x+y}{X}, \qquad v^2 = \frac{x+y}{Y}, \qquad c^2 = \frac{xy}{6\alpha}, \\
X &= x(x+\alpha) - \frac{2\mu}{x^2}, \quad Y = y(\alpha - y) + \frac{2\nu}{y^2},
\end{aligned}
\tag{4.1}
$$

Completely regular examples were discussed in [25].

Calabi-Yau structure

We can define the vielbein basis

$$e^1 = \tfrac{1}{2}u\,dx\,, \qquad e^2 = -\frac{1}{u}\Big(d\tau+\frac{y}{3\alpha}(\sigma_3+\tilde\sigma_3)\Big)\,, \qquad e^3 = \tfrac{1}{2}v\,dy\,, \tag{4.2}$$

$$e^4 = \frac{1}{v}\Big(d\tau-\frac{x}{3\alpha}(\sigma_3+\tilde\sigma_3)\Big)\,, \quad e^5 = c\,\sigma_1\,, \quad e^6 = c\,\sigma_2\,, \qquad e^7 = c\,\tilde\sigma_1\,, \quad e^8 = c\,\tilde\sigma_2\,,$$

and then the complex vielbein

$$\epsilon_1 = e^1+\mathrm{i}\,e^2\,, \qquad \epsilon_2 = e^3+\mathrm{i}\,e^4\,, \qquad \epsilon_3 = e^5+\mathrm{i}\,e^6\,, \qquad \epsilon_4 = e^7+\mathrm{i}\,e^8\,. \tag{4.3}$$

The Kähler 2-form and holomorphic $(4,0)$-form are given by

$$J = \tfrac{\mathrm{i}}{2}\epsilon^i\wedge\bar\epsilon^i\,, \tag{4.4}$$

and

$$G_{(4)} = e^{-4\mathrm{i}\tau}\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\wedge\epsilon^4\,. \tag{4.5}$$

Harmonic $(2,2)$-forms

We find four self-dual $(2,2)$-forms; they are given by

$$\Phi_1 = \frac{(\bar\epsilon_1\wedge\epsilon_1+\bar\epsilon_2\wedge\epsilon_2)\wedge(\bar\epsilon_3\wedge\epsilon_3+\bar\epsilon_4\wedge\epsilon_4)-2(\bar\epsilon_1\wedge\epsilon_1\wedge\bar\epsilon_2\wedge\epsilon_2+\bar\epsilon_3\wedge\epsilon_3\wedge\bar\epsilon_4\wedge\epsilon_4)}{x^3y^3}$$

$$\Phi_2 = \frac{(\bar\epsilon_1\wedge\epsilon_1-\bar\epsilon_2\wedge\epsilon_2)\wedge(\bar\epsilon_3\wedge\epsilon_3-\bar\epsilon_4\wedge\epsilon_4)}{xy(x+y)^2}\,,$$

$$\Phi_3 = \frac{e^{-4\mathrm{i}\tau}\,(\bar\epsilon_1\wedge\bar\epsilon_2\wedge\epsilon_3\wedge\epsilon_4+\epsilon_1\wedge\epsilon_2\wedge\bar\epsilon_3\wedge\bar\epsilon_4)}{x^2y^2\,XY}\,,$$

$$\Phi_4 = \frac{(\bar\epsilon_1\wedge\epsilon_2-\epsilon_1\wedge\bar\epsilon_2)\wedge(\bar\epsilon_3\wedge\epsilon_3+\bar\epsilon_4\wedge\epsilon_4)}{xy\sqrt{XY}}\,. \tag{4.6}$$

Notice that Φ_1 and Φ_2 are square integrable, in that they are well behaved at both small and large asymptotic distance. For the cases in which the eight-dimensional Calabi-Yau spaces are regular [25], these harmonic forms can be used to construct completely non-singular M2-brane solutions to eleven-dimensional supergravity.

