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**T-duality and U-duality in toroidally-compactified strings**H. Lü<sup>†</sup> and C.N. Pope<sup>‡1</sup>

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ABSTRACT

We address the issue of T-duality and U-duality symmetries in the toroidally-compactified type IIA string. It is customary to take as a starting point the dimensionally-reduced maximal supergravity theories, with certain field strengths dualised such that the classical theory exhibits a global  $E_{n(n)}$  symmetry, where  $n = 11 - D$  in  $D$  dimensions. A discrete subgroup then becomes the conjectured U-duality group. In dimensions  $D \leq 6$ , these necessary dualisations include NS-NS fields, whose potentials, rather than merely their field strengths, appear explicitly in the couplings to the string worldsheet. Thus the usually-stated U-duality symmetries act non-locally on the fundamental fields of perturbative string theory. At least at the perturbative level, it seems to be more appropriate to consider the symmetries of the versions of the lower-dimensional supergravities in which no dualisations of NS-NS fields are required, although dualisations of the R-R fields are permissible since these couple to the string through their field strengths. Taking this viewpoint, the usual T-duality groups survive unscathed, as one would hope since T-duality is a perturbative symmetry, but the U-duality groups are modified in  $D \leq 6$ .

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Eleven-dimensional supergravity has enjoyed a chequered history since its discovery in 1978 [1]. Since it occupies the distinguished position of being the highest-dimensional supergravity theory, it has long been thought likely to play an important rôle in fundamental physics. However, earlier attempts to exploit it as a starting-point for a unified theory foundered for a variety of reasons, including its non-renormalisability and the difficulties of extracting a realistic-looking four-dimensional theory from it. Nonetheless, the developments that were made in the process of trying to achieve a realistic theory established many of the ideas that have subsequently been used extensively in string theory.

One of the striking features of  $D = 11$  supergravity is that its dimensional reduction on a circle gives rise, after truncation to the massless sector, to precisely the low-energy effective limit of the ten-dimensional type IIA string. The true significance of this fact really only emerged with the observation by Witten that the  $D = 11$  theory can be viewed as describing the degrees of freedom of the type IIA string in the limiting regime where its coupling constant tends to infinity [2]. This can be seen in the Kaluza-Klein description of the reduction from  $D = 11$  to  $D = 10$ , in which the Einstein-frame metrics of the two theories are related by

$$ds_{11}^2 = e^{\frac{1}{6}\phi} ds_E^2 + e^{-\frac{4}{3}\phi} (dz + \mathcal{A})^2, \quad (1)$$

where  $\phi$  is the dilaton of the IIA string,  $z$  is the eleventh coordinate which is compactified on a circle, and  $\mathcal{A} = \mathcal{A}_M dx^M$  is the Ramond-Ramond vector potential of the IIA string. Thus we see that in the limit where the string coupling constant  $g = e^{-\phi_0}$  becomes large, the radius  $R_{11} = e^{-2/3\phi_0}$  of the eleventh dimension enlarges such that the description of the IIA string theory effectively becomes eleven dimensional. On the other hand, the eleventh dimension becomes invisible in perturbative string theory, where the coupling  $g$  is small. Of course it is no longer believed that supergravity itself is the fundamental theory in  $D = 11$ ; rather, there is an as-yet undiscovered M-theory whose low-energy limit is described by  $D = 11$  supergravity.

The M-theory conjecture would be further strengthened if one found also that the physical degrees of freedom in  $D = 10$  had their origin in  $D = 11$ . In particular, massless states in  $D = 11$  will give rise not only to massless states but also to massive states upon circular compactification to  $D = 10$ , where the masses of the particles are given by  $n/R_{11}$ . Indeed, there do exist extremal black hole solutions carrying R-R electric charges in type IIA supergravity, which correspond to Kaluza-Klein charges from the eleven-dimensional point of view. In order to discuss the masses of these black holes, for comparison with the massive Kaluza-Klein states of M-theory, it is necessary first to specify the metric with respect to

which the masses are being measured. The metric  $ds_E^2$  in (1) is the Einstein-frame metric in  $D = 10$ , which we shall denote by  $G_E$ . In ten-dimensional string theory, there are two more metrics that are relevant: one is the string metric  $G_S$ , related to the eleven-dimensional metric by  $ds_{11}^2 = e^{\frac{2}{3}\phi} ds_S^2 + e^{-\frac{4}{3}\phi} (dz + \mathcal{A})^2$ , and the other is the same as the eleven-dimensional metric  $G_M$  without any conformal rescaling, *i.e.*  $ds_{11}^2 = ds_M^2 + e^{-\frac{4}{3}\phi} (dz + \mathcal{A})^2$ . Thus we have

$$G_M = e^{\frac{2}{3}\phi} G_S = e^{\frac{1}{6}\phi} G_E . \quad (2)$$

For a generic extremal  $p$ -brane, if the mass per unit  $p$ -volume is  $m_A$  in the metric  $G_A$  and  $m_B$  in the metric  $G_B$ , and if the two metrics are conformally related by  $G_A = \Omega^2 G_B$ , then from simple dimensional analysis we have

$$m_B = \Omega_0^{p+1} m_A , \quad (3)$$

where  $\Omega_0$  is the asymptotic value of  $\Omega$  at infinity. The black hole solutions in the  $D = 10$  type IIA theory have masses given by  $e^{\frac{3}{4}\phi_0} Q$  in the Einstein-frame metric [3], where  $Q$  denotes the electric charge. It follows from (2) and (3) that the masses are  $Q/g$  in the string metric  $G_S$ , and  $Q/R_{11}$  in the eleven-dimensional metric [2]. Thus from the string-metric point of view, the black hole solutions are intrinsically non-perturbative, and thus are absent from the perturbative string spectrum. From the eleven-dimensional point of view, the masses are indeed precisely the same as those of the Kaluza-Klein massive modes coming from M-theory.

There are also other extremal  $p$ -brane solutions in  $D = 10$  type IIA supergravity for other values of  $p$ , whose masses in the Einstein-frame metric can easily be obtained [3]. To be precise, if the coupling of a field strength to the dilaton is given by  $e^{-a\phi} F^2$  in the Einstein-frame metric, then the mass per unit  $p$ -volume for the corresponding electrically-charged solution is given by  $e^{\frac{1}{2}a\phi_0} Q$ , whilst for the magnetically-charged solution it is given by  $e^{-\frac{1}{2}a\phi_0} Q$ . From these masses, we can enumerate the masses in the string metric  $G_S$  and in the eleven-dimensional metric  $G_M$  for all the  $p$ -branes in type IIA supergravity. They are summarised in table 1:

$D = 11$		membrane		5-brane		pp-wave	twisted
$D = 11$ mass		$Q_2$		$Q_5$		$Q_{pp}$	$Q_t$
$D = 10$		string	2-brane	4-brane	5-brane	0-brane	6-brane
$D = 10$ mass	$G_M$	$R_{11}Q_2$	$Q_2$	$R_{11}Q_5$	$Q_5$	$R_{11}^{-1}Q_{pp}$	$R_{11}^2Q_t$
$D = 10$ mass	$G_S$	$Q_2$	$g^{-1}Q_2$	$g^{-1}Q_5$	$g^{-2}Q_5$	$g^{-1}Q_{pp}$	$g^{-1}Q_t$
$D = 10$ mass	$G_E$	$e^{-\frac{1}{2}\phi_0}Q_2$	$e^{\frac{1}{4}\phi_0}Q_2$	$e^{-\frac{1}{4}\phi_0}Q_5$	$e^{\frac{1}{2}\phi_0}Q_5$	$e^{\frac{3}{4}\phi_0}Q_{pp}$	$e^{-\frac{3}{4}\phi_0}Q_t$

