Published for SISSA by Springer

Received: August 19, 2015
Revised: December 17, 2015
Accepted: January 11, 2016
Published: January 22, 2016

JHEP01(2016)139

Scalar-vector bootstrap

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Abstract: We work out all of the details required for implementation of the conformal bootstrap program applied to the four-point function of two scalars and two vectors in an abstract conformal field theory in arbitrary dimension. This includes a review of which tensor structures make appearances, a construction of the projectors onto the required mixed symmetry representations, and a computation of the conformal blocks for all possible operators which can be exchanged. These blocks are presented as differential operators acting upon the previously known scalar conformal blocks. Finally, we set up the bootstrap equations which implement crossing symmetry. Special attention is given to the case of conserved vectors, where several simplifications occur.

Keywords: Conformal and W Symmetry, Global Symmetries

ArXiv ePrint: 1508.02676

doi:10.1007/JHEP01(2016)139

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1 Introduction

The data of an abstract unitary conformal field theory (CFT) in D dimensions is encoded by the spectrum of primary operators and their OPEs, which are in turn specified by a finite number of real constants for each triplet of primary operators. From this information, we can in principle compute any correlation function by iteratively performing OPEs to reduce the correlator to a two-point function. This procedure should not depend on which order we perform OPEs, and the equivalence of different procedures puts constraints on which sets of data can correspond to consistent CFTs. In particular, for a four-point function we can divide the four operators into pairs in three different ways, or channels. Equivalence between these channels is called crossing symmetry, and the general endeavor of exploring the constraints on CFT data which are imposed by crossing symmetry is known as the conformal bootstrap program.

Since the revival of the conformal bootstrap program in recent years [1], several research groups have obtained both numerical (bounds on operator dimensions, OPE coefficients and central charges) [2–13] and analytical results (determination of anomalous dimensions and OPE coefficients) [14–19], as well as studies in theories with global symmetries [20, 23–27] or supersymmetries [6, 28–39]. So far these results have arisen from bootstrapping 4-point functions of scalar (in relation to the Lorentz subgroup of the conformal group) operators,[1] whose conformal blocks were computed in [41–43] for $D = 2, 4, 6$ (in any even dimension they can be computed recursively) with numerical approximations given in [21–23] for any dimension D.

[1]There is a notable recent exception of [40], which considers bootstrapping fermionic operators in 3D.

However, the consistency conditions from scalar correlators are only a small part of the (infinitely) many conditions that the bootstrap program imposes. One expects more interesting and universal bounds to arise from bootstrapping 4-point functions of operators with spin, such as the stress-tensor or conserved currents. The main obstacle in tackling these problems is that the full set of conformal blocks for spinning correlators is not readily available yet. Partial progress has been made in this direction. In [44] it was observed that there is a class of conformal blocks of tensor 4-point functions that can be related (via differential operators) to the well known scalar blocks of [41–43]. However, the class of conformal blocks derived in this way is associated to the exchange of traceless symmetric operators $\mathcal{O}$, whereas tensor correlator bootstrap requires, in addition, the exchange of mixed-symmetric operators $\mathcal{A}$. Later, in [45, 46] it was shown that conformal blocks associated to $\mathcal{A}$ can be calculated as a (finite) sum of scalar blocks evaluated at zero spin which, in principle, can be done by a computer. However, the numerical evaluation of these blocks is quite resource intensive due to the fact that the number of terms in the sum increases rapidly with the spin of $\mathcal{A}$. In numerical computations one might get away with if the maximum spin of $\mathcal{A}$ is not too large, but this approach is hopeless in the analytic bootstrap, where one needs to have control over the conformal blocks at very high spin [14, 15]. Therefore the objective of this paper is to start building explicit closed form expressions of spinning conformal blocks that can be used in the analytic bootstrap and for efficient numerical evaluation.[2]

To start with, in section 2 we classify all of the tensor structures which can appear in the three- and four-point functions which concern us in this paper (namely three-point functions with either two scalars or a scalar and a vector, along with a third operator, and four-point functions with four, three, or two scalars with zero, one, or two vectors). We pay special attention to the information obtained from exchanging two operators, especially when the operators are identical. In section 2.5 we work out the extra information available when the vector operator is conserved.

Section 3 reviews the shadow formalism, and computes the three-point coefficients for shadow operators in terms of the three-point functions of the original operators, the results of which are needed for the computation of the conformal blocks.

Section 4 is the heart of the paper, in which we compute the conformal blocks which are needed to implement the bootstrap program with two scalars and two vectors. Using the shadow formalism and the results from appendix A (various identities obeyed by the building blocks of our correlators), appendix B (where we compute the projection operators corresponding to the Lorentz representations of exchanged operators; in particular the results of appendix B.2 for the mixed symmetry exchange are some of the novel ingredients which really allows us to compute the required blocks), and appendix D (where we evaluate all of the required basic integrals), we compute the required integrals and perform the monodromy projection to finally obtain the conformal blocks. The new result in this section is the computation of the mixed-symmetric blocks, which relies on the contraction

[2]During the preparation of this draft, [47] appeared which generalizes [44] and proposes a relation between spinning (not necessarily bosonic) blocks associated to mixed-symmetric exchange, to more basic "seed" conformal blocks in 4D. However, the "seed" blocks were not presented yet. In the language of that paper, our work provides the "seed" blocks for the $[k+1,1]$ representation.

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formula (4.76). Further details on how this formula is derived, are given in appendix C where we also present one of the two new contractions that appear in conformal blocks of four vectors, as evidence that our methods can be applied in more general situations. Our results are all written in terms of differential operators acting on scalar conformal blocks.

Then in section 5 we set up the bootstrap program for four-point functions of two scalars and two vectors. In particular, we examine the case of conserved vectors, in which case several simplifications occur. Finally, we summarize our results and look forward to future directions in section 6.

The next step in this program is to use the results of this paper to obtain bounds (numerical or analytical) on the data of a general class of CFTs, and in particular for a CFT with a conserved primary vector operator and an associated continuous global symmetry. More formally, one would like to use the techniques developed in the present work to set up the bootstrap for even more complicated four-point functions. In particular, the cases of four vectors (in particular conserved currents), or correlators involving conserved stress tensors, would be of great interest. The real prize would be to implement the bootstrap with four conserved stress-energy tensors, thus gleaning extremely general information about the space of consistent unitary CFTs.

2 Tensor structures

Conformal invariance places strong constraints on the form of correlation functions. We will focus on correlations of primary operators. Correlation functions of descendants can of course be obtained from those of primaries. For two-point functions of primary operators, conformal invariance fixes the result up to an overall constant, and it is conventional to normalize the primary operators themselves to remove that remaining ambiguity. Each three-point function is determined up to a finite number of constants, each one multiplying a different tensor structure. These same constants appear in the operator product expansion (OPE). For four-point functions (and higher, though we won't go beyond four-point in this paper), there are a finite number of tensor structures. These tensor structures are multiplied not by constants in general, but by functions of the conformally invariant cross-ratios. Since the four-point function can in principle be evaluated by splitting into two pairs of operators and then using operator product expansions to reduce the problem to a sum of two-point functions, it follow that the functions multiplying the tensor structures are determined by the spectrum of primary operators and the constants which appear in their three-point functions.

In this section we will determine the tensor structures which can appear in the four-point function of scalars and up to two vectors, and in the three-point functions which act as intermediate stages in the evaluation. The techniques are well established [41–43], and especially in [48], but we give a self-contained presentation in order to establish our conventions and to put emphasis on the properties that will be most relevant for our purposes. In subsequent sections we will compute the functions which multiply these tensor structures in terms of the underlying data of the CFT.

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2.1 Embedding space

When considering the consequences of conformal invariance, it is often useful to make use of embedding space. This is a $(D+2)$-dimensional space, with coordinates P^A and metric

$$ds^2 = \eta_{AB} dP^A dP^B = -dP^+ dP^- + \delta_{ab} dP^a dP^b, \tag{2.1}$$

on which the conformal group $\mathrm{SO}(D+1,1)$ acts linearly (we will be working with Euclidean signature in physical space throughout this paper). The D-dimensional physical space is identified with a null projective surface. The map to physical coordinates is given by

$$x^a = P^a/P^+, \tag{2.2}$$

while we can do the inverse map by sending a point in physical space to a particular point on the projective null line,

$$P^A(x) = (P^+, P^-, P^a) = (1, x^2, x^a), \qquad \eta_{AB} P^A(x) P^B(x) = 0. \tag{2.3}$$

Now consider a tensor function of three coordinates (to serve as an example) on embedding space,

$$F_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}(P_1, P_2, P_3), \tag{2.4}$$

which is homogeneous in each variable (so that it is well defined on projective hypersurfaces),

$$F_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}(\lambda_1 P_1, \lambda_2 P_2, \lambda_3 P_3) = \lambda_1^{-\Delta_1}\lambda_2^{-\Delta_2}\lambda_3^{-\Delta_3} F_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}(P_1, P_2, P_3), \tag{2.5}$$

and is transverse in the sense that

$$\begin{aligned} &P_1^{A_1} F_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}(P_1, P_2, P_3) = \cdots = P_1^{A_k} F_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}(P_1, P_2, P_3) \\ &= P_2^{B_1} F_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}(P_1, P_2, P_3) = \cdots = P_2^{B_\ell} F_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}(P_1, P_2, P_3) \\ &= P_3^{C_1} F_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}(P_1, P_2, P_3) = \cdots = P_3^{C_m} F_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}(P_1, P_2, P_3) \\ &= 0. \end{aligned} \tag{2.6}$$

We can map this function to a tensor function on physical space by

$$\begin{aligned} &f_{a_1\cdots a_k, b_1\cdots b_\ell, c_1\cdots c_m}(x_1, x_2, x_3) \\ &= \frac{\partial P_1^{A_1}}{\partial x_1^{a_1}} \cdots \frac{\partial P_1^{A_k}}{\partial x_1^{a_k}} \frac{\partial P_2^{B_1}}{\partial x_2^{b_1}} \cdots \frac{\partial P_2^{B_\ell}}{\partial x_2^{b_\ell}} \frac{\partial P_3^{C_1}}{\partial x_3^{c_1}} \cdots \frac{\partial P_3^{C_m}}{\partial x_3^{c_m}} F_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}(P_1, P_2, P_3), \end{aligned} \tag{2.7}$$

where we use the map (2.3). Because we are mapping from a null hypersurface, different embedding space tensors can map to the same physical tensor if they are related by

$$F'_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m} = F_{A_1\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m} + P_1\, {}_{A_1} \Lambda_{A_2\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}, \tag{2.8}$$

for any choice $\Lambda_{A_2\cdots A_k, B_1\cdots B_\ell, C_1\cdots C_m}$, and similarly for each of the other indices. We will sometimes refer to this redundancy (somewhat sloppily) as gauge freedom.

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The resulting function $f_{a_1\cdots a_k,b_1\cdots b_\ell,c_1\cdots c_m}(x_1,x_2,x_3)$ transforms as a conformal tensor of weights Δ_1, Δ_2, and Δ_3 under conformal transformations of x_1, x_2, or x_3 respectively. It turns out that a converse is also true; any function which transforms as a tensor of weights Δ_i can be obtained from a homogenous (of weights Δ_i) transverse tensor in embedding space, unique up to equivalences of the form (2.8).

Thus, in order to determine the possible form of correlation functions of given operators, we need only determine the homogenous transverse tensors in embedding space up to the equivalences. In embedding space there are not many different objects we can build. Any scalar must be built out of scalar products of distinct P_i's, and it will be useful to define

$$P_{ij} = -2\eta_{AB}P_i^A P_j^B. \tag{2.9}$$

In physical space, this simply projects down to x_{ij}^2, where $x_{ij}^a = x_i^a - x_j^a$. To ensure that free indices are transverse, it will also be useful to define (for distinct i, j, and k)

$$K_A^{(ijk)} = \frac{P_{ik}P_{j\,A} - P_{ij}P_{k\,A}}{(P_{ij}P_{ik}P_{jk})^{1/2}}, \tag{2.10}$$

which is transverse with respect to P_i^A, and antisymmetric in j and k, and projects down to

$$k_a^{(ijk)} = \frac{x_{ij}^2\,(x_{ik})_a - x_{ik}^2\,(x_{ij})_a}{\left(x_{ij}^2x_{ik}^2x_{jk}^2\right)^{1/2}}, \tag{2.11}$$

and for distinct i and j, both

$$N_{A_1A_2}^{(ij)} = \eta_{A_1A_2} + \frac{2}{P_{ij}}\left(P_{i\,A_1}P_{j\,A_2} + P_{j\,A_1}P_{i\,A_2}\right), \tag{2.12}$$

which is transverse in both indices with respect to P_i (or P_j) and projects to δ_{ab}, and

$$M_{AB}^{(ij)} = \eta_{AB} + \frac{2}{P_{ij}}P_{j\,A}P_{i\,B}, \tag{2.13}$$

which is transverse to P_i^A in the first index and P_j^B in the second index, and projects down to

$$m_{ab}^{(ij)} = \delta_{ab} - \frac{2}{x_{ij}^2}\,(x_{ij})_a\,(x_{ij})_b\,. \tag{2.14}$$

Note that these building blocks, particularly (2.10) are defined to be scale invariant.

Finally, note that if $F_{A_1\cdots A_k}$ (we suppress other indices for now) transforms in a given way under permutations, then its projection $f_{a_1\cdots a_k}$ will inherit the same transformation and will thus transform as the corresponding representation of the rotation group $\mathrm{SO}(D)$. For example, if $F_{A_1\cdots A_k}$ is invariant under permutations of its indices, then $f_{a_1\cdots a_k}$ will be a symmetric tensor. If $F_{A_1\cdots A_k}$ is also traceless, then so will be $f_{a_1\cdots a_k}$.