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4.2 Cohomogeneity-four metrics on resolved cones over L^{pqrs}

We now turn to the general cohomogeneity-four metrics on resolved Calabi-Yau cones over the seven-dimensional Einstein-Sasaki spaces L^{pqrs}, which can be written as [22]

$$
\begin{aligned}
ds_8^2 &= \tfrac{1}{4}(u_1^2\,dx_1^2+u_2^2\,dx_2^2+u_3^2\,dx_3^2+u_4^2\,dx_4^2) \\
&\quad +\frac{1}{u_1^2}\,[d\tau+(x_2+x_3+x_4)d\phi+(x_2x_3+x_2x_4+x_3x_4)d\psi+x_2x_3x_4d\chi]^2 \\
&\quad +\frac{1}{u_2^2}\,[d\tau+(x_1+x_3+x_4)d\phi+(x_1x_3+x_1x_4+x_3x_4)d\psi+x_1x_3x_4d\chi]^2 \\
&\quad +\frac{1}{u_3^2}\,[d\tau+(x_1+x_2+x_4)d\phi+(x_1x_2+x_1x_4+x_2x_4)d\psi+x_1x_2x_4d\chi]^2 \\
&\quad +\frac{1}{u_4^2}\,[d\tau+(x_1+x_2+x_3)d\phi+(x_1x_2+x_1x_3+x_2x_3)d\psi+x_1x_2x_3d\chi]^2\,, \quad (4.7)
\end{aligned}
$$

where

$$
\begin{aligned}
u_1^2 &= \frac{(x_2-x_1)(x_3-x_1)(x_4-x_1)}{X_1}\,, \qquad u_2^2 = \frac{(x_1-x_2)(x_3-x_2)(x_4-x_2)}{X_2}\,, \\
u_3^2 &= \frac{(x_1-x_3)(x_2-x_3)(x_4-x_3)}{X_3}\,, \qquad u_4^2 = \frac{(x_1-x_4)(x_2-x_4)(x_3-x_4)}{X_4}\,, \\
X_1 &= x_1(a-x_1)(b-x_1)(c-x_1)-2M_1\,, \\
X_2 &= x_2(a-x_2)(b-x_2)(c-x_2)-2M_2\,, \\
X_3 &= x_3(a-x_3)(b-x_3)(c-x_3)-2M_3\,, \\
X_4 &= x_4(a-x_4)(b-x_4)(c-x_4)-2M_4\,.
\end{aligned}
\qquad (4.8)
$$

<u>Calabi-Yau structure</u>

We shall choose the vielbein basis

$$
\begin{aligned}
e^1 &= \tfrac{1}{2}u_1\,dx_1\,, \quad u_3=\tfrac{1}{2}u_2\,dx_2\,, \quad e^5=\tfrac{1}{2}u_3\,dx_3\,, \quad e^7=\tfrac{1}{2}u_4\,dx_4\,, \\
e^2 &= \frac{1}{u_1}\,[d\tau+(x_2+x_3+x_4)d\phi+(x_2x_3+x_2x_4+x_3x_4)d\psi+x_2x_3x_4d\chi]\,, \\
e^4 &= \frac{1}{u_2}\,[d\tau+(x_1+x_3+x_4)d\phi+(x_1x_3+x_1x_4+x_3x_4)d\psi+x_1x_3x_4d\chi]\,, \\
e^6 &= \frac{1}{u_3}\,[d\tau+(x_1+x_2+x_4)d\phi+(x_1x_2+x_1x_4+x_2x_4)d\psi+x_1x_2x_4d\chi]\,, \\
e^8 &= \frac{1}{u_4}\,[d\tau+(x_1+x_2+x_3)d\phi+(x_1x_2+x_1x_3+x_2x_3)d\psi+x_1x_2x_3d\chi]\,. \quad (4.9)
\end{aligned}
$$

The holomorphic vielbein are then given by

$$
\epsilon^1 = e^1+\mathrm{i}\,e^2\,, \quad \epsilon^2 = e^3+\mathrm{i}\,e^4\,, \quad \epsilon^3 = e^5+\mathrm{i}\,e^6\,, \quad \epsilon^4 = e^7+\mathrm{i}\,e^8\,. \qquad (4.10)
$$

