

# Consistent Pauli Sphere Reductions and the Action

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## ABSTRACT

It is a commonly held belief that a consistent dimensional reduction ansatz can be equally well substituted into either the higher-dimensional equations of motion or the higher-dimensional action, and that the resulting lower-dimensional theories will be the same. This is certainly true for Kaluza-Klein circle reductions and for DeWitt group-manifold reductions, where group-invariance arguments guarantee the equivalence. In this paper we address the question in the case of the non-trivial consistent Pauli coset reductions, such as the  $S^7$  and  $S^4$  reductions of eleven-dimensional supergravity. These always work at the level of the equations of motion. In some cases the reduction ansatz can only be given at the level of field strengths, rather than the gauge potentials which are the fundamental fields in the action, and so in such cases there is certainly no question of being able to substitute instead into the action. By examining explicit examples, we show that even in cases where the ansatz can be given for the fundamental fields appearing in an action, substituting it into the higher-dimensional action may not give the correct lower-dimensional theory. This highlights the fact that much remains to be understood about the way in which Pauli reductions work.

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# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Consistent Reduction on <math>S^7</math></b>	<b>6</b>
2.1	Truncation to gravity, scalar and pseudoscalar . . . . .	6
2.2	Truncation to the gravity plus scalar subsystem . . . . .	8
2.3	Dualisation in a toy model . . . . .	8
2.4	The gravity plus scalar and pseudoscalar truncation . . . . .	11
<b>3</b>	<b><math>S^4</math> Reduction of <math>D = 11</math> Supergravity</b>	<b>11</b>
3.1	Reduction of the field equations . . . . .	11
3.2	Substitution into the eleven-dimensional action . . . . .	14
<b>4</b>	<b>Discussion and Conclusions</b>	<b>15</b>

## 1 Introduction

The idea of dimensional reduction in general relativity was pioneered by Kaluza in 1921 [1]. He observed that by taking the metric in five dimensions to be independent of the fifth coordinate, one obtained a four-dimensional system describing electromagnetism with an additional scalar field, coupled to four-dimensional gravity. It was later observed by Klein that one could take the fifth coordinate to be compactified into a circle, which would become effectively unobservable if its radius were taken to be sufficiently small [2].<sup>1</sup> By a straightforward process of iteration, one can immediately extend the idea of Kaluza-Klein reduction to the case of compactification of a higher-dimensional theory on the  $n$ -dimensional torus,  $T^n$ . The lower-dimensional theory will include the gauge fields of  $U(1)^n$ , associated with the  $U(1)^n$  isometry group of  $T^n$ .

The next significant development in the idea of obtaining gauge fields from the isometry group of a compact internal space was the proposal by Pauli, in 1953, that one might obtain the non-abelian gauge fields of the group  $SO(3)$  by compactifying on the 2-sphere [4]. He had in mind the idea of reducing six-dimensional general relativity to four dimensions.

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<sup>1</sup>However, at the same time Klein took the retrograde step of setting the scalar field to a constant, in an attempt to obtain pure Einstein-Maxwell theory in four dimensions. (See, for example, [3].) The problem with doing this is that it contradicts the scalar field's own equation of motion, which has a source term built from the square of the Maxwell field strength. This is the *ur* example of an inconsistent reduction ansatz, although the remedy in this case is simple; do not set the scalar to a constant.

More generally, one could envisage obtaining the Yang-Mills gauge group  $G$  by means of a compactification on the coset space  $G/H$ . Following the discussion in [5], we shall refer to any such reduction as a *Pauli Reduction*. Although Pauli proposed reductions of this kind, he also recognised that there would be difficulties in realising them in practice. These difficulties are associated with what is nowadays called an “inconsistency of the reduction ansatz.” For example, if one attempts to implement Pauli’s original idea of reducing six-dimensional gravity on  $S^2$ , one finds that one cannot obtain consistent four-dimensional equations of motion by substituting the ansatz into the six-dimensional Einstein equations.

The definition of a “consistent reduction ansatz” is one that can be substituted into the higher-dimensional equations of motion, yielding a consistent system of equations for the lower-dimensional system, with the consequent property that any solution of the lower-dimensional equations gives rise to a solution of the original higher-dimensional equations. The obstacle to achieving such a consistent reduction, in a situation of the type envisaged by Pauli, is that one will not in general be able to “factor out” the dependence on the coordinates of the internal reduction space when one substitutes into the higher-dimensional equations of motion.

Another way of characterising the inconsistency is as follows. One could always choose to perform an expansion of all the higher-dimensional fields in terms of complete sets of appropriate harmonic fields on the internal space. This would just give a generalised Fourier expansion of the original theory, resulting in a theory containing infinite sets of lower-dimensional fields. This would necessarily be consistent, in the sense defined above. The lower-dimensional fields would essentially comprise a finite set of massless fields, coupled to infinite towers of massive fields, with masses unbounded above. In the type of reduction envisaged by Pauli, the reduction ansatz would only retain the massless fields (or, at least, some finite subset, including the metric and the Yang-Mills gauge fields of the isometry group of the internal space). The crucial question, therefore, is whether setting the remainder of the infinite towers of fields to zero is consistent with their own equations of motion. The answer, in the case of a “Pauli reduction” of a generic higher-dimensional theory, is an unequivocal “No.” One cannot, in general, consistently set the infinite towers to zero, and so the Pauli reduction will in general be inconsistent.<sup>2</sup>

The inconsistency of a generic Pauli reduction can be attributed to the fact that the

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<sup>2</sup>Note that we are concerned here with exact mathematical statements. We are not interested here in the question of whether the massive towers can be neglected, or “integrated out,” in some physically-motivated low-energy approximation.

internal-space harmonics associated with the infinite massive towers are generated by products of the internal-space harmonics of the retained “massless” sector. This implies, in view of the non-linear nature of the theory one is reducing, that there will be source terms in the equations of motion for the massive towers, comprised of products of the purely massless fields. Thus the massive towers cannot be consistently set to zero. If one starts at the outset with an ansatz involving only the massless sector, the inconsistency will reveal itself as an inability to extract purely lower-dimensional equations of motion with the coordinate dependence of the internal space factored out. Instead, one obtains an over-constrained system of equations with no sensible lower-dimensional content.

