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Spectrum-generating Symmetries for BPS Solitons[◇]

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ABSTRACT

We show that there exist nonlinearly realised duality symmetries that are independent of the standard supergravity global symmetries, and which provide active spectrum-generating symmetries for the fundamental BPS solitons. The additional ingredient, in any spacetime dimension, is a single scaling transformation that allows one to map between BPS solitons with different masses. Without the inclusion of this additional transformation, which is a symmetry of the classical equations of motion, but not the action, it is not possible to find a spectrum-generating symmetry. The necessity of including this scaling transformation highlights the vulnerability of duality multiplets to quantum anomalies. We argue that fundamental BPS solitons may be immune to this threat.

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1 Introduction

The study of BPS-saturated solitons in the supergravities that describe the low-energy limits of string theories has proved to be a valuable tool for elucidating the non-perturbative structures of these theories. These solitons are solutions in which infinite p-branes occupy a longitudinal submanifold in spacetime, with the fields depending on the coordinates of the transverse space. These fields include one or more of the antisymmetric tensor field strengths in the supergravity theory, which carry either electric or magnetic type charges (or both, in the case of dyonic p-branes). In fact, these solutions are characterised by the configuration of non-vanishing charges, and by the asymptotic values of the scalar fields at infinity. These asymptotic values can be thought of as the modulus parameters for the solution. If we restrict attention to BPS-saturated solutions, then the mass is not an independent parameter, but is instead some function of the charges and the scalar moduli. It is useful to try to organise the various solutions into multiplets, by making use of the global symmetries of the supergravity theory.

There indeed exist continuous global symmetries in supergravity theories [1, 2], which act linearly on the charges, nonlinearly on the scalars, and which leave the Einstein-frame metric invariant. In the following, we shall refer to these symmetries as the standard supergravity global symmetries. The orbits of these standard global symmetry groups G yield families of p-brane solutions. In particular, the maximal compact subgroup H of G is the stability group of a point in modulus space; this allows the set of charges to be rotated while holding the asymptotic values of the scalars fixed. This rotation in the vector space of the charges holds an invariant quadratic expression in the charge vectors fixed, whose square root may be thought of as an H-invariant "length." Such a rotation, and indeed any transformation under the standard global symmetry group G, preserves the mass of the solution. Put another way, the expression for the mass as a function of the charges and the scalar moduli is invariant under G. This can be immediately seen from the fact that G leaves the Einstein-frame metric invariant. The implication of these observations is that the standard global supergravity symmetry groups are insufficient for the purpose of generating complete sets of p-brane solitons.

The problem of finding a solution-generating symmetry arises in a more severe form at the quantum level in string theory. It has been argued that in the quantised string theory only a discrete subgroup of the classical supergravity symmetry G can be consistent with the Dirac quantisation condition, which allows only a discrete lattice of charges for any given vacuum.[1] The discrete subgroup $G(\mathbb{Z})$ is known as U-duality, and is conjectured to survive as an exact symmetry even at the non-perturbative level [3]. In fact, in string theory it has been proposed that states should be *identified* under U-duality. Thus, not only does the U-duality group $G(\mathbb{Z})$ suffer from the same deficiency as the classical group G for generating independent solutions, but also those solutions that it can relate are taken to be identical, and so the $G(\mathbb{Z})$ orbit consists of one single state. This is not what one could call a satisfactory spectrum-generating symmetry.

Despite these shortcomings, there is a certain sense in which the orbit of the U-duality group is associated with the spectrum of distinct BPS quantum states. If one looks only at the action of the U-duality group on the charge lattice, and ignores its action on the scalar moduli, then it does map between allowed charge vectors. If these charge vectors were taken to be associated with a single fixed vacuum, then one would indeed have the spectrum of physically-distinct states. In Ref. [3], a procedure of "analytic continuation in the moduli" was proposed, to return the moduli after a U-duality transformation to their initial values. This procedure, however, does not make clear whether there is an actual symmetry transformation in the theory that can implement this analytic continuation, and so it does not clearly give a proper spectrum-generating symmetry.

In this paper, we shall show that there exists a different G symmetry (although with the same abstract group G), realised nonlinearly on the fields of the theory, that holds the scalar moduli fixed while transforming the charge vectors in a linear fashion. This is the true spectrum-generating group, which we shall call the *active* G symmetry group. The key point is to recognise that the actual classical global symmetry group in any supergravity theory is larger than is customarily presented, and includes an additional scaling transformation, which is a symmetry of the equations of motion corresponding to a homogeneous scaling of the action. A well-known example of such a symmetry is in pure Einstein gravity, where there is a global scaling symmetry of the equations of motion under the transformation $g_{\mu\nu} \to \lambda^2\, g_{\mu\nu}$. This symmetry can be used to explain the organisation of solutions into one-parameter families, such as the Schwarzschild solution, where the mass is a free parameter. Analogous scaling symmetries exist in all supergravity theories. Because they allow one to scale magnitudes in and out, we shall call these scaling symmetries "trombone" symmetries. At the classical level, taken together with the standard supergravity G symmetry,[2] they allow one to reach the entire parameter space of BPS solitons that preserve half

[1] Actually, as we shall argue below, it is really the spectrum-generating symmetry, and not the standard supergravity symmetry G, that is discretised by virtue of the Dirac quantisation condition.

[2] Note that at least at the classical level the trombone symmetry is as much a valid symmetry of the

the supersymmetry. Thus, the entire classical family of such fundamental BPS solutions can be reached by the application of symmetries of the theory. More precisely, it is the stability group H of the moduli together with the trombone symmetry, that allow one to reach arbitrary points in the charge-vector space while holding the scalar moduli fixed.

At the quantum level, the discussion becomes more involved. One might think that one could simply take the direct product of the trombone symmetry with the H subgroup of the standard G symmetry, and then make a restriction to a discrete linearly-realised subgroup of this product that is compatible with the charge lattice required by the Dirac quantisation condition. However, there is no such group. The reason for this is that the trombone scaling symmetry and the H symmetry would need to be independently restricted to discrete subgroups, and there is in general no way to do this for either factor in such a way as to ensure that only allowed charge-lattice points can be related. What one must do instead is to construct a nonlinear realisation of the full $G(\mathbb{Z})$ group, linearly realised on H, using the trombone symmetry as a compensating transformation.

It is worth emphasising that the Dirac quantisation condition *by itself* requires that the classical spectrum-generating symmetry of a supergravity theory be restricted to a discrete $G(\mathbb{Z})$ subgroup. This is quite different from the situation for the standard supergravity global symmetry groups, since the latter move not only the charges but also the scalar moduli, and the Dirac quantisation condition *by itself* does not require that the charge lattices for different points in the modulus space must coincide (even though the lattice at each modulus point must respect the quantisation condition). Thus the discretisation of the U-duality groups in [3] must arise for reasons that go beyond the Dirac quantisation condition. In fact it has been argued that since the T-duality groups are subgroups of the U-duality groups, then the discretisation of the former (based on perturbative string-theoretic considerations) provides supporting evidence for the discretisation of the latter [3].

We shall be concerned with p-brane solitons in $D = 11$ supergravity, type IIB supergravity, and their dimensional reductions. All of these theories have analogous scaling symmetries; in fact, in the dimensionally-reduced theories they are a direct consequence of the corresponding symmetries of the higher-dimensional $D = 11$ or type IIB theories. In $D = 11$, the bosonic Lagrangian is [5]

$$\mathcal{L} = eR - \tfrac{1}{48} e\, F_4^2 + \tfrac{1}{6} * (dA_3 \wedge dA_3 \wedge A_3) \ , \tag{1}$$

where $F_4 = dA_3$, A_3 is the 3-form potential, and e is the determinant of the vielbein. The

theory as the standard global symmetry G. In fact in even dimensions, the latter shares with the trombone symmetry the feature that it is a symmetry only of the equations of motion, but not of the action.

corresponding trombone symmetry is

$$g_{\mu\nu} \longrightarrow \lambda^2\, g_{\mu\nu}\ , \qquad A_3 \longrightarrow \lambda^3\, A_3\ , \tag{2}$$

under which the Lagrangian scales as $\mathcal{L} \to \lambda^9\, \mathcal{L}$. The equations of motion scale homogeneously, and thus (2) is a symmetry of the equations of motion. This symmetry is responsible for the extremal membrane and 5-brane classical soliton solutions occurring in one-parameter families with arbitrary values of the charge. The symmetry (2) is preserved under Kaluza-Klein dimensional reduction, after which the metric still scales as $g_{\mu\nu} \to \lambda^2\, g_{\mu\nu}$, and in addition all n-index potentials scale with a factor λ^n, while all scalar fields are left invariant. (Trombone symmetries were also used in the rheonomy approach to supersymmetry; see, for example [6].) The same rescaling rules apply to the trombone symmetry of the type IIB supergravity theory. We shall see that this rather humble scaling symmetry plays a central rôle in permitting the construction of solution-generating symmetry transformations that map actively between physically-inequivalent soliton solutions.