Defining

$$J = \frac{i}{2} \left(\bar{\epsilon}^1 \wedge \epsilon^1 + \bar{\epsilon}^2 \wedge \epsilon^2 + \bar{\epsilon}^3 \wedge \epsilon^3 + \bar{\epsilon}^4 \wedge \epsilon^4 \right),$$
$$\Omega = e^{i\nu} \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4, \tag{4.11}$$

where

$$\nu = 4\tau + 3(a+b+c)\,\phi + 2(ab+bc+ca)\,\psi + abc\,\chi, \tag{4.12}$$

it is straightforward to verify that

$$dJ = 0, \qquad d\Omega = 0, \tag{4.13}$$

and hence that the metric is indeed Ricci-flat Kähler, with J being the Kähler form and Ω the holomorphic $(4,0)$-form.

Harmonic $(3,1)$-forms

We find that harmonic $(3,1)$-forms can be constructed as follows. First, it can be verified that

$$G_{(3,1)} = \frac{1}{X_1}\, e^{i\nu}\, \bar{\epsilon}^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4 \tag{4.14}$$

is closed, and hence harmonic. Next, we define the function

$$\gamma = \sqrt{\frac{X_2 X_3 X_4}{X_1}}\, e^{i\nu}, \tag{4.15}$$

which can be shown to satisfy the relation

$$
\begin{aligned}
d\gamma = {}& \frac{u_1\, e^{i\nu}}{u_2 u_3 u_4\,(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} \Big(u_1\,(x_2 - x_3)(x_2 - x_4)(x_4 - x_3)X_1'\,\bar{\epsilon}^1 \\
& - u_2\,(x_3 - x_1)(x_3 - x_4)(x_4 - x_1)\,X_2'\,\epsilon^2 + u_3\,(x_1 - x_2)(x_4 - x_1)(x_4 - x_2)\,X_3'\,\epsilon^3 \\
& + u_4\,(x_1 - x_2)(x_3 - x_1)(x_2 - x_3)\,X_4'\,\epsilon^4 \Big),
\end{aligned}
\tag{4.16}
$$

where X_i' denotes the derivative of X_i with respect to its argument x_i. It therefore follows that $d\gamma \wedge G_{(3,1)} = 0$, and so

$$\Phi_{(3,1)} = f(\gamma)\, G_{(3,1)} \tag{4.17}$$

is a harmonic $(3,1)$-form for any function f. In particular, we have a family of harmonic $(3,1)$-forms given by

$$\Psi_{(3,1)} = \frac{X_2^\delta\, X_3^\delta\, X_4^\delta}{X_1^{\delta+1}}\, e^{(2\delta+1)\,i\nu}\, \bar{\epsilon}^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^4 \tag{4.18}$$

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for any constant δ. For nonzero δ, these forms preserve only a $U(1)^3$ subgroup of the $U(1)^4$ isometry of the eight-dimensional space. Note that $\Psi_{(3,1)}$ has a singularity only at short distance if $\delta \geq 0$, where we have taken x_1 to be the radial direction. Additional harmonic $(3,1)$-forms can be constructed by permuting the x_i directions, but these forms have singularities for all x_1. They are analogous to the $(2,1)$-forms Φ_1, Φ_2 and Φ_3 in (2.27) for a six-dimensional space, and they do not support nontrivial flux.

Harmonic $(2,2)$-forms

We can also construct harmonic $(2,2)$-forms as follows. We define $(2,2)$-forms

$$
\begin{aligned}
G_{(2,2)} \;=\;& f \,(\bar\epsilon^1 \wedge \epsilon^1 \wedge \bar\epsilon^2 \wedge \epsilon^2 + \bar\epsilon^3 \wedge \epsilon^3 \wedge \bar\epsilon^4 \wedge \epsilon^4) \\
&+ g \,(\bar\epsilon^1 \wedge \epsilon^1 \wedge \bar\epsilon^3 \wedge \epsilon^3 + \bar\epsilon^2 \wedge \epsilon^2 \wedge \bar\epsilon^4 \wedge \epsilon^4) \\
&+ h \,(\bar\epsilon^1 \wedge \epsilon^1 \wedge \bar\epsilon^4 \wedge \epsilon^4 + \bar\epsilon^2 \wedge \epsilon^2 \wedge \bar\epsilon^3 \wedge \epsilon^3)\,,
\end{aligned} \tag{4.19}
$$