The inconsistencies described above are of course avoided in the Kaluza-Klein reductions on  $S^1$  or  $T^n$ . An easy way to see this is that the massive fields in the Kaluza-Klein towers are all *charged* under the  $U(1)^n$  gauge group, whilst the massless modes are uncharged (i.e. they are independent of the toroidal coordinates). Since products of the uncharged massless fields can never carry charge, it is impossible for them to act as sources for the massive charged fields.

Another situation where one is guaranteed a consistent reduction is in the scheme introduced by DeWitt in 1963, in which a reduction is performed on the compact group manifold  $G$  [6]. The bi-invariant metric on  $G$  has the isometry group  $G_L \times G_R$ , denoting the independent left and right actions of  $G$  on the group manifold. Thus in a generalised Fourier expansion one would obtain massless Yang-Mills gauge fields of the full  $G_L \times G_R$  isometry group in the lower dimension. In the DeWitt reduction, however, only the gauge fields of one copy, say  $G_L$ , are retained. In fact, the full set of fields retained in the DeWitt reduction comprise precisely *all* those which are *singlets* under  $G_R$ . Since  $G_R$  acts transitively on  $G$ , this means that the retained fields will be finite in number. The DeWitt reduction will necessarily be a consistent one, for a group theoretic reason analogous to the one discussed above for Kaluza-Klein  $T^n$  reductions. Namely, since all the retained fields are singlets under  $G_R$ , it is impossible for products of these fields to act as sources for the massive towers that are set to zero, since by definition all the fields in those towers are non-singlets under  $G_R$ .

It is almost a truism in any of the consistent reductions that has such a group-theoretic explanation that one could equally well choose to substitute the reduction ansatz into the higher-dimensional action, integrate over the internal group manifold, and thereby obtain a lower-dimensional action whose equations of motion coincide with those one would obtain by instead substituting the ansatz into the higher-dimensional equations of motion. It

is presumably for this reason that a “folklore” belief has arisen that in any consistent reduction, whether or not the consistency has a group-theoretic explanation, one should be able equivalently to substitute the ansatz into either the higher-dimensional equations of motion or instead into the higher-dimensional action.

In fact in the case of consistent Pauli reductions, where no group-theoretic explanation for the consistency exists, it is not in general true that one obtains the same lower-dimensional theory by substituting the reduction ansatz into either the equations of motion or the action of the higher-dimensional theory. Indeed, in many cases the basic premise underlying the question is false; one may well not even be able to write the ansatz in a form that would allow it to be substituted into the action, and so the question of the equivalence or otherwise of the two procedures does not even arise. Such a situation typically occurs if the ansatz can be expressed only at the level of the field strengths of the higher-dimensional theory, but not in terms of the underlying gauge potentials that are the fundamental fields in the action. Even if one looks at examples where the ansatz can be given for the gauge potentials themselves, it can still be the case that substitution into the higher-dimensional action fails to give the correct lower-dimensional action.<sup>3</sup>

The purpose of the present paper is to explore in detailed examples this question of the equivalence, or otherwise, of substituting a consistent Pauli reduction ansatz into the higher-dimensional equations of motion or action. Examples of consistent Pauli reductions are very few and far between, and in fact most of the known cases are associated with supergravity theories. The reason for the paucity of examples can be understood using an argument given in [7]. If the consistent Pauli reduction of a particular theory on  $S^n$  is possible, then by definition it will yield a lower-dimensional theory with Yang-Mills fields gauging the group  $SO(n+1)$ . By turning off the gauge coupling, which amounts to sending the radius of the sphere to infinity, the lower-dimensional theory will become the one that would instead result from a Kaluza-Klein reduction on  $T^n$ . From this standpoint, the  $S^n$ -reduced theory can then be viewed as a gauging of an  $SO(n+1)$  global symmetry of the  $T^n$ -reduced theory. A *necessary* condition for the consistency of the  $S^n$  Pauli reduction is therefore that if the higher-dimensional theory is instead reduced on  $T^n$ , it must give rise to a lower-dimensional theory with a global symmetry group  $G$  that is *at least* large enough to contain  $SO(n+1)$ . Now, a generic theory reduced on  $T^n$  will have only a  $GL(n, \mathbb{R})$

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<sup>3</sup>An unambiguous definition of the “correct” lower-dimensional equations is the ones such that, for any solution of these equations, the reduction ansatz yields a solution of the higher-dimensional equations of motion.

global symmetry, and this has as maximal compact subgroup  $SO(n)$ , which of course does not contain  $SO(n+1)$ . The upshot is that only in the case of a higher-dimensional theory whose  $T^n$  reduction gives a theory with a *global symmetry enhancement*, to a group  $G$  that contains  $SO(n+1)$ , can there be any chance that a Pauli reduction on  $S^n$  is possible. In practice, most of the examples where such a global symmetry enhancement occurs are found in supergravity, such as the  $T^n$  reductions of  $D = 11$  supergravity, which yield maximal supergravities in dimension  $D = 11 - n$  with  $E_n$  global symmetry.

Three important examples of Pauli reductions are the  $S^4$  and  $S^7$  reductions of eleven-dimensional supergravity, and the  $S^5$  reduction of type IIB supergravity. Each of these satisfies the necessary condition discussed above; the  $S^n$  isometry groups,  $T^n$ -reduced global symmetry groups and maximal compact subgroups are as follows:

$$\begin{aligned}
S^4 : \quad & SO(5) \subset SO(5) \subset SL(5, \mathbb{R}), \\
S^7 : \quad & SO(8) \subset SU(8) \subset E_7, \\
S^5 : \quad & SO(6) \subset USp(8) \subset E_6.
\end{aligned} \tag{1.1}$$

The consistency of the  $S^4$  Pauli reduction was demonstrated in [8] and the consistency of the  $S^7$  Pauli reduction was demonstrated in [9]. The consistency of the  $S^5$  Pauli reduction of type IIB supergravity has never been demonstrated, although various non-trivial sub-cases have been explicitly shown to be consistent (See, for example, [10].)