The simplest non-trivial example of a solution-generating global symmetry is when $G = SL(2,\mathbb{R})$. We shall accordingly consider in detail the example of string solitons in the type IIB supergravity theory, for which $SL(2,\mathbb{R})$ is the symmetry group. (Another example would be provided by S-duality in the $D = 4$ heterotic string theory [4].) The layout of the rest of this paper is as follows. In section 2, we describe the type IIB supergravity theory and the action of the standard $G = SL(2,\mathbb{R})$ symmetry on its fields. In section 3, we construct the nonlinearly realised active $SL(2,\mathbb{Z})$ symmetry, and we discuss its group structure in section 4. In section 5, we give a group-theoretical interpretation of the charge spectrum, using some elements of the earlier construction. In section 6 we generalise the discussion to the lower-dimensional cases with larger symmetry groups, and in section 7 we consider the problem of quantum anomalies in the spectrum-generating symmetry. We end with a conclusion in section 8.

2 Type IIB in $D = 10$

The low-energy effective theory for the type IIB string is type IIB supergravity, whose bosonic fields comprise the metric, a dilaton ϕ, an axion χ, two 2-form potentials $A_2^{(i)}$, and a 4-form potential whose associated field strength is self dual. The 4-form potential, χ, and $A_2^{(2)}$ are R-R fields, and the remainder are NS-NS. Owing to the self-duality of the 5-form field strength, there is no simple way to write a covariant Lagrangian for these fields alone. However, by adding extra degrees of freedom, namely by removing the self-duality condition, one can write a Lagrangian whose equations of motion yield the type IIB equations after imposing by hand the self-duality constraint as a consistent truncation [7]. Thus, our starting point is the Lagrangian

$$\begin{aligned}
\mathcal{L} &= eR + \tfrac{1}{4}e\,\mathrm{tr}(\partial_\mu \mathcal{M}^{-1}\,\partial^\mu \mathcal{M}) - \tfrac{1}{12}e\,H_3^T\,\mathcal{M}\,H_3 - \tfrac{1}{240}e\,H_5^2 \\
&\quad - \tfrac{1}{2\sqrt{2}}\epsilon_{ij}\,*(B_4 \wedge dA_2^{(i)} \wedge dA_2^{(j)})\ , && (3)\\
&= eR - \tfrac{1}{2}e\,(\partial\phi)^2 - \tfrac{1}{2}e\,e^{2\phi}\,(\partial\chi)^2 - \tfrac{1}{12}e\,e^{-\phi}\,(F_3^{(1)})^2 - \tfrac{1}{12}e\,e^{\phi}\,(F_3^{(2)})^2 \\
&\quad - \tfrac{1}{240}e\,H_5^2 - \tfrac{1}{2\sqrt{2}}\epsilon_{ij}\,*(B_4 \wedge dA_2^{(i)} \wedge dA_2^{(j)})\ , && (4)
\end{aligned}$$

where

$$\mathcal{M} = \begin{pmatrix} e^{-\phi} + \chi^2\,e^{\phi} & \chi\,e^{\phi} \\ \chi\,e^{\phi} & e^{\phi} \end{pmatrix}\ , \qquad H_3 = \begin{pmatrix} dA_2^{(1)} \\ dA_2^{(2)} \end{pmatrix}\ . \tag{5}$$

The field strengths appearing in (4) are defined as follows:

$$F_3^{(1)} = dA_2^{(1)}\ , \qquad F_3^{(2)} = dA_2^{(2)} + \chi\,dA_2^{(1)}\ , \qquad H_5 = dB_4 + \tfrac{1}{2\sqrt{2}}\epsilon_{ij}A_2^{(i)} \wedge dA_2^{(j)}\ . \tag{6}$$

The equations of motion following from (4) are

$$\begin{aligned}
R_{\mu\nu} &= \tfrac{1}{2}\partial_\mu\phi\,\partial_\nu\phi + \tfrac{1}{2}e^{2\phi}\,\partial_\mu\chi\,\partial_\nu\chi + \tfrac{1}{48}(H_{\mu\nu}^2 - \tfrac{1}{10}H_5^2\,g_{\mu\nu}) && (7)\\
&\quad + \tfrac{1}{4}e^{-\phi}((F^{(1)})_{\mu\nu}^2 - \tfrac{1}{12}(F_3^{(1)})^2 g_{\mu\nu}) \\
&\quad + \tfrac{1}{4}e^{\phi}((F^{(2)})_{\mu\nu}^2 - \tfrac{1}{12}(F_3^{(2)})^2 g_{\mu\nu})\ , \\
\nabla^\mu\,H_{\mu\nu\rho\sigma\lambda} &= \tfrac{1}{72\sqrt{2}}\,\epsilon_{ij}\,\epsilon_{\nu\rho\sigma\lambda}{}^{\mu_1\cdots\mu_6}\,F_{\mu_1\mu_2\mu_3}^{(i)}\,F_{\mu_4\mu_5\mu_6}^{(j)}\ , && (8)\\
\nabla^\mu\,(e^{-\phi}\,F_{\mu\nu\rho}^{(1)} - e^{\phi}\,\chi\,F_{\mu\nu\rho}^{(2)}) &= -\tfrac{1}{6\sqrt{2}}\,H_{\nu\rho}{}^{\mu\lambda\sigma}\,(F_{\mu\lambda\sigma}^{(2)} + \chi\,F_{\mu\lambda\sigma}^{(1)})\ , && (9)\\
\nabla^\mu\,(e^{\phi}\,F_{\mu\nu\rho}^{(2)}) &= \tfrac{1}{6\sqrt{2}}\,H_{\nu\rho}{}^{\mu\lambda\sigma}\,F_{\mu\lambda\sigma}^{(1)}\ , && (10)\\
\nabla^\mu(e^{2\phi}\,\partial_\mu\chi) &= \tfrac{1}{6}\,e^{\phi}\,F_{\mu\nu\rho}^{(1)}\,F^{(2)\mu\nu\rho}\ , && (11)\\
\Box\phi &= e^{2\phi}\,(\partial\chi)^2 + \tfrac{1}{12}\,e^{-\phi}\,(F_3^{(1)})^2 - \tfrac{1}{12}\,e^{\phi}\,(F_3^{(2)})^2\ , && (12)
\end{aligned}$$

Note that the equation for H_5 can be rewritten as $d * H_5 = \frac{1}{2\sqrt{2}}\,\epsilon_{ij}\,F_3^{(i)} \wedge F_3^{(j)}$. Since we also have the Bianchi identity $dH_5 = \frac{1}{2\sqrt{2}}\,\epsilon_{ij}\,F_3^{(i)} \wedge F_3^{(j)}$, we see that we can consistently impose the self-duality condition $H_5 = *H_5$. After doing this, the equations (7-12) become precisely the field equations of type IIB supergravity [8]. The Lagrangian (4) is manifestly $SL(2,\mathbb{R})$ invariant.

The action of $SL(2,\mathbb{R})$ on the fields can be expressed as

$$H_3 \longrightarrow (\Lambda^T)^{-1}\,H_3\ , \qquad \mathcal{M} \longrightarrow \Lambda\,\mathcal{M}\,\Lambda^T\ , \tag{13}$$

where

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{14}$$

and $ad - bc = 1$. Defining the complex scalar field $\tau = \chi + i\, e^{-\phi}$, the transformation on $\mathcal{M}$ can be seen to imply that

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} . \tag{15}$$

Note that since H_5 is a singlet under $SL(2,\mathbb{R})$, the self-duality constraint which is imposed by hand also preserves the $SL(2,\mathbb{R})$ symmetry.

In addition to the $SL(2,\mathbb{R})$ symmetry described above, there is also a trombone scaling symmetry, as we anticipated in the Introduction:

$$g_{\mu\nu} \longrightarrow \lambda^2\, g_{\mu\nu} , \qquad A_2^{(i)} \longrightarrow \lambda^2\, A_2^{(i)} , \qquad H_5 \longrightarrow \lambda^4\, H_5 . \tag{16}$$

It is important to note that this rescaling leaves the scalar fields ϕ and χ invariant. This is also true of the scalar fields of all the lower-dimensional supergravities that we shall consider. Thus the full global symmetry group of type IIB supergravity is $GL(2,\mathbb{R})$.

We shall first consider classical string soliton solutions. These are characterised by two electric charges $Q_e = (p, q)$ (carried by the NS-NS and R-R 3-form field strengths), and by the two scalar moduli ϕ_0 and χ_0, corresponding to the asymptotic values at infinity of the dilaton ϕ and the axion χ. The string coupling constant is given by $g = e^{\phi_0}$. The full 4-parameter family of solutions can be generated starting from a specific solution, for example from a pure NS-NS string with vanishing moduli ϕ_0 and χ_0, by acting with the $SL(2,\mathbb{R})$ and trombone symmetries of the equations of motion.