Table 1:  $p$ -brane masses in the various  $D = 11$  and  $D = 10$  metrics

Note that although the string tension vanishes in both the Einstein-frame metric  $G_E$  and the eleven-dimensional metric  $G_M$  when the radius  $R_{11}$ , and hence the coupling constant  $g$ , become small, it remains a constant in the string metric  $G_S$ . (In fact, all the  $p$ -brane tensions are independent of the dilaton coupling in their own metric.) In the string metric, for all the  $p$ -branes with R-R charges, and the 5-brane, the mass per unit  $p$ -volume goes to infinity when  $g$  becomes small, implying that they describe non-perturbative degrees of freedom. The masses of the ten-dimensional string and membrane in the eleven-dimensional metric  $G_M$  are  $R_{11}Q_2$  and  $Q_2$ , consistent with the fact that they are obtained from double [4] and vertical [5] dimensional reduction of the eleven-dimensional membrane. The same analysis applies to the ten-dimensional 4-brane and 5-brane. All the  $p$ -branes in type IIA supergravity describe physical degrees of freedom of the string theory, since they are BPS saturated states which will survive quantisation. As we saw, these  $p$ -branes all have natural eleven-dimensional explanations. This implies that the membranes, 5-branes, *etc.* in  $D = 11$  are all part of the physical degrees of freedom of M-theory.

The above discussion relates the type IIA string in ten dimensions with an eleven-dimensional theory. Previous work, notably that by Hull and Townsend, had already provided strong indications that  $D = 11$  supergravity has a rôle to play in describing the properties of type IIA or IIB strings compactified to  $D \leq 9$  dimensions on a torus [6]. In particular, it was conjectured that the long-established Cremmer-Julia global symmetry group  $E_{n(n)}$  of maximal supergravity in  $D = 11 - n$  dimensions [7, 8] survives in the associated toroidally-compactified string theories in the form of a discrete subgroup  $E_{n(n)}(Z)$  that is an exact U-duality symmetry at the full quantum non-perturbative level [6]. This U-duality symmetry is the closure of the  $D_{n-1} \sim O(n-1, n-1)$  subgroup of perturbative T-duality symmetries and an  $SL(2, R)$  subgroup which describes a non-perturbative duality symmetry that interchanges the NS-NS and R-R fields of the theory. By contrast the

T-duality symmetry, which is valid order-by-order in string perturbation theory, acts only within the NS-NS and R-R sectors themselves, but does not mix between them.

One of the issues arising in the discussion of U-duality symmetries in compactified string theories is the question of whether these symmetries are simply artefacts of the compactification procedure, or whether they are providing insights into the symmetries of the ten-dimensional strings themselves. In fact the answer seems to lie somewhere in between. One illustration of this is provided by considering the example of the low-energy limit of the type IIA string. In ten dimensions this supergravity theory has no Cremmer-Julia type symmetry group; however, upon compactification on a circle to  $D = 9$  one finds an  $SL(2, R)$  global symmetry (actually  $GL(2, R)$  [8], including an additional  $R$  symmetry that is not relevant for our immediate discussion). Now  $SL(2, R)$  is a symmetry that is characteristic of the Kaluza-Klein reduction on a 2-torus of more or less any field theory that includes gravity. Thus we see that we actually learn something about ten-dimensional IIA supergravity by compactifying it on a circle, namely that its global symmetries are strongly suggestive of an eleven-dimensional origin.<sup>1</sup> Further toroidal compactification to  $D \leq 8$  dimensions, however, seems to provide us with lesser further insights into the properties of IIA supergravity itself; the successive enlargements of the global symmetry groups as one descends through the dimensions seem to be telling us more about what happens under compactification than about the ten-dimensional theory in its own right.

In the case of the full ten-dimensional type IIA string theory, the above discussion becomes, of course, more complicated. In essence, the main additional ingredient in the argument presented in [6] is that there exist BPS-saturated states which can be expected to be protected from quantum corrections by their supersymmetry. They correspond to  $p$ -brane soliton solutions in the lower-dimensional supergravities that preserve half the supersymmetry. Necessarily, these solutions form multiplets under the Cremmer-Julia  $E_{n(n)}$  symmetry, and after taking the analogue of the Dirac quantisation condition between electric  $p$ -branes and magnetic  $(D - p - 4)$ -branes into account, one can argue that the appropriate  $E_{n(n)}(Z)$  discrete subgroup of the Cremmer-Julia group should also be preserved in the full quantum theory. The electrically-charged NS-NS  $p$ -branes are identified with elementary states in the perturbative string spectrum, while the magnetically-charged ones and those with R-R charges will be non-perturbative states. Thus one conjectures an exact non-perturbative

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<sup>1</sup>There are, of course, other more direct ways of seeing the eleven-dimensional origin of IIA supergravity, but we are interested here in considering the problem from the point of view of what can be learned by toroidal compactification.

U-duality symmetry group  $E_{n(n)}(Z)$  [6]. This suggests that the full toroidally-compactified IIA string theory in  $D \leq 9$  has an eleven-dimensional origin.

As in our previous illustrative discussion of the classical low-energy theories, it is not clear that the argument above acquires significantly greater strength by considering toroidal compactifications to  $D \leq 8$  rather than simply the compactification to  $D = 9$  on a circle. Further toroidal compactifications below  $D = 9$  introduce the additional complexities of lower-dimensional theories while providing lesser further insights into the structure of the IIA string itself. In fact, one might argue that some of the issues arising from the complexities of the lower-dimensional theories may be serving to obscure rather than clarify the relevant discussion. One aspect of this can be seen by considering the Cremmer-Julia global symmetry groups for maximal supergravities in  $D \leq 9$  dimensions. Although these are normally said to be  $E_{n(n)}$  in  $D = 11 - n$ , it should be borne in mind that in  $D \leq 7$  it is only true after dualising certain of the antisymmetric tensor fields.<sup>2</sup> To be precise,  $E_{n(n)}$  is the symmetry group if every field strength in  $D \leq 7$  is dualised whenever this results in a field strength of a smaller degree. For example, the version of  $D = 7$  supergravity that has an  $E_{4(4)} \sim SL(5, R)$  symmetry is the one that comes from the dimensional reduction of  $D = 11$  supergravity (or type IIA supergravity) followed by a dualisation of the 4-form field strength  $F_4$  in  $D = 7$  to give a 3-form field strength  $F_3 = *F_4$ . Strictly speaking, this is a different theory from the one in which the 4-form is not dualised. In particular, the potential  $A_3$  for  $F_4$  in this original version is non-locally related to the potential  $B_2$  for  $F_3 = *F_4$  in the dualised version. Similar considerations apply in lower dimensions too, with a rapidly-growing variety of versions of the supergravity theories, whose potentials are non-locally related.