The three Pauli reductions mentioned above are all of considerable complexity. However, it turns out that we can consider further (consistent) truncations within these examples, where only a small number of lower-dimensional fields are retained, which are nevertheless still of a sufficiently non-trivial nature to be able to establish the results that we wish to demonstrate. Specifically, we shall first consider a truncation of the  $S^7$  reduction of eleven-dimensional supergravity in which only the four-dimensional metric and a single scalar field are retained. This will enable us to provide an explicit example in which it is impossible to give a reduction ansatz that is expressed in terms of the fundamental fields of the higher-dimensional action, and so it is not possible even in principle to discuss whether the lower-dimensional theory obtained by substituting the ansatz into the action agrees with that obtained by substituting instead into the equations of motion.

As a second example, we shall elaborate the first case a little, by including also a pseudo-scalar field in the truncated Pauli reduction. Finally, for a third example, we shall consider a consistent truncation of the  $S^4$  Pauli reduction of eleven-dimensional supergravity. In this truncation, the only seven-dimensional fields that are retained are the metric, a scalar,

and a 3-form potential that satisfies a first-order “odd-dimensional self-duality” equation in  $D = 7$ . In this case, we shall find that it *is* possible to express the ansatz in terms of the fundamental fields of an eleven-dimensional action, but nevertheless one gets the incorrect seven-dimensional equations of motion if one substitutes this ansatz into the action. By contrast, in this as in all the examples of consistent Pauli reductions, one does obtain the correct lower-dimensional equations of motion by substituting the ansatz into the higher-dimensional equations of motion.

## 2 Consistent Reduction on $S^7$

### 2.1 Truncation to gravity, scalar and pseudoscalar

The most complete presentation of the ansatz for the reduction of the field equations of eleven-dimensional supergravity on  $S^7$ , yielding the field equations of maximal gauged  $SO(8)$  supergravity in four dimensions, was given in [9]. This contained an essentially complete proof of the consistency of the reduction, the complete ansatz for the reduction of the fermions and the metric reduction, and part of the reduction ansatz for the 4-form field strength. Even had the 4-form ansatz been complete, the complexity of the reduction procedure would make it very difficult to proceed with an explicit discussion of substitution into the eleven-dimensional action. A very much simpler situation was considered in [11], where the consistent reduction on  $S^7$  that yields the truncation to  $N = 4$  gauged  $SO(4)$  supergravity was considered. A complete and fully explicit ansatz for the reduction of the metric and 4-form of eleven-dimensional supergravity was obtained, and it was demonstrated that it consistently gave the required four-dimensional equations of motion when substituted into the eleven-dimensional equations of motion. These equations follow from the eleven-dimensional Lagrangian

$$\mathcal{L}_{11} = \hat{R} * \mathbb{1} - \frac{1}{2} \hat{*} \hat{F}_{(4)} \wedge \hat{F}_4 + \frac{1}{6} \hat{F}_{(4)} \wedge \hat{F}_{(4)} \wedge \hat{A}_{(3)}, \quad (2.1)$$

where  $\hat{F}_{(4)} \equiv d\hat{A}_{(3)}$ . The bosonic fields in the four-dimensional gauged  $SO(4)$  supergravity comprise the metric, a scalar and a pseudoscalar, and the  $SO(4)$  Yang-Mills fields.

In order to simplify the discussion still further, while still retaining a highly non-trivial reduction scheme, we shall begin by setting the  $SO(4)$  gauge fields to zero. It is, of course, essential that this truncation is itself a consistent one. This follows from a simple group-theoretic argument; since the metric, scalar and pseudoscalar are  $SO(4)$  singlets, it is necessarily consistent to set the  $SO(4)$  gauge fields to zero. The reduction ansatz for the

remaining fields then becomes (see [11])

$$ds_{11}^2 = \Delta^{2/3} ds_4^2 + 4g^{-2} \Delta^{2/3} \left( d\xi^2 + \frac{c^2}{\Omega} d\Sigma_3^2 + \frac{s^2}{\tilde{\Omega}} d\tilde{\Sigma}_3^2 \right), \quad (2.2)$$

$$\hat{F}_{(4)} = -gU\epsilon_{(4)} - 2scg^{-1} *d\phi \wedge d\xi + 2scg^{-1} \chi X^4 *d\chi \wedge d\xi + \hat{F}'_{(4)}, \quad (2.3)$$

where  $\epsilon_{(4)}$  is the volume form of the four-dimensional metric  $ds_4^2$ ,

$$\begin{aligned} X &= e^{\frac{1}{2}\phi}, & \tilde{X} &= qX^{-1}, & q^2 &= 1 + \chi^2 X^4, \\ \Omega &= c^2 X^2 + s^2, & \tilde{\Omega} &= s^2 \tilde{X}^2 + c^2, & \Delta^2 &= \Omega \tilde{\Omega}, & U &= c^2 X^2 + s^2 \tilde{X}^2 + 2, \\ c &= \cos \xi, & s &= \sin \xi, \end{aligned} \quad (2.4)$$

and  $d\Sigma_3^2$  and  $d\tilde{\Sigma}_3^2$  are metrics on two unit 3-spheres. The extra term  $\hat{F}'_{(4)}$  in the expression for  $\hat{F}_{(4)}$  is given by

$$\hat{F}'_{(4)} = d\hat{A}'_{(3)}, \quad \hat{A}'_{(3)} = f\epsilon_{(3)} + \tilde{f}\tilde{\epsilon}_{(3)}, \quad (2.5)$$

where  $\epsilon_{(3)}$  and  $\tilde{\epsilon}_{(3)}$  are the volume forms for the two 3-sphere metrics, and

$$f = \frac{c^4 \chi X^2}{g^3 \Omega}, \quad \tilde{f} = -\frac{s^4 \chi X^2}{g^3 \tilde{\Omega}}. \quad (2.6)$$

It is worth remarking that the metric on the unit round 7-sphere is given by

$$d\Sigma_7^2 = d\xi^2 + c^2 d\Sigma_3^2 + s^2 d\tilde{\Sigma}_3^2, \quad (2.7)$$

which means that  $\xi$  is a ‘‘latitude coordinate,’’ with  $0 \leq \xi \leq \frac{1}{2}\pi$ . The surfaces of constant  $\xi$  are  $S^3 \times S^3$  with one  $S^3$  shrinking to zero size at  $\xi = 0$ , whilst the other  $S^3$  shrinks to zero size at  $\xi = \frac{1}{2}\pi$ . The scalar  $\phi$  and pseudoscalar  $\chi$  thus parameterise very specific deformations of the 7-sphere, which maintain the homogeneity of the  $S^3 \times S^3$  surfaces, in the reduction ansatz (2.2).