Appendix A contains several useful formulae and identities for these structures in physical space.

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2.2 Two-point functions

As we will see in the next subsection, the primary operators we will need in this paper fall into two classes of irreducible representations of the rotation group $\mathrm{SO}(D)$. We either have totally symmetric traceless tensors of spin ℓ, $\mathcal{O}_{a_1\cdots a_\ell}(x)$, which includes scalars and vectors as special cases, or we have mixed symmetry tensors $\mathcal{A}_{a_1a_2b_1\cdots b_k}(x)$ which are completely traceless, are antisymmetric in a_1 and a_2, are totally symmetric in the b_i, and which vanish when antisymmetrized over any three indices. In terms of Young tableaux, the $\mathcal{O}_{a_1\cdots a_\ell}$ are represented by a horizontal row of ℓ boxes, while $\mathcal{A}_{a_1a_2b_1\cdots b_k}$ are represented by one row of $k+1$ boxes and a second row with only one box (equivalently one column with two boxes and k columns of one box each). For each of these cases we construct projectors onto the given representation in appendix B. For $\mathcal{O}_{a_1\cdots a_\ell}$ and $\mathcal{A}_{a_1a_2b_1\cdots b_k}$ we use projectors

$$\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}, \qquad \text{and} \qquad \widetilde{\Pi}^{(k)\,c_1c_2d_1\cdots d_k}_{a_1a_2b_1\cdots b_k}, \tag{2.15}$$

given in (B.8) and (B.25) respectively. We can also write the projectors in embedding space by simply taking the expressions in appendix B and replacing each δ_{ab} with $N^{(ij)}_{AB}$, with i labeling the operator being projected, and j being an arbitrarily chosen other variable (the choice is not physically relevant and can be changed by a gauge transformation (2.8).

It is well known that we can diagonalize the space of primary operators with respect to the two-point correlation functions, so we will only need to compute the two-point function of either a pair or $\mathcal{O}$ operators or a pair of $\mathcal{A}$ operators. Indeed, if we have

$$h_{a_1\cdots a_\ell;b_1\cdots b_\ell}(x_1,x_2) = \langle \mathcal{O}_{a_1\cdots a_\ell}(x_1)\mathcal{O}_{b_1\cdots b_\ell}(x_2)\rangle\,, \tag{2.16}$$

then this must descend from a tensor $H_{A_1\cdots A_\ell,B_1\cdots B_\ell}$ in embedding space. In order to get the symmetric traceless representation, we must be able to put the indices on projectors $\Pi^{(\ell)}$. By transversality, each A index must be carried by either $P_{1\,A}$, $N^{(12)}_{AA'}$, or $M^{(12)}_{AB}$. The first possibility is pure gauge and can be discarded. The second possibility, which projects down to $\delta_{aa'}$ will be eliminated when multiplied by the projector $\Pi^{(\ell)}$, and so can also be discarded (though it will appear in the projectors themselves). This leaves only the third possibilty. In order to get the correct homogeneity property we must include the appropriate power of P_{12}. Finally, then, we are left with the form

$$H_{A_1\cdots A_\ell,B_1\cdots B_\ell}(P_1,P_2) = P_{12}^{-\Delta_\mathcal{O}}\Pi^{(\ell)\,C_1\cdots C_\ell}_{A_1\cdots A_\ell}\Pi^{(\ell)\,D_1\cdots D_\ell}_{B_1\cdots B_\ell}M^{(12)}_{C_1D_1}\cdots M^{(12)}_{C_\ell D_\ell}, \tag{2.17}$$

which projects down to

$$h_{a_1\cdots a_\ell,b_1\cdots b_\ell}(x_1,x_2) = \left(x_{12}^2\right)^{-\Delta_\mathcal{O}}\,\Pi^{(\ell)\,c_1\cdots c_\ell}_{a_1\cdots a_\ell}\Pi^{(\ell)\,d_1\cdots d_\ell}_{b_1\cdots b_\ell}m^{(12)}_{c_1d_1}\cdots m^{(12)}_{c_\ell d_\ell}. \tag{2.18}$$

The same reasoning gives[3]

$$\begin{aligned}&\langle \mathcal{A}_{a_1a_2b_1\cdots b_k}(x_1)\mathcal{A}_{c_1c_2d_1\cdots d_k}(x_2)\rangle\\ &\qquad = \left(x_{12}^2\right)^{-\Delta_\mathcal{A}}\widetilde{\Pi}^{(k)\,e_1e_2f_1\cdots f_k}_{a_1a_2b_1\cdots b_k}\widetilde{\Pi}^{(k)\,g_1g_2h_1\cdots h_k}_{c_1c_2d_1\cdots d_k}m^{(12)}_{e_1g_1}m^{(12)}_{e_2g_2}m^{(12)}_{f_1h_1}\cdots m^{(12)}_{f_kh_k}.\end{aligned} \tag{2.19}$$

[3]Making use of the symmetries of $\widetilde{\Pi}^{(k)}$, one can show that any other arrangement of indices on the $m^{(12)}$'s, e.g. replacing $m^{(12)}_{e_2g_2}m^{(12)}_{f_1h_1}$ by $m^{(12)}_{e_2h_1}m^{(12)}_{f_1g_2}$, is equivalent to the one given.

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2.3 Three-point functions

Similarly, three-point correlation functions can be lifted to embedding space. If all operators are in irreducible representations, then $N^{(ij)}_{A_1A_2}$ should again only appear in projectors, so all indices will be carried by either $K^{(ijk)}_A$ or by $M^{(ij)}_{AB}$.

2.3.1 $\langle SS\mathcal{O}\rangle$

If the first two operators are scalars, then all the indices of the remaining operator (after projection) must be carried by $K^{(312)}_A$. This will be vanishing for any irreducible representation except for the symmetric traceless representation. Then the three-point correlator $\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle$ lifts to an embedding space tensor

$$\begin{aligned}&F_{A_1\cdots A_\ell}(P_1,P_2,P_3)\\&=\lambda_{12\mathcal{O}}P_{12}^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_\mathcal{O})}P_{13}^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_\mathcal{O})}P_{23}^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_\mathcal{O})}\Pi^{(\ell)\,B_1\cdots B_\ell}_{A_1\cdots A_\ell}K^{(312)}_{B_1}\cdots K^{(312)}_{B_\ell},\end{aligned}\tag{2.20}$$

which projects down to

$$\begin{aligned}\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle=\lambda_{12\mathcal{O}}\left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_\mathcal{O})}\left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_\mathcal{O})}\\\times\left(x_{23}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_\mathcal{O})}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}k^{(312)}_{b_1}\cdots k^{(312)}_{b_\ell}.\end{aligned}\tag{2.21}$$

Here $\lambda_{12\mathcal{O}}$ is a constant real number (in a unitary CFT), which is otherwise arbitrary.

If the two scalars are identical, then the the result has the form

$$\langle\phi(x_1)\phi(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle=\lambda_{\phi\phi\mathcal{O}}\left(x_{12}^2\right)^{\frac{1}{2}\left(-2\Delta_\phi+\Delta_\mathcal{O}\right)}\left(x_{13}^2x_{23}^2\right)^{-\frac{1}{2}\Delta_\mathcal{O}}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}k^{(312)}_{b_1}\cdots k^{(312)}_{b_\ell},\tag{2.22}$$

and this result should be invariant under the exchange of x_1 and x_2, which in turn forces ℓ to be even (otherwise the result changes sign under this exchange, since we get one factor of -1 from each $k^{(312)}$).

2.3.2 $\langle SV\mathcal{O}\rangle$

Next we consider the three-point function with one scalar ϕ, one vector v_a, and one other operator. This correlator will lift to an embedding tensor $F_{AB_1\cdots B_m}(P_1,P_2,P_3)$ where A is transverse to P_2 and the B indices are transverse to P_3. The A index can only be carried by either $K^{(213)}_A$ or $M^{(23)}_{AB}$, and then the remaining B indices (after being projected by the appropriate rotation group projector) must be carried by $K^{(312)}_B$. In the latter case, no two of the indices carried by $K^{(312)}_B$ can be antisymmetric. So the third operator can only be either totally symmetric $\mathcal{O}$, or it can be an $\mathcal{A}$ in the mixed symmetry representation described above, where one of the first two B indices is carried by $M^{(23)}_{AB}$, and the rest by $K^{(312)}_B$'s.

In the case where the third operator is totally symmetric, then there are two structures which can arise, with embedding space form

$$\begin{aligned}F_{AB_1\cdots B_\ell}(P_1,P_2,P_3)=P_{12}^{\frac{1}{2}\left(-\Delta_\phi-\Delta_v+\Delta_\mathcal{O}\right)}P_{13}^{\frac{1}{2}\left(-\Delta_\phi+\Delta_v-\Delta_\mathcal{O}\right)}P_{23}^{\frac{1}{2}\left(\Delta_\phi-\Delta_v-\Delta_\mathcal{O}\right)}\\\times\Pi^{(\ell)\,C_1\cdots C_\ell}_{B_1\cdots B_\ell}\left[\alpha_{\phi v\mathcal{O}}K^{(213)}_AK^{(312)}_{B_1}\cdots K^{(312)}_{B_\ell}+\beta_{\phi v\mathcal{O}}M^{(23)}_{AB_1}K^{(312)}_{B_2}\cdots K^{(312)}_{B_\ell}\right],\end{aligned}\tag{2.23}$$

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which projects down to

$$\langle\phi(x_1)v_a(x_2)\mathcal{O}_{b_1\cdots b_\ell}(x_3)\rangle = \left(x_{12}^2\right)^{\frac{1}{2}\left(-\Delta_\phi-\Delta_v+\Delta_\mathcal{O}\right)}\left(x_{13}^2\right)^{\frac{1}{2}\left(-\Delta_\phi+\Delta_v-\Delta_\mathcal{O}\right)}\left(x_{23}^2\right)^{\frac{1}{2}\left(\Delta_\phi-\Delta_v-\Delta_\mathcal{O}\right)} \times \Pi_{b_1\cdots b_\ell}^{(\ell)\,c_1\cdots c_\ell}\left[\alpha_\mathcal{O} k_a^{(213)}k_{c_1}^{(312)}\cdots k_{c_\ell}^{(312)} + \beta_\mathcal{O} m_{ac_1}^{(23)}k_{c_2}^{(312)}\cdots k_{c_\ell}^{(312)}\right]. \quad (2.24)$$

If $\ell = 0$, then we only have the first term labeled by a constant $\alpha_{\phi v\mathcal{O}}$. If $\ell > 0$, then we have two distinct possible tensor structures labeled by two real constant numbers $\alpha_{\phi v\mathcal{O}}$ and $\beta_{\phi v\mathcal{O}}$.

2.3.3 $\langle SV\mathcal{A}\rangle$

Similar considerations for the case where the third operator has mixed symmetry show that the three-point correlation function will have the form

$$\langle\phi(x_1)v_a(x_2)\mathcal{A}_{b_1b_2c_1\cdots c_k}(x_3)\rangle = \gamma_{\phi v\mathcal{A}}\left(x_{12}^2\right)^{\frac{1}{2}\left(-\Delta_\phi-\Delta_v+\Delta_\mathcal{A}\right)}\left(x_{13}^2\right)^{\frac{1}{2}\left(-\Delta_\phi+\Delta_v-\Delta_\mathcal{A}\right)} \times \left(x_{23}^2\right)^{\frac{1}{2}\left(\Delta_\phi-\Delta_v-\Delta_\mathcal{A}\right)}\widetilde{\Pi}_{b_1b_2c_1\cdots c_k}^{(k)\,d_1d_2e_1\cdots e_k} m_{ad_1}^{(23)}k_{d_2}^{(312)}k_{e_1}^{(312)}\cdots k_{e_k}^{(312)}, \quad (2.25)$$

with $\gamma_{\phi v\mathcal{A}}$ as a real constant.

2.4 Four-point functions

The case of four-point functions proceeds similarly, with the main difference being that there are cross-ratios

$$U = \frac{P_{12}P_{34}}{P_{13}P_{24}}, \qquad V = \frac{P_{14}P_{23}}{P_{13}P_{24}}, \quad (2.26)$$

in embedding space, or

$$u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}, \qquad v = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}, \quad (2.27)$$

in physical space. Then each tensor structure is accompanied by a function of the cross-ratios rather than by just a constant.

2.4.1 $\langle SSSS\rangle$

For the case of four scalars, we have

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}(\Delta_3-\Delta_4)}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}(\Delta_1-\Delta_2)}\left(x_{12}^2\right)^{-\frac{1}{2}(\Delta_1+\Delta_2)}\left(x_{34}^2\right)^{-\frac{1}{2}(\Delta_3+\Delta_4)}q(u,v), \quad (2.28)$$

where $q(u,v)$ is an (a priori) arbitrary function of the cross-ratios u and v. The factor multiplying $q(u,v)$, which does the work in ensuring that the correlator scales correctly, will appear often, and so it is convenient to abbreviate it. Thus, we define

$$X_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} = \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}(\Delta_3-\Delta_4)}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}(\Delta_1-\Delta_2)}\left(x_{12}^2\right)^{-\frac{1}{2}(\Delta_1+\Delta_2)}\left(x_{34}^2\right)^{-\frac{1}{2}(\Delta_3+\Delta_4)}, \quad (2.29)$$

and sometimes we will simply write X_{Δ_i} for short.

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In the case that all four scalars are identical, we have

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \left(x_{12}^2 x_{34}^2\right)^{-\Delta_\phi} q(u,v), \tag{2.30}$$

and invariance under exchange of x_1 with x_2 implies that

$$q(u,v) = q(u/v, 1/v), \tag{2.31}$$

while under exchange of x_1 and x_3 we have

$$q(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} q(v,u). \tag{2.32}$$

Other permutations of the x_i give no new information about the function $q(u,v)$.