The action of the $SL(2,\mathbb{R})$ symmetry on the parameters of the solutions is

$$Q_e \equiv \begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow \Lambda \begin{pmatrix} p \\ q \end{pmatrix} , \qquad \tau_0 \longrightarrow \frac{a\,\tau_0 + b}{c\,\tau_0 + d} . \tag{17}$$

The $SL(2,\mathbb{R})$ transformation rule for the electric charges follows from the fact that the field equations (9) and (10) for the two 3-form field strengths can be rewritten as

$$d * (\mathcal{M}\, H_3) = -\tfrac{1}{\sqrt{2}}\, H_5 \wedge \Omega H_3 , \qquad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{18}$$

and so the canonical electric Noether charges are given by

$$Q_e = \begin{pmatrix} p \\ q \end{pmatrix} = \int \Big(* \mathcal{M} H_3 + \tfrac{1}{3\sqrt{2}}\, \Omega (2 B_4 \wedge H_3 - H_5 \wedge A_2) \Big) . \tag{19}$$

(This expression is in fact invariant under the gauge transformations $\delta A_2^{(i)} = d\Lambda_1^{(i)}$, despite the appearance of bare 2-form potentials, since the gauge invariance of H_5 requires a compensating transformation $\delta B_4 = -\frac{1}{2\sqrt{2}} \epsilon_{ij}\, \Lambda_1^{(i)} \wedge dA_2^{(j)}$, as can be seen from (6).) From the transformation rules (13) for $\mathcal{M}$ and H_3, and from the invariance $\Omega = \Lambda\Omega\Lambda^T$ of Ω, defined in (18), the $SL(2,\mathbb{R})$ transformation rule given in (17) for the electric charges follows. Note that the magnetic charges, by contrast, are defined by

$$Q_m = \frac{1}{2\pi}\int H_3\ , \tag{20}$$

and so they transform as $Q_m \longrightarrow (\Lambda^T)^{-1}\, Q_m$ under $SL(2,\mathbb{R})$. (This is indeed consistent with the fact that the Dirac quantisation condition is $Q_m^T\, Q_e =$ integer, and must be preserved under the standard $SL(2,\mathbb{R})$.) In the rest of this paper, we shall be considering only electric charges, and shall therefore drop the subscript 'e' on Q.

The action of the trombone symmetry on the charge and modulus parameters is simply incorporated by relaxing the condition $ad - bc = 1$; in other words, the full symmetry has the group structure $GL(2,\mathbb{R})$. This indeed has four parameters, consistent with the fact that an arbitrary BPS string solution can be obtained from any specific one by a $GL(2,\mathbb{R})$ transformation.

At the quantum level, due account must be taken of the Dirac quantisation conditions between electrically-charged strings and magnetically-charged 5-branes. These conditions imply that the electric charges Q in a given fixed scalar vacuum should lie on a lattice, of which the simplest choice is to take $Q = (p,q)$, with p and q integers. (An equally valid choice would be the lattice obtained from this by acting on the charge vectors with an arbitrary fixed $SL(2,\mathbb{R})$ transformation with the magnetic charge lattice transforming in the appropriate contragedient fashion.) If the standard global $SL(2,\mathbb{R})$ symmetry had acted only on the charges, and not also on the scalar moduli, it would be discretised by the Dirac quantisation condition since only an $SL(2,\mathbb{Z})$ subgroup would preserve the chosen charge lattice. Since the standard $SL(2,\mathbb{R})$ actually moves the scalar moduli at the same time as moving the charges, the Dirac quantisation condition by itself does not imply that this group must be discretised. In fact, as we shall show in the next section, there is a nonlinearly realised active $SL(2,\mathbb{R})$ symmetry that does move just the charges, while holding the moduli fixed, and it is the discretisation to $SL(2,\mathbb{Z})$ of this symmetry that can be deduced from the Dirac quantisation condition. In the simple case where the electric charges are integers, $Q = (p,q)$, this active $SL(2,\mathbb{Z})$ is defined by the requirement that the entries a, b, c and d be integers. These active $SL(2,\mathbb{Z})$ transformations can generate the entire charge lattice starting just from the set of charges $Q = (n,0)$. Starting from $Q = (1,0)$, and acting with an active $SL(2,\mathbb{Z})$ transformation of the form (14), but with a, b, c and d now integers satisfying $ad-bc = 1$, one may verify that one generates integer pairs (p,q) with p and q relatively prime. Starting instead from $(n,0)$ gives the set (p,q) for all integer pairs with a common factor n. Since solitonic p-brane solutions with charges $(n, 0)$ can be viewed as coincident superpositions of n p-branes with charge $(1, 0)$, the spectrum of elementary single p-branes will be taken to be the irreducible active $SL(2, \mathbb{Z})$ orbit of solutions containing the $(1, 0)$ solution. We shall call this the elementary $SL(2, \mathbb{Z})$ orbit.

3 Construction of the active $SL(2, \mathbb{Z})$

The standard supergravity $SL(2, \mathbb{R})$ symmetry has the property of transforming the charges and the scalar moduli at the same time. In string theory, states that are related by an $SL(2, \mathbb{Z})$ subgroup of this standard $SL(2, \mathbb{R})$ symmetry are treated as equivalent. In other words, this $SL(2, \mathbb{Z})$ is interpreted as a local or "gauged" symmetry. In particular, this implies that the full charge lattice of solutions in a given vacuum (ϕ_0, χ_0) is identified with the full charge lattice of solutions for any other vacuum whose scalar moduli are related to (ϕ_0, χ_0) by a gauged $SL(2, \mathbb{Z})$ transformation. However, in string theory, the physical spectrum of distinct string solitons is described by the full charge lattice of solutions for a given *fixed* vacuum, *i.e.* with (ϕ_0, χ_0) fixed. Clearly, the $SL(2, \mathbb{Z})$ subgroup of the standard $SL(2, \mathbb{R})$ cannot generate this multiplet of distinct charge states in a given vacuum. In particular, at fixed (ϕ_0, χ_0), the masses of the various strings at different charge-lattice points will in general be different, and so it is evident that no subgroup of the standard classical $SL(2, \mathbb{R})$ can possibly relate them. In fact, the extra ingredient that is needed in order to construct the multiplets of physically-distinct solutions is the trombone symmetry, which does rescale the masses.

Classically, it is easy to see that the trombone symmetry, together with the $H = SO(2)$ maximal compact subgroup of $G = SL(2, \mathbb{R})$, can be used to generate a solution with arbitrary charges $Q = (p, q)$, while holding (ϕ_0, χ_0) fixed. This is because, as we remarked previously, the H subgroup rotates the charge vector while keeping its invariant length fixed. Combined with the rescaling trombone symmetry, which also preserves the scalar moduli, the entire plane of charges can be reached. (In fact the H subgroup, together with a rescaling of the charges, was used at the level of string solutions in [9] to obtain the general family of $Q = (p, q)$ string solitons for arbitrary fixed scalar moduli in the type IIB theory.)

At the quantum level, one might be tempted to try to take the direct product of the H subgroup times the trombone scaling symmetry, acting linearly on the charge-vector space, and then to search for a subgroup of this product that maps only between the points on the charge lattice which are allowed by the Dirac quantisation condition. Unfortunately, there is no subgroup that does this. For the trombone symmetry, this can be seen because this linearly-realised transformation acts multiplicatively on the charges, and only by multiplying by the integers could one ensure that only maps between lattice points occur. However, the integers with multiplication taken as the composition operation do not form a group, except for a Z_2 generated by -1, which leaves the mass invariant. This is clearly insufficient for the purposes of generating the spectrum. Similarly, the $H = SO(2)$ stability group does not in general admit a discrete subgroup that preserves the given charge lattice. An exceptional situation, where a lattice-preserving discrete subgroup of H does occur, arises for a specific point in the modulus space, and for those points related to it by $SL(2,\mathbb{Z})$ transformations that preserve the given charge lattice. For these special moduli there is a lattice-preserving Z_2 subgroup of H. For example, for the case of the integer charge lattice, the special modulus point is the self-dual value $\tau_0 = i$, for which $\chi_0 = \phi_0 = 0$. In fact this Z_2 subgroup is the Weyl group of $SL(2,\mathbb{R})$, and in lower dimensions, where the global symmetry groups G are larger, the lattice-preserving subgroup of H for appropriate special values of the scalar moduli again turns out to be the Weyl group of G [10, 11]. In any case, however, the Weyl group is not sufficiently large to allow the whole charge lattice to be generated, even at these special modulus values. This is because the action of the Weyl group is to permute the axes of the charge-vector space, but it does not allow any intermediate rotations.

The way to solve this quantum-level problem is to change over to a nonlinear realisation of $SL(2,\mathbb{Z})$, acting on the fields of the theory through $H = SO(2)$ together with compensating trombone transformations, in a way which happens to coincide with the linearly-realised $SL(2,\mathbb{Z})$ when acting on the charge vectors, but leaving the vacuum modulus fixed. Since this symmetry coincides with the known action of $SL(2,\mathbb{Z})$ on the charge vectors, it clearly preserves the charge lattice.

Let us begin by trying to make use of the trombone symmetry to generate the multiplet of quantum-level (p,q) strings in a fixed vacuum. The idea will be to perform an $SL(2,\mathbb{Z})$ transformation on a given charge pair, for example $Q = (1,0)$, and then to follow this with a compensating transformation that preserves the new charges, but restores the transformed scalar moduli back to their original values. This compensating transformation can in fact be decomposed as a product of two factors. First, there is a certain subgroup of the $SL(2,\mathbb{R})$ symmetry group, obtained by conjugating its Borel subgroup by the denominator group $H = SO(2)$, that preserves the charge vector up to an overall scale,[3] while allowing the

[3]This property of the Borel subgroup of $SL(2,\mathbb{R})$ also plays an important role in the twistor-based formulation of a superparticle in $d = 2+1$ dimensions; if one views $SL(2,\mathbb{R})$ as $Spin(2,1)$, this construction

scalar moduli to be transformed to arbitrary values. We can use this subgroup to restore the scalar moduli to their original values. Secondly, we can apply a trombone rescaling to restore the rotated charges to their proper normalisations, while leaving the scalar moduli fixed at their now-restored values. The group structure of the combined Borel-trombone transformations, a subgroup of $GL(2,\mathbb{R})$, is isomorphic to that of the Borel subgroup of $SL(2,\mathbb{R})$. The net effect of the $SL(2,\mathbb{Z})$ transformation followed by the Borel-trombone compensator is to give a solution at a new point on the charge lattice, while remaining in the original vacuum. The full set of such compensated $SL(2,\mathbb{Z})$ transformations will generate the entire irreducible $SL(2,\mathbb{Z})$ multiplet of (p,q) strings in a given vacuum. Note that the compensator transformations do not need to be integer-valued, since they leave the charges fixed.