It is easy to see that the Bianchi identity  $d\hat{F}_{(4)} = 0$  is not satisfied identically; rather, it implies a specific combination of the scalar and pseudoscalar equations of motion. Substitution of the ansatze (2.2) and (2.3) into the full set of  $D = 11$  equations of motion following from (2.1), namely

$$\hat{R}_{MN} = \frac{1}{12}(\hat{F}_{MN}^2 - \frac{1}{12}\hat{F}_{(4)}^2 \hat{g}_{MN}), \quad (2.8)$$

$$d*\hat{F}_{(4)} = \frac{1}{2}\hat{F}_{(4)} \wedge \hat{F}_{(4)}, \quad (2.9)$$

and Bianchi identity

$$d\hat{F}_{(4)} = 0, \quad (2.10)$$



leads consistently to a set of four-dimensional equations of motion, which can themselves be derived from the Lagrangian

$$\mathcal{L}_4 = R*\mathbf{1} - \frac{1}{2}*d\phi \wedge d\phi - \frac{1}{2}e^{2\phi} *d\chi \wedge d\chi - V*\mathbf{1}, \quad (2.11)$$

where the scalar potential  $V$  is given by

$$V = -g^2(4 + 2 \cosh \phi + \chi^2 e^\phi). \quad (2.12)$$

## 2.2 Truncation to the gravity plus scalar subsystem

We can make a further consistent truncation of the system described above, in which the pseudoscalar  $\chi$  is set to zero. The ansatz for the metric and 4-form then reduce to

$$d\hat{s}_{11}^2 = \Delta^{2/3} ds_4^2 + 4g^{-2} \Delta^{2/3} d\xi^2 + 4g^{-2} \Delta^{-1/3} \left( c^2 e^{-\frac{1}{2}\phi} d\Sigma_3^2 + s^2 \epsilon^{\frac{1}{2}\phi} d\tilde{\Sigma}_3^2 \right), \quad (2.13)$$

$$\hat{F}_{(4)} = -gU\epsilon_{(4)} - 2scg^{-1} *d\phi \wedge d\xi, \quad (2.14)$$

where now  $\Delta$  and  $U$  have become

$$\Delta = c^2 X + s^2 X^{-1}, \quad U = c^2 X^2 + s^2 X^{-2} + 2, \quad (2.15)$$

with again  $X = e^{\phi/2}$ .

We see that the Bianchi identity (2.10) implies the four-dimensional equation of motion for the scalar field  $\phi$ , namely

$$d*d\phi = g^2(e^\phi - e^{-\phi})\epsilon_{(4)}. \quad (2.16)$$

Since the 4-form field strength  $\hat{F}_{(4)}$  in (2.14) is not closed *identically*, but only modulo the use of the four-dimensional scalar field equation (2.16), it is clearly impossible to re-express the 4-form ansatz in terms of its 3-form potential, since one would need to be able to write  $\hat{A}_{(3)}$  off-shell. Thus we have exhibited, in this relatively simple truncation of the  $S^7$  reduction ansatz, the fact that it cannot be written in such a way that it can be substituted into the eleven-dimensional supergravity action.

## 2.3 Dualisation in a toy model

It is nonetheless interesting to observe that this particular highly-truncated reduction ansatz can in fact be reinterpreted as an ansatz for a “toy” eleven-dimensional theory with a 7-form field strength rather than a 4-form field strength, for which one can then re-express the reduction at the level of a fundamental 6-form potential. The toy theory, it must be

emphasised, is *not* a dual formulation of eleven-dimensional supergravity; it is well known that it is *impossible* to rewrite eleven-dimensional supergravity in terms of a dual formulation that involves only a 6-form potential with no 3-form potential. The obstacle to any such rewriting is the Chern-Simons term  $\hat{F}_{(4)} \wedge \hat{F}_{(4)} \wedge \hat{A}_{(3)}$ , for which there is no way to avoid having  $\hat{A}_{(3)}$  as a fundamental field.

The reason why we can give a reformulation in terms of a toy dual theory in the present case of the truncation that retains just gravity and the scalar field in four dimensions is that the Chern-Simons term in the field equation (2.9) for  $\hat{F}_{(4)}$  vanishes for this ansatz, and so we are left with just  $d\hat{*}\hat{F}_{(4)} = 0$  from this equation of motion. Thus in this truncated example we could take the “original” eleven-dimensional theory to be described by the simpler Lagrangian

$$\mathcal{L}_{11} = \hat{R} \hat{*}\mathbb{1} - \frac{1}{2} \hat{*}\hat{F}_{(4)} \wedge \hat{F}_{(4)}. \quad (2.17)$$

From the expressions in [11], the dual of  $\hat{F}_{(4)}$  in our present truncation is given by

$$\hat{*}\hat{F}_{(4)} = \frac{2s^3 c^3 U}{g^6 \Delta^2 U} d\xi \wedge \epsilon_{(3)} \wedge \tilde{\epsilon}_{(3)} - \frac{s^4 c^4}{g^6 \Delta^2} d\phi \wedge \epsilon_{(3)} \wedge \tilde{\epsilon}_{(3)}. \quad (2.18)$$

A simple calculation shows that the exterior derivative of  $\hat{*}\hat{F}_{(4)}$  vanishes identically in this truncated situation. This means that it is natural to introduce the dual 7-form field strength  $\hat{F}_{(7)} = \hat{*}\hat{F}_{(4)}$  here, since then we can integrate the associated ansatz for  $\hat{F}_{(7)}$ . We find that  $\hat{F}_{(7)} = d\hat{A}_{(6)}$  where we can write

$$\hat{A}_6 = \frac{s^4}{g^6} \left( \frac{c^2 X^{-1}}{\Delta} + \frac{1}{2} \right) \epsilon_{(3)} \wedge \tilde{\epsilon}_{(3)}. \quad (2.19)$$