2.4.2 $\langle SVSS\rangle$ or $\langle SSSV\rangle$

Let us now consider the four-point function of three scalars and one vector (in the second position to start). In principle the free index could be carried, in embedding space, by any of the three possibilities $K_A^{(213)}$, $K_A^{(214)}$, or $K_A^{(234)}$, but it turns out that there is a linear relation (A.4)

$$K_A^{(213)} = V^{1/2} K_A^{(214)} - U^{1/2} K_A^{(234)}, \tag{2.33}$$

so only two of the combinations are independent, and we can write (after projecting to physical space)

$$\langle\phi_1(x_1)v_a(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = X_{\Delta_1,\Delta_v,\Delta_3,\Delta_4}\left[q_1(u,v)k_a^{(214)} + q_2(u,v)k_a^{(234)}\right]. \tag{2.34}$$

If ϕ_3 and ϕ_4 are identical, then symmetry under exchange of x_3 and x_4 implies that

$$q_1(u/v,1/v) = v^{\frac{1}{2}(\Delta_v-\Delta_1-1)} q_1(u,v),\ q_2(u/v,1/v) = -v^{\frac{1}{2}(\Delta_v-\Delta_1)}\left(q_2(u,v) + \left(\frac{u}{v}\right)^{\frac{1}{2}} q_1(u,v)\right), \tag{2.35}$$

while if ϕ_1 and ϕ_3 are identical, then we have

$$q_2(u,v) = \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_v)} q_1(v,u). \tag{2.36}$$

If all three scalars are identical, then both sets of constraints hold.

The situation when the vector is in fourth position is completely analogous (we put primes on the q_i' to distinguish them from the SVSS functions),

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)v_a(x_4)\rangle = X_{\Delta_1,\Delta_2,\Delta_3,\Delta_v}\left[q_1'(u,v)k_a^{(412)} + q_2'(u,v)k_a^{(432)}\right]. \tag{2.37}$$

2.4.3 $\langle SVSV\rangle$

Finally, consider a four-point function of two scalars and two vectors,

$$f_{ab}(x_1,x_2,x_3,x_4) = \langle\phi_1(x_1)v_{2\,a}(x_2)\phi_3(x_3)v_{4\,b}(x_4)\rangle\,. \tag{2.38}$$

In embedding space, the indices of the corresponding tensor can be either carried by $M_{AB}^{(24)}$ or else both indices are carried by K's. There are two independent choices of K possible for each index, and so there are five possible tensor structures altogether,

$$f_{ab} = X_{\Delta_i}\Big[q_0(u,v)m_{ab}^{(24)} + q_{11}(u,v)k_a^{(214)}k_b^{(412)} + q_{12}(u,v)k_a^{(214)}k_b^{(432)} + q_{21}(u,v)k_a^{(234)}k_b^{(412)} + q_{22}(u,v)k_a^{(234)}k_b^{(432)}\Big]. \quad (2.39)$$

If the two scalars are identical, then x_1-x_3 exchange gives constraints

$$q_0(v,u) = \left(\frac{v}{u}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_2)} q_0(u,v), \quad (2.40)$$

$$q_{21}(u,v) = \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_2)} q_{12}(v,u), \quad (2.41)$$

$$q_{22}(u,v) = \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_2)} q_{11}(v,u). \quad (2.42)$$

If the two vectors are identical, then exchanging x_2 and x_4 while also exchanging the indices a and b, gives

$$q_0(v,u) = \left(\frac{v}{u}\right)^{\frac{1}{2}(\Delta_3+\Delta_v)} q_0(u,v), \quad (2.43)$$

$$q_{11}(v,u) = \left(\frac{v}{u}\right)^{\frac{1}{2}(\Delta_3+\Delta_v)} q_{11}(u,v), \quad (2.44)$$

$$q_{21}(u,v) = \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_3+\Delta_v)} q_{12}(v,u), \quad (2.45)$$

$$q_{22}(v,u) = \left(\frac{v}{u}\right)^{\frac{1}{2}(\Delta_3+\Delta_v)} q_{22}(u,v). \quad (2.46)$$

Finally if we have two identical scalars and two identical vectors, then we can combine the constraints and determine

$$q_0(v,u) = \left(\frac{v}{u}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_v)} q_0(u,v), \quad (2.47)$$

$$q_{11}(v,u) = \left(\frac{v}{u}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_v)} q_{11}(u,v), \quad (2.48)$$

$$q_{21}(u,v) = \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_v)} q_{12}(v,u), \quad (2.49)$$

$$q_{22}(u,v) = q_{11}(u,v). \quad (2.50)$$

Thus in this case we have one unconstrained function $q_{12}(u,v)$, and two constrained functions $q_0(u,v)$ and $q_{11}(u,v)$, with $q_{21}(u,v)$ and $q_{22}(u,v)$ determined in terms of the others.

2.5 Conserved vectors

Many of the structures discussed above simplify somewhat if we are dealing with conserved vectors, which obey $\partial^a v_a(x) = 0$ inside correlation functions. From the vector-vector two-point function, we have

$$0 = \partial_1^b \langle v_b(x_1)v_a(x_2)\rangle = \partial_1^b\left[\left(x_{12}^2\right)^{-\Delta_v} m_{ba}^{(12)}\right]$$

$$
\begin{aligned}
&= \left(x_{12}^2\right)^{-\Delta_v}\left[-\frac{2\Delta_v x_{12}^b}{x_{12}^2}m_{ba}^{(12)} - \frac{2\,(D-1)\,x_{12\,a}}{x_{12}^2}\right] \\
&= 2\left(x_{12}^2\right)^{-\Delta_v-1}(\Delta_v - D + 1)\,x_{12\,a}.
\end{aligned} \tag{2.51}
$$

Thus we conclude that $\Delta_v = D-1$ for a conserved vector in D-dimensions, i.e. it saturates the unitarity bound.

Turning next to three-point functions, we have (for $\ell > 0$)

$$
\begin{aligned}
0 &= \partial_2^b \langle \phi(x_1) v_b(x_2) \mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle \\
&= \partial_2^b \Big\{ \left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_\phi+\Delta_\mathcal{O}-D+1)} \left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_\phi-\Delta_\mathcal{O}+D-1)} \left(x_{23}^2\right)^{\frac{1}{2}(\Delta_\phi-\Delta_\mathcal{O}-D+1)} \\
&\qquad \times \Pi_{b_1\cdots b_\ell}^{(\ell)\,c_1\cdots c_\ell}\left[\alpha_{\phi v\mathcal{O}} k_a^{(213)} k_{c_1}^{(312)}\cdots k_{c_\ell}^{(312)} + \beta_{\phi v\mathcal{O}} m_{ac_1}^{(23)} k_{c_2}^{(312)}\cdots k_{c_\ell}^{(312)}\right]\Big\} \\
&= \left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_\phi+\Delta_\mathcal{O}-D)} \left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_\phi-\Delta_\mathcal{O}+D)} \left(x_{23}^2\right)^{\frac{1}{2}(\Delta_\phi-\Delta_\mathcal{O}-D)} \\
&\qquad \times \left[\alpha_{\phi v\mathcal{O}}\,(\Delta_\phi - \Delta_\mathcal{O}) + \beta_{\phi v\mathcal{O}}\,(\Delta_\phi - \Delta_\mathcal{O} + D + \ell - 2)\right] \Pi_{a_1\cdots a_\ell}^{(\ell)\,b_1\cdots b_\ell} k_{b_1}^{(312)}\cdots k_{b_\ell}^{(312)}.
\end{aligned} \tag{2.52}
$$

where we have made use of the fact that $\Delta_v = D-1$. For this expression to vanish, we require

$$
(\Delta_\phi - \Delta_\mathcal{O})\,\alpha_{\phi v\mathcal{O}} + (\Delta_\phi - \Delta_\mathcal{O} + D + \ell - 2)\,\beta_{\phi v\mathcal{O}} = 0. \tag{2.53}
$$

For $\ell = 0$, we simply set $\beta_{\phi v\mathcal{O}} = 0$ in the above equation, and require either $\alpha_{\phi v\mathcal{O}} = 0$ or $\Delta_\phi = \Delta_\mathcal{O}$. Actually, we can assign some more physical significance to this case by first recalling that we expect each conserved primary vector operator to be associated to a one parameter continuous global symmetry of our CFT. Now pick a particular conformal weight Δ and consider all scalar operators ϕ_i that have that weight. Form a matrix α_{ij} by taking three point functions with the conserved vector v_a,

$$
\langle \phi_i(x_1) v_a(x_2) \phi_j(x_3)\rangle = \left(x_{12}^2 x_{23}^2\right)^{-\frac{1}{2}(D-1)} \left(x_{13}^2\right)^{\frac{1}{2}(D-1)-\Delta} \alpha_{ij} k_a^{(213)}. \tag{2.54}
$$

Then α_{ij} is antisymmetric in its indices. Since we are free to make orthonormal (with respect to the normalized two-point functions) rotations on the space of ϕ_i, we can always take a basis in which α_{ij} is block diagonal,

$$
\alpha_{ij} = \begin{pmatrix}
0 & -Q_1 & & & & & \\
Q_1 & 0 & \cdots & & & & \\
 & \vdots & \ddots & & & & \\
 & & & 0 & -Q_n & & \\
 & & & Q_n & 0 & \cdots & \\
 & & & & & 0 & \\
 & & & & \vdots & \ddots & \\
 & & & & & & 0
\end{pmatrix}. \tag{2.55}
$$

Here n is just the number of charged scalars with weight Δ. In this basis, we say that for $i > 2n$, ϕ_i is neutral under the global symmetry. We can combine the others into complex combinations $\varphi_i = \phi_{2i-1} + i\phi_{2i}$, and we can say that φ_i has charge[4] Q_i.

For the other three-point functions, $\langle \phi v \mathcal{A} \rangle$, a similar calculation shows that conservation is automatic once we impose that $\Delta_v = D-1$. Conservation gives no other constraints in this case.

For four-point functions with conserved vectors, the coefficient functions must obey linear differential equations. For example, in the $\langle SVSS \rangle$ amplitude, if the vector is conserved then the functions q_1 and q_2 must obey

$$\begin{aligned} 0 = &\left[\Delta_1 - 2u\partial_u + \left(-uv^{-1} - 1 + v^{-1}\right) v\partial_v\right] q_1 + \left(\frac{u}{v}\right)^{\frac{1}{2}} \left[\frac{\Delta_1}{2}\left(1 + u^{-1}v - u^{-1}\right)\right. \\ &\left. + \frac{D-1}{2}\left(-1 + u^{-1}v - u^{-1}\right) + \left(-1 - u^{-1}v + u^{-1}\right) u\partial_u - 2v\partial_v\right] q_2. \end{aligned} \tag{2.56}$$

And in the case of $\langle SVSV \rangle$, if the vector at x_4 is conserved (so in particular $\Delta_4 = D-1$), then we have

$$\begin{aligned} 0 = &\left(\Delta_1 - \Delta_2 - \Delta_3 + D - 1\right) q_0 \\ &+ \left[\left(\Delta_1 - \Delta_2\right) + \frac{1}{2}\left(\Delta_3 - 1\right)\left(-1 - u^{-1} + u^{-1}v\right) + \frac{D}{2}\left(1 - u^{-1} + u^{-1}v\right)\right] q_{11} \\ &+ \left[\frac{1}{2}\left(\Delta_1 - \Delta_2 + D\right)\left(u^{1/2}v^{-1/2} + u^{-1/2}v^{1/2} - u^{-1/2}v^{-1/2}\right)\right. \\ &\left. + \frac{1}{2}\Delta_3\left(-u^{1/2}v^{-1/2} + u^{-1/2}v^{1/2} + u^{-1/2}v^{-1/2}\right)\right] q_{12} - u^{-1/2}v^{1/2} q_{21} - 2v\partial_v q_0 \\ &+ \left(1 - u - v\right)\partial_u q_{11} - 2v\partial_v q_{11} - 2u^{1/2}v^{1/2}\partial_u q_{12} \\ &+ \left(-u^{1/2}v^{1/2} - u^{-1/2}v^{3/2} + u^{-1/2}v^{1/2}\right)\partial_v q_{12}, \end{aligned} \tag{2.57}$$

and

$$\begin{aligned} 0 = &\left(\Delta_3 + D - 1\right) q_0 - u^{1/2}v^{-1/2} q_{12} \\ &+ \left[\left(\Delta_1 - \Delta_2\right) u^{1/2}v^{-1/2} + \frac{1}{2}\Delta_3\left(-u^{1/2}v^{-1/2} + u^{-1/2}v^{1/2} - u^{-1/2}v^{-1/2}\right)\right. \\ &\left. + \frac{D}{2}\left(u^{1/2}v^{-1/2} + u^{-1/2}v^{1/2} - u^{-1/2}v^{-1/2}\right)\right] q_{21} \\ &+ \left[\frac{1}{2}\left(\Delta_1 - \Delta_2 + D - 1\right)\left(1 + uv^{-1} - v^{-1}\right) + \frac{1}{2}\Delta_3\left(1 - uv^{-1} + v^{-1}\right)\right] q_{22} \\ &- 2u\partial_u q_0 + \left(-u^{3/2}v^{-1/2} - u^{1/2}v^{1/2} + u^{1/2}v^{-1/2}\right)\partial_u q_{21} - 2u^{1/2}v^{1/2}\partial_v q_{21} \\ &- 2u\partial_u q_{22} + \left(1 - u - v\right)\partial_v q_{22}. \end{aligned} \tag{2.58}$$

[4] We have chosen a normalization for the charge that is convenient from the point of view of an abstract CFT, since it is given simply by the three-point function of primary fields (which have themselves been normalized by their two-point functions). However, it may well differ from other well-motivated normalizations. For example, in the case of a free complex scalar in $D > 2$ dimensions, and the usual global U(1) symmetry, our definition gives the scalar a charge of $\sqrt{\frac{D-2}{2}}$.