where f, g and h are functions of (x_1, x_2, x_3, x_4). Imposing the closure of $G_{(2,2)}$ leads to three independent solutions for f, g and h, namely

$$
f = g = h = 1\,, \tag{4.20}
$$

$$
\begin{aligned}
f &= \frac{1}{(x_1-x_2)^2(x_1-x_3)(x_2-x_4)(x_3-x_4)^2}\,, \\
g &= \frac{x_1\,(2x_4-x_2-x_3)+x_2\,(2x_3-x_4)-x_3\,x_4}{(x_1-x_2)^2(x_1-x_3)^2(x_2-x_4)^2(x_3-x_4)^2}\,, \\
h &= \frac{1}{(x_1-x_2)(x_1-x_3)^2(x_2-x_4)^2(x_3-x_4)}\,,
\end{aligned} \tag{4.21}
$$

and

$$
\begin{aligned}
f &= \frac{1}{(x_1-x_3)(x_2-x_3)^2(x_1-x_4)^2(x_2-x_4)}\,, \\
g &= \frac{1}{(x_1-x_3)^2(x_2-x_3)(x_1-x_4)(x_2-x_4)^2}\,, \\
h &= \frac{x_1\,(x_3+x_4-2x_2)+x_2\,(x_3+x_4)-2x_3\,x_4}{(x_1-x_3)^2(x_2-x_3)^2(x_1-x_4)^2(x_2-x_4)^2}\,.
\end{aligned} \tag{4.22}
$$

These forms are somewhat analogous to the $(2,1)$-forms Φ_4 and Φ_5 given in (2.27) for a six-dimensional space. The first solution, (4.20), is just the harmonic $(2,2)$-form

$J{\wedge}J$. It follows from (4.19) that $J{\wedge}G_{(2,2)}$ is proportional to $(f{+}g{+}h)$, and so $J{\wedge}G_{(2,2)}$ is non-zero for (4.20). However, each of the solutions (4.21) and (4.22) satisfies $f{+}g{+}h = 0$, and so these two harmonic $(2,2)$-forms satisfy the supersymmetric condition

$$J{\wedge}G_{(2,2)} = 0\,. \tag{4.23}$$

Notice also that these harmonic $(2,2)$-forms are square integrable. These can be used to construct modified M2-brane solutions, which have only orbifold-type singularities. Note that none of these cohomogeneity-four Calabi-Yau spaces are completely regular [25].

4.3   M2-brane solutions

We can use these eight-dimensional spaces, and the harmonic 4-forms which they support, to construct a modified M2-brane solution to eleven-dimensional supergravity, given by

$$
\begin{aligned}
ds_{11}^2 &= H^{-2/3}(-dt^2{+}dx_1^2{+}dx_2^2){+}H^{1/3}ds_8^2\,, \\
F_{(4)} &= dt{\wedge}dx_1{\wedge}dx_2{\wedge}dH^{-1}{+}m\,L_{(4)}\,,
\end{aligned}
\tag{4.24}
$$

where

$$\Box H = -\frac{1}{48}m^2 L_{(4)}^2\,, \tag{4.25}$$

and $L_{(4)}$ is an (anti)self-dual harmonic 4-form on the eight-dimensional space with the metric ds_8^2.

Let us first consider the case with $m = 0$, for which the Laplace equation on the Calabi-Yau metric is separable. The solution for general dimensionality is presented in the appendix B. Here we just give a solution for the eight-dimensional case that depends only on the radial variable x_1; it is given by

$$H = \int^{x_1} \frac{3Q}{X(x_1')}dx_1'\,. \tag{4.26}$$

Thus in the asymptotic region at large x_1, the function H has the behavior

$$H = \frac{Q}{x_1^3}\left(1{+}\frac{c_2}{x_1}{+}\cdots\right)\,, \qquad \text{where}\;\; c_2 = \frac{3}{4}(\alpha{+}\beta{+}\gamma)\,. \tag{4.27}$$

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We have taken an arbitrary additive constant to zero, so that the geometry is asymptotically $\text{AdS}_4 \times L^{pqrs}$. Since x_1 has dimension two, we see that there is a non-mesonic dimension-two scalar operator being turned on with expectation value c_2.