In this simplified situation, therefore, we can consider a dualised theory where the 4-form field strength is replaced by a 7-form field strength, and write the eleven-dimensional Lagrangian

$$\mathcal{L}_{11} = \hat{R} \hat{*}\mathbb{1} - \frac{1}{2} \hat{*}\hat{F}_{(7)} \wedge \hat{F}_{(7)}, \quad (2.20)$$

where  $\hat{F}_{(7)} \equiv d\hat{A}_{(6)}$ . This may also be written in terms of the Lagrangian density

$$\mathcal{L}_{11} = \sqrt{-\hat{g}} \left( \hat{R} - \frac{1}{2 \cdot 7!} \hat{F}_{(7)}^2 \right). \quad (2.21)$$

Since we have an explicit ansatz for the fundamental 6-form potential  $\hat{A}_{(6)}$ , we may now investigate what happens if we substitute the ansatz into the dualised 11-dimensional action following from (2.20) or (2.21).

Substituting the ansatz (2.13) and (2.19) into (2.21), we obtain results as follows. Firstly, we find that

$$\sqrt{-\hat{g}} \hat{R} = 2s^3 c^3 g^{-7} Y \sqrt{-g} R - \frac{2}{3} s^3 c^3 g^{-7} Y \sqrt{-g} \left( 1 - \frac{2s^2}{X\Delta} \right) \square\phi \quad (2.22)$$

$$\begin{aligned}
& -s^3 c^3 g^{-7} Y \sqrt{-g} \left( 1 + \frac{s^2}{3X\Delta} - \frac{s^4}{3X^2\Delta^2} \right) (\partial\phi)^2 \\
& + s c g^{-5} Y \sqrt{-g} \left( 3s^2 c^2 (X^2 + X^{-2}) + \frac{2}{3} s^2 (1 + 21c^2) - \frac{s^2(3 - 4c^2)}{3X\Delta} + \frac{s^4}{3X^2\Delta^2} \right),
\end{aligned}$$

where  $Y$  denotes the product of the square roots of the determinants of the two 3-sphere metrics  $d\Sigma_3^2$  and  $d\tilde{\Sigma}_3^2$ .

From the field strength, we get

$$-\frac{1}{2 \cdot 7!} \sqrt{-\hat{g}} \hat{F}_{(7)}^2 = -s^5 c^5 g^{-7} Y \sqrt{-g} (\partial\phi)^2 - s^3 c^3 Y \sqrt{-g} \frac{U^2}{g^5 \Delta^2}. \quad (2.23)$$

It should be noted that unlike in a standard toroidal reduction, we cannot drop the  $\square\phi$  term in (2.22) as a total derivative, because there is  $\phi$  dependence in the prefactor. We can, however, perform an integration by parts in the action, and this improves the appearance of the scalar kinetic terms considerably. However, we wish to keep track of the  $\xi$  dependence of the integrand in the action, and so for now we shall integrate only over the four-dimensional spacetime. Thus we find

$$\int d^4 x \mathcal{L}_{11} = 2s^3 c^3 g^{-7} Y \int d^4 x \sqrt{-g} (R - \frac{1}{2} (\partial\phi)^2) + g^{-5} Y \int d^4 x \sqrt{-g} Q, \quad (2.24)$$

where

$$Q = -\frac{4}{3} s^3 c (1 - 9c^2) + 2s^3 c^3 (X^2 + X^{-2}) - \frac{8s^3 c^3}{3X\Delta} + \frac{4s^5 c}{3X^2\Delta^2}. \quad (2.25)$$

At this stage, we see that although the integrand of the terms in the action involving the lower-dimensional Einstein-Hilbert and scalar kinetic terms has a uniform  $\xi$  dependence, namely a prefactor  $s^3 c^3$ , the same is not true of the terms involving the scalar potential. However, although this part of the integrand, given in (2.25), has an extremely complicated  $\xi$  dependence, it turns out that after integrating over  $\xi$  all the terms in the action assemble into the hoped-for four-dimensional action, namely

$$\int d^{11} x \mathcal{L}_{11} = \frac{128\pi^4}{3g^7} \int d^4 x \sqrt{-g} \left[ R - \frac{1}{2} (\partial\phi)^2 + g^2 (4 + e^\phi + e^{-\phi}) \right]. \quad (2.26)$$

This can be seen to be equivalent to the four-dimensional Lagrangian (2.11), after truncating out the pseudoscalar  $\chi$ , which was itself obtained by requiring that it reproduce the consistently-derived equations of motion.

The upshot of this discussion is that in the consistent truncation where only the four-dimensional metric and scalar are retained, one can in fact substitute the eleven-dimensional reduction ansatz into the ‘‘toy’’ dualised eleven-dimensional action given by (2.20) or (2.21), and obtain the correct four-dimensional action. The way in which this works is rather non-trivial, requiring cancels and ‘‘conspiracies’’ between terms in order to give the correct four-dimensional action. It is also noteworthy that it is only after integrating over the internal  $S^7$

directions that one obtains a simple result; prior to this integration, the eleven-dimensional integrand is not of the form of a single overall  $S^7$ -dependent function multiplying a four-dimensional Lagrangian.

## 2.4 The gravity plus scalar and pseudoscalar truncation

If we now go back to the somewhat larger reduction given in (2.2) and (2.3), where the pseudoscalar is included as well as the scalar and the metric, we can see that neither the Bianchi identity nor the field equation for the 4-form  $\hat{F}_{(4)}$  is identically satisfied. (The Chern-Simons term in eleven dimensions does now give a contribution to the equation of motion for  $\hat{F}_{(4)}$ .) Each of the eleven-dimensional Bianchi identity and equation of motion implies that certain combinations of the four-dimensional equations of motion must hold. Thus one could not even construct a “toy model” for this larger truncation, analogous to the one in described in section 2.3, within which one could address the possibility of substituting the ansatz into the action.

## 3 $S^4$ Reduction of $D = 11$ Supergravity

### 3.1 Reduction of the field equations

The complete reduction of  $D = 11$  supergravity on  $S^4$ , yielding  $\mathcal{N} = 4$   $SO(5)$  gauged supergravity in  $D = 7$ , was obtained in [8]. The simpler case of the reduction to  $\mathcal{N} = 2$   $SU(2)$  gauged supergravity in  $D = 11$  was obtained in [12].