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3 Shadow formalism

As an intermediate step in the calculation of conformal blocks, we will need to define *shadow operators*. Given any local primary operator $\mathcal{O}_{a_1\cdots a_n}(x)$ of conformal weight Δ, we can define its shadow operator

$$\widetilde{\mathcal{O}}_{a_1\cdots a_\ell}(x_1) = \Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell} \int \frac{d^D x_0}{\left(x_{01}^2\right)^{D-\Delta}} m^{(01)}{}_{b_1}^{\ c_1} \cdots m^{(01)}{}_{b_\ell}^{\ c_\ell} \mathcal{O}_{c_1\cdots c_\ell}(x_0), \tag{3.1}$$

which is a non-local operator that transforms as a primary operator of weight $D-\Delta$ under conformal transformations, and under $\mathrm{SO}(D)$ rotations transforms in the same way as $\mathcal{O}$. When we insert $\widetilde{\mathcal{O}}(x_1)$ in a correlation function, the prescription is to insert $\mathcal{O}(x_0)$, evaluate the correlation function, and then perform the integral above.

3.1 Mixing matrices

We would like to compute the constants which appear in three-point functions involving shadow operators. Since $\widetilde{\mathcal{O}}$ is linearly related to $\mathcal{O}$, the constant or constants appearing in a three-point function of $\widetilde{\mathcal{O}}$ with two other operators will be linear combinations of the constants in the three-point function of $\mathcal{O}$ with those same two operators. We would like to determine the matrices which encode these linear combinations.

3.1.1 $\langle SS\widetilde{\mathcal{O}}\rangle$

Consider first the case where $\mathcal{O}_{a_1\cdots a_\ell}$ is symmetric traceless, and the other two operators are scalars ϕ_1 and ϕ_2. The three-point function with $\mathcal{O}$ is fixed up to a single constant $\lambda_{12\mathcal{O}} = \lambda_{\mathcal{O}}$,

$$\begin{aligned}&\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle \\ &= \lambda_{\mathcal{O}} \left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_{\mathcal{O}})} \left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_{\mathcal{O}})} \left(x_{23}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_{\mathcal{O}})} \Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell} k^{(312)}_{b_1}\cdots k^{(312)}_{b_\ell},\end{aligned} \tag{3.2}$$

and we expect that the shadow operator will similarly have

$$\begin{aligned}\left\langle\phi_1(x_1)\phi_2(x_2)\widetilde{\mathcal{O}}_{a_1\cdots a_\ell}(x_3)\right\rangle &= \lambda_{\widetilde{\mathcal{O}}} \left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_{\widetilde{\mathcal{O}}})} \left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_{\widetilde{\mathcal{O}}})} \\ &\quad\times \left(x_{23}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_{\widetilde{\mathcal{O}}})} \Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell} k^{(312)}_{b_1}\cdots k^{(312)}_{b_\ell},\end{aligned} \tag{3.3}$$

where $\Delta_{\widetilde{\mathcal{O}}} = D-\Delta_{\mathcal{O}}$. Inserting the definition of the shadow operator (3.1) and performing the integral leads to[5]

$$\begin{aligned}\lambda_{\widetilde{\mathcal{O}}} = \pi^{D/2}&\frac{\Gamma(\Delta_{\mathcal{O}}-\frac{D}{2})\Gamma(\Delta_{\mathcal{O}}+\ell-1)}{\Gamma(\Delta_{\mathcal{O}}-1)\Gamma(D-\Delta_{\mathcal{O}}+\ell)} \\ &\times \frac{\Gamma(\frac{1}{2}(D+\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell))\Gamma(\frac{1}{2}(D-\Delta_1+\Delta_2-\Delta_{\mathcal{O}}+\ell))}{\Gamma(\frac{1}{2}(\Delta_1-\Delta_2+\Delta_{\mathcal{O}}+\ell))\Gamma(\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}+\ell))}\lambda_{\mathcal{O}}.\end{aligned} \tag{3.4}$$

[5]Details on the computation of these integrals are given in appendix E.

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3.1.2 $\langle SV\widetilde{\mathcal{O}}\rangle$

Next, we consider symmetric traceless $\mathcal{O}(x_3)$, but in a three-point function with a scalar $\phi(x_1)$ and a vector $v_a(x_2)$. In this case, both OPE contribute:

$$\begin{aligned}\alpha_{\widetilde{\mathcal{O}}} &= \pi^{D/2}\frac{\Gamma(\frac{1}{2}(D+\Delta_\phi-\Delta_v-\Delta_{\mathcal{O}}+\ell+1))\Gamma(\frac{1}{2}(D-\Delta_\phi+\Delta_v-\Delta_{\mathcal{O}}+\ell-1))\Gamma(\Delta_{\mathcal{O}}-\frac{D}{2})}{\Gamma(\frac{1}{2}(\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}}+\ell+1))\Gamma(\frac{1}{2}(-\Delta_\phi+\Delta_v+\Delta_{\mathcal{O}}+\ell+1))\Gamma(\Delta_{\mathcal{O}})} \\ &\quad\times\frac{\Gamma(\Delta_{\mathcal{O}}+\ell-1)}{\Gamma(D-\Delta_{\mathcal{O}}+\ell)}\left[\frac{1}{2}\left((\Delta_{\mathcal{O}}+\ell-1)(D-\Delta_{\mathcal{O}}-1)-(\Delta_{\mathcal{O}}-1)(\Delta_\phi-\Delta_v)\right)\alpha_{\mathcal{O}}\right. \\ &\quad\left.-\left(\Delta_{\mathcal{O}}-\frac{D}{2}\right)(\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}}+\ell-1)\,\beta_{\mathcal{O}}\right], \qquad (3.5)\\ \beta_{\widetilde{\mathcal{O}}} &= \pi^{D/2}\frac{\Gamma(\frac{1}{2}(D+\Delta_\phi-\Delta_v-\Delta_{\mathcal{O}}+\ell-1))\Gamma(\frac{1}{2}(D-\Delta_\phi+\Delta_v-\Delta_{\mathcal{O}}+\ell-1))\Gamma(\Delta_{\mathcal{O}}-\frac{D}{2})}{\Gamma(\frac{1}{2}(\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}}+\ell+1))\Gamma(\frac{1}{2}(-\Delta_\phi+\Delta_v+\Delta_{\mathcal{O}}+\ell+1))\Gamma(\Delta_{\mathcal{O}})} \\ &\quad\times\frac{\Gamma(\Delta_{\mathcal{O}}+\ell-1)}{\Gamma(D-\Delta_{\mathcal{O}}+\ell)}\left[\frac{\ell}{2}\left(\Delta_{\mathcal{O}}-\frac{D}{2}\right)(\Delta_\phi-\Delta_v)\,\alpha_{\mathcal{O}}+\frac{1}{4}(\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}}+\ell-1)\right. \\ &\quad\left.\times\left((\Delta_{\mathcal{O}}-1)(D-\Delta_{\mathcal{O}}+\ell-1)-(D-\Delta_{\mathcal{O}}-1)(\Delta_\phi-\Delta_v)\right)\beta_{\mathcal{O}}\right]. \qquad (3.6)\end{aligned}$$

As a nice check on this result, we can show that in the case that v_a is conserved, so $\alpha_{\mathcal{O}}$ and $\beta_{\mathcal{O}}$ obey (2.53) for $\ell > 0$ (the $\ell = 0$ case has a subtlety but can also be shown to be consistent), then $\alpha_{\widetilde{\mathcal{O}}}$ and $\beta_{\widetilde{\mathcal{O}}}$ obey the corresponding equation with $\Delta_{\widetilde{\mathcal{O}}}$. For future reference we will write

$$\alpha_{\phi v\widetilde{\mathcal{O}}} = M_\alpha{}^\alpha\alpha_{\phi v\mathcal{O}} + M_\alpha{}^\beta\beta_{\phi v\mathcal{O}}, \qquad \beta_{\phi v\widetilde{\mathcal{O}}} = M_\beta{}^\alpha\alpha_{\phi v\mathcal{O}} + M_\beta{}^\beta\beta_{\phi v\mathcal{O}}, \qquad (3.7)$$

where the constants $M_r{}^s$ can be read off from (3.5) and (3.6).

3.1.3 $\langle SV\widetilde{\mathcal{A}}\rangle$

Finally, we turn to the mixed symmetry operator $\mathcal{A}_{b_1b_2c_1\cdots c_k}(x_3)$ and its shadow

$$\begin{aligned}&\widetilde{\mathcal{A}}_{b_1b_2c_1\cdots c_k}(x_3) \qquad (3.8)\\ &= \widetilde{\Pi}^{(k)\,d_1d_2e_1\cdots e_k}_{b_1b_2c_1\cdots c_k}\int\frac{d^Dx_0}{(x_{03}^2)^{D-\Delta_{\mathcal{A}}}}m^{(03)}{}_{d_1}^{\ f_1}m^{(03)}{}_{d_2}^{\ f_2}m^{(03)}{}_{e_1}^{\ g_1}\cdots m^{(03)}{}_{e_k}^{\ g_k}\mathcal{A}_{f_1f_2g_1\cdots g_k}(x_0),\end{aligned}$$

which can appear in a three-point function with a scalar $\phi(x_1)$ and a vector $v_a(x_2)$.

Similar techniques to those employed above give the relation between the three-point coefficients,

$$\begin{aligned}\gamma_{\widetilde{\mathcal{A}}} &= \pi^{D/2}\frac{\Gamma(\Delta_{\mathcal{A}}+k)\Gamma(\Delta_{\mathcal{A}}-\frac{D}{2})\Gamma(\frac{1}{2}(D+\Delta_\phi-\Delta_v-\Delta_{\mathcal{A}}+k+1))}{\Gamma(\Delta_{\mathcal{A}})\Gamma(D-\Delta_{\mathcal{A}}+k+1)\Gamma(\frac{1}{2}(\Delta_\phi-\Delta_v+\Delta_{\mathcal{A}}+k+1))} \\ &\quad\times\frac{\Gamma(\frac{1}{2}(D-\Delta_\phi+\Delta_v-\Delta_{\mathcal{A}}+k+1))}{\Gamma(\frac{1}{2}(-\Delta_\phi+\Delta_v+\Delta_{\mathcal{A}}+k+1))}(\Delta_{\mathcal{A}}-2)\,\gamma_{\mathcal{A}}. \qquad (3.9)\end{aligned}$$

3.2 Shadow projectors

Given a primary operator $\mathcal{O}$, define a shadow projector

$$P_{\mathcal{O}} = \mathcal{N}_{\mathcal{O}}\int d^Dx_0\,|\mathcal{O}_{a_1\cdots a_\ell}(x_0)\rangle\left\langle\widetilde{\mathcal{O}}^{a_1\cdots a_\ell}(x_0)\right|. \qquad (3.10)$$

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This should be interpreted as an operator that gets inserted into a correlation function, separating it into two correlation functions with an integral. When inserted into a given channel in a correlation function, it is designed to pick out the contribution of $\mathcal{O}$ and its descendants. $\mathcal{N}_{\mathcal{O}}$ is a normalization constant that we fix by demanding

$$\langle\varphi_1(x_1)\varphi_2(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle = \langle\mathcal{O}_{a_1\cdots a_\ell}(x_3)P_{\mathcal{O}}\varphi_1(x_1)\varphi_2(x_2)\rangle \tag{3.11}$$

and so we need to take (see appendix E)

$$\mathcal{N}_{\mathcal{O}} = \pi^{-D}\frac{(\Delta_{\mathcal{O}}+\ell-1)\,(D-\Delta_{\mathcal{O}}+\ell-1)\,\Gamma(\Delta_{\mathcal{O}}-1)\Gamma(D-\Delta_{\mathcal{O}}-1)}{\Gamma(\Delta_{\mathcal{O}}-\frac{D}{2})\Gamma(\frac{D}{2}-\Delta_{\mathcal{O}})}. \tag{3.12}$$

Note that $\mathcal{N}_{\mathcal{O}}$ is independent of Δ_1 and Δ_2, as it should be.