To be specific, consider the case of a single NS-NS string, with electric charges $Q=(p,q)$. For simplicity, begin with the scalar vacuum defined by $\phi_0=0$, $\chi_0=0$, *i.e.* at the self-dual point $\tau_0=i$. The Dirac quantization condition can be satisfied by choosing to restrict Q to be of the form $Q=(p,q)$, where p and q are arbitrary integers. In order for this to lie on the elementary $SL(2,\mathbb{Z})$ orbit we must take p and q to be relatively prime. Application of $SL(2,\mathbb{Z})$ transformations will then map transitively around the irreducible lattice of elementary charge states. Starting from $Q_1=(p_1,q_1)^{\rm T}$, one thus arrives at the charge

$$Q_2=\begin{pmatrix} p_2 \\ q_2 \end{pmatrix}=\Lambda Q_1\ . \tag{21}$$

In the process, however, the scalar moduli generally become shifted, according to the rule (17). In order to restore the moduli to the original vacuum $\tau_0=i$, while keeping $Q_2=(p_2,q_2)^{\rm T}$ fixed, we must act with a compensating Borel-trombone transformation. Indeed such a (modulus and charge dependent) transformation exists, and is unique. The detailed form of the compensating transformation needed to return τ_0 to its initial value is somewhat complicated, and we shall not present its general form here. As an example, however, one may consider a simpler special case, where the initial charge vector is $Q_1=(1,0)^{\rm T}$, and the $SL(2,\mathbb{Z})$ matrix Λ mapping from Q_1 to Q_2 has $a=p_2$, $c=q_2$. For this special case, the Borel-trombone compensator is

$$\mathbf{B}\,t=\begin{pmatrix} dp_2+q_2^2 & -p_2b-p_2q_2 \\ dq_2-p_2q_2 & p_2^2-bq_2 \end{pmatrix}\ . \tag{22}$$

In the general case of an $SL(2,\mathbb{Z})$ mapping between arbitrary Q_1 and Q_2 on the elementary charge lattice, we note that the Iwasawa decomposition for $SL(2,\mathbb{R})$ allows one to

also naturally generalises to Lorentz groups in higher dimensions, treating the Lorentz group $Spin(d-1,1)$ as the conformal group of the sphere $\mathcal{S}^{d-2}$ [12].

factorize $\Lambda \in SL(2,\mathbb{Z})$ as

$$\Lambda = \tilde{\mathbf{B}}\,\mathbf{H}\ , \tag{23}$$

where $\tilde{\mathbf{B}}$ is an element of the Borel group that leaves Q_2 invariant up to scaling and $\mathbf{H}$ is an element of the stability group $H = SO(2)$ of the point $\tau_0 = i$. This stability group is at the same time the linearly-realized subgroup of the standard classical $G = SL(2,\mathbb{R})$ symmetry group, with the scalar fields taking their values in G/H. Clearly, it is only the $\mathbf{B}$ transformation that actually causes τ_0 to move, so the Borel-transformation part of the compensator must be simply $\mathbf{B} = \tilde{\mathbf{B}}^{-1}$. Consequently, the Borel-transformation parts of the compensator and of Λ cancel out, and one is left simply with

$$\mathbf{B}\,t\,\Lambda = t\,\mathbf{H}\ , \tag{24}$$

i.e. the compensated $SL(2,\mathbb{Z})$ transformation may be realised as a specific $SO(2)$ transformation $\mathbf{H}$ times a trombone rescaling t. It should be noted here that the matrix $\mathbf{H}$ (and also the product $t\,\mathbf{H}$) is not generally an integer-valued matrix.

We shall call the compensated $SL(2,\mathbb{Z})$ transformations the *active* $SL(2,\mathbb{Z})$, in order to distinguish them from the standard supergravity $SL(2,\mathbb{R})$ transformations which move both the modulus τ_0 and the charges at the same time, and which in string theory are discretised to $SL(2,\mathbb{Z})$ and interpreted as a local, or *gauged* symmetry. In other words, states related by the gauged $SL(2,\mathbb{Z})$ are identified in string theory. In fact, the gauged symmetry is what is customarily called the U-duality symmetry [3]. The gauged $SL(2,\mathbb{Z})$ acts linearly on the charges and field strengths of the theory, according to the standard transformation rules. The active $SL(2,\mathbb{Z})$ transformations, on the other hand, maintain a fixed vacuum value of the modulus τ_0, and map the charge vectors between physically-distinct values on the charge lattice in the given fixed vacuum. The active $SL(2,\mathbb{Z})$ manages to avoid moving the scalar modulus by means of the compensation construction, using specific details of the modulus τ_0 that is to be maintained and of the final charge Q that is reached (since the Borel group used is the Borel group for this specific Q). Consequently, the active $SL(2,\mathbb{Z})$ is in general realised *nonlinearly* on the variables of the theory, not only on the scalar fields but also on the 3-form field strengths and the metric. When the action of the active $SL(2,\mathbb{Z})$ transformations is considered specifically on the charges Q, however, this generally nonlinear transformation simplifies to the linear transformation (21). The action on fields in general, however, is nonlinear. Note that although the moduli, *i.e.* the asymptotic values of the scalar fields, are held fixed, the scalar fields throughout spacetime do generally transform. A crucial point, which emphasises the distinction between the gauged and the active $SL(2,\mathbb{Z})$ symmetries, is that while the gauged $SL(2,\mathbb{Z})$ preserves the value of the mass, the active $SL(2,\mathbb{Z})$ changes it, since the metric is also transformed. Although an identification of states under the gauged symmetry is perfectly consistent, it would clearly be inconsistent to identify states which can have different masses under the active symmetry. The active $SL(2,\mathbb{Z})$ is a genuine spectrum-generating symmetry, and should not be confused with the standard U-duality, which cannot generate the spectrum since it cannot change the masses.

The detailed expression for the active transformation can be given without restricting it to the Dirac-quantized $SL(2,\mathbb{Z})$, since the compensation construction given above can be carried out also for all classically-allowed $SL(2,\mathbb{R})$ transformations. The key to deriving the specific form of the t and $\mathbf{H}$ parts of this transformation in (24) is to note that, when acting specifically on the charges Q, the transformation becomes linear, so that in mapping from a charge vector $Q_i = (p_i, q_i)$ to a charge vector Q_j, corresponding to the application of an $SL(2,\mathbb{R})$ matrix Λ, one must have

$$t_{ji}\,\mathbf{H}_{ji}\,Q_i = Q_j = \Lambda Q_i \ , \tag{25}$$

where the i, j subscripts are not indices, but labels corresponding to the charge configurations Q_i, $i = 1, 2, \ldots$ reached.

The trombone scaling part of the transformation (25) is given by

$$t_{ij} = \frac{m_i}{m_j} \ , \tag{26}$$

where m_i is the mass, which is given, in the $\tau_0 = i$ vacuum that we are initially considering here, by

$$m_i = \sqrt{p_i^2 + q_i^2} \ . \tag{27}$$

This expression for m_i is an example of an H-invariant "length" for a charge vector, as referred to earlier, specialised to the $\tau_0 = i$ vacuum. We shall give the corresponding expression for general τ_0 shortly.

The $H = SO(2)$ part of the transformation is given by

$$\mathbf{H}_{ij} = \begin{pmatrix} \cos\theta_{ij} & \sin\theta_{ij} \\ -\sin\theta_{ij} & \cos\theta_{ij} \end{pmatrix} \ , \qquad \theta_{ij} = \theta_i - \theta_j \ , \tag{28}$$

where $\tan\theta_i = p_i/q_i$. Note that the product $t_{ij}\,\mathbf{H}_{ij}$ is given by

$$t_{ij}\,\mathbf{H}_{ij} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \ , \tag{29}$$

where

$$\alpha = \frac{p_1\, p_2 + q_1\, q_2}{p_1^2 + q_1^2}\ , \qquad \beta = \frac{p_2\, q_1 - p_1\, q_2}{p_1^2 + q_1^2}\ . \tag{30}$$

It is straightforward to generalise the above discussion to the case where a generic point $\tau_0 = \chi_0 + i/g$, rather than $\tau_0 = i$, is taken to be the vacuum modulus ($g = e^{\phi_0}$ is the string coupling constant). This can be done by noting that the stability group H of τ_0 leaves the matrix $\mathcal{M}_0$ invariant, *i.e.* $\mathcal{M}_0 = \mathbf{H}\, \mathcal{M}_0\, \mathbf{H}^T$, where $\mathcal{M}_0$ is the matrix $\mathcal{M}$ defined in (5), but with the scalar fields replaced by their asymptotic values $\phi_0 = \log g$ and χ_0, and $\mathbf{H}$ denotes an element of H. We can write $\mathcal{M}_0 = V\, V^T$, where the "vielbein" V is given by the $SL(2,\mathbb{R})$ matrix

$$V = \frac{1}{\sqrt{g}} \begin{pmatrix} 1 & g\, \chi_0 \\ 0 & g \end{pmatrix}\ . \tag{31}$$