It is especially interesting to construct M2-brane solutions using one of the square-integrable harmonic $(2, 2)$-forms that we found previously, since this guarantees that with the appropriate integration constants the only singularities are of orbifold type. This is because the 4-form prevents the blown-up 4-cycle from collapsing. Moreover, examples of regular eight-dimensional Calabi-Yau spaces that have been discussed in [25] can be used to construct completely non-singular M2-brane solutions. The resulting geometry smoothly interpolates between $\text{AdS}_4 \times L^{pqrs}$ asymptotically, and a direct product of Minkowski$_3$ and a compact space at short distance. Many examples of cohomogeneity-one solutions of this type were constructed in [49, 50, 51]. Although not much is known even about the UV conformal fixed point of the dual three-dimensional $\mathcal{N} = 2$ super Yang-Mills field theory, based on the geometrical properties of the supergravity background it flows to a confining phase in the IR region.

5 Harmonic forms on higher-dimensional resolved cones

In this section, we extend some of the constructions of harmonic middle-dimension forms to the case of higher-dimensional metrics on the resolutions of cones over Einstein-Sasaki spaces. We take as our starting point the local Ricci-flat Kähler metrics in dimension $D = 2n+4$ that were considered in [25]:

$$
\begin{aligned}
d\tilde{s}^2 &= \frac{x+y}{4X}dx^2 + \frac{x+y}{4Y}dy^2 + \frac{X}{x+y}\left[d\tau + \frac{y}{\alpha}\sigma\right]^2 + \frac{Y}{x+y}\left[d\tau - \frac{x}{\alpha}\sigma\right]^2 + \frac{xy}{\alpha}d\Sigma_n^2 \\
\sigma &= d\psi + A, \qquad X = x(x+\alpha) - \frac{2\mu}{x^n}, \qquad Y = y(\alpha-y) + \frac{2\nu}{y^n},
\end{aligned} \tag{5.1}
$$

where $d\Sigma_n^2$ is a metric on a $2n$-dimensional Einstein-Kähler space Z, satisfying $R_{ab} = 2(n+1)\,g_{ab}$, with Kähler form $J = \frac{1}{2}dA$. (We have made some minor changes of coordinates compared to the metric presented in [25].) For convenience, we shall set the constant α to unity. This can always be done, when $\alpha \neq 0$, by means of coordinate scalings together with an overall rescaling of the Ricci-flat metric. The special case

$\alpha = 0$ can be recovered via a limiting procedure.

Next, we define the 2-forms

$$\omega_x = \tfrac{1}{2}dx \wedge (d\tau + y\,\sigma)\,, \qquad \omega_y = \tfrac{1}{2}dy \wedge (d\tau - x\,\sigma)\,, \qquad \omega = xy\,J\,. \qquad (5.2)$$

It can easily be verified that $\hat{J} \equiv \omega_x - \omega_y + \omega$ is closed and, in fact, this is the Kähler form of the Ricci-flat Kähler metric (5.1). In the case that n is even ($n = 2m$), we find that the middle-degree form

$$G_{(2m+2)} = \frac{1}{(xy)^{2m+1}}\left[\omega_x \wedge \omega_y \wedge \omega^{m-1} + \frac{1}{m+1}\left(\omega_x - \omega_y\right) \wedge \omega^m - \frac{1}{m(m+1)}\,\omega^{m+1}\right] \qquad (5.3)$$

is closed. Since it is also self-dual, it follows that it is a harmonic form. This generalises the harmonic $(2,2)$-form Φ_1 in eight dimensions given in (4.6) and is somewhat analogous to the $(2,1)$-forms Φ_4 and Φ_5 given in (2.27) for a six dimensions.