Similarly, for the mixed symmetry case we can define a projector,

$$P_{\mathcal{A}} = \mathcal{N}_{\mathcal{A}}\int d^Dx_0\,|\mathcal{A}_{a_1a_2b_1\cdots b_k}(x_0)\rangle\left\langle\widetilde{\mathcal{A}}^{a_1a_2b_1\cdots b_k}\right|, \tag{3.13}$$

and $\mathcal{N}_{\mathcal{A}}$ can be computed to be

$$\mathcal{N}_{\mathcal{A}} = \pi^{-D}\frac{(\Delta_{\mathcal{A}}+k)\,(D-\Delta_{\mathcal{A}}+k)\,\Gamma(\Delta_{\mathcal{A}})\Gamma(D-\Delta_{\mathcal{A}})}{(\Delta_{\mathcal{A}}-2)\,(D-\Delta_{\mathcal{A}}-2)\,\Gamma(\Delta_{\mathcal{A}}-\frac{D}{2})\Gamma(\frac{D}{2}-\Delta_{\mathcal{A}})}. \tag{3.14}$$

4 Conformal blocks

We next turn to four-point functions. These can be evaluated by first performing operator product expansions (OPEs) of the first two operators and the last two operators, and then evaluating the remaining two-point functions. Consider first a general OPE. Let's use notation where $\bar{a}$ represents a multi-index, transforming as some representation of SO(D). Then the OPE of two arbitrary operators has the form

$$\phi_{1\,\bar{a}}(x_1)\phi_{2\,\bar{b}}(x_2) = \sum_{\mathcal{U}} f^{\bar{c}}_{12\mathcal{U}\,\bar{a}\bar{b}}(x_{12})\mathcal{U}_{\bar{c}}(x_2), \tag{4.1}$$

where the sum is in principle over all local operators $\mathcal{U}_{\bar{c}}(x)$ in the theory, and the coefficients $f^{\bar{c}}_{12\mathcal{U}\,\bar{a}\bar{b}}$ are functions of x_{12}. Actually, for fixed representations only a finite number of tensor structures are compatible with the symmetries, so we can write this as a sum over tensor structures labeled by r,

$$f^{\bar{c}}_{12\mathcal{U}\,\bar{a}\bar{b}}(x_{12}) = \sum_r \lambda_{12\mathcal{U}\,r}\,s^{r\,\bar{c}}_{\bar{a}\bar{b}}(x_{12}), \tag{4.2}$$

where the three-point tensor structures $s^{r\,\bar{c}}_{\bar{a}\bar{b}}(x_{12})$ are universal quantities which depend on the conformal representations (meaning both the SO(D) representations and the conformal weights) involved, but are otherwise independent of the theory or the particular operators. That dependence is entirely contained in the constants $\lambda_{12\mathcal{U}\,r}$. Finally, there is one further simplification, which is that when $\mathcal{U}$ is a descendent of a primary operator $\mathcal{O}$ (and thus corresponds to some differential operator acting on $\mathcal{O}$), then its coefficients in the $\phi_1\times\phi_2$

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OPE is determined linearly in terms of the coefficients of $\mathcal{O}$ in the OPE. Thus the OPE can in fact be written as a sum over primary operators $\mathcal{O}$,

$$\phi_{1\,\bar{a}}(x_1)\phi_{2\,\bar{b}}(x_2) = \sum_{\mathcal{O}} \sum_{r} \lambda_{12\mathcal{O}\,r} C^{r\,\bar{c}}_{\bar{a}\bar{b}}(x_{12}, \partial_2)\mathcal{O}_{\bar{c}}(x_2). \tag{4.3}$$

Again the differential operators $C^{r\,\bar{c}}_{\bar{a}\bar{b}}(x_{12}, \partial_2)$ are universal in the same sense as above.

Now inserting this form of the OPE into the four-point function, we can write

$$\langle \phi_{1\,\bar{a}}(x_1)\phi_{2\,\bar{b}}(x_2)\phi_{3\,\bar{c}}(x_3)\phi_{4\,\bar{d}}(x_4)\rangle = \sum_{\mathcal{O}} \sum_{r,s} \lambda_{12\mathcal{O}\,r}\lambda_{34\mathcal{O}\,s} W^{rs}_{\bar{a}\bar{b}\bar{c}\bar{d}}(x_1, x_2, x_3, x_4). \tag{4.4}$$

The functions $W^{rs}_{\bar{a}\bar{b}\bar{c}\bar{d}}$ depend only on the SO(D) representations and conformal weights of the ϕ_i and of $\mathcal{O}$. These functions are often called conformal partial waves (though this nomenclature is not universal). Conformal invariance can actually be used to further restrict the form of the W's, so that we can write

$$W^{rs}_{\bar{a}\bar{b}\bar{c}\bar{d}}(x_1, x_2, x_3, x_4) = X_{\Delta_i} \sum_{p} g^{rs}_p(u, v) t^p_{\bar{a}\bar{b}\bar{c}\bar{d}}(x_i). \tag{4.5}$$

Here the sum p runs over allowed tensor structures, and the four-point tensor structures $t^p_{\bar{a}\bar{b}\bar{c}\bar{d}}(x_i)$ depend only on the SO(D) representations of the external operators, not the conformal dimensions, while the functions $g^{rs}_p(u, v)$ depend on the full conformal representations (i.e. both SO(D) representations and conformal weights), but are themselves scalar functions of the cross-ratios u and v. These $g^{rs}_p(u, v)$ are called conformal blocks, and our task in the rest of this section is to compute them for the scalar and vector four-point functions of interest.

4.1 General discussion

Our primary purpose in this paper involves specific examples of four-point functions, but let us first have a very brief general discussion. Roughly, the idea is that by inserting the projector $P_{\mathcal{O}}$ into the correlator (4.4), we should pick out only the contribution from the primary $\mathcal{O}$ and its descendants. This is not quite correct, as explained in [45] and elsewhere; rather we must insert the projector and then pick out only the terms of the result which transform with a phase $e^{2\pi i(\Delta_{\mathcal{O}}-\Delta_1-\Delta_2)}$ as we send $x^2_{12} \to e^{4\pi i} x^2_{12}$. The remaining terms will transform with a phase $e^{-2\pi i(\Delta_{\mathcal{O}}+\Delta_1+\Delta_2)}$ under this rotation and these terms should be thrown away. This procedure is called monodromy projection. In practice, we can write the result before the monodromy projection as a certain double integral over Feynman-Schwinger parameters, and then the monodromy projection can be implemented simply as a modification of the integration region, along with some insertions of signs in the integrand. Once we have successfully picked out the contributions from $\mathcal{O}$ and its descendants, we can read off $g^{rs}_p(u, v)$ from the terms proportional to $\lambda_{12\mathcal{O}\,r}\lambda_{34\mathcal{O}\,s} t^p_{\bar{a}\bar{b}\bar{c}\bar{d}}$.

If we write the general four-point function as

$$\langle \phi_{1\,\bar{a}}(x_1)\phi_{2\,\bar{b}}(x_2)\phi_{3\,\bar{c}}(x_3)\phi_{4\,\bar{d}}(x_4)\rangle = X_{\Delta_i} \sum_{p} q_p(u, v) t^p_{\bar{a}\bar{b}\bar{c}\bar{d}}(x_i), \tag{4.6}$$

then we have

$$q_p(u, v) = \sum_{\mathcal{O}} \sum_{r,s} \lambda_{12\mathcal{O}\,r}\lambda_{34\mathcal{O}\,s} g^{rs}_p(u, v). \tag{4.7}$$

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4.2 Scalars and vectors

Let's understand what we should then be computing for our examples of interest. First we will review the case of four scalars. In this case the exchanged primary must be traceless symmetric, with its representation labeled by a spin ℓ and dimension $\Delta_{\mathcal{O}}$. There is a unique three-point tensor structure for each ℓ,

$$s_{a_1\cdots a_\ell} = \Pi^{(\ell)\, b_1\cdots b_\ell}_{a_1\cdots a_\ell} k^{(012)}_{b_1}\cdots k^{(012)}_{b_\ell}, \tag{4.8}$$

and a unique four-point tensor structure $t = 1$. In other words, the correlator should take the form

$$\begin{aligned}\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle &= \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} W(x_i;\Delta_i;\ell,\Delta_{\mathcal{O}}) \\ &= X_{\Delta_i}\sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, g(u,v;\Delta_i;\ell,\Delta_{\mathcal{O}}).\end{aligned} \tag{4.9}$$

Here we have shown on which parameters the conformal partial waves W or conformal blocks g can depend; often we will not indicate this explicitly. In terms of the function $q(u,v)$ introduced in (2.28) we have

$$q(u,v) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, g(u,v). \tag{4.10}$$

Next suppose we have three scalars and a vector v in the second position. The only possibility for exchanged operators are again traceless symmetric $\mathcal{O}$ of spin ℓ. There is only one tensor structure which can appear in the $\langle\phi_3\phi_4\mathcal{O}\rangle$ three-point function, with coefficient $\lambda_{34\mathcal{O}}$, but there are two possible tensor structures in the $\langle\phi_1(x_1)v(x_2)\mathcal{O}(x_0)\rangle$ three-point function,

$$s^{\alpha}_{ab_1\cdots b_\ell} = \Pi^{(\ell)\, c_1\cdots c_\ell}_{b_1\cdots b_\ell} k^{(210)}_a k^{(012)}_{c_1}\cdots k^{(012)}_{c_\ell}, \quad s^{\beta}_{ab_1\cdots b_\ell} = \Pi^{(\ell)\, c_1\cdots c_\ell}_{b_1\cdots b_\ell} m^{(20)}_{ac_1} k^{(012)}_{c_1}\cdots k^{(012)}_{c_\ell}, \tag{4.11}$$

whose coefficients we will label $\alpha_{12\mathcal{O}}$ and $\beta_{12\mathcal{O}}$. There are also two four-point tensor structures[6] $t^1_a = k^{(214)}_a$ and $t^2_a = k^{(234)}_a$, with coefficient functions $q_1(u,v)$ and $q_2(u,v)$ respectively, and these are related to the conformal blocks $g_1^{\alpha\lambda}$, $g_2^{\alpha\lambda}$, $g_1^{\beta\lambda}$, and $g_2^{\beta\lambda}$ by

$$q_i(u,v) = \sum_{\mathcal{O}} \lambda_{34\mathcal{O}}\left(\alpha_{12\mathcal{O}} g_i^{\alpha\lambda}(u,v) + \beta_{12\mathcal{O}} g_i^{\beta\lambda}(u,v)\right), \tag{4.12}$$

for $i = 1, 2$. Each of the four conformal block functions will depend on the conformal weights Δ_1, $\Delta_2 = \Delta_v$, Δ_3, Δ_4, and $\Delta_{\mathcal{O}}$, as well as the spin ℓ of $\mathcal{O}$.

Having the vector in the fourth position is essentially the same upon interchanging $(12) \leftrightarrow (34)$, with $t^1_a = k^{(412)}_a$, $t^2_a = k^{(432)}_a$, and

$$q'_i(u,v) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\left(\alpha_{34\mathcal{O}} g_i^{\lambda\alpha}(u,v) + \beta_{34\mathcal{O}} g_i^{\lambda\beta}(u,v)\right). \tag{4.13}$$

[6] From the point of view of this correlator these might not be the first choice of tensor structures; we might prefer $k^{(213)}$ and $k^{(214)}$ to be more symmetric between x_3 and x_4. However we will also being using these correlators and tensor structures as intermediate expressions in computing the SVSV conformal blocks, where the symmetry we will want to maintain is between x_2 and x_4.

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Finally, for the case of two scalars and two vectors we found in (2.39) five four-point tensor structures,

$$t^0_{ab} = m^{(24)}_{ab}, \quad t^{11}_{ab} = k^{(214)}_a k^{(412)}_b, \; t^{12}_{ab} = k^{(214)}_a k^{(432)}_b, \; t^{21}_{ab} = k^{(234)}_a k^{(412)}_b, \; t^{22}_{ab} = k^{(234)}_a k^{(432)}_b, \tag{4.14}$$

with associated coefficient functions q_0 and q_{ij}. In this case the exchanged operator can either be traceless symmetric $\mathcal{O}$ of spin ℓ, or it can be a mixed-symmetry operator $\mathcal{A}$ whose representation is labeled by k. In the former case, each of the three point function has two tensor structures s^α and s^β, while in the latter case there is a unique three-point tensor structure

$$s^\gamma_{ab_1b_2c_1\cdots c_k} = \widetilde{\Pi}^{(k)\,d_1d_2e_1\cdots e_k}_{b_1b_2c_1\cdots c_k} m^{(20)}_{ad_1} k^{(012)}_{d_2} k^{(012)}_{e_1} \cdots k^{(012)}_{e_k}. \tag{4.15}$$

Hence, for generic (not necessarily identical) scalars and vectors, we have

$$q_0 = \sum_{\mathcal{O}} \left(\alpha_{12\mathcal{O}}\alpha_{34\mathcal{O}} g_0^{\alpha\alpha} + \alpha_{12\mathcal{O}}\beta_{34\mathcal{O}} g_0^{\alpha\beta} + \beta_{12\mathcal{O}}\alpha_{34\mathcal{O}} g_0^{\beta\alpha} + \beta_{12\mathcal{O}}\beta_{34\mathcal{O}} g_0^{\beta\beta} \right) + \sum_{\mathcal{A}} \gamma_{12\mathcal{A}}\gamma_{34\mathcal{A}} g_0^{\gamma\gamma}, \tag{4.16}$$

$$q_{ij} = \sum_{\mathcal{O}} \left(\alpha_{12\mathcal{O}}\alpha_{34\mathcal{O}} g_{ij}^{\alpha\alpha} + \alpha_{12\mathcal{O}}\beta_{34\mathcal{O}} g_{ij}^{\alpha\beta} + \beta_{12\mathcal{O}}\alpha_{34\mathcal{O}} g_{ij}^{\beta\alpha} + \beta_{12\mathcal{O}}\beta_{34\mathcal{O}} g_{ij}^{\beta\beta} \right) + \sum_{\mathcal{A}} \gamma_{12\mathcal{A}}\gamma_{34\mathcal{A}} g_{ij}^{\gamma\gamma}. \tag{4.17}$$

Altogether there are twenty-five conformal block functions. $2 \times 2 \times 5 = 20$ of them are associated with symmetric traceless exchange and will depend on the spin ℓ as well as the conformal weights Δ_i and $\Delta_{\mathcal{O}}$, while the other five are associated to mixed symmetry exchange, and will depend on Δ_i, $\Delta_{\mathcal{A}}$ and k, which labels the mixed symmetry representation.

4.3 Exchange symmetries

As in the classification of tensor structures, the structure of conformal blocks can simplify significantly when some of the operators are identical, so that we have extra symmetry from exchanging those operators. Note however, that since the conformal block decomposition picks out a particular exchange channel, not all exchanges will give us constraints on individual conformal block functions. An exchange that results in a different exchange channel is called a crossing symmetry, and will constrain only the full sum of conformal blocks, not the individual conformal blocks themselves. Crossing symmetry is the subject of the next section, when we will set up the bootstrap. In the current subsection, however, we will consider the exchanges which don't mix channels, and so can constrain the blocks themselves. These can involve exchange of operator 1 with operator 2, of operator 3 with operator 4, or exchanging the pair (12) with the pair (34).