The previous modulus point $\tau_0 = i$ corresponds to $\mathcal{M}_0 = 1$. The matrix V (31) may also be viewed as an $SL(2,\mathbb{R})$ element that maps the vacuum modulus from i to τ_0. Thus the stability group matrices for a generic τ_0 will be of the form (28), but conjugated with V, so that the group element $\mathbf{H}_{ij}$ is now given by

$$\mathbf{H}_{ij} = V \begin{pmatrix} \cos\tilde\theta_{ij} & \sin\tilde\theta_{ij} \\ -\sin\tilde\theta_{ij} & \cos\tilde\theta_{ij} \end{pmatrix} V^{-1}\ , \qquad \tilde\theta_{ij} = \tilde\theta_i - \tilde\theta_j\ , \tag{32}$$

where

$$\tilde\theta_{ij} = \tilde\theta_i - \tilde\theta_j \tag{33}$$

$$\tan\tilde\theta_i = \tilde p_i / \tilde q_i\ . \tag{34}$$

The quantities $\tilde Q_i = (\tilde p_i, \tilde q_i)^T$ are related to the charges[4] $Q_i = (p_i, q_i)^T$ by

$$\tilde Q_i = V^{-1}\, Q_i\ . \tag{35}$$

[4]We continue to take the electric charges $Q = (p, q)$ to be integers in the generic vacuum. This is purely for convenience, and any other choice allowed by the Dirac quantisation condition will lead to identical conclusions about the nonlinearly-realised spectrum-generating group. In fact the Dirac quantisation condition does not by itself completely determine the lattice of allowed charge states; any $SL(2,\mathbb{R})$ transformation of the integer lattice would be equally valid (provided of course that the lattice of magnetic 5-brane charges is transformed contragediently at the same time). In order to fix the lattice, one needs also to specify a set of standard charge states in each given vacuum. In the present purely electric-charge system, it is natural to make this choice without regard to the values of the scalar moduli, following Ref. [9]. This choice preserves the symmetry between the two 3-form field strengths $F_3^{(i)}$, and has the consequence that the $SL(2,\mathbb{Z})$ group that preserves the charge lattice is always represented by integer-valued matrices, irrespective of the value of the scalar modulus τ_0.

This implies that

$$\tan\tilde{\theta}_i = g\,\tan\theta_i - g\,\chi_0\ . \tag{36}$$

The trombone scaling t_{ij} is again given by (26), but the mass m_i is now given by

$$m_i^2 = Q_i^T\,\mathcal{M}_0^{-1}\,Q_i = \tilde{Q}_i^T\,\tilde{Q}_i\ . \tag{37}$$

The product $t_{ji}\,\mathbf{H}_{ji}$ transforms the charges according to (25), while leaving the generic vacuum τ_0 invariant.

The mass defined in (37) is also the H-invariant "length" of the charge vector for the vacuum specified by τ_0. It has the property of being preserved in form under H transformations, which hold the moduli fixed.

4 Group structure of the active $SL(2,\mathbb{Z})$

As is clear from the fact that the compensated active transformation constructed above is generated solely by the $SO(2)$ and trombone combination (24,26,32), this cannot be a linear realisation of $SL(2,\mathbb{R})$ or $SL(2,\mathbb{Z})$. Nonetheless, we have in the continuous-parameter case a perfectly proper realisation of $SL(2,\mathbb{R})$, and this may then be restricted to its discrete $SL(2,\mathbb{Z})$ subgroup. We shall focus on the continuous-parameter case in demonstrating that this is an acceptable group realisation. The nonlinear dependence of the transformation on the scalar moduli and on the initial values of the charges $Q_1 = (p_1, q_1)$, as can be seen in (32), requires care in establishing that this is a proper realisation of $SL(2,\mathbb{R})$.

In order to have a proper group action of $SL(2,\mathbb{R})$ on a set X, in which an invertible operator $\mathcal{O}(\Lambda)$ maps the point $x \in \mathsf{X}$ into $\mathcal{O}(\Lambda)(x)$, it is necessary that the composition of two such transformations respect the $SL(2,\mathbb{R})$ group composition rules, *i.e.*

$$\mathcal{O}(\Lambda_2)\Big(\mathcal{O}(\Lambda_1)(x)\Big) = \mathcal{O}(\Lambda_2\Lambda_1)(x)\ , \tag{38}$$

where the product $\Lambda_2\Lambda_1$ is the ordinary matrix product of two $SL(2,\mathbb{R})$ matrices. The set X being acted upon here consists of the full set of fields of type IIB supergravity, since all of them, including the metric, transform either under the $SO(2)$ part (32) or the trombone part (26) of the transformation (24). More explicitly, in a notation that indicates the action on the charges, we may write the composition law as

$$\mathcal{O}(\Lambda_2, \Lambda_1 Q_1)\,\mathcal{O}(\Lambda_1, Q_1) = \mathcal{O}(\Lambda_2\Lambda_1, Q_1)\ . \tag{39}$$

For the nonlinear realisation (24), establishing the composition property (38) amounts to verifying that the trombone part (26) and the $SO(2)$ part (32) separately respect the

$SL(2,\mathbb{R})$ group composition rules. We consider mapping from a configuration labelled 1 to a configuration labelled 2 and then on to a configuration labelled 3. For the trombone part of the transformation, the check of $SL(2,\mathbb{R})$ composition is straightforward:

$$t_{32}\, t_{21} = \frac{m_3}{m_2}\cdot\frac{m_2}{m_1} = t_{31}\ , \tag{40}$$

where the masses m_i for general τ_0 are given by (37). For the $SO(2)$ part of the transformation, the check of $SL(2,\mathbb{R})$ composition starts by combining the matrices (32):

$$\begin{aligned} \mathbf{H}_{32}\,\mathbf{H}_{21} &= V\begin{pmatrix} \cos\tilde\theta_{32} & \sin\tilde\theta_{32}\\ -\sin\tilde\theta_{32} & \cos\tilde\theta_{32}\end{pmatrix} V^{-1}\, V\begin{pmatrix} \cos\tilde\theta_{21} & \sin\tilde\theta_{21}\\ -\sin\tilde\theta_{21} & \cos\tilde\theta_{21}\end{pmatrix} V^{-1}\\ &= V\begin{pmatrix} \cos(\tilde\theta_{32}+\tilde\theta_{21}) & \sin(\tilde\theta_{32}+\tilde\theta_{21})\\ -\sin(\tilde\theta_{32}+\tilde\theta_{21}) & \cos(\tilde\theta_{32}+\tilde\theta_{21})\end{pmatrix} V^{-1} \end{aligned} \tag{41}$$

Next, the sum of the $SO(2)$ angles $\tilde\theta_{32}+\tilde\theta_{21}$ combines to produce a single angle $\tilde\theta_{31}$, as one can see from (33). Thus we have the desired composition law

$$\mathbf{H}_{32}\,\mathbf{H}_{21} = \mathbf{H}_{31}\ . \tag{42}$$

In terms of the notation in the group composition rule (39), this becomes

$$\Big(\tilde\theta(\Lambda_2\Lambda_1 Q_1) - \tilde\theta(\Lambda_1 Q_1)\Big) + \Big(\tilde\theta(\Lambda_1 Q_1) - \tilde\theta(Q_1)\Big) = \tilde\theta(\Lambda_2\Lambda_1 Q_1) - \tilde\theta(Q_1)\ , \tag{43}$$

where the $\tilde\theta(Q)$ is given by (34) and (35).

Consequently, when acting on any field of the type IIB supergravity theory, the non-linearly-realised transformation (24) respects the $SL(2,\mathbb{R})$ group composition rule, so we do in fact have a proper $SL(2,\mathbb{R})$ realisation. One may then simply restrict this to the $SL(2,\mathbb{Z})$ discrete subgroup in order to obtain the transformations that map between the allowed solutions corresponding to points on the Dirac-quantised elementary charge lattice.

It should again be emphasised that the discretisation of the active spectrum-generating $SL(2,\mathbb{R})$ is a consequence purely of the Dirac quantisation condition, and involves no additional input. The discretisation of the standard $SL(2,\mathbb{R})$ symmetry to the conjectured $SL(2,\mathbb{Z})$ U-duality group is quite a different matter, however, and is one that can be settled only with some additional input, for example from string theory; the Dirac quantisation condition for BPS states is not by itself enough to imply a discretisation of the standard $SL(2,\mathbb{R})$. The fact that in the type IIB theory the standard $SL(2,\mathbb{R})$ is a purely non-perturbative symmetry makes the issue of its discretisation particularly tricky to study, and emphasises the importance of not confusing it with the active spectrum-generating

$SL(2,\mathbb{R})$, whose discretisation is easily established. In order to clarify this point, let us try to see at what stage one is in a position to deduce that the standard $SL(2,\mathbb{R})$ symmetry is discretised to $SL(2,\mathbb{Z})$. As we have previously observed, the Dirac quantisation condition itself can be satisfied by any charge lattice that is generated by an arbitrary $SL(2,\mathbb{R})$ transformation of a given allowed lattice. On the other hand, at the classical level we have a global $SL(2,\mathbb{R})$ symmetry that allows us to view a modulus point τ_0^i with charge lattice C^i as merely a relabelling of a modulus point τ_0^j with charge lattice C^j if the moduli and charges are simply related to one another by an $SL(2,\mathbb{R})$ symmetry transformation. Thus to avoid an overcounting of (τ_0^i, C^i) pairs, we should mod out by this $SL(2,\mathbb{R})$, viewed as a relabelling symmetry, and only count as distinct those pairs that are not related by $SL(2,\mathbb{R})$ symmetry transformations. In other words, the set of all possible charge lattices above every point on the $SL(2,\mathbb{R})/SO(2)$ modulus space is equivalent to a principle fibre bundle with an $SL(2,\mathbb{R})$ fibre above each modulus point. Modding out by the relabelling amounts to choosing a cross-section of this bundle.