Further harmonic forms can be obtained if one takes the Einstein-Kähler base metric $d\Sigma_n^2$ to be a product of Einstein-Kähler metrics. For example, if we choose it to be the product of metrics on two copies of $\mathbb{CP}^m$ (recall that we are considering the case where $n = 2m$ is even), with Kähler forms J_1 and J_2 respectively (so $J = J_1 + J_2$), then defining

$$\omega_1 = xy\,J_1\,, \qquad \omega_2 = xy\,J_2\,, \qquad (5.4)$$

we find that

$$\widetilde{G}_{(2m+2)} = \frac{1}{(x+y)^2\,(xy)^m}\,(\omega_x + \omega_y) \wedge \sum_{p=0}^{m}(-1)^p\,\omega_1^{m-p} \wedge \omega_2^p \qquad (5.5)$$

is closed and self-dual, and therefore it is harmonic.

6   Conclusions

We have investigated the Kähler moduli associated with blowing up a 2-cycle or 4-cycle on Calabi-Yau cones over the L^{abc} spaces. This yields a countably infinite number of backgrounds with ALE singularities on which perturbative string dynamics is well-defined. Although adding D3-branes induces a power-law type singularity at short distance, one can still use the AdS/CFT dictionary to relate the blown-up cycles to deformations of the dual quiver gauge theory close to the UV conformal

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fixed point. In particular, we identify the non-mesonic dimension-two real scalar operator that acquires a vev, thereby generalizing the state/operator correspondence for the resolved conifold over T^{11} [14] and $T^{11}/\mathbb{Z}_2$ [18] to resolved cones over the L^{abc} spaces. On the other hand, blowing up a 4-cycle corresponds to a dimension-six non-mesonic scalar operator getting a vev.

The resolved cones over the cohomogeneity-two L^{abc} spaces support various harmonic $(2,1)$-forms, some of which depend nontrivially on three non-azimuthal coordinate directions. These forms can be further generalized by a multiplicative function, so long as the exterior derivative of this function satisfies a certain orthogonality condition. In particular, there are harmonic $(2,1)$-forms which depend on continuous parameters. 3-forms carrying nontrivial flux correspond to fractional D3-branes, while those which do not correspond to giving a vev to a dimension-six operator.

For the D3-brane solutions constructed with resolved cones over L^{abc}, we have restricted ourselves to the case in which the D3-branes are smeared over the blown-up cycle. As we already mentioned, this yields to a power-law singularity at short distance. For solutions involving a 3-form field, one may be able to smooth out this singularity by a complex deformation of the Calabi-Yau space that results in a blown-up 3-cycle. Although it has been shown that there are obstructions to the existence of complex deformations of cones over Y^{pq} spaces, there are other subsets of the L^{abc} cones which do allow for complex structure deformations [37, 38, 12]. It would be useful to construct the explicit metrics describing these deformed L^{abc} cones, as well as the non-singular supergravity solutions that describe fractional D3-branes on these spaces.

Alternatively, one can consider stacking the D3-branes at a single point. For the case of the resolved conifold, this has been shown to yield a completely regular solution [26]. Perhaps there are analogous constructions with the resolved cones over L^{abc}. With the exceptions of $T^{1,1}$, $T^{1,1}/\mathbb{Z}_2$ and $Y^{2,1}$, the resolved L^{abc} cones have orbifold singularities. Although these singularities will remain there when D3-branes are stacked at a single point, perturbative string dynamics is well-defined on such backgrounds.

One can also consider fibering a D3-brane worldvolume direction (which need not be compact) over a resolved L^{abc} cone in such a way that the resulting geometry

only has orbifold-type singularities. For the case of the resolved conifold, such a D3-brane solution has already been constructed and is completely regular, and it is also supersymmetric [46]. The corresponding D3-brane solutions for the resolved L^{abc} cones are currently being investigated [47].