For example, consider the case of four scalars, with its unique conformal block function $g(u, v; \Delta_1, \Delta_2, \Delta_3, \Delta_4; \ell, \Delta_{\mathcal{O}})$, where for this section we will show explicit dependence on parameters. The four-point function will be invariant if we simultaneously exchange x_1 with x_2 and Δ_1 with Δ_2. This leads to a constraint on the conformal blocks,

$$g(u, v; \Delta_1, \Delta_2, \Delta_3, \Delta_4; \ell, \Delta_{\mathcal{O}}) = v^{-\frac{1}{2}(\Delta_3 - \Delta_4)} g(u/v, 1/v; \Delta_2, \Delta_1, \Delta_3, \Delta_4; \ell, \Delta_{\mathcal{O}}). \tag{4.18}$$

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Similarly, for $3 \leftrightarrow 4$ exchange,

$$g(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) = v^{\frac{1}{2}(\Delta_1-\Delta_2)} g(u/v,1/v;\Delta_1,\Delta_2,\Delta_4,\Delta_3;\ell,\Delta_{\mathcal{O}}), \tag{4.19}$$

and for $(12) \leftrightarrow (34)$ exchange,

$$g(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) = v^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_3+\Delta_4)} g(u,v;\Delta_3,\Delta_4,\Delta_1,\Delta_2;\ell,\Delta_{\mathcal{O}}). \tag{4.20}$$

These relations are most useful when some of the scalars are really identical. For example, if all four scalars are identical with weight Δ, then we have

$$g(u,v;\Delta;\ell,\Delta_{\mathcal{O}}) = g(u/v,1/v;\Delta;\ell,\Delta_{\mathcal{O}}). \tag{4.21}$$

In the case of three scalars and a vector, we have a couple of options. If the vector is in the second position, then we have the $3 \leftrightarrow 4$ exchange of scalars, which tells us that

$$g_1^{r\lambda}(u,v;\Delta_1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) = v^{\frac{1}{2}(\Delta_1-\Delta_v+1)} g_1^{r\lambda}(u/v,1/v;\Delta_1,\Delta_v,\Delta_4,\Delta_3;\ell,\Delta_{\mathcal{O}}), \tag{4.22}$$

and

$$\begin{aligned} &g_2^{r\lambda}(u,v;\Delta_1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) \\ &= -v^{\frac{1}{2}(\Delta_1-\Delta_v)} \left(u^{\frac{1}{2}} g_1^{r\lambda}(u/v,1/v;\Delta_1,\Delta_v,\Delta_4,\Delta_3;\ell,\Delta_{\mathcal{O}}) + g_2^{r\lambda}(u/v,1/v;\Delta_1,\Delta_v,\Delta_4,\Delta_3) \right), \end{aligned} \tag{4.23}$$

where r is either α or β. Note that in deriving these relations, we needed to transform the t_a^p under this exchange and then reëxpress the result in terms of our basis t_a^p again. Since our chosen basis $t_a^1 = k_a^{(214)}$, $t_a^2 = k^{(234)}$ does not behave particularly nicely (rather we chose it to make later computations with two scalars and two vectors slightly nicer), the resulting expressions are slightly messier than they would be in a basis like $t'^1_a = k_a^{(213)}$, $t'^2_a = k_a^{(214)}$ which simply gets exchanged under $3 \leftrightarrow 4$. Performing a $(12) \leftrightarrow (34)$ exchange relates the SVSS conformal blocks to the SSSV conformal blocks,

$$g_1^{r\lambda}(u,v;\Delta_1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) = v^{\frac{1}{2}(\Delta_1-\Delta_v-\Delta_3+\Delta_4)} g_2^{\lambda r}(u,v;\Delta_3,\Delta_4,\Delta_1,\Delta_v;\ell,\Delta_{\mathcal{O}}), \tag{4.24}$$

and

$$g_2^{r\lambda}(u,v;\Delta_1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) = v^{\frac{1}{2}(\Delta_1-\Delta_v-\Delta_3+\Delta_4)} g_1^{\lambda r}(u,v;\Delta_3,\Delta_4,\Delta_1,\Delta_v;\ell,\Delta_{\mathcal{O}}). \tag{4.25}$$

Finally, for the SVSV case, the only useful exchange is $(12) \leftrightarrow (34)$, which tells us

$$g_{22}^{rs}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) = v^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_3+\Delta_4)} g_{11}^{sr}(u,v;\Delta_3,\Delta_4,\Delta_1,\Delta_2;\ell,\Delta_{\mathcal{O}}), \tag{4.26}$$

and

$$g_p^{rs}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) = v^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_3+\Delta_4)} g_p^{sr}(u,v;\Delta_3,\Delta_4,\Delta_1,\Delta_2;\ell,\Delta_{\mathcal{O}}), \tag{4.27}$$

for r and s being α or β, and for p being 0, 12, or 21. Similarly

$$g_{22}^{\gamma\gamma}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}}) = v^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_3+\Delta_4)} g_{11}^{\gamma\gamma}(u,v;\Delta_3,\Delta_4,\Delta_1,\Delta_2;k,\Delta_{\mathcal{A}}), \tag{4.28}$$

$$g_p^{\gamma\gamma}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}}) = v^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_3+\Delta_4)} g_p^{\gamma\gamma}(u,v;\Delta_3,\Delta_4,\Delta_1,\Delta_2;k,\Delta_{\mathcal{A}}). \tag{4.29}$$

In particular, if we have identical scalars and identical vectors, then the g_0, g_{12}, and g_{21} are only constrained to be symmetric in their upper indices (i.e. $g_p^{\alpha\beta} = g_p^{\beta\alpha}$), while the g_{22} functions are determined by the g_{11}'s,

$$g_{22}^{rs}(u,v) = g_{11}^{sr}(u,v). \tag{4.30}$$

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4.4 Computing the blocks

At the risk of cluttering notation, we will add a hat to the conformal block functions to denote the result obtained from insertion of the shadow projector,

$$\langle\phi_{1\,\bar{a}}(x_1)\phi_{2\,\bar{b}}(x_2)P_{\mathcal{O}}\phi_{3\,\bar{c}}(x_3)\phi_{4\,\bar{d}}(x_4)\rangle = X_{\Delta_i}\lambda_{12\mathcal{O}\,r}\lambda_{34\mathcal{O}\,s}\,\widehat{g}_p^{rs}(u,v)t^p_{\bar{a}\bar{b}\bar{c}\bar{d}}. \tag{4.31}$$

The actual conformal blocks $g_p^{rs}(u,v)$ themselves are then obtained from the $\widehat{g}_p^{rs}(u,v)$ by a monodromy projection, which now picks out the terms in $\widehat{g}_p^{rs}(u,v)$ which transform with a phase $e^{2\pi i\Delta_{\mathcal{O}}}$ as $u\to e^{4\pi i}u$, and throws away the terms which transform as $e^{-2\pi i\Delta_{\mathcal{O}}}$.

We will call the process of implementing the monodromy projection, going from $\widehat{g}_p^{rs}$ to g_p^{rs} (i.e. removing the hat), *doffing*.

4.4.1 $\langle SSSS\rangle$

We'll start by reviewing the computation of the conformal blocks for four scalar operators. Here, on insertion of the shadow projector we have

$$\begin{aligned}
&\langle\phi_1(x_1)\phi_2(x_2)P_{\mathcal{O}}\phi_3(x_3)\phi_4(x_4)\rangle\\
&\quad=\mathcal{N}_{\mathcal{O}}\int d^Dx_0\,\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_0)\rangle\left\langle\widetilde{\mathcal{O}}^{a_1\cdots a_\ell}(x_0)\phi_3(x_3)\phi_4(x_4)\right\rangle\\
&\quad=\mathcal{N}_{\mathcal{O}}\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}\int d^Dx_0\left[\lambda_{12\mathcal{O}}\left(x_{01}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_{\mathcal{O}})}\left(x_{02}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_{\mathcal{O}})}\left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_{\mathcal{O}})}\right.\\
&\qquad\left.\times k_{a_1}^{(012)}\cdots k_{a_\ell}^{(012)}\right]\left[\lambda_{34\widetilde{\mathcal{O}}}\left(x_{03}^2\right)^{\frac{1}{2}(-\Delta_3+\Delta_4+\Delta_{\mathcal{O}}-D)}\left(x_{04}^2\right)^{\frac{1}{2}(\Delta_3-\Delta_4+\Delta_{\mathcal{O}}-D)}\right.\\
&\qquad\left.\times\left(x_{34}^2\right)^{\frac{1}{2}(D-\Delta_3-\Delta_4-\Delta_{\mathcal{O}})}k^{(034)\,b_1}\cdots k^{(034)\,b_\ell}\right]\\
&\quad=X_{\Delta_i}\left[\mathcal{N}_{\mathcal{O}}\lambda_{12\mathcal{O}}\lambda_{34\widetilde{\mathcal{O}}}\left(x_{12}^2\right)^{\frac{1}{2}\Delta_{\mathcal{O}}}\left(x_{13}^2\right)^{\frac{1}{2}(\Delta_3-\Delta_4)}\left(x_{14}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_3+\Delta_4)}\left(x_{24}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2)}\right.\\
&\qquad\times\left(x_{34}^2\right)^{\frac{1}{2}(D-\Delta_{\mathcal{O}})}\int d^Dx_0\left(x_{01}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_{\mathcal{O}})}\left(x_{02}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_{\mathcal{O}})}\left(x_{03}^2\right)^{\frac{1}{2}(-\Delta_3+\Delta_4+\Delta_{\mathcal{O}}-D)}\\
&\qquad\left.\times\left(x_{04}^2\right)^{\frac{1}{2}(\Delta_3-\Delta_4+\Delta_{\mathcal{O}}-D)}\left(k_{a_1}^{(012)}\cdots k_{a_\ell}^{(012)}\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}k^{(034)\,b_1}\cdots k^{(034)\,b_\ell}\right)\right].
\end{aligned}\tag{4.32}$$

from which we can identify $\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\widehat{g}(u,v)$ with the quantity in square brackets.

As shown in appendix B.1, we can write

$$k_{a_1}^{(012)}\cdots k_{a_\ell}^{(012)}\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}k^{(034)\,b_1}\cdots k^{(034)\,b_\ell}=p_{D,\ell}(t), \tag{4.33}$$

where $p_{D,\ell}(t)$ is a polynomial of degree ℓ whose properties are explained in the appendix, and

$$\begin{aligned}
t&=k^{(012)}\cdot k^{(034)} \qquad (4.34)\\
&=\frac{1}{2}\left(x_{01}^2x_{02}^2x_{03}^2x_{04}^2x_{12}^2x_{34}^2\right)^{-1/2}\left(-x_{01}^2x_{03}^2x_{24}^2+x_{01}^2x_{04}^2x_{23}^2+x_{02}^2x_{03}^2x_{14}^2-x_{02}^2x_{04}^2x_{13}^2\right).
\end{aligned}$$

Let us now define integrals

$$I^{(\ell)}_{\alpha,\beta,\gamma,\delta}=\int\frac{d^Dx_0\,p_{D,\ell}(t)}{\left(x_{01}^2\right)^{\alpha}\left(x_{02}^2\right)^{\beta}\left(x_{03}^2\right)^{\gamma}\left(x_{04}^2\right)^{\delta}}. \tag{4.35}$$

For $\ell=0$ this integral is evaluated in (D.12).

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With this definition and the expressions (3.12) and (3.4), we have

$$\begin{aligned}
&\widehat{g}(u,v;\Delta_i;\ell,\Delta_{\mathcal{O}}) \qquad (4.36)\\
&= \pi^{-D/2}\frac{\Gamma(\frac{1}{2}(D+\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell))\Gamma(\frac{1}{2}(D-\Delta_3+\Delta_4-\Delta_{\mathcal{O}}+\ell))\Gamma(\Delta_{\mathcal{O}}+\ell)}{\Gamma(\frac{1}{2}(\Delta_3-\Delta_4+\Delta_{\mathcal{O}}+\ell))\Gamma(\frac{1}{2}(-\Delta_3+\Delta_4+\Delta_{\mathcal{O}}+\ell))\Gamma(D-\Delta_{\mathcal{O}}+\ell-1)}\\
&\quad\times\frac{\Gamma(D-\Delta_{\mathcal{O}}-1)}{\Gamma(\frac{D}{2}-\Delta_{\mathcal{O}})}\left(x_{12}^2\right)^{\frac{1}{2}\Delta_{\mathcal{O}}}\left(x_{13}^2\right)^{\frac{1}{2}(\Delta_3-\Delta_4)}\left(x_{14}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_3+\Delta_4)}\left(x_{24}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2)}\\
&\quad\times\left(x_{34}^2\right)^{\frac{1}{2}(D-\Delta_{\mathcal{O}})}I^{(\ell)}_{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_{\mathcal{O}}),\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}),\frac{1}{2}(D+\Delta_3-\Delta_4-\Delta_{\mathcal{O}}),\frac{1}{2}(D-\Delta_3+\Delta_4-\Delta_{\mathcal{O}})}.
\end{aligned}$$

Note that the prefactor $(x_{12}^2)^{\Delta_{\mathcal{O}}/2}$ already has the desired behavior under the monodromy projection, so we will want to pick out the terms from the integral which are invariant under the monodromy.