For many purposes the different versions of maximal supergravity in a given dimension can be viewed as being essentially equivalent, if one does not wish to relate the potentials in one version to those in another. The situation is a little different in string theory, however, since at least at the perturbative level a particular significance is attached to the 2-form potential  $A_{MN}$  and the metric tensor  $g_{MN}$  in ten dimensions, namely that these NS-NS fields themselves are the ones that couple to the worldsheet of the string. By contrast, the  $A_M$  and  $A_{MNP}$  potentials, which are R-R fields, couple to the string only *via* their field strengths. Thus when discussing the T-duality symmetry of the toroidally-compactified IIA string, it is important that whatever other fields may require dualisations in order to implement the

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<sup>2</sup>For  $n \leq 6$ , the  $E_n$  groups in their non-compact versions have the following isomorphisms:  $E_5 \sim SO(5, 5)$ ,  $E_4 \sim SL(5, R)$ ,  $E_3 \sim SL(3, R) \times SL(2, R)$  and  $E_2 \sim GL(2, R)$ .

symmetry, the field strengths associated with the NS-NS fields  $A_{MN}$  and  $g_{MN}$  should be left intact and undualised. (Related issues in the context of the four-dimensional heterotic string have been considered in [9].) It might further be argued that in all the discussions of duality symmetries of the toroidally-compactified IIA string, one should not require any of the field strengths originating from the 2-form potential or the metric in  $D = 10$  to be dualised. In order to explore this further, we shall consider the global symmetries and the T-duality of the toroidal compactifications in detail.

It is convenient to describe the toroidally-compactified type IIA supergravities in a notation derived from the eleven-dimensional supergravity that can be viewed as their progenitor. Thus starting from the metric  $g_{MN}$  and 3-form potential  $A_{MNP}$  in  $D = 11$ , we may consider a set of successive 1-step reductions on circles. In each reduction step from  $(D + 1)$  to  $D$  dimensions, the metric in  $(D + 1)$  will give rise to a metric, a Kaluza-Klein vector potential  $\mathcal{A}_M$ , and a “dilaton” scalar field  $\phi$  in  $D$  dimensions. An  $n$ -index gauge potential in  $(D + 1)$  dimensions will give rise to an  $n$ -index gauge potential and an  $(n - 1)$ -index gauge potential in  $D$  dimensions. Thus it is easy to see that eventually after descending from  $D = 11$  to  $D$  dimensions, we shall have the following structure of  $D$ -dimensional bosonic fields:

$$\begin{aligned} g_{MN} &\longrightarrow g_{MN}, & \vec{\phi}, & \mathcal{A}_1^{(i)}, & \mathcal{A}_0^{(ij)}, \\ A_3 &\longrightarrow A_3, & A_2^{(i)}, & A_1^{(ij)}, & A_0^{(ijk)}, \end{aligned} \quad (4)$$

where the indices  $i, j, k$  run over the  $11 - D$  internal toroidally-compactified dimensions, starting from  $i = 1$  for the step from  $D = 11$  to  $D = 10$ . The subscripts on gauge potentials denote the number of spacetime indices they carry. The potentials  $A_1^{(ij)}$  and  $A_0^{(ijk)}$  are automatically antisymmetric in their internal indices, whereas the 0-form potentials  $\mathcal{A}_0^{(ij)}$  that come from the subsequent dimensional reductions of the Kaluza-Klein vector potentials  $\mathcal{A}_1^{(i)}$  are defined only for  $j > i$ . The quantity  $\vec{\phi}$  denotes the  $(11 - D)$ -vector of dilatonic scalar fields coming from the diagonal components of the internal metric.

The Lagrangian for the bosonic  $D$ -dimensional toroidal compactification of eleven-dimensional supergravity then takes the form [10]

$$\begin{aligned} \mathcal{L} = & eR - \frac{1}{2}e(\partial\vec{\phi})^2 - \frac{1}{48}e e^{\vec{a}\cdot\vec{\phi}} F_4^2 - \frac{1}{12}e \sum_i e^{\vec{a}_i\cdot\vec{\phi}} (F_3^i)^2 - \frac{1}{4}e \sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}} (F_2^{ij})^2 \\ & - \frac{1}{4}e \sum_i e^{\vec{b}_i\cdot\vec{\phi}} (\mathcal{F}_2^i)^2 - \frac{1}{2}e \sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}} (F_1^{ijk})^2 - \frac{1}{2}e \sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}} (\mathcal{F}_1^{ij})^2 + \mathcal{L}_{FFA}, \end{aligned} \quad (5)$$

where the “dilaton vectors”  $\vec{a}$ ,  $\vec{a}_i$ ,  $\vec{a}_{ij}$ ,  $\vec{a}_{ijk}$ ,  $\vec{b}_i$ ,  $\vec{b}_{ij}$  are constants that characterise the couplings of the dilatonic scalars  $\vec{\phi}$  to the various gauge fields. The field strengths are associated

with the gauge potentials in the obvious way; for example  $F_4$  is the field strength for  $A_3$ ,  $F_3^{(i)}$  is the field strength for  $A_2^{(i)}$ , *etc.* In general, the field strengths appearing in the kinetic terms are not simply the exterior derivatives of their associated potentials, but have Chern-Simons type corrections as well. On the other hand the terms included in  $\mathcal{L}_{FFA}$ , which denotes the dimensional reduction of the  $F_4 \wedge F_4 \wedge A_3$  term in  $D = 11$ , are expressed purely in terms of the potentials and their exterior derivatives. The complete details of all the terms, and the dilaton vectors, are given in [10].

It is important for our purposes to distinguish between the NS-NS and the R-R fields in the lower-dimensional theories, viewed as compactifications of the type IIA string. In  $D = 10$ , in the above notation, we have NS-NS fields  $g_{MN}$ ,  $\phi$  and  $A_2^{(1)}$ , and R-R fields  $A_3$  and  $\mathcal{A}_1^{(1)}$ . Upon reduction to  $D \leq 9$ , each of these fields gives rise to sets of fields that have the same NS-NS or R-R characteristics as their  $D = 10$  progenitors. Thus if we split the  $i, j, \dots$  indices as  $i = (1, \alpha)$ , *etc.*, where  $\alpha, \beta, \dots$  run from 2 to  $11 - D$ , we have the following assignments of NS-NS and R-R fields in  $D$  dimensions [12]:

$$\begin{array}{ll}
\text{NS-NS :} & g_{MN} , \quad \vec{\phi} , \quad A_2^{(1)} , \quad A_1^{(1\alpha)} , \quad A_0^{(1\alpha\beta)} , \quad \mathcal{A}_1^{(\alpha)} , \quad \mathcal{A}_0^{(\alpha\beta)} \\
\text{R-R :} & A_3 , \quad A_2^{(\alpha)} , \quad A_1^{(\alpha\beta)} , \quad A_0^{(\alpha\beta\gamma)} , \quad \mathcal{A}_1^{(1)} , \quad \mathcal{A}_0^{(1\alpha)}
\end{array} \tag{6}$$

The multiplicities of gauge potentials with one, two or three  $\alpha$ -type indices are clearly  $10 - D$ ,  $\frac{1}{2}(10 - D)(9 - D)$  or  $\frac{1}{6}(10 - D)(9 - D)(8 - D)$  respectively.