Different choices of cross-section should not generally be thought of as being physically equivalent. In particular, different choices, all equally allowed by the Dirac quantisation condition, can give differing unbroken residual symmetries. For example, there is a family of cross-sections obtained by taking an integer charge lattice at some given modulus point, and then, at other modulus points that are obtained from the given one by operation with an $SL(2,\mathbb{R})$ transformation Λ, choosing the charge lattice to be Λ rotations of the initial integer lattice. Such a cross-section leaves the full classical $SL(2,\mathbb{R})$ symmetry unbroken, by construction. Another example is to choose the *same* charge lattice C for every point τ_0^i in the modulus space. In this case, there is a residual $SL(2,\mathbb{Z})$ symmetry of the (τ_0^i, C) pairs. The specific charge lattice chosen in this second example need not be an integer lattice, but in this case, there will be a relabelling of the (τ_0^i, C) pairs *via* a fixed $SL(2,\mathbb{R})$ transformation that transforms the charge lattices into integer lattices. The surviving $SL(2,\mathbb{Z})$ symmetry for a given cross-section of this type will be given by conjugation of the integer-valued representation of $SL(2,\mathbb{Z})$ by the fixed $SL(2,\mathbb{R})$ element. In such a case, with a surviving $SL(2,\mathbb{Z})$, one has the option of dividing out by this surviving discrete symmetry, an option that has been argued to be taken up in string theory. This has the effect of restricting the modulus space to its fundamental domain.

For a generic cross-section, the classical $SL(2,\mathbb{R})$ will be completely broken down to the identity. The conclusion is that the standard classical $SL(2,\mathbb{R})$ symmetry is not discretised by virtue of the Dirac quantisation condition alone. In order to determine the unbroken symmetry, one must choose a cross-section of the bundle of lattices over modulus space. This choice must be made using additional information over and beyond the Dirac quantisation condition. Frequently in the litterature, one encounters the additional requirement that purely electric states exist in the lattice, for example. And string theory introduces modifications to the classical theory that appear to select the $SL(2,\mathbb{Z})$-preserving cross-sections. Evidence for the discretisation of the standard $SL(2,\mathbb{R})$ in string theory has been offered in [13], where candidate counterterms that preserve only an $SL(2,\mathbb{Z})$ subgroup of the standard $SL(2,\mathbb{R})$ have been given.

In referring to the $SL(2,\mathbb{Z})$ "gauged" subgroup of the standard supergravity $SL(2,\mathbb{R})$ in this paper, we are making an implied choice of the second kind of lattice-bundle cross-section described above, for which the same lattice of canonical charges is chosen at each point in modulus space. Then, by an $SL(2,\mathbb{R})$ transformation and relabelling of the points of modulus space, these lattices may also be taken to be integer-valued. This choice appears to be the standard one in string-theory discussions [9], but the additional assumptions lying beneath this choice should be more carefully inspected.

5 Group-theoretical structure of the spectrum

The active $SL(2,\mathbb{R})$ transformation acts transitively on the charge-vector space of the type IIB theory's string soliton solutions in a given vacuum. This space is the two-dimensional Euclidean plane with the origin excluded, $\mathbb{E}^2\backslash\{\vec{0}\}$ (the origin is excluded if one wants to focus attention only on solutions having the same degree of unbroken supersymmetry, thus excluding the zero-charge pure Minkowski-space solution). This transitive group action makes it straightforward to identify the spectrum from a group-theoretical perspective. Whenever a group's orbit coincides with the whole of the set on which it acts, *i.e.* whenever the group acts transitively on the set, the realisation is equivalent to that on a coset space G/S_{x_0}, where S_{x_0} is the stability group of any particular chosen point x_0 on the orbit.

In the classical $SL(2,\mathbb{R})$ case with a continuous charge-vector space, the stability group of a given charge Q is isomorphic to the strict Borel group, $S_Q \cong B_{\rm strict}$, whose entries are purely upper-triangular (*i.e.* excluding the Cartan subalgebra). For the present case with $G = SL(2,\mathbb{R})$, the coset space $SL(2,\mathbb{R})/B_{\rm strict}$ is indeed equivalent to $\mathbb{E}^2\backslash\{\vec{0}\}$. In order to see this, consider the equivalence relation implied by membership in a given $SL(2,\mathbb{R})/B_{\rm strict}$ coset. Taking the charge vector $(1,0)$ to correspond to the chosen point x_0 above, the corresponding stability group has elements

$$B_{\text{strict}} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} , \tag{44}$$

where $k \in \mathbb{R}$ is an arbitrary real number. The equivalence relation between coset members is then

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\sim \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & ak+b \\ c & ck+d \end{pmatrix} , \end{aligned} \tag{45}$$

implying that *all* $SL(2,\mathbb{R})$ matrices with given a and c are equivalent. The natural label for this equivalence class is the vector (a,c), whose entries cannot both vanish because a, b, c, d must satisfy $ad-bc=1$. (In the discrete case, the equivalence relation holds for all $SL(2,\mathbb{Z})$ matrices with given a and c.) Indeed, this labelling establishes an equivariant map between $SL(2,\mathbb{R})/B_{\text{strict}}$ and $\mathbb{E}^2\backslash\{\vec{0}\}$, thus establishing the equivalence. (In other words, this is a one-to-one map between $SL(2,\mathbb{R})/B_{\text{strict}}$ and $\mathbb{E}^2\backslash\{\vec{0}\}$ that preserves the action of the $SL(2,\mathbb{R})$ group.)

In the quantum case, where the active symmetry group is reduced to $SL(2,\mathbb{Z})$, the space of states allowed by the Dirac quantisation condition is the charge lattice (p,q) with p and q non-vanishing integers. The action of $SL(2,\mathbb{Z})$ on this set is not transitive, however. As we have observed earlier, the various disjoint orbits of this discrete group may be characterised by the points $(n,0)$ that they contain. We have identified the irreducible orbit containing the point $(1,0)$ as the "elementary" orbit. When the charge lattice is restricted to this sub-lattice, for which the integers p and q are always relatively prime, the action of $SL(2,\mathbb{Z})$ once again becomes transitive. We may then make a group-theoretical identification of this elementary discrete orbit. The standard charge vector $(1,0)$ lies on the elementary $SL(2,\mathbb{Z})$ orbit, and for this chosen point the stability group $B_{\text{strict}}(\mathbb{Z})$ is once again of the form (44), but with k now restricted to be an integer, $k\in\mathbb{Z}$. Then the equivalence relation between coset elements in $SL(2,\mathbb{Z})/B_{\text{strict}}(\mathbb{Z})$ is once again of the form (45), but now with $k\in\mathbb{Z}$. The natural label for this equivalence class is again (a,c), but with a and c now integers. These charge vectors coincide precisely with the points on the elementary $SL(2,\mathbb{Z})$ orbit because $SL(2,\mathbb{Z})$ matrices cannot satisfy the constraint $ad-bc=1$ unless the integers a and c are relatively prime.

In the continuous-charge classical case, the strict Borel group (44) is isomorphic to $\mathbb{R}$, so the classical charge spectrum may be identified as $SL(2,\mathbb{R})/\mathbb{R}$. In the quantum case, the strict Borel group is isomorphic to $\mathbb{Z}$, so the elementary orbit of string solitons may be identified as $SL(2,\mathbb{Z})/\mathbb{Z}$.

6 Lower-dimensional cases

Up until now, our discussion has focussed on the type IIB supergravity in $D = 10$. As we shall now show, many of the features that we encountered there persist in lower-dimensional examples. However, there are also some additional subtleties that need to be considered.

In the $D = 10$ type IIB case, the only transformation beyond the standard $SL(2,\mathbb{R})$ symmetry that was needed in order to construct the active symmetry group that leaves the scalar modulus fixed was a single trombone scaling transformation. Indeed, this trombone symmetry of the equations of motion is the only readily available symmetry that one has in any dimension which can be used in the compensation construction of such fixed-modulus active transformations. Although it might not seem immediately apparent that this is all that is necessary in lower dimensions, where the symmetry groups are larger, this does in fact prove to be the case, at least for the construction of symmetries acting on multiplets of "fundamental" supergravity solitons, in a sense that we shall now define.