Here, we shall discuss a subset of the  $\mathcal{N} = 2$  reduction, in which we consistently truncate out the  $SU(2)$  Yang-Mills fields, leaving just the metric, a dilaton and a 3-form gauge potential in  $D = 7$ .

After rescaling the gauge-coupling constant  $g$  in [12] by sending  $g \rightarrow g/\sqrt{2}$  for convenience, the reduction ansatz for the remaining fields can be written as

$$d\hat{s}_{11}^2 = \Delta^{1/3} ds_7^2 + 4g^{-2} X^3 \Delta^{1/3} d\xi^2 + g^{-2} \Delta^{-2/3} X^{-1} c^2 \sigma_i^2, \quad (3.1)$$

$$\hat{A}_{(3)} = s A_{(3)} + f \epsilon_{(3)}, \quad (3.2)$$

where  $\sigma_i$  are left-invariant 1-forms of  $SU(2)$ ,  $X = \exp(-\phi/\sqrt{10})$  where  $\phi$  is the dilaton field, and  $\epsilon_{(3)} = \sigma_1 \wedge \sigma_2 \wedge \sigma_3$ . The functions  $\Delta$  and  $f$  are given by

$$\Delta = X^{-4} s^2 + X c^2, \quad (3.3)$$

$$f = g^{-3} (2s + s c^2 \Delta^{-1} X^{-4}), \quad (3.4)$$

and the symbols  $s$  and  $c$  are shorthand for

$$s = \sin \xi, \quad c = \cos \xi. \quad (3.5)$$

In [12], the ansatz for the antisymmetric tensor was supplemented by the requirement that the seven-dimensional 3-form satisfy the first-order condition

$$F_{(4)} = \frac{1}{2}gX^{-4} *A_{(3)}. \quad (3.6)$$

This could be viewed as part of the specification of the consistent reduction ansatz. An alternative approach, as we shall discuss below, involves making a modification to the anti-symmetric tensor ansatz so that it is expressed directly on the field strength  $\hat{F}_{(4)}$  but can no longer be given for the potential  $\hat{A}_{(3)}$ . One or other of these approaches is inevitable, because the 3-form in seven dimensions should satisfy a first-order ‘‘odd-dimensional self-duality’’ equation, and it is not possible to derive a first-order equation for  $A_{(3)}$  by substituting an ansatz for  $\hat{A}_{(3)}$  into the second-order eleven-dimensional equation of motion for  $\hat{A}_{(3)}$ .

First, we note that the required  $D = 7$  equations of motion for the truncated gauged supergravity are

$$X^4 *F_{(4)} = -\frac{1}{2}g A_{(3)}, \quad (3.7)$$

$$d(X^{-1} *dX) = \frac{1}{5}X^4 *F_{(4)} \wedge F_{(4)} - \frac{1}{10}g^2(X^{-8} + 2X^2 - 3X^{-3}) \epsilon_{(7)}, \quad (3.8)$$

$$R_{\mu\nu} = 5X^{-2} \partial_\mu X \partial_\nu X + \frac{1}{12}(F_{\mu\nu}^2 - \frac{3}{20}F_{(4)}^2) g_{\mu\nu}, \quad (3.9)$$

where  $F_{(4)} = dA_{(3)}$ . Note that (3.7) is the ‘‘first-order self-duality’’ equation for the 3-form field. By substituting (3.1) and (3.2) into (2.8) and (2.9), we derive the scalar field equation (3.8) and the Einstein equation (3.9).

As mentioned above, the first-order equation (3.7) is not itself *derivable* purely by using the ansatz (3.2). To see this, we note that (3.2) and (3.1) imply

$$\begin{aligned} \hat{F}_{(4)} &= sF_{(4)} + cd\xi \wedge A_{(3)} + \frac{\partial f}{\partial \xi} d\xi \wedge \epsilon_{(3)} + \frac{\partial f}{\partial X} dX \wedge \epsilon_{(3)}, \\ &= sF_{(4)} + cd\xi \wedge A_{(3)} - 5g^{-3}sc^4\Delta^{-2}X^{-4}dX \wedge \epsilon_{(3)} \\ &\quad + g^{-3}c^3\Delta^{-2}X^{-8}[X^5(1 + 2X^5)c^2 + (4X^5 - 1)s^2]d\xi \wedge \epsilon_{(3)}, \\ \hat{*}\hat{F}_{(4)} &= 2g^{-4}sc^3\Delta^{-1} *F_{(4)} \wedge d\xi \wedge \epsilon_{(3)} - \frac{1}{2}g^{-2}c^4\Delta^{-1}X^{-3} *A_{(3)} \wedge \epsilon_{(3)} \\ &\quad + \frac{1}{2}g^4c^{-3}\Delta^2 \frac{\partial f}{\partial \xi} \epsilon_{(7)} - 2g^2\Delta^2X^3c^{-3} \frac{\partial f}{\partial X} *dX \wedge d\xi. \end{aligned} \quad (3.10)$$

Substituting into the equation of motion for  $\hat{F}_{(4)}$  (2.9), we obtain the seven-dimensional equations:

$$d(X^{-4} *A_{(3)}) = 0,$$

$$\begin{aligned}
dX \wedge (F_{(4)} - \frac{1}{2}gX^{-4} *A_{(3)}) &= 0, \\
5dX \wedge (*F_{(4)} + \frac{1}{2}gX^{-4} A_3) + gX^{-3} (1 - X^5)(F_{(4)} - \frac{1}{2}gX^{-4} *A_{(3)}) &= 0, \\
2d(X^4 *F_{(4)}) + 2g^2 X *A_{(3)} - g(4X^5 - 1)F_{(4)} &= 0, \\
d(X^{-1} *dX) + \frac{1}{10}g^2(X^{-8} + 2X^2 - 3X^{-3})\epsilon_{(7)} + \frac{1}{10}gF_4 \wedge A_{(3)} &= 0. \quad (3.11)
\end{aligned}$$

(These equations arise from equating all the independent  $\xi$ -dependent structures to zero.) The first four equations in (3.11) are consistent with the first-order equation (3.7) (note that the dual of the first-order equation gives  $X^4 F_{(4)} = +\frac{1}{2}g *A_{(3)}$ ). However, the first four equations in (3.11) do not actually *imply* the first-order equation (3.7). Finally, we can see that the last equation in (3.11) becomes, after the use of the first-order equation, equivalent to the scalar equation (3.8). For this reason, the ansatz in [12] was supplemented by the condition (3.6). This was sufficient to provide a consistent embedding of solutions of the seven-dimensional supergravity equations in eleven dimensions.