It is known that in the customary dualised formulations of the toroidally-compactified theories, where field strengths of higher rank are dualised to ones of lower rank wherever possible, there is a T-duality symmetry  $D_{10-D} \sim O(10 - D, 10 - D)$ , which is valid order by order in string perturbation theory [11]. Since in low dimensions the customary discussion includes a dualisation of the field strength  $F_3^{(1)}$  associated with the NS-NS 2-form potential  $A_2^{(1)}$ , which in our discussion we wish to leave undualised, our first task will be to verify that T-duality continues to operate in the same manner if we insist that  $F_3^{(1)}$  be left intact and undualised. Likewise, we should verify that none of the other NS-NS gauge fields have to undergo dualisations in order to achieve the symmetry under T-duality transformations. In order to do this, it will be convenient to discuss first a certain discrete subgroup of the putative T-duality group, namely its Weyl group  $WD_{10-D}$ . The Weyl groups of the T-duality and U-duality groups for the toroidally-compactified strings were discussed in detail in [12]. They can be viewed as capturing the essence of the full continuous symmetry groups of the supergravity theories, and correspond precisely to the discrete subgroups that implement exact permutations between the various field strengths, which, for more general

continuous group elements, would instead be mixed together in more complicated ways. (For example, in a case such as an electric-magnetic duality in  $D = 4$ , the Weyl group would be the discrete  $Z_2$  subgroup of duality rotations that implemented an exact interchange of the electric and magnetic fields.) If the theory is to be invariant under such permutations of the field strengths, then, as can be seen from (5), the dilaton vectors associated with the field strengths must permute at the same time. Thus the Weyl group of the T-duality group can be identified by seeking discrete sets of permutations that exchange certain sets of dilaton vectors.

From the details of the dilaton vectors given in [10], it is relatively straightforward to find their multiplet structures under the relevant T-duality symmetries. The procedure was described in detail in [12], and the only difference here is that we are paying close attention to the question of which fields must be dualised in order to achieve the  $WD_{10-D}$  symmetry, and which can be left undualised. To begin, we tabulate the multiplicities of the various gauge potentials appearing in the toroidally-compactified theories:

$D$	NS-NS					R-R					
	$A_2^{(1)}$	$A_1^{(1\alpha)}$	$\mathcal{A}_1^{(\alpha)}$	$A_0^{(1\alpha\beta)}$	$\mathcal{A}_0^{(\alpha\beta)}$	$A_3$	$A_2^{(\alpha)}$	$A_1^{(\alpha\beta)}$	$\mathcal{A}_1^{(1)}$	$A_0^{(\alpha\beta\gamma)}$	$\mathcal{A}_0^{(1\alpha)}$
8	1	2	2	1	1	1	2	1	1	–	2
7	1	3	3	3	3	1	3	3	1	1	3
6	1	4	4	6	6	1	4	6	1	4	4
5	1	5	5	10	10	1	5	10	1	10	5
4	1	6	6	15	15	1	6	15	1	20	6
3	1	7	7	21	21	–	7	21	1	35	7

Table 2: Multiplicities of gauge potentials in compactified IIA strings

At this stage, the discussion separates into two parts, one for the NS-NS potentials and the other for the R-R potentials. The situation is simpler for the NS-NS potentials since, as remarked previously, it turns out that no dualisations are necessary in order to assemble these fields into multiplets under T-duality. Thus we find that the multiplets under the  $WD_{10-D}$  Weyl group for the NS-NS potentials are as follows:

$D$	Weyl Gp	$A_2^{(1)}$	$\{A_1^{(1\alpha)}, \mathcal{A}_1^{(\alpha)}\}$	$\{A_0^{(1\alpha\beta)}, \mathcal{A}_0^{(\alpha\beta)}\}$
8	$WD_2$	(1,1)	(2,2)	(1,2) + (2,1)
7	$WD_3$	1	6	12
6	$WD_4$	1 + 1	8	24
5	$WD_5$	1	10	40
4	$WD_6$	1	12 + 12	60
3	$WD_7$	1	14	84

Table 3: T-duality Weyl-group multiplets for NS-NS potentials in IIA strings

Some comments about the results in this table are in order. First, we note that the T-duality group in  $D = 8$  dimensions is  $D_2 \sim O(2,2)$ , which is a product group,  $D_2 \sim D_1 \times D_1 \sim O(2,1) \times O(2,1)$ . Consequently, the multiplicities in  $D = 8$  are given in terms of their dimensions under the two factors. Secondly, as discussed in [12], the multiplicities for 0-form potentials acquire a doubling when the Weyl-group multiplets are assembled. This is because there is a discrete symmetry  $\vec{\phi} \rightarrow -\vec{\phi}$  and  $A_0 \rightarrow e^{\vec{c}\cdot\vec{\phi}} A_0$ , where  $\vec{c}$  denotes the dilaton vector associated with the 0-form potential  $A_0$ . Thus, for example, although we see from Table 2 that there are three NS-NS 0-forms  $A_0^{(1\alpha\beta)}$  and three NS-NS 0-forms  $\mathcal{A}_0^{(\alpha\beta)}$  in  $D = 7$ , the permutations of their dilaton vectors under the  $WD_3$  Weyl group form an irreducible 12-element representation, involving the six dilaton vectors and their negatives. Thirdly, a doubling of multiplicities occurs for a different reason in the case of the 2-form potential  $A_2^{(1)}$  in  $D = 6$  and the 1-form potentials in  $D = 4$ . These doublings really only occur at the level of solutions, and arise because in these cases the field strengths can carry either electric or magnetic charges. Note however that in each dimension, including  $D = 6$ , the 2-form potential  $A_2^{(1)}$  itself is a singlet under the Weyl group.

The discussion of the T-duality for the R-R potentials is more complicated, because here we find that certain dualisations of field strengths can become necessary in dimensions  $D \leq 8$ , in order to assemble the Weyl-group multiplets. Since this is a dimension-dependent procedure, we shall adopt a different format for presenting the results here. Grouping together R-R potentials into multiplets under the T-duality Weyl group, we find the following:

$D = 8 :$	$\{A_3\}_{(1,2)}$	$\{A_2^{(\alpha)}\}_{(2,1)}$	$\{A_1^{(\alpha\beta)}\}_{(1,1)}$	$\{\mathcal{A}_1^{(1)}\}_{(1,1)}$	$\{\mathcal{A}_0^{(1\alpha)}\}_{(1,2)+(2,1)}$
$D = 7 :$	$\{A_3, A_2^{(\alpha)}\}_4$	$\{A_1^{(\alpha\beta)}, \mathcal{A}_1^{(1)}\}_4$	$\{A_0^{(\alpha\beta\gamma)}, \mathcal{A}_0^{(1\alpha)}\}_{4+4}$		
$D = 6 :$	$\{A_3, A_1^{(\alpha\beta)}, \mathcal{A}_1^{(1)}\}_8$	$\{A_2^{(\alpha)}\}_8$	$\{A_0^{(\alpha\beta\gamma)}, \mathcal{A}_0^{(1\alpha)}\}_{8+8}$		
$D = 5 :$	$\{A_3, A_0^{(\alpha\beta\gamma)}, \mathcal{A}_0^{(1\alpha)}\}_{16+16}$	$\{A_2^{(\alpha)}, A_1^{(\alpha\beta)}, \mathcal{A}_1^{(1)}\}_{16}$			
$D = 4 :$	$\{A_3\}_1$	$\{A_2^{(\alpha)}, A_0^{(\alpha\beta\gamma)}, \mathcal{A}_0^{(1\alpha)}\}_{32+32}$	$\{A_1^{(\alpha\beta)}, \mathcal{A}_1^{(1)}\}_{32}$		
$D = 3 :$	$\{A_2^{(\alpha)}\}_{14}$	$\{A_1^{(\alpha\beta)}, \mathcal{A}_1^{(1)}, A_0^{(\alpha\beta\gamma)}, \mathcal{A}_0^{(1\alpha)}\}_{64+64}$			

Table 4: T-duality Weyl-group multiplets for R-R potentials in IIA strings

The subscripts denote the dimensions of the representations under the relevant T-duality Weyl groups given previously. In cases where gauge potentials of different degrees are grouped together within a set of braces, this implies that dualisations must be performed.

To summarise the story so far, we have seen that the Weyl groups of the T-duality symmetries of toroidally-compactified type IIA strings, which are  $WD_{10-D}$  in the usual dualised versions with  $E_{11-D}$  U-duality, continue to be the same if we work instead with the the versions of the theories in which no NS-NS fields are dualised.<sup>3</sup> This is an important point since T-dualities are perturbative symmetries, and at least at the perturbative level, the bare NS-NS fields  $g_{MN}$  and  $A_2^{(1)}$  have intrinsic significance as the fields that couple directly to the string worldsheet. By contrast, obtaining the  $WD_{10-D}$  symmetry does require that some of the R-R field strengths be dualised; this does not present any difficulty since the associated R-R-potentials couple to the string only *via* their field strengths, and thus the theory could equally well be formulated in such a way that it is the relevant dualised field strengths that couple to the string.

Having discussed the T-dualities of the compactified theories, we now turn to a consideration of their U-dualities. As we remarked previously, there are many different versions of  $D$ -dimensional maximal supergravity, corresponding to different choices of duality complexions for the various field strengths. At the level of the gauge potentials, which are the basic field variables in the actions, these different versions are related to one another by non-local field redefinitions. The global symmetry groups for the different versions will be different too, and the conventional  $E_{11-D}$  Cremmer-Julia symmetries are associated with

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<sup>3</sup>By arguments analogous to those discussed in [12], this will continue to be true for the full T-duality groups themselves.

the versions where field strengths are always dualised if it results in a lowering of their degrees. Here, we shall pursue the investigation of what happens to the U-duality if we again opt for versions of the compactified theories in which NS-NS fields remain undualised.

Let us recall that the Cremmer-Julia symmetry can first of all be thought of as the global symmetry group of the scalar sector of the supergravity Lagrangian, *i.e.* the part that involves just the dilatonic scalars  $\vec{\phi}$  and the axionic scalars  $A_0^{(ijk)}$  and  $\mathcal{A}_0^{(ij)}$ . The invariance of the entire theory under this symmetry can then be understood as the consequence of a covariance of the Lagrangian, with the scalars coupling to the other fields *via* the metric or vielbeins on the scalar manifold. For example, in  $D = 9$  the scalar sector of the Lagrangian takes the form

$$e^{-1} \mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{-2\phi} (\partial\chi)^2, \quad (7)$$

where in our type IIA notation  $\chi$  is the axion  $\mathcal{A}_0^{(1,2)}$ , and  $\varphi$  and  $\phi$  are related to  $\phi_1$  and  $\phi_2$  by  $\varphi = -\frac{1}{4}\sqrt{7}\phi_1 - \frac{3}{4}\phi_2$  and  $\phi = \frac{3}{4}\phi_1 - \frac{1}{4}\sqrt{7}\phi_2$ . This has a  $GL(2, R) \sim SL(2, R) \times R$  global symmetry, where  $SL(2, R)$  acts on  $\tau = \chi + ie^\phi$  by fractional linear transformations, and the additional  $R$  symmetry comes from constant shifts of  $\varphi$ . Thus included in this  $GL(2, R)$  Cremmer-Julia symmetry is the “gauge transformation”  $\chi \rightarrow \chi + \text{const.}$  for the 0-form potential (corresponding to  $\tau \rightarrow \tau + \text{const.}$  in  $SL(2, R)$ ). Unlike a gauge transformation for a higher-rank potential, this gauge transformation is “demoted” to a global symmetry, since the only closed 0-form is a constant. The purpose of making this observation is to emphasise that when we look for global symmetries in lower dimensions, we should recognise that such global shift symmetries of 0-form potentials should be included in the symmetry group, whereas the local gauge transformations of higher-rank potentials should not.

First, we shall consider the global symmetries for the versions of the toroidally compactified  $D = 11$  theory in which no dualisations at all are performed. It is not hard to see that in  $D$  dimensions there will be an “obvious”  $GL(11 - D, R)$  symmetry, generalising the  $GL(2, R)$  symmetry in  $D = 9$ . There will however be more than this, since as we descend through the dimensions we accumulate more 0-form potentials than are needed for the invariances of the  $GL(11 - D, R)$ -symmetric scalar manifold. The excess 0-forms, with global shift symmetries, will contribute additional  $R$  factors in the global symmetry group of the entire scalar manifold. In fact the excess 0-forms are precisely the axions  $A_0^{(ijk)}$  coming from the dimensional reduction of  $A_3$  in  $D = 11$ , of which there will be  $p = (1, 4, 10, 20, 35, 56)$  in  $D = (8, 7, 6, 5, 4, 3)$  respectively. Thus in their totally undualised formulations, the maximal supergravities in  $D$  dimensions have global  $R^p * GL(11 - D, R)$ . Here  $*$  denotes a semi-direct product, which arises rather than a direct product because the scalar fields  $A_0^{(ijk)}$

carry  $GL(11 - D, R)$  indices, and thus transform (linearly) under  $GL(11 - D, R)$ .

In the above discussion, all the axionic fields  $A_0^{(ijk)}$  are placed on an equal footing. Not surprisingly, this runs into difficulties with the T-duality that we discussed earlier, since we were required to put together the R-R axions  $A_0^{(\alpha\beta\gamma)}$  and  $\mathcal{A}_0^{(1\alpha)}$  in order to form representations under the  $O(10 - D, 10 - D)$  T-duality group. However, it can be shown that although it is indeed possible to make a choice of field variables in which all the  $A_0^{(ijk)}$  axions are simultaneously covered everywhere by derivatives, and are thus subject to manifest shift symmetries, the price that is paid is that no other axions can also at the same time be covered with derivatives everywhere [13]. In particular, in the totally undualised formulation the R-R axions  $A_0^{(\alpha\beta\gamma)}$  can be covered by derivatives everywhere but then their would-be T-duality multiplet partners  $\mathcal{A}_0^{(1\alpha)}$  cannot be also covered everywhere. This would break the T-duality symmetry.<sup>4</sup> Another way of seeing this is to note that the  $GL(11 - D, R)$  symmetry of the totally undualised formulation mixes the  $i = 1$  index value with the  $i = \alpha$  values, and thus it interchanges NS-NS fields with R-R fields. The  $GL(10 - D, R)$  subgroup that acts only on the  $i = \alpha$  internal indices, and thus preserves the NS-NS and R-R sectors independently, is not large enough to contain the  $O(10 - D, 10 - D)$  T-duality group.