The key point to recognise in dealing with the lower-dimensional cases is that for any of the maximally-noncompact supergravity symmetry groups G shown in Table 1, one has an Iwasawa decomposition of a general group element Λ, specialised to the vacuum point on the scalar modulus manifold G/H and to the charges Q defining a given fundamental soliton solution:

$$\Lambda = \mathbf{B}\,\mathbf{H}\ , \tag{46}$$

where $\mathbf{H}$ is an element of the stability group H of the vacuum modulus point $\mathcal{M}_0$ on G/H and $\mathbf{B}$ is an element of the Borel subgroup corresponding to Q. (The Iwasawa decomposition and the Borel subgroups (which leave highest weight vectors invariant up to rescaling) of the global supergravity symmetry groups E_{11-D} were extensively studied in [17].) In writing this Iwasawa decomposition, we are making an important assumption that the supergravity p-brane soliton is "fundamental," in the sense of lying on the same symmetry orbit as a single-charge solution, for which only one of the theory's field strengths is non-vanishing. In this case, $\mathbf{B}$ can be taken to belong to a subgroup of G that is isomorphic to the canonical upper-triangular Borel subgroup, *via* a similarity transformation using an appropriate element of H, such that all the elements of this subgroup leave the charge Q invariant up to an overall scaling. In this case, it is clear that all that is necessary to construct a nonlinearly-realised active symmetry transformation is a single trombone transformation that can compensate for the overall scaling of Q. Indeed, even though we have more complicated group structures in lower dimensions, it should be noted that there exists precisely one independent trombone symmetry, since if there were two, we could find one combination that left the Einstein-frame metric invariant, and hence would be part of the standard supergravity global symmetry group.

D	G	H
9	$GL(2,\mathbb{R})$	$SO(2)$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$
7	$SL(5,\mathbb{R})$	$SO(5)$
6	$SO(5,5)$	$SO(5) \times SO(5)$
5	$E_{6(+6)}$	$USp(8)$
4	$E_{7(+7)}$	$SU(8)$
3	$E_{8(+8)}$	$SO(16)$

Table 1: Supergravity symmetry groups.

The restriction to fundamental solutions in this discussion derives from the requirement that there exist a Borel subgroup leaving the charge configuration Q invariant up to an overall scaling. If only a single charge component is turned on, it is easy to see that such a Borel subgroup will exist. Moreover, such a Borel subgroup will exist for any charge configuration on the same orbit as the single-charge configuration, obtainable by an appropriate similarity transformation. To see this in detail, we first note that the single-charge solutions supported by field strengths of a given rank form an irreducible representation under the Weyl group [10] of the standard supergravity symmetry group as shown in Table 1. One of these single-charge solutions will be a highest-weight state [14], say with charge $Q_{\rm h}$, and will therefore be invariant up to scaling under the canonical upper-triangular Borel group[5] $B_{\rm c}$. Any solution, with charge $Q = \Lambda Q_{\rm h}$, lying on the same orbit as the highest-weight solution, will have a corresponding Borel group B, obtained from the canonical one by the similarity transformation[6] $B = \Lambda B_{\rm c} \Lambda^{-1}$.

[5]The algebra of the canonical Borel group comprises the positive-root generators and the Cartan generators of the supergravity symmetry group.

[6]Note that if one makes an Iwasawa decomposition of Λ into an element of the vacuum stability group H and an element of $B_{\rm c}$, then only the element of H is effective in moving $B_{\rm c}$ into B, so the similarity transformation rotating $B_{\rm c}$ into B is effectively made by an element of H.

Once one has made the decomposition (46) of a general group element, the Borel-group factor is the one that should be cancelled out in constructing the compensated active symmetry transformation. In order to see that the Borel group B is precisely the part of G that is effective in moving the scalar moduli, note that the scalar manifold G/H can be parametrised in the fashion

$$\mathcal{M} = VV^{\#} \; , \tag{47}$$

where $V^{\#} = \tau(V^{-1})$ and τ is the Cartan involution, whose fixed-point set is the maximal compact subgroup H. In simple cases, where H has a regular embedding in G, $V^{\#} = V^{\mathrm{T}}$ if H is orthogonal, $V^{\#} = V^{\dagger}$ if H is unitary and $V^{\#} = \Omega V^{\dagger}$ if H is a USp group, with Ω its invariant symplectic matrix. The matrix V generalises the vielbein (31) of the $SL(2,\mathbb{R})$ case, and is an element of the Borel group. The scalar moduli are then characterised by the matrix $\mathcal{M}_0$, which is the asymptotic form of $\mathcal{M}$. Thus, the Borel group is precisely what is needed in order to move the scalar moduli around on G/H, since given an element $\mathbf{B} \in B$, the matrix $\mathcal{M}$ (which in general transforms according to $\mathcal{M} \to \Lambda \mathcal{M} \Lambda^{\#}$) transforms to

$$\begin{aligned} \mathcal{M}' &= \mathbf{B} \mathcal{M} \mathbf{B}^{\#} \\ &= \mathbf{B} V (\mathbf{B} V)^{\#} \; , \end{aligned} \tag{48}$$

so that V transforms to $\mathbf{B}V$, which is just another element of the same Borel group.

In the solitonic-string example that we began with in $D = 10$ type IIB supergravity, all of the solitonic solutions are of the fundamental type. In dimensions 9 and lower, however, "multi-charge" solutions are also found, for which the construction of an active symmetry transformation using a single trombone scaling transformation is not possible. These multi-charge solutions[7] are all characterised by a lower degree of supersymmetry preservation than the fundamental solutions. The fundamental solutions all preserve 1/2 of the rigid supersymmetry of the supergravity theory, but the multi-charge solutions preserve 1/4 or less. Such multi-charge solutions also possess static generalisations in which the charge centers corresponding to the independent field strengths are separated, so the multi-charge solutions may be interpreted as "bound states at threshold" of single-charge solutions [18, 19, 20]. It is not yet known whether there is a group-theoretical origin to the various moduli of the multi-charge solutions. If there is, including a transformation that generalises the trombone symmetry of this paper but such that the various independent charge components

[7] In the classification of Refs [15, 16], the single-charge solutions are characterised by a parameter Δ that takes the value 4, and multi-charge solutions are characterised by $\Delta = 4/N$, with $N > 1$.

are independently scaled, then the above compensation construction of a fixed-modulus active symmetry could also be carried out for the multi-charge solutions.

7 Anomalies

The trombone scaling symmetry that forms a crucial ingredient in the above discussion arose as a symmetry of the classical equations of motion. If the BPS soliton multiplets are to survive at the quantum level, it is important to establish that there is no scaling anomaly that destroys the trombone symmetry. However, this symmetry is extremely sensitive to quantum corrections. For generic solutions of the classical equations, it is clear that quantum counterterms will certainly spoil this symmetry, because they have different scaling dimensions from that of the classical Lagrangian. Nonetheless, the possibility of having soliton multiplets incorporating the trombone scaling is not by this fact immediately ruled out for the BPS-saturated p-brane solutions that we have been studying.

The relevant question for the BPS solutions is whether they persist as solutions with arbitrary scale size in the quantum-corrected effective equations of motion. To illustrate the problem, consider the first dangerous perturbative counterterm, *i.e.* the first one that cannot be absorbed just by a renormalisation of the fields of the effective theory. This is the first supersymmetric invariant that does not vanish when subjected to the classical-level equations of motion. The counterterm contains a term quartic in Riemann tensors, plus superpartners for the other fields; the purely gravitational part of the invariant is the square of the Bel-Robinson tensor [21]. In all extended supergravities, there are extended-supersymmetric generalisations of this counterterm [22, 23, 24, 25, 26]. In all $D=4$ supergravity cases, these counterterms are expected to occur with infinite coefficients at the three-loop order [27]; in the sigma-model beta-function approach to string effective field theories, the corresponding finite contributions to the effective action occur at the $(\alpha')^3$ order [28].

In $N=2$, $D=4$ superfields, the quartic counterterm takes the form [23, 24]

$$\Delta I_3 = \int d^4x d^8\theta E W_{\alpha\beta} W^{\alpha\beta} \bar{W}_{\dot{\alpha}\dot{\beta}} \bar{W}^{\dot{\alpha}\dot{\beta}} \ , \tag{49}$$

where E is the determinant of the supervielbein, $W_{\alpha\beta}$ is the $N=2$ supergravity conformal-field-strength superfield, and all two-component indices α, $\dot{\alpha}$ are referred to the Lorentz-covariant tangent space. As one can see, this $D=4$ counterterm scales like λ^{-2} under the trombone scaling (2), whereas the $D=4$ classical action scales as λ^2, so the presence of this counterterm certainly ruins the trombone symmetry as far as general field configurations are concerned. Nonetheless, one might still hope to find the integration constants, implied at the classical level by the trombone symmetry (2), to be present in *certain classes* of solutions to the full theory, such as the class of BPS states.

The question of whether the counterterm (49) vanishes in BPS backgrounds has been discussed in [29], where it was shown that the counterterm itself vanishes for one of the classic BPS solutions, *i.e.* for the original extreme Reissner-Nordstrom solution [30]. Unfortunately, it is not enough to establish just that counterterms themselves vanish in BPS backgrounds, so the analysis of Ref. [29] remains incomplete. In order for the BPS solutions to be stable against quantum corrections, it is necessary to ensure that the *variations* of counterterms vanish in BPS backgrounds, *i.e.* that the BPS field configurations remain good solutions when substituted into the counterterm-corrected equations of motion.