We can in fact write a reduction ansatz that genuinely allows one to *derive* the seven-dimensional equations of motion, including the first-order equation for  $A_{(3)}$ . To achieve this, we must instead write an ansatz for  $\hat{F}_{(4)}$  that *cannot* be written identically as  $d\hat{A}_{(3)}$ , i.e. an ansatz that does not satisfy the Bianchi identity  $d\hat{F}_{(4)} = 0$  identically. We do this by replacing  $\hat{F}_{(4)}$  in (3.10) by

$$\begin{aligned}
\hat{F}_{(4)} &= \frac{1}{2}gsX^{-4} *A_{(3)} + cd\xi \wedge A_{(3)} - 5g^{-3}sc^4\Delta^{-2}X^{-4}dX \wedge \epsilon_{(3)} \\
&\quad + g^{-3}c^3\Delta^{-2}X^{-8}[X^5(1 + 2X^5)c^2 + (4X^5 - 1)s^2]d\xi \wedge \epsilon_{(3)}. \quad (3.12)
\end{aligned}$$

(Note that what we have done here is to use the first-order equation (3.7) to replace  $F_{(4)}$  by  $\frac{1}{2}gX^{-4} *A_{(3)}$  in the first term.) It is easy to see that the Bianchi identity  $d\hat{F}_{(4)} = 0$  indeed now implies (as well as previously-obtained equations) the first-order equation (3.7). In all other respects, the modified ansatz yields the same conclusions as previously, and so we now have a reduction scheme from which one can consistently *derive* the lower-dimensional equations of motion. It should be emphasised, though, that this is a reduction in which the ansatz is now given for  $\hat{F}_{(4)}$ , but it cannot be given for  $\hat{A}_{(3)}$ .

Clearly this modified reduction ansatz cannot be substituted into the usual eleven-dimensional action given by (2.1), precisely because it can no longer be given as an ansatz for the 3-form potential  $\hat{A}_{(3)}$ . One could, of course, substitute the ansatz in its original form (3.1) and (3.2) into the action, but since this ansatz does not even give a *derivation* of the first-order equation for  $A_{(3)}$  when substituted into the eleven-dimensional field equations, one can hardly expect it to work when substituted into the action. Indeed, an explicit calcu-

lation shows that one does not obtain a sensible seven-dimensional action upon integrating over the 4-sphere.

### 3.2 Substitution into the eleven-dimensional action

A new first-order formulation of eleven-dimensional supergravity was constructed in [8]. This entails writing the bosonic Lagrangian as

$$\mathcal{L} = \hat{R}\hat{*}\mathbf{1} + \frac{1}{2}\hat{*}\hat{\mathcal{F}}_{(4)} \wedge \hat{\mathcal{F}}_{(4)} - \hat{*}\hat{\mathcal{F}}_{(4)} \wedge d\hat{A}_{(3)} + \frac{1}{6}d\hat{A}_{(3)} \wedge d\hat{A}_{(3)} \wedge \hat{A}_{(3)}. \quad (3.13)$$

where  $\hat{\mathcal{F}}_{(4)}$  and  $\hat{A}_{(3)}$  are treated as independent fields. The equation of motion for  $\hat{\mathcal{F}}_{(4)}$  implies

$$\hat{\mathcal{F}}_{(4)} = d\hat{A}_{(3)}, \quad (3.14)$$

while the equation of motion for  $\hat{A}_{(3)}$  implies

$$d\hat{*}\mathcal{F}_{(4)} = \frac{1}{2}d\hat{A}_{(3)} \wedge d\hat{A}_{(3)}. \quad (3.15)$$

We could now take the ansatz for  $\hat{A}_{(3)}$  to be given by (3.2), and the ansatz for  $\hat{\mathcal{F}}_{(4)}$  to be given by (3.12), i.e.

$$\begin{aligned} \hat{\mathcal{F}}_{(4)} &= \frac{1}{2}gsX^{-4} *A_{(3)} + cd\xi \wedge A_{(3)} - 5g^{-3}sc^4\Delta^{-2}X^{-4}dX \wedge \epsilon_{(3)} \\ &\quad + g^{-3}c^3\Delta^{-2}X^{-8}[X^5(1 + 2X^5)c^2 + (4X^5 - 1)s^2]d\xi \wedge \epsilon_{(3)}. \end{aligned} \quad (3.16)$$

It is certainly the case that if one substitutes these ansätze into the equations of motion (3.14) and (3.15) following from (3.13), then one derives the correct seven-dimensional equations of motion (including the first-order equation (3.7)).

However, if we substitute the ansätze (3.1), (3.2) and (3.16) into the first-order Lagrangian (3.13), we find (focusing for now on just the terms involving  $A_{(3)}$ ) the terms

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}g^{-2}c^3X^{-4} *A_{(3)} \wedge A_{(3)} \wedge d\xi \wedge \epsilon_{(3)} \\ &\quad + g^{-3}(s^2c^3\Delta^{-1}X^{-4} - \frac{2}{3}s^4c\Delta^{-2}X^{-8})F_{(4)} \wedge A_{(3)} \wedge d\xi \wedge \epsilon_{(3)} + \dots, \end{aligned} \quad (3.17)$$

where the ellipses indicate terms involving fields other than  $A_{(3)}$ . If the integration over  $\xi$  (and, trivially, over the  $S^3$ ) were to yield appropriate quantities, this would have the possibility to yield a seven-dimensional Lagrangian with terms of the desired form

$$\mathcal{L}_7 \sim -\frac{1}{2}X^{-4} *A_{(3)} \wedge A_{(3)} + g^{-1}F_{(4)} \wedge A_{(3)} + \dots. \quad (3.18)$$

The equation of motion from this Lagrangian would produce the desired first-order equation (3.7). What actually happens, however, is that the integration over the prefactor of  $F_{(4)} \wedge A_{(3)}$  in (3.17) yields precisely 0. In fact, therefore, one obtains only

$$\mathcal{L}_7 = -\frac{1}{2}X^{-4} *A_{(3)} \wedge A_{(3)} + \dots . \quad (3.19)$$

Thus we obtain a seven-dimensional Lagrangian that does *not* produce the proper seven-dimensional equations of motion. This is a clear-cut example where we have an explicit ansatz for the fundamental fields in a higher-dimensional action, which nevertheless fails to give the correct lower-dimensional action.