Note that if we expand the polynomial $p_{D,\ell}(t)$ using the explicit formulae in appendix B.1 then the integral is simply a sum of terms of a form computed in appendix D.2. For example, in the case $\ell = 0$, then $p_{D,0}(t) = 1$, and we have (restoring the explicit parameter dependence)

$$\begin{aligned}
&\widehat{g}(u,v;\Delta_i;0,\Delta_{\mathcal{O}}) \qquad (4.37)\\
&= \frac{\Gamma(\Delta_{\mathcal{O}})\Gamma(\frac{1}{2}(D-\Delta_3+\Delta_4-\Delta_{\mathcal{O}}))}{\Gamma(\frac{D}{2}-\Delta_{\mathcal{O}})\Gamma(\frac{1}{2}(\Delta_1-\Delta_2+\Delta_{\mathcal{O}}))\Gamma(\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}))\Gamma(\frac{1}{2}(-\Delta_3+\Delta_4+\Delta_{\mathcal{O}}))}u^{\frac{1}{2}\Delta_{\mathcal{O}}}\\
&\times v^{\frac{1}{2}(-\Delta_3+\Delta_4-\Delta_{\mathcal{O}})}\widehat{f}_{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_{\mathcal{O}}),\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}),\frac{1}{2}(D+\Delta_3-\Delta_4-\Delta_{\mathcal{O}}),\frac{1}{2}(D-\Delta_3+\Delta_4-\Delta_{\mathcal{O}})}(uv^{-1},v^{-1}),
\end{aligned}$$

where $\widehat{f}$ is defined in (D.15). Since the $u^{\Delta_{\mathcal{O}}/2}$ factor already behaves correctly under the monodromy projection, then to obtain the conformal block $g(u,v)$ we must restrict to the monodromy invariant piece of $\widehat{f}$, and this is given simply by a function f defined in (D.16). Then $g(u,v;\Delta_i;0,\Delta_{\mathcal{O}})$ is given by doffing the expression (4.37), replacing $\widehat{f}$ by f. Note also that this formula shows explicity that $g(u,v;\Delta_i;\ell,\Delta_{\mathcal{O}})$ doesn't depend on all four of the Δ_i individually, but only on the differences $\Delta_1-\Delta_2$ and $\Delta_3-\Delta_4$. Because of this we can adopt some condensed notation that will be useful below, defining functions that are related to the standard blocks by shifting these two differences by integer amounts P and Q,

$$g_{\ell;P,Q}(u,v) = g(u,v;\Delta_1+P,\Delta_2,\Delta_3+Q,\Delta_4;\ell,\Delta_{\mathcal{O}}). \qquad (4.38)$$

In this notation (which can also be used for $\widehat{g}$) the dependence on the Δ_i and $\Delta_{\mathcal{O}}$ is left implicit.

In even dimensions[7] we can evaluate the integrals in f explicitly, with the result

$$\begin{aligned}
g_{0;0,0}(u,v) = &\frac{\Gamma(\Delta_{\mathcal{O}})\Gamma(\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}-D+2))\Gamma(\frac{1}{2}(\Delta_3-\Delta_4+\Delta_{\mathcal{O}}-D+2))}{\Gamma(\Delta_{\mathcal{O}}-\frac{D}{2}+1)\Gamma(\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}))\Gamma(\frac{1}{2}(\Delta_3-\Delta_4+\Delta_{\mathcal{O}}))} \qquad (4.39)\\
&\times(x\bar{x})^{\frac{1}{2}\Delta_{\mathcal{O}}}\left(\frac{1}{x-\bar{x}}(x\partial_x-\bar{x}\partial_{\bar{x}})\right)^{\frac{D}{2}-1}
\end{aligned}$$

[7] In arbitrary dimensions there exists a closed form for the $\ell = 0$ conformal block in terms of Appel functions [41], but for even dimensions the result can be expressed using the much more familiar ${}_2F_1$ hypergeometric functions.

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$$\cdot\left[{}_2F_1\left(\frac{-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}-D+2}{2},\frac{\Delta_3-\Delta_4+\Delta_{\mathcal{O}}-D+2}{2},\Delta_{\mathcal{O}}-\frac{D}{2}+1;x\right)\right.$$
$$\left.\times{}_2F_1\left(\frac{-\Delta_1+\Delta_2+\Delta_{\mathcal{O}}-D+2}{2},\frac{\Delta_3-\Delta_4+\Delta_{\mathcal{O}}-D+2}{2},\Delta_{\mathcal{O}}-\frac{D}{2}+1;\bar{x}\right)\right],$$

where the variables x and $\bar{x}$ are related to u and v via

$$u=x\bar{x},\qquad v=(1-x)(1-\bar{x}). \tag{4.40}$$

What about $\ell>0$? As indicated, for any fixed small ℓ we can of course expand $p_{D,\ell}(t)$ into monomials and proceed as above. But in fact we can be a bit more clever than that and exploit the recursion relations (B.15) to expand the numerator of the integrand in (4.35). In the recursion relation we also need to expand t according to (4.34), and reabsorb the powers of (x_{0i}^2) as shifts of the external operator dimensions. Finally, passing to the monodromy-projected answer, the result is [41]

$$\begin{aligned}
&g_{\ell;0,0}(u,v) \qquad\qquad (4.41)\\
&=\frac{\Delta_{\mathcal{O}}+\ell-1}{D-\Delta_{\mathcal{O}}+\ell-2}\left[\frac{1}{2}\frac{D+\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell-2}{\Delta_3-\Delta_4+\Delta_{\mathcal{O}}+\ell-2}u^{-1/2}\left(g_{\ell-1;1,-1}(u,v)-g_{\ell-1;-1,-1}(u,v)\right)\right.\\
&\quad+\frac{1}{2}\frac{D-\Delta_3+\Delta_4-\Delta_{\mathcal{O}}+\ell-2}{-\Delta_3+\Delta_4+\Delta_{\mathcal{O}}+\ell-2}u^{-1/2}\left(vg_{\ell-1;-1,1}(u,v)-g_{\ell-1;1,1}(u,v)\right)\\
&\quad-\frac{(\Delta_{\mathcal{O}}+\ell-2)(D+\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell-2)(D-\Delta_3+\Delta_4-\Delta_{\mathcal{O}}+\ell-2)}{(D-\Delta_{\mathcal{O}}+\ell-3)(\Delta_3-\Delta_4+\Delta_{\mathcal{O}}+\ell-2)(-\Delta_3+\Delta_4+\Delta_{\mathcal{O}}+\ell-2)}\\
&\quad\left.\times\frac{(\ell-1)(D+\ell-4)}{(D+2\ell-4)(D+2\ell-6)}g_{\ell-2;0,0}(u,v)\right]
\end{aligned}$$

This recursion holds in any dimension. In $D=2$ the recursion can actually be solved explicitly to get a closed form expression for $g(u,v;\Delta_i;\ell,\Delta_{\mathcal{O}})$ in terms of elementary hypergeometric functions, and in higher even dimensions solutions can also be constructed (by using a relation between the blocks in $D+2$ dimensions and those in D dimensions). For example, in $D=4$,

$$g(u,v;\Delta_i;\ell,\Delta_{\mathcal{O}})=\left(-\frac{1}{2}\right)^{\ell}\frac{x\bar{x}}{x-\bar{x}}\left[k_{\Delta_{\mathcal{O}}+\ell}(x)k_{\Delta_{\mathcal{O}}-\ell-2}(\bar{x})-k_{\Delta_{\mathcal{O}}-\ell-2}(x)k_{\Delta_{\mathcal{O}}+\ell}(\bar{x})\right], \tag{4.42}$$

where

$$k_\beta(x)=x^{\beta/2}{}_2F_1\left(\frac{\beta-\Delta_1+\Delta_2}{2},\frac{\beta+\Delta_3-\Delta_4}{2},\beta;x\right). \tag{4.43}$$

This is a good moment to make a point about normalizations. The Casimir differential equation implies that in the limit $\bar{x},x\to 0$ the conformal block should behave approximately as $c_\ell x^{\frac{1}{2}(\Delta_{\mathcal{O}}+\ell)}\bar{x}^{\frac{1}{2}(\Delta_{\mathcal{O}}-\ell)}$ for some constants c_ℓ. Our blocks are defined according to (4.4) and (4.5), and this turns out to imply $c_\ell=(-1/2)^\ell$. Some authors prefer different normalizations, say with $c_\ell=1$. It is always easy to go back and forth between conventions, as long as one is aware of them.

An alternative approach is to expand the polynomials $p_{D,\ell}(t)$ in the integrals $I^{(\ell)}$ to obtain an expression for conformal blocks with $\ell>0$ as a sum of $\ell=0$ blocks. This result (with or without hats) is

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$$\widehat{g}_{\ell;0,0}(u,v) = 2^{-\ell}\frac{\Gamma(\frac{1}{2}(D+\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell))\Gamma(\frac{1}{2}(D-\Delta_3+\Delta_4-\Delta_{\mathcal{O}}+\ell))}{\Gamma(\frac{1}{2}(\Delta_3-\Delta_4+\Delta_{\mathcal{O}}+\ell))\Gamma(\frac{1}{2}(-\Delta_3+\Delta_4+\Delta_{\mathcal{O}}+\ell))}\frac{\Gamma(\Delta_{\mathcal{O}}+\ell)}{\Gamma(\Delta_{\mathcal{O}})}$$
$$\times\frac{\Gamma(D-\Delta_{\mathcal{O}}-1)}{\Gamma(D-\Delta_{\mathcal{O}}+\ell-1)}\sum_{i=0}^{\lfloor\ell/2\rfloor}\sum_{A=0}^{\ell-2i}\sum_{B=0}^{\ell-2i-A}\sum_{C=0}^{\ell-2i-A-B}(-1)^{\ell+i+B+C}\frac{\ell!}{i!A!B!C!(\ell-2i-A-B-C)!}$$
$$\times\frac{\Gamma(\frac{1}{2}(\Delta_3-\Delta_4+\Delta_{\mathcal{O}}+\ell)-i-A-C)\Gamma(\frac{1}{2}(-\Delta_3+\Delta_4+\Delta_{\mathcal{O}}-\ell)+i+A+C)}{\Gamma(\frac{1}{2}(D+\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell)-i-A-C)\Gamma(\frac{1}{2}(D-\Delta_3+\Delta_4-\Delta_{\mathcal{O}}-\ell)+i+A+C)}$$
$$\times\frac{\Gamma(\frac{D}{2}+\ell-i-1)}{\Gamma(\frac{D}{2}+\ell-1)}u^{i-\frac{\ell}{2}}v^B\widehat{g}_{0;\ell-2(i+A+B),\ell-2(i+A+C)}(u,v) \tag{4.44}$$

At any rate, in subsequent sections we will assume that these SSSS conformal blocks are some known functions, and we will endeavor to compute the new conformal blocks in terms of these.

4.4.2 $\langle SVSS\rangle$

Let's now move to the case with one vector. The most efficient way to proceed is to first note that we can relate the three-point function of a scalar, a vector, and a symmetric traceless tensor to the three-point function of two scalars and a symmetric traceless tensor [44]. Explicitly, we can define

$$\begin{aligned}&S^{\lambda}_{a_1\cdots a_\ell}(x_i;\Delta_i)\\&=(x_{12}^2)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_{\mathcal{O}})}(x_{13}^2)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_{\mathcal{O}})}(x_{23}^2)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_{\mathcal{O}})}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}k^{(312)}_{b_1}\cdots k^{(312)}_{b_\ell},\end{aligned} \tag{4.45}$$

so that

$$\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle=\lambda_{12\mathcal{O}}S^{\lambda}_{a_1\cdots a_\ell}(x_i;\Delta_i). \tag{4.46}$$

Then we can write

$$\langle\phi_1(x_1)v_a(x_2)\mathcal{O}_{b_1\cdots b_\ell}(x_3)\rangle=\alpha_{12\mathcal{O}}S^{\alpha}_{a\,b_1\cdots b_\ell}(x_i;\Delta_i)+\beta_{12\mathcal{O}}S^{\beta}_{a\,b_1\cdots b_\ell}(x_i;\Delta_i), \tag{4.47}$$

where

$$\begin{aligned}&S^{\alpha}_{a\,b_1\cdots b_\ell}(x_i;\Delta_\phi,\Delta_v,\Delta_{\mathcal{O}})\\&=(x_{12}^2)^{\frac{1}{2}(-\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}})}(x_{13}^2)^{\frac{1}{2}(-\Delta_\phi+\Delta_v-\Delta_{\mathcal{O}})}(x_{23}^2)^{\frac{1}{2}(\Delta_\phi-\Delta_v-\Delta_{\mathcal{O}})}\Pi^{(\ell)\,c_1\cdots c_\ell}_{b_1\cdots b_\ell}k^{(213)}_{a}k^{(312)}_{c_1}\cdots k^{(312)}_{c_\ell}\\&=\frac{1}{2(1-\Delta_{\mathcal{O}})}\left[m^{(12)}{}_a{}^c\left(\frac{\partial}{\partial x_1^c}+2(\Delta_\phi-1)\frac{(x_{12})_c}{x_{12}^2}\right)S^{\lambda}_{b_1\cdots b_\ell}(x_i;\Delta_\phi-1,\Delta_v,\Delta_{\mathcal{O}})\right.\\&\quad\left.+\left(\frac{\partial}{\partial x_2^a}-2(\Delta_v-1)\frac{(x_{12})_a}{x_{12}^2}\right)S^{\lambda}_{b_1\cdots b_\ell}(x_i;\Delta_\phi,\Delta_v-1,\Delta_{\mathcal{O}})\right],\end{aligned} \tag{4.48}$$

and

$$\begin{aligned}&S^{\beta}_{a\,b_1\cdots b_\ell}(x_i;\Delta_\phi,\Delta_v,\Delta_{\mathcal{O}})\\&=(x_{12}^2)^{\frac{1}{2}(-\Delta_\phi-\Delta_v+\Delta_{\mathcal{O}})}(x_{13}^2)^{\frac{1}{2}(-\Delta_\phi+\Delta_v-\Delta_{\mathcal{O}})}(x_{23}^2)^{\frac{1}{2}(\Delta_\phi-\Delta_v-\Delta_{\mathcal{O}})}\Pi^{(\ell)\,c_1\cdots c_\ell}_{b_1\cdots b_\ell}m^{(23)}_{ac_1}k^{(312)}_{c_2}\cdots k^{(312)}_{c_\ell}\\&=\frac{\Delta_\phi-\Delta_v-\Delta_{\mathcal{O}}+\ell+1}{\ell}S^{\alpha}_{a\,b_1\cdots b_\ell}(x_i;\Delta_\phi,\Delta_v,\Delta_{\mathcal{O}})\end{aligned} \tag{4.49}$$

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$$- \frac{1}{\ell}\left(\frac{\partial}{\partial x_2^a} - 2\left(\Delta_v - 1\right)\frac{(x_{12})_a}{x_{12}^2}\right) S^{\lambda}_{b_1\cdots b_\ell}(x_i;\Delta_\phi,\Delta_v-1,\Delta_{\mathcal{O}}),$$

as can be verified by explicit computation.