We also discussed the geometry of higher-dimensional Calabi-Yau spaces with blown-up cycles, as well as the various harmonic forms which live on them. In particular, we have found that eight-dimensional resolved cones over the L^{pqrs} spaces support harmonic 4-forms that are square integrable. They can be used to construct M2-brane solutions of eleven-dimensional supergravity which have only orbifold-type singularities. Unfortunately, not much is known about the dual three-dimensional $\mathcal{N} = 2$ gauge theories, other than that they flow from a UV conformal fixed point to a confining phase in the IR region.

Lastly, the type IIB supergravity backgrounds dual to certain marginal deformations (β deformations) of the conformal fixed point of the Y^{pq} and L^{abc} quiver gauge theories were obtained in [48, 52]. The solution-generating method works for any gravity solution with $U(1) \times U(1)$ global symmetry. It might be interesting to see if these deformations can be applied to the gravity solutions discussed in this paper, since they possess the necessary global symmetry.

Acknowledgments

A   Complex structure and first-order equations

In this appendix, we construct Ricci-flat Kähler spaces in dimension $D = 2n+4$, built over an Einstein-Kähler base space of real dimension $2n$ with metric $d\Sigma_n^2$. We normalise this metric so that it satisfies $R_{ij} = 2(n+1)g_{ij}$. Its Kähler form will be

written as $J = \frac{1}{2}dA$. We may also assume that it admits a holomorphic $(n,0)$-form Ω, satisfying (see, for example, section 4 of [23])

$$d\Omega = \mathrm{i}\,(n{+}1)A\wedge\Omega\,. \tag{A.1}$$

The ansatz for the $(3n{+}4)$-dimensional Ricci-flat Kähler metrics will be

$$d\hat{s}^2 = u^2 dx^2 + v^2 dy^2 + a^2(d\tau + f_1\sigma)^2 + b^2(d\tau + f_2\sigma)^2 + c^2 d\Sigma_n^2\,, \tag{A.2}$$

where a, b, c, u, v, f_1 and f_2 are functions of x and y, and

$$\sigma = d\psi + A\,. \tag{A.3}$$

We define the vielbein

$$\hat{e}^1 = u\,dx\,, \qquad \hat{e}^2 = a(d\tau + f_1\sigma)\,, \qquad \hat{e}^3 = v\,dy\,, \qquad \hat{e}^4 = b(d\tau + f_2\sigma)\,, \qquad \hat{e}^i = c\,e^i\,, \tag{A.4}$$

where e^i is a vielbein for the Einstein-Kähler base metric $d\Sigma_n^2$.

We make the ansatz

$$\hat{J} = e^1\wedge e^2 + e^3\wedge e^4 + c^2 J \tag{A.5}$$

for the Kähler form. It is then natural to define a complex vielbein by

$$\hat{\epsilon}^1 = \hat{e}^1 + \mathrm{i}\,\hat{e}^2\,, \qquad \hat{\epsilon}^2 = \hat{e}^3 + \mathrm{i}\,\hat{e}^4\,, \qquad \hat{\epsilon}^i = c\,\epsilon^i\,, \tag{A.6}$$

where ϵ^i is a complex vielbein for the base metric $d\Sigma_n^2$. We also make the ansatz

$$\hat{\Omega} = e^{\mathrm{i}\alpha\tau + \mathrm{i}\beta\psi}\,c^n\,\hat{\epsilon}^1\wedge\hat{\epsilon}^2\wedge\Omega \tag{A.7}$$

for the holomorphic $(n{+}2,0)$-form. The conditions for $d\hat{s}^2$ to be Ricci flat and Kähler are then given by

$$d\hat{J} = 0\,, \qquad d\hat{\Omega} = 0\,. \tag{A.8}$$

One immediately finds that the constant β should be chosen to be

$$\beta = n{+}1\,. \tag{A.9}$$

However, the constant α can be left arbitrary.