In order to try to find an extended symmetry group, possibly with the inclusion of non-perturbative generators, that includes the T-duality group, we may instead opt for a choice of field variables in which all the R-R potentials are simultaneously covered by derivatives. In fact it was shown in [13] that there always exists such a choice of field variables, when necessary R-R fields are dualised. This means that we can still dualise the R-R fields as required for the T-duality discussion, and at the same time have all the R-R axions subject to shift symmetries. Note that T-duality, as a perturbative symmetry, is usually described in the context of heterotic string theory, or the NS-NS sector of type II strings, where there are no R-R fields and so the symmetry is concerned purely with perturbative degrees of freedom. In the absence of R-R fields, there is a maximal  $R^{(10-D)(9-D)/2}$  symmetry contained within the T-duality group  $O(10 - D, 10 - D)$ , in that there are a total of  $(10 - D)(9 - D)/2$

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<sup>4</sup>T-duality implies that each of the R-R scalars  $A_0^{(\alpha\beta\gamma)}$  and  $\mathcal{A}_0^{1\alpha}$  has a shift R symmetry. Since these R symmetries are commuting, it follows that there must exist a choice of field variables such that these R symmetries are manifest, *i.e.* all the R-R scalars are simultaneously covered by derivatives. However, this is not possible in the undualised form of supergravities, and hence the T-duality is broken. On the other hand, in the R-R dualised form, it is possible to redefine field variables such that all the R-R fields are covered by exterior derivatives. Thus it is worth emphasising that field redefinitions can make the global symmetries manifest, but do not alter whether or not they are present. It is the dualisation procedure that alters the global symmetries in the different versions of the supergravities.

NS-NS scalars appearing in the Lagrangian only *via* derivatives. In type II theories the introduction of R-R fields, which are believed to be still organised by T-duality despite their non-perturbative nature, has a subtle effect in modifying the T-duality group. It was shown in [13] that although all the R-R potentials can be covered by derivatives, this is at the price that none of the NS-NS scalars appears in the full Lagrangian only through a derivative any more. Thus the previously manifest  $R^{(10-D)(9-D)/2}$  symmetry in the T-duality group  $O(10 - D, 10 - d)$  of the heterotic string becomes no longer manifest with the introduction of the R-R fields in the type IIA theory. Of course, the R symmetries in  $O(10 - D, 10 - D)$  are actually still preserved, although non-manifestly, even in the presence of the R-R fields since the R-R axions, as in the case of all higher-form gauge potentials, transform linearly under the T-duality group  $O(10 - D, 10 - D)$ . Since now there are  $q = (2, 4, 8, 16, 32, 64)$  R-R axions in  $D = (8, 7, 6, 5, 4, 3)$  dimensions, which appear in the Lagrangian through their 1-form field strengths only, we therefore have the global symmetries  $R^q * O(10 - D, 10 - D)$  in these R-R dualised versions of the lower-dimensional maximal supergravities. Here  $*$  denotes a semi-direct product. The reason why it is a semi-direct product rather than a direct product is that the R-R axions rotate (linearly) among themselves under the T-duality  $O(10 - D, 10 - D)$ , in the same manner as higher-form gauge potentials, and thus the  $R^q$  symmetry does not commute with  $O(10 - D, 10 - D)$ .<sup>5</sup>

As a consistency check, we may verify that with the global symmetry groups taken to be  $G = R^q * O(10 - D, 10 - D)$ , we indeed get the correct counting of scalars if we augment them with extra scalars in the adjoint representation of the maximal compact subgroup  $H$  of  $G$ , such that the physical scalars are in the coset  $G/H$ . Thus we have  $H = O(10 - D) \times O(10 - D)$ , implying that  $G/H$  has dimension  $\{6, 13, 24, 41, 68, 113\}$  in  $D = \{8, 7, 6, 5, 4, 3\}$ . This is exactly coincident with the total numbers of scalars in each dimension, where axions, dilatonic scalars and the dualisations of rank- $(D - 1)$  R-R field strengths (where appropriate) are included, but the dualisations of any NS-NS field strengths are excluded. (The numbers of axions and R-R fields dual to axions can be read off from table 2.) Furthermore, the  $D = 10$  dilaton itself is left out of the counting, since it does not participate in the perturbative symmetries described by the  $R^q * O(10 - D, 10 - D)$  group. The counting of the various scalars in the cosets is summarised in table 5 below, where, in the columns headed by NS-NS and R-R, the numbers of axions of each type are listed:

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<sup>5</sup>We are grateful to E. Cremmer for extensive discussions on the semi-direct product structure of the symmetry groups.

	Coset		Numbers of scalars		
$D$	$G/H$	Dimension	NS-NS	R-R	Dilatons
8	$R^2 * O(2, 2)/O(2) \times O(2)$	6	2	2	2
7	$R^4 * O(3, 3)/O(3) \times O(3)$	13	6	4	3
6	$R^8 * O(4, 4)/O(4) \times O(4)$	24	12	8	4
5	$R^{16} * O(5, 5)/O(5) \times O(5)$	41	20	16	5
4	$R^{32} * O(6, 6)/O(6) \times O(6)$	68	30	32	6
3	$R^{64} * O(7, 7)/O(7) \times O(7)$	113	42	64	7

Table 5: Counting of scalars in  $G/H$  for R-R dualisation

To recapitulate the situation so far, we have seen that by working in the formulations of the supergravity theories where all the R-R fields are covered with derivatives, we have perturbative symmetry groups  $R^q * O(10 - D, 10 - D)$  in each dimension. We may now discuss the possible enlargements of these symmetry groups to include non-perturbative generators as well. However, in keeping with the spirit of our previous discussions, we shall require that these non-perturbative symmetries be achieved by dualising only R-R fields, but not NS-NS fields. The results for these non-perturbative global symmetry groups may be presented in the following table, below which we shall discuss the various entries:

	Global Symmetry Groups			T-duality
$D$	No dualisation	R-R dualisation	Full dualisation	
9	$GL(2, R)$	$GL(2, R)$	$GL(2, R)$	–
8	$R * GL(3, R)$	$SL(3, R) \times SL(2, R)$	$SL(3, R) \times SL(2, R)$	$O(2, 2)$
7	$R^4 * GL(4, R)$	$SL(5, R)$	$SL(5, R)$	$O(3, 3)$
6	$R^{10} * GL(5, R)$	$R^8 * O(4, 4)$	$SO(5, 5)$	$O(4, 4)$
5	$R^{20} * GL(6, R)$	$R^{16} * O(5, 5)$	$E_{6(6)}$	$O(5, 5)$
4	$R^{35} * GL(7, R)$	$R^{32} * O(6, 6)$	$E_{7(7)}$	$O(6, 6)$
3	$R^{56} * GL(8, R)$	$R^{64} * O(7, 7)$	$E_{8(8)}$	$O(7, 7)$