We shall not carry out a full nonlinear analysis of this question for the counterterm (49), but shall be content to indicate how its variation manages to vanish for BPS backgrounds using a linearised superfield analysis [26] considering, for example, an asymptotic region where the deviations from flat empty space are small. This analysis is actually made simpler by the dangerousness of the counterterm (49), which is built using precisely those field-strength combinations that do not vanish when subjected to the classical Einstein-supergravity field equations, *i.e.* from the field strengths of $D = 4$, $N = 2$ *conformal* supergravity. Of course, as one can see from its non-trivial scaling dimension, (49) is not really a conformal supergravity invariant. However, at the leading order in deviations from flat space, it becomes a linearised superconformal invariant, and so it depends to this order only on the non-gauge parts of the $N = 2$ supergravity multiplet with respect to $N = 2$ conformal supergravity. At the linearised level, the superfield $W_{\alpha\beta}$ may be expressed in terms of an $N = 2$ conformal supergravity prepotential V [26, 31]

$$W_{\alpha\beta} = i\bar{D}^4 D_{\alpha\beta} V \ , \tag{50}$$

where $D_{\alpha\beta}$ is given in terms of the basic superspace covariant derivatives by $D_{\alpha\beta} = \frac{1}{2}\epsilon_{ij} D^i_\alpha D^j_\beta$. In varying the lowest-order term of (49), which is quartic in small quantities and thus depends only on the linearised superconformally-invariant $W_{\alpha\beta}$ superfield, it is sufficient to vary the conformal prepotential V (*i.e.* the variation of (49) at lowest order can have contributions only from the variables actually present in (49) at that order, and these are the non-gauge parts of the conformal supergravity multiplet). The result is a contribution to the equation of motion for the singlet scalar auxiliary field of the $N = 2$ multiplet [32] (*i.e.* to the field conjugate to the lowest component of V). Using the supergravity constraints and equations of motion $\bar{D}_{\dot{\alpha}\, i} W_{\alpha\beta} = 0$, $D^i_\alpha W_{\beta\gamma} = D^i_{(\alpha} W_{\beta\gamma)}$ (where the brackets denote symmetrisation with strength one), for a purely bosonic field configuration, together with the condition characterising the BPS extreme Reissner-Nordstrom solution [29]

$$C_{\alpha\beta\gamma\delta} = -\nabla^{\dot\rho}_{(\alpha} W_{\beta\gamma} \hat K_{\delta)\dot\rho}\ , \tag{51}$$

where $\hat K_{\alpha\dot\beta}$ is the normalised timelike Killing vector of the static BPS solution with indices referred to the tangent space, one straightforwardly verifies that the variation of (49) with respect to V vanishes.

The $D = 4$ extreme Reissner-Nordstrom solution thus manages to escape from this threat of a perturbative anomaly in the trombone scaling symmetry, but the anomalies arising from (49) and from higher-order terms certainly do affect more general non-BPS solutions. In fact, the extreme Reissner-Nordstrom black-hole solution is known from $d = 2$ string sigma-model considerations to give a fully conformally invariant sigma model [33], thus managing to be an exact solution to the full effective supergravity field theory with arbitrary scale size to all orders in α'. This example serves to illustrate how important the BPS saturation conditions are for the existence of duality multiplets. In view of the anomalies in the trombone scaling symmetry at the quantum level for general field configurations, there is no reason to expect non-BPS solutions to form duality multiplets such as the $SL(2,\mathbb{Z})$ lattice that describes the BPS solutions.

8 Conclusion

In this paper, we have shown that the full set of integration constants for fundamental BPS supergravity p-brane solutions preserving half of the supersymmetry may be associated to symmetries of the theory. The essential element that goes beyond the standard supergravity symmetry groups G shown in Table 1 is the trombone scaling transformation; this allows one to reach solutions at different mass levels by the application of symmetry transformations. Provided attention is restricted to the fundamental solutions preserving half of the supersymmetry, this single extra transformation is all that is necessary in order for one to be able reach the full BPS parameter space, for supergravity in any spacetime dimension. For “multi-charge solutions” preserving less than half of the supersymmetry, it remains unclear whether the full parameter space of the solutions can be reached by symmetry transformations. However, since the multi-charge solutions may be interpreted as superpositions of the fundamental ones, the most essential class of BPS p-branes may be comprehensively discussed from a group-theoretical basis.

The moduli of the scalar fields of a supergravity theory form a special class of integration constants, for they include the various vacuum angles and coupling constants for the theory in a given vacuum, and in effect define the vacuum. When one restricts attention to a given vacuum, the BPS solutions are entirely characterised by their charges. We have shown that there exists a nonlinear realisation of the symmetry group G, which is quite distinct from the standard one in that it makes essential use of the trombone scaling transformation and hence can map between solutions at different mass levels, while transforming the scalar fields in such a way as to hold fixed their asymptotic values, *i.e.* while holding the scalar moduli fixed. This nonlinearly realised symmetry is distinguished from other ways of covering the charge-vector space in that it allows a restriction at the quantum level to a discretised group $G(\mathbb{Z})$ that maps only between states permitted by the Dirac quantisation condition. This nonlinear realisation of $G(\mathbb{Z})$ is thus the true spectrum-generating symmetry of the theory, which we have called the "active" $G(\mathbb{Z})$.

The only known way consistently to quantise supergravity theories is superstring theory, of course, so one needs to take into account the string-theory modifications to the above group-theoretical picture. One important class of such modifications is the infinite series of corrections to the effective field theory Lagrangian, of the same general forms as the infinite counterterms of supergravity when quantised as a field theory, but with individually finite coefficients when obtained from superstring theory. These perturbative counterterms have different scaling behaviour from the classical supergravity Lagrangian, and hence threaten the existence of the active $G(\mathbb{Z})$ multiplets. When attention is restricted to BPS solutions, however, we have argued that the active $G(\mathbb{Z})$ multiplet structure is maintained, because at least this class of solutions to the effective theory persists with arbitrary scale size even in the presence of the perturbative counterterms. What happens in the light of non-perturbative corrections to the effective field theory is a different question that should also be carefully considered. We cannot currently shed much light on this question, but would hope that the active discrete $G(\mathbb{Z})$ multiplet structure would persist also in the face of the non-perturbative corrections. Of course if the scaling symmetry were to be broken for BPS states, this would be a problem not only for the trombone symmetry *per se*, but for the whole idea of the existence of $G(\mathbb{Z})$ multiplets of BPS states in a given vacuum.

Another important modification that superstring theory makes to the above group-theoretical picture is the identification of states related by the standard $G(\mathbb{Z})$ transformations which move the p-brane charges and also the scalar moduli at the same time. Because these standard $G(\mathbb{Z})$ transformations are interpreted in this local, or gauged fashion, we have called this group the "gauged" $G(\mathbb{Z})$. Such an identification is possible but not required at the level of supergravity field theory, but at least those parts of this group that coincide with string theory T-duality transformations need to be given a local interpretation, expressing identifications between modulus/charge configurations related by such transformations. In Ref. [3], it was suggested that this local interpretation be extended to the full gauged $G(\mathbb{Z})$ group. This local interpretation marks another important distinction to be drawn between the gauged $G(\mathbb{Z})$ and the spectrum-generating active $G(\mathbb{Z})$ that we have introduced. After the identifications, the gauged $G(\mathbb{Z})$ orbits consist of single points, whereas the active $G(\mathbb{Z})$ continues to map transitively around the entire charge lattice of elementary BPS states in a given vacuum. As for the active $G(\mathbb{Z})$ symmetry, one must also consider whether the *gauged* $G(\mathbb{Z})$ symmetry survives quantum corrections. At first sight this symmetry, like the trombone symmetry, might also seem to be at risk from quantum anomalies. If one wishes to identify states under the gauged $G(\mathbb{Z})$ then it is important that it survive as a symmetry of the full theory, and not merely when restricted to BPS solutions. Evidence that $G(\mathbb{Z})$ may survive in non-perturbative string effective field theories, while the standard continuous classical G symmetry is broken, has recently been offered in [13].

The string-theory-induced identifications may have hidden the necessity for the active $G(\mathbb{Z})$ transformations that we have described. Indeed, the identification of scalar moduli under the gauged $G(\mathbb{Z})$ in the true vacuum sector of the theory, *i.e.* for vanishing p-brane charges, or in other words for flat-space metrics, makes it tempting to simply overlook the requirement of coordinated transformations of charges at the same time as scalar moduli that characterises the standard gauged $G(\mathbb{Z})$ transformations. For example in Ref. [34], a proposal was made to generate the charge lattice of BPS solitons using just the gauged $G(\mathbb{Z})$, by effectively declaring that the scalar moduli being mapped between are identified, while the charges being mapped at the same time are not. We must admit to being puzzled by this proposal, since the only consistent way to implement a discrete group such as $G(\mathbb{Z})$ as a gauged or local symmetry is to declare that the states are actually equivalence classes with respect to the action of the group. Since this action maps the charges and scalar moduli together, it would seem that only the definition of equivalence classes under this joint action could be mathematically consistent. The distinct and independent active $G(\mathbb{Z})$ transformations introduced in the present paper make unnecessary such a construction, in any case. Moreover, the high degree of sensitivity of the active $G(\mathbb{Z})$ transformations to quantum corrections, owing to their trombone-scaling-transformation content, makes clear a feature that would not otherwise be prominent: only the BPS solutions enjoy a degree of protection from quantum anomalies that would otherwise seem certain to obliterate the $G(\mathbb{Z})$ multiplet structure.

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