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We now obtain the first-order equations:

$$d\hat{J} = 0: \quad (bv)' - (au)\dot{} = 0, \qquad (c^2)' - 2au f_1 = 0, \qquad (c^2)\dot{} - 2bv f_2 = 0,$$

$$d\hat{\Omega} = 0: \quad \alpha\, uv c^n - (av c^n)' - (bu c^n)\dot{} = 0,$$

$$\alpha b u c^n f_2 - (n+1)bu c^n + [abc^n(f_1 - f_2)]' = 0,$$

$$\alpha a v c^n f_1 - (n+1)av c^n - [abc^n(f_1 - f_2)]\dot{} = 0. \tag{A.10}$$

The constant α appearing in the first-order equations (A.10) is always trivial, in the sense that it can be set to any chosen non-zero value without loss of generality. To see this, we perform the following rescaling of coordinates and functions:

$$x \to \lambda x, \qquad y \to \lambda y, \qquad \tau \to \lambda \tau,$$

$$c \to \lambda c, \qquad f_1 \to \lambda f_1 \qquad f_2 \to \lambda f_2, \tag{A.11}$$

whilst leaving the functions a, b, u and v unscaled. It can be seen that the effect of these rescalings is to scale the metric $d\hat{s}^2$ in (A.2) according to

$$d\hat{s}^2 \to \lambda^2\, d\hat{s}^2. \tag{A.12}$$

The rescalings have the effect of replacing α by $\lambda\alpha$ in the first-order equations (A.10), thus giving

$$d\hat{J} = 0: \quad (bv)' - (au)\dot{} = 0, \qquad (c^2)' - 2au f_1 = 0, \qquad (c^2)\dot{} - 2bv f_2 = 0,$$

$$d\hat{\Omega} = 0: \quad \lambda\alpha\, uv c^n - (av c^n)' - (bu c^n)\dot{} = 0,$$

$$\lambda\alpha\, bu c^n f_2 - (n+1)bu c^n + [abc^n(f_1 - f_2)]' = 0,$$

$$\lambda\alpha\, av c^n f_1 - (n+1)av c^n - [abc^n(f_1 - f_2)]\dot{} = 0. \tag{A.13}$$

Since a rescaling of a Ricci-flat metric by a non-zero constant leaves it Ricci-flat, it follows that the constant λ can be chosen at will, and so no generality is lost by setting α to any desired finite and non-zero value.

B   Separability of Laplacian on Calabi-Yau metrics

We consider the Calabi-Yau metrics obtained in [21, 22]. The metric can be expressed as

$$ds^2 \;=\; \sum_{\mu=1}^{n} \Big[\frac{U_\mu \, dx_\mu^2}{4X_\mu} + \frac{X_\mu}{U_\mu} \big(\sum_{i=0}^{n-1} W_i d\phi_i)^2 \Big] ,$$

$$X_\mu \;=\; x_\mu \prod_{i=1}^{n-1} (\alpha_i - x_\mu) - 2\ell_\mu , \qquad U_\mu = \prod_{\nu=1}^{n}{}' (x_\nu - x_\mu) , \tag{B.1}$$

where W_i is defined by

$$\prod_{\mu=1}^{n} (1 + q x_\mu) \equiv \sum_{i=0}^{n-1} W_i \, q^{i+1} . \tag{B.2}$$

It turns out that the equation $\Box H = 0$ is separable in the x_μ coordinates, where $\Box$ is the Laplacian taken on the above metric. (The separability for the more general non-extremal Kerr-NUT-AdS metrics was shown explicitly in [43, 44, 45]. Making the ansatz

$$H = \Big(\prod_{\mu=1}^{n} H_\mu(x_\mu) \Big) \, \exp \Big(2\mathrm{i} \sum_{i=0}^{n-1} (-1)^i a_i \phi_{n-1-i} \Big) , \tag{B.3}$$

for the harmonic function, we find that the $H_\mu(x_\mu)$ satisfy

$$(X_\mu H_\mu')' - \Big(\frac{(\sum_{i=0}^{n-1} a_i x_\mu^i)^2}{X_\mu} + \sum_{i=1}^{n-2} b_i x_\mu^i \Big) H_\mu = 0 , \tag{B.4}$$

where a prime on H_μ or X_μ denotes a derivative with respect to its argument x_μ. The system thus has $2n-1$ independent separation constants $a_0, a_1, \ldots a_{n-1}$ and $b_0, b_1, \ldots, b_{n-2}$.

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