Table 6: Global symmetry groups for maximal supergravities

No dualisation is necessary in  $D = 9$ , whilst for  $D = 8$  and  $7$  the only field that needs to be dualised is the R-R 4-form field strength. Thus for  $D \geq 7$ , the full non-perturbative

Cremmer-Julia group  $E_{11-D}$  is consistent with the requirement that no NS-NS fields should be dualised. This implies in particular that  $R^q * O(10-D, 10-D)$  is a perturbative subgroup of the non-perturbative  $E_{11-D}$  group in these cases. For  $D \leq 6$ , the story becomes more complicated, in that the existence of the  $E_{11-D}$  symmetry requires the dualisation of NS-NS fields as well as R-R fields. If we instead insist that only R-R fields can be dualised, which is consistent with T-duality, we find that the symmetry group is simply  $R^q * O(10-D, 10-D)$ , which is perturbative only. Adding any non-perturbative generators, such as rotations between NS-NS and R-R fields, would naturally force the NS-NS fields to be dualised, since the R-R fields form multiplets together with their duals. Thus it seems that for  $D \leq 6$ , the requirement that no NS-NS fields be dualised rules out the possibility of having any enlargement of the perturbative  $R^q * O(10-D, 10-D)$  symmetry to include non-perturbative generators. In fact, the only way to find a non-perturbative symmetry, while leaving the NS-NS fields undualised, is in the case where no fields at all undergo dualisations. (We shall show presently that this is consistent with a possible eleven-dimensional supermembrane [14] origin.) The  $GL(11-D, R)$  and  $O(10-D, 10-D)$  groups are subgroups of  $E_{11-D}$ , and indeed the closure of the two groups generates the full  $E_{11-D}$  group. However, the  $R^p$  and  $R^q$  symmetries are not contained within  $E_{11-D}$  in lower dimensions. In fact the maximal numbers of shift symmetries in the  $E_{11-D}$ -symmetric versions of the supergravity theories have been studied in [15]. These numbers in  $D = \{8, 7, 6, 5, 4, 3\}$  are  $\{3, 6, 10, 16, 27, 44\}$ , of which  $2^{8-D} = \frac{1}{2}q$  are for R-R axions, with the remainder being for NS-NS axions. In particular, it is evident that in the lower dimensions these numbers are less than the numbers  $p$  or  $q$  of shift symmetries in the undualised or R-R dualised formulations, demonstrating that the symmetry groups in these versions of the supergravities cannot be subgroups of the symmetries of the  $E_{11-D}$ -symmetric versions.

Note that a similar consistency check to that which we performed for the supergravities with R-R dualisation can be carried out also for the  $G = R^p * GL(11-D, R)$  global symmetries of the totally undualised supergravity theories. In this case the corresponding maximal compact subgroups of  $G$  are given by  $H = O(11-D, R)$ , and so the coset  $G/H$  has dimension  $\{3, 7, 14, 25, 41, 63, 92\}$  in  $D = \{8, 7, 6, 5, 4, 3\}$ . From the results in table 2, we see that these dimensions are indeed equal to the numbers of axions plus dilatons in each dimension, where in this case no dualisations at all of  $(D-1)$ -form field strengths are performed. Note also that in this case, since the  $G = R^p * GL(11-D, R)$  symmetry is non-perturbative, the  $D = 10$  dilaton itself is included in the counting of dilatons. The counting of the various scalars in the cosets is presented in table 7 below:

	Coset		Numbers of scalars		
$D$	$G/H$	Dimension	NS-NS	R-R	Dilatons
8	$R * GL(3, R)/O(3)$	7	2	2	3
7	$R^4 * GL(4, R)/O(4)$	14	6	4	4
6	$R^{10} * GL(5, R)/O(5)$	25	12	8	5
5	$R^{20} * GL(6, R)/O(6)$	41	20	15	6
4	$R^{35} * GL(7, R)/O(7)$	63	30	26	7
3	$R^{56} * GL(8, R)/O(8)$	92	42	42	8

Table 7: Counting of scalars in  $G/H$  for no dualisation

Thus in summary, if we insist that no dualisations of NS-NS fields be performed, we can follow two possible routes. In one of them, we look for the largest symmetry group that includes the perturbative T-duality symmetry  $O(10 - D, 10 - D)$ . We find that this is the perturbative symmetry group  $R^q * O(10 - D, 10 - D)$ , where  $q = 2^{9-D}$  is the number of R-R axions, for general values of  $D$ . In  $D \geq 7$  it can be enlarged to include non-perturbative generators, since these still do not require the dualisation of NS-NS fields. Below seven dimensions, no enlargement to such a non-perturbative group is possible. The results are summarised in the ‘‘R-R dualisation’’ column in table 6. A second, alternative, route is to sacrifice the T-duality symmetry as a subgroup, in order to find a non-perturbative symmetry group in lower dimensions. Indeed we can then find non-perturbative symmetries that exist also in  $D \leq 6$ , by choosing the versions of the supergravity theories in which no dualisations at all are performed. The results in this case are summarised in the ‘‘No dualisation’’ column in table 6. In  $D \leq 8$ , this is achieved at the price of no longer having the perturbative  $O(10 - D, 10 - D)$  T-duality symmetry as a subgroup.

It should be emphasised that in the versions of the  $D$ -dimensional supergravities where no dualisations at all are performed, the  $R^p * GL(11 - D, R)$  global groups are symmetries of the Lagrangian for all values of  $D$ . On the other hand, in the R-R dualised versions of the supergravities, the  $R^q * O(10 - D, 10 - D)$  groups are symmetries of the Lagrangian itself only when  $D$  is odd. When  $D$  is even, the symmetries are valid only at the level of the equations of motion. This is similar to the situation for the  $E_{11-D}$ -symmetric versions of the supergravities.

A general remark about the nature of the various symmetry groups is perhaps in order here. One might have thought that at the level of the equations of motion all the potentials

would be occurring *via* their field strengths, and that therefore there should be a unique answer for the global symmetry group of the theory. As we have seen, this is in fact not the case. The explanation lies in the fact that as a result of the Chern-Simons modifications to the field strengths, bare potentials that are not covered by derivatives appear even in the field equations. By performing field redefinitions, it is possible to ensure that all the fields that need to be dualised in the usual  $E_{11-D}$ -symmetric formulations of the supergravity theories are everywhere covered with derivatives. However, if instead some of these fields are “sacrificed,” it is possible to perform field redefinitions that transfer the derivatives onto other potentials, thereby gaining new symmetries at the cost of losing the previous ones. (Once field redefinitions are performed that cause potentials to be exposed without derivatives, symmetries that require dualisations of the associated field strengths no longer act locally on the fields of the theory.) Thus for example in four dimensions the usual fully-dualised theory has a global  $E_7$  symmetry, whose maximal abelian invariant subalgebra has dimension 27 (*i.e.* shift symmetries for 16 R-R axions and 11 NS-NS axions) [15]. If the symmetries involving dualisations of the NS-NS field strengths are sacrificed, the derivatives on their potentials can be transferred to other fields, including all 32 R-R axions. The  $R^{32} * O(6, 6)$  symmetry for this choice is manifestly not a subgroup of  $E_7$ . If all symmetries involving dualisations, both for NS-NS and R-R fields are sacrificed, the derivatives can be transferred so as to cover all of the 35 axions  $A_0^{(ijk)}$ , and the resulting global symmetry group  $R^{35} * GL(7, R)$  is neither contained in, nor does it contain, either of the previous two groups. Thus the global symmetry group of the theory is not uniquely determined until the fields on which it is to act locally are specified. For example in four dimensions, if it is to act only on potentials then the symmetry is  $R^{35} * GL(7, R)$ ; it is instead  $R^{32} * O(6, 6)$  if it is to act on NS-NS potentials but R-R field strengths; and it is  $E_7$  if it is to act on field strengths of both types. String theory, at least in a perturbative formulation, favours the first or second possibilities. Since perturbative string theory the R-R fields coupling only *via* their field strengths, the second possibility seems to be the most natural one.