## 4 Discussion and Conclusions

A consistent Pauli reduction is defined to be a dimensional reduction on a coset space  $G/H$  which retains a finite number of lower-dimensional fields including the gauge bosons of the isometry group  $G$ . Substitution of the reduction ansatz into the higher-dimensional equations of motion yields a consistent system of equations of motion for the lower-dimensional fields. Conversely, any solution of the lower-dimensional equations of motion will lift to give a solution of the original higher-dimensional equations of motion.

In this paper, we have used several relatively simple examples to study the question of whether a consistent reduction ansatz of this type can alternatively be substituted into the higher-dimensional action, thereby yielding a lower-dimensional action that correctly reproduces the lower-dimensional equations of motion obtained as described above. This question is non-trivial for a variety of reasons, not the least of which is that there is no known group-theoretical explanation for why consistent Pauli reductions have to work; they are very much the exception rather than the rule, and their success depends on very special features and “conspiracies” that arise in specific higher-dimensional theories. In the absence of a group-invariance argument of the type one has in a more traditional Kaluza-Klein circle reduction or DeWitt group-manifold reduction, there is apparently no *a priori* reason why a substitution of the Pauli reduction ansatz into the higher-dimensional action should produce a correct lower-dimensional action.

In order to render the computations manageable, our examples have been chosen to be further truncations of certain highly non-trivial Pauli reductions, namely the  $S^7$  and  $S^4$  reductions of eleven-dimensional supergravity.

In some cases, there are clear-cut reasons why one cannot even begin to discuss the substitution of the reduction ansatz into the action. Our first example, the truncation of



the  $S^7$  reduction to just the four-dimensional metric and one scalar field, illustrates this rather clearly. The reduction ansatz for the antisymmetric tensor of eleven-dimensional supergravity can only be given for the field strength  $\hat{F}_{(4)}$ , and there is no way to write an off-shell expression for the fundamental gauge potential  $\hat{A}_{(3)}$ . Since there is no formulation of an action for eleven-dimensional supergravity that does not include  $\hat{A}_{(3)}$  as a fundamental field (because of the  $\hat{F}_{(4)} \wedge \hat{F}_{(4)} \wedge \hat{A}_{(3)}$  Chern-Simons term), one therefore has no possibility of substituting an ansatz for this system into the action. Of course, the conclusion applies all the more to the case where one retains the full set of fields of the  $S^7$  Pauli reduction (i.e. the fields of four-dimensional maximal gauged supergravity).

In our example discussed above, where only the four-dimensional metric and a scalar field are retained, it so happens that the eleven-dimensional Chern-Simons term plays no rôle in the equations of motion. In this case, therefore, one can consider a “toy” theory, namely the bosonic sector of eleven-dimensional supergravity with the Chern-Simons term omitted. It is furthermore the case for this gravity plus scalar truncation that although one cannot give a reduction ansatz for the potential  $\hat{A}_{(3)}$  (since the Bianchi identity for  $\hat{F}_{(4)}$  in the field-strength reduction ansatz is not identically satisfied, but instead implies the lower-dimensional scalar equation of motion), the higher-dimensional equation of motion for  $\hat{F}_{(4)}$  is identically satisfied. This means that within the framework of the “toy” theory one can dualise and re-express it in terms of a 6-form potential  $\hat{A}_{(6)}$  instead, and then one can give an explicit reduction ansatz for the dualised gauge potential  $\hat{A}_{(6)}$  itself. Having done this, one can investigate what happens if this is substituted into the dualised “toy” action; it does in fact turn out to yield (in a rather non-trivial way) the correct four-dimensional action.

This trick of passing to a toy theory where a dualisation is possible is very specific to the highly-truncated gravity plus scalar system. We next showed that if one considers instead a slightly less extreme truncation of the  $S^7$  reduction ansatz, where a pseudoscalar is retained as well as the metric and the scalar field, then neither the Bianchi identity nor the field equation for  $\hat{F}_{(4)}$  is identically satisfied, the Chern-Simons term of eleven-dimensional supergravity now plays an essential rôle, and there is no possibility of recasting the ansatz in a manner that would allow it to be substituted into the eleven-dimensional action.

One might have drawn the conclusion from the examples above that at least in those cases where a Pauli reduction ansatz could be given for the fundamental fields appearing in the higher-dimensional action, then substitution of the ansatz into the action would give the correct lower-dimensional action (as in the case of the gravity plus scalar truncation in

the toy theory described above). A third example that we studied appears to invalidate this conclusion. We considered a truncation of the  $S^4$  Pauli reduction of eleven-dimensional supergravity, in which just the metric, a scalar field, and a 3-index antisymmetric tensor  $A_{(3)}$  are retained. The 3-form  $A_{(3)}$  satisfies a first-order “odd-dimensional self-duality equation” in seven dimensions. We showed that the consistent reduction ansatz could be written explicitly in terms of the fundamental fields of a recent first-order reformulation of eleven-dimensional supergravity given in [8]. Nonetheless, upon substitution into the eleven-dimensional action, this failed to produce the correct seven-dimensional action.

The moral to be drawn from these examples seems to be that there is much that remains to be understood about the remarkable examples where consistent Pauli reductions occur. The usual expectation, derived from experience with Kaluza-Klein or DeWitt group-manifold reductions where group-invariance arguments apply, apparently do not extend to the much less well understood Pauli reductions. This leaves many open questions and interesting avenues for further investigation.

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