One further remark that should be made is that the above discussions of the various possibilities for non-perturbative symmetry groups that do not require dualisations of NS-NS fields do not in any way conflict with the known multiplet structures of BPS-saturated soliton solutions. In particular, the  $p$ -brane solutions that carry a single type of electric or magnetic charge fall into multiplets under the Weyl group of the usual U-duality group, an example being the black holes in  $D = 4$ , which form an irreducible 56-component representation under the Weyl group of  $E_7$  [12]. Under the Weyl group of  $R^{32} * O(6, 6)$  they will

instead form a reducible  $32 + 12 + 12$  component representation, while under  $R^{35} * GL(7, R)$  they will form a reducible  $7 + 7 + 21 + 21$  component representation. In all of these cases, the total number of distinct black hole species will be 56. Similar considerations apply to all the other species of  $p$ -brane solitons.

With the proposal that the degrees of freedom of string theory are more appropriately described by an eleven-dimensional theory in general, with the string providing a useful limiting description in the perturbative regime, it is of interest to consider how the discussions of T-duality and U-duality in this paper might apply in M-theory itself. It has sometimes been argued that the eleven-dimensional supermembrane [14] should be viewed as a fundamental entity in its own right, whose quantisation might give rise to an eleven-dimensional theory which has  $D = 11$  supergravity as its low-energy limit. There are many objections to this viewpoint, centering on the apparently insurmountable difficulties in setting up a sensible perturbative quantisation scheme. One signal of this problem is the absence of any candidate for a loop-counting parameter, analogous to the coupling constant  $e^{-\phi_0}$  in string theory, owing to the fact that there is no dilaton in the eleven-dimensional theory. If, as therefore seems likely, there is no perturbative regime for the quantisation of a supermembrane, it is not clear that there would be any utility to attempting a description of some region of the modulus space of  $M$ -theory in terms of a fundamental supermembrane. Indeed, it can be argued that string theory is useful only insofar as it enables a region of the modulus space of M-theory to be described perturbatively [2].

Another indication of the intrinsically non-perturbative nature of the supermembrane is provided by considering T-duality in terms of an hypothetical  $D = 11$  supermembrane description. We have seen above that in the case of string theory, the usual T-duality groups of the toroidally-compactified theory are perfectly compatible with a formulation of the theory in which none of the NS-NS fields needs to be dualised. It is important that this should be possible because T-duality is a perturbative symmetry, and thus should be adequately describable in terms of string perturbation theory. In this theory, the NS-NS gauge potentials have a fundamental significance since it is they, and not their field strengths, that couple to the worldsheet of the string. However, the situation would be quite different if the lower-dimensional theories were to be described as toroidal compactifications of a quantised supermembrane. The difference lies in the fact that in such a supermembrane theory it is the 3-form potential  $A_3$  of  $D = 11$  supergravity, rather than the 2-form potential  $A_2^{(1)}$  of  $D = 10$  supergravity, that would now be playing a distinguished rôle since  $A_3$  couples directly to the world-volume of the membrane. But as we have seen, in order to realise the

T-duality symmetries it is necessary that some of the Kaluza-Klein descendants of  $F_4$ , which from the string point of view are R-R fields, must be dualised. This means that T-duality must act non-locally on the potential  $A_3$  in eleven dimensions, and thus it would be incompatible with a perturbative description in terms of fundamental membranes.

The eleven-dimensional supermembrane action implies that the symmetry group of the theory compactified on a torus should be  $R^p \times GL(11 - D, R)$ , obtained from the supergravity theories where no dualisation is performed. It is non-perturbative in nature. However there is a subgroup of  $GL(11 - D, R)$ , namely  $O(10 - D)$ , which rotates NS-NS and R-R field within themselves. This  $O(10 - D)$  subgroup is perturbative, since it is the intersection of  $G(11 - D, R)$  and  $O(10 - D, 10 - D)$ . This perturbative symmetry of the membrane action on a torus may provide some clue as to how to quantise the theory, if it is quantisable at all.

In conclusion, we have seen that various different viewpoints are possible concerning the global symmetry groups in toroidally-compactified type IIA strings in lower dimensions. In particular, when  $D$  is less than seven the symmetry groups obtained for the theories with only R-R dualisations are different from, and are not contained in, the usual  $E_{n(n)}$  groups that arise for the theories where dualisations of NS-NS fields are also performed. At least at the level of the perturbative T-duality symmetry it seems reasonable to avoid such NS-NS dualisations, since it is the NS-NS potentials  $A_{MN}$  and  $g_{MN}$  themselves, rather than gauge-invariant field strengths built from them, that couple directly to the string worldsheet. If this reasoning is extended to the discussion of non-perturbative symmetries too then it might be argued that since the possible symmetry groups under the various viewpoints are inequivalent, then it is not inconceivable that it is the symmetries in the formulations that do not require NS-NS dualisations that should be regarded as more fundamental.<sup>6</sup> Such reasoning need not be in conflict with any of the results about the relation between compactifications of M-theory and of the type IIA string, since, as we observed in the introduction to the paper, the key results can already be seen in the compactifications to nine dimensions. Indeed, the fact that the relations are robust with respect to the different choices for the dualisations that should be performed in the lower-dimensional theories serves to emphasise that the basic relations between the higher-dimensional theories can be established without needing to invoke the details of the lower-dimensional compactifications.

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<sup>6</sup>Another way to express the observation is that non-perturbative symmetries in string theory are associated with non-local symmetries in the perturbative description. Thus for example the non-perturbative  $SL(2, R)$  S-duality of four-dimensional string theory, which preserves the NS-NS and R-R sectors but is associated with dualisations of the 2-form field strengths [9], acts non-locally on the NS-NS potentials.

Finally, we remark that our discussions have concentrated on the supergravity theories, rather than the full quantum string theories. Thus for example although we have used the language of string theory to distinguish between symmetries that would act perturbatively, and those that would act non-perturbatively, at the level of our discussion the symmetry groups have all been the classical continuous groups of the Cremmer-Julia kind. It would be interesting to study the implications of the quantisation conditions for electrically and magnetically charged solitons, to uncover the likely discrete subgroups that would survive in the full string theory.

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