The conformal blocks will be computed by the expression

$$\begin{aligned} & X_{\Delta_1,\Delta_v,\Delta_3,\Delta_4}\lambda_{34\mathcal{O}}\left[\left(\alpha_{12\mathcal{O}}\widehat{g}_1^{\alpha\lambda}+\beta_{12\mathcal{O}}\widehat{g}_1^{\beta\lambda}\right)k_a^{(214)}+\left(\alpha_{12\mathcal{O}}\widehat{g}_2^{\alpha\lambda}+\beta_{12\mathcal{O}}\widehat{g}_2^{\beta\lambda}\right)k_a^{(234)}\right] \\ & = \mathcal{N}_{\mathcal{O}}\int d^Dx_0\,\langle\phi_1(x_1)v_a(x_2)\mathcal{O}_{b_1\cdots b_\ell}(x_0)\rangle\left\langle\phi_3(x_3)\phi_4(x_4)\widetilde{\mathcal{O}}^{b_1\cdots b_\ell}(x_0)\right\rangle \\ & = \mathcal{N}_{\mathcal{O}}\int d^Dx_0\left(\alpha_{12\mathcal{O}}S^{\alpha}_{a\,b_1\cdots b_\ell}(x_1,x_2,x_0;\Delta_1,\Delta_v,\Delta_{\mathcal{O}})+\beta_{12\mathcal{O}}S^{\beta}_{a\,b_1\cdots b_\ell}(x_1,x_2,x_0;\Delta_1,\Delta_v,\Delta_{\mathcal{O}})\right) \\ & \quad\times\lambda_{34\widetilde{\mathcal{O}}}S^{\lambda\,b_1\cdots b_\ell}(x_3,x_4,x_0;\Delta_3,\Delta_4,D-\Delta_{\mathcal{O}}). \end{aligned} \tag{4.50}$$

On the other hand, we have

$$\begin{aligned} & X_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\widehat{g}(u,v;\Delta_i;\ell,\Delta_{\mathcal{O}}) \\ & = \mathcal{N}_{\mathcal{O}}\lambda_{12\mathcal{O}}\lambda_{34\widetilde{\mathcal{O}}}\int d^Dx_0 S^{\lambda}_{a_1\cdots a_\ell}(x_1,x_2,x_0;\Delta_1,\Delta_2,\Delta_{\mathcal{O}})S^{\lambda\,a_1\cdots a_\ell}(x_3,x_4,x_0;\Delta_3,\Delta_4,D-\Delta_{\mathcal{O}}). \end{aligned} \tag{4.51}$$

By expressing S^α and S^β in terms of S^λ, and pulling the differential operators outside of the integral, we can express $\widehat{g}_i^{r\lambda}$ in terms of differential operators acting on $\widehat{g}$.

For example, to compute $\widehat{g}_i^{\alpha\lambda}$ we get

$$\begin{aligned} & \widehat{g}_1^{\alpha\lambda}k_a^{(214)}+\widehat{g}_2^{\alpha\lambda}k_a^{(234)} = \frac{1}{2\left(1-\Delta_{\mathcal{O}}\right)}X^{-1}_{\Delta_1,\Delta_v,\Delta_3,\Delta_4} \\ & \times\left[m^{(12)}{}_a{}^c\left(\frac{\partial}{\partial x_1^c}+2\left(\Delta_1-1\right)\frac{(x_{12})_c}{x_{12}^2}\right)\left(X_{\Delta_1-1,\Delta_v,\Delta_3,\Delta_4}\widehat{g}(x_i;\Delta_\phi-1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}})\right)\right. \\ & \left.+\left(\frac{\partial}{\partial x_2^a}-2\left(\Delta_v-1\right)\frac{(x_{12})_a}{x_{12}^2}\right)\left(X_{\Delta_1,\Delta_v-1,\Delta_3,\Delta_4}\widehat{g}(x_i;\Delta_1,\Delta_v-1,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}})\right)\right]. \end{aligned} \tag{4.52}$$

This leads to

$$\begin{aligned} \widehat{g}_1^{\alpha\lambda} = \frac{1}{2\left(1-\Delta_{\mathcal{O}}\right)}\left[\left(1-\Delta_1+\Delta_v+\left(1-v\right)\left(\Delta_3-\Delta_4\right)+2v\left(1-v\right)\partial_v-2uv\partial_u\right)\right. \\ \times\widehat{g}(u,v;\Delta_1-1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) \\ \left.+\left(1+\Delta_1-\Delta_v-2u\partial_u\right)\widehat{g}(u,v;\Delta_1+1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}})\right], \end{aligned} \tag{4.53}$$

$$\begin{aligned} \widehat{g}_2^{\alpha\lambda} = \frac{\sqrt{uv}}{2\left(1-\Delta_{\mathcal{O}}\right)}\left[\left(\Delta_3-\Delta_4+2u\partial_u+2v\partial_v\right)\widehat{g}(u,v;\Delta_1-1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}})\right. \\ \left.-2\partial_v\widehat{g}(u,v;\Delta_1+1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}})\right], \end{aligned} \tag{4.54}$$

Similarly,

$$\begin{aligned} \widehat{g}_1^{\beta\lambda} = \frac{\Delta_1-\Delta_v-\Delta_{\mathcal{O}}+\ell+1}{\ell}\widehat{g}_1^{\alpha\lambda}(u,v;\Delta_1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) \\ -\frac{1}{\ell}\left(1+\Delta_1-\Delta_v-2u\partial_u\right)\widehat{g}(u,v;\Delta_1+1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}), \end{aligned} \tag{4.55}$$

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$$\widehat{g}_2^{\beta\lambda} = \frac{\Delta_1 - \Delta_v - \Delta_{\mathcal{O}} + \ell + 1}{\ell} \widehat{g}_2^{\alpha\lambda}(u, v; \Delta_1, \Delta_v, \Delta_3, \Delta_4; \ell, \Delta_{\mathcal{O}}) + \frac{2\sqrt{uv}}{\ell} \partial_v \widehat{g}(u, v; \Delta_1 + 1, \Delta_v, \Delta_3, \Delta_4; \ell, \Delta_{\mathcal{O}}). \quad (4.56)$$

Note that as with the scalar blocks, the expressions only depend on the difference $\Delta_1 - \Delta_v$ and $\Delta_3 - \Delta_4$, not on the weights individually. The other crucial property of these expressions is that the operators which act on the $\widehat{g}$ on the right hand side involve only integer powers of $\sqrt{u}$, so in particular they are all invariant under the monodromy projection. This means that when we implement the monodromy projection, all we have to do is remove the hats from the scalar blocks on the right hand side and from the new blocks on the left hand side. After these expressions have been thus doffed, we have relations between the full $g_i^{r\lambda}$ blocks and the scalar blocks g.

4.4.3 $\langle SSSV \rangle$

The case when the vector is in the fourth position is very similar. We have

$$\begin{aligned} & X_{\Delta_1,\Delta_2,\Delta_3,\Delta_v} \lambda_{12\mathcal{O}} \left[\left(\alpha_{34\mathcal{O}} \widehat{g}_1^{\lambda\alpha} + \beta_{34\mathcal{O}} \widehat{g}_1^{\lambda\beta} \right) k_a^{(412)} + \left(\alpha_{34\mathcal{O}} \widehat{g}_2^{\lambda\alpha} + \beta_{34\mathcal{O}} \widehat{g}_2^{\lambda\beta} \right) k_a^{(432)} \right] \quad (4.57) \\ & = \mathcal{N}_{\mathcal{O}} \int d^D x_0 \, \langle \phi_1(x_1) \phi_2(x_2) \mathcal{O}_{b_1 \cdots b_\ell}(x_0) \rangle \left\langle \phi_3(x_3) v_a(x_4) \widetilde{\mathcal{O}}^{b_1 \cdots b_\ell}(x_0) \right\rangle \\ & = \mathcal{N}_{\mathcal{O}} \int d^D x_0 \lambda_{12\mathcal{O}} S^{\lambda}_{b_1 \cdots b_\ell}(x_1, x_2, x_0; \Delta_1, \Delta_2, \Delta_{\mathcal{O}}) \\ & \times \left(\alpha_{34\widetilde{\mathcal{O}}} S^{\alpha}{}_{a}{}^{b_1 \cdots b_\ell}(x_3, x_4, x_0; \Delta_3, \Delta_v, D - \Delta_{\mathcal{O}}) + \beta_{34\widetilde{\mathcal{O}}} S^{\beta}{}_{a}{}^{b_1 \cdots b_\ell}(x_3, x_4, x_0; \Delta_3, \Delta_v, D - \Delta_{\mathcal{O}}) \right). \end{aligned}$$

Note that because $\alpha_{34\widetilde{\mathcal{O}}}$ is not simply proportional to $\alpha_{34\mathcal{O}}$ (we should expand it using (3.5)), and similarly for the β's, it will now be the case (unlike for SVSS) that each conformal block will get contributions from both terms on the right-hand side.

The results (after also doffing the expressions) are

$$\begin{aligned} g_1^{\lambda\alpha} = \frac{\sqrt{u}}{2(\Delta_{\mathcal{O}} - 1)} & \left[(1 - \Delta_3 + \Delta_v - 2u\partial_u - 2v\partial_v) \, g(u, v; \Delta_1, \Delta_2, \Delta_3 - 1, \Delta_v; \ell, \Delta_{\mathcal{O}}) \right. \\ & \left. + (1 - \Delta_1 + \Delta_2 + \Delta_3 - \Delta_v + 2v\partial_v) \, g(u, v; \Delta_1, \Delta_2, \Delta_3 + 1, \Delta_v; \ell, \Delta_{\mathcal{O}}) \right], \quad (4.58) \end{aligned}$$

$$\begin{aligned} g_2^{\lambda\alpha} = \frac{\sqrt{v}}{2(1 - \Delta_{\mathcal{O}})} & \left[(1 - \Delta_3 + \Delta_v - 2u\partial_u + 2(1 - v)\partial_v) \, g(u, v; \Delta_1, \Delta_2, \Delta_3 - 1, \Delta_v; \ell, \Delta_{\mathcal{O}}) \right. \\ & \left. + (1 + \Delta_3 - \Delta_v - 2u\partial_u) \, g(u, v; \Delta_1, \Delta_2, \Delta_3 + 1, \Delta_v; \ell, \Delta_{\mathcal{O}}) \right], \quad (4.59) \end{aligned}$$

$$\begin{aligned} g_1^{\lambda\beta} = & \frac{\Delta_3 - \Delta_v - \Delta_{\mathcal{O}} + \ell + 1}{\ell} g_1^{\lambda\alpha}(u, v; \Delta_1, \Delta_2, \Delta_3, \Delta_v; \ell, \Delta_{\mathcal{O}}) \\ & + \frac{\sqrt{u}}{\ell} (1 - \Delta_1 + \Delta_2 + \Delta_3 - \Delta_v + 2v\partial_v) \, g(u, v; \Delta_1, \Delta_2, \Delta_3 + 1, \Delta_v; \ell, \Delta_{\mathcal{O}}), \quad (4.60) \end{aligned}$$

$$\begin{aligned} g_2^{\lambda\beta} = & \frac{\Delta_3 - \Delta_v - \Delta_{\mathcal{O}} + \ell + 1}{\ell} g_2^{\lambda\alpha}(u, v; \Delta_1, \Delta_2, \Delta_3, \Delta_v; \ell, \Delta_{\mathcal{O}}) \\ & - \frac{\sqrt{v}}{\ell} (1 + \Delta_3 - \Delta_v - 2u\partial_u) \, g(u, v; \Delta_1, \Delta_2, \Delta_3 + 1, \Delta_v; \ell, \Delta_{\mathcal{O}}), \quad (4.61) \end{aligned}$$

again only depending on the differences $\Delta_1 - \Delta_2$ and $\Delta_3 - \Delta_v$.

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4.4.4 $\langle SVSV \rangle$

At last we turn to the case of primary interest; two scalars and two vectors. We will label the scalars 1 and 3, and the vectors 2 and 4. As we have seen, the exchange operator can be either traceless symmetric $\mathcal{O}$ or a mixed symmetry operator $\mathcal{A}$.

We'll start with the symmetric exchange. There are twenty different conformal blocks which can arise, g_p^{rs}, where r and s run over α and β (for $\ell > 0$, or only α for $\ell = 0$) and p runs over the five tensor structures of the four-point function, which we have labeled 0, 11, 12, 21, and 22. By inserting the shadow projector, we can get all the symmetric exchange blocks as contractions of S^α's and S^β's, which we can in turn write as differential operators acting on S^λ's. Finally, we impose the monodromy projection by doffing all expressions. In fact, the resulting expressions are more compact if we write them in terms of either $g_i^{\alpha\lambda}$ or $g_i^{\lambda\alpha}$. For the $\alpha\alpha$ blocks we'll use the former representation, and we will further introduce shorthand

$$g^{\alpha\lambda}_{i;\ell;P,Q} = g_i^{\alpha\lambda}(u,v;\Delta_1+P,\Delta_2,\Delta_3+Q,\Delta_4;\ell,\Delta_{\mathcal{O}}). \tag{4.62}$$

The final expressions are

$$g_0^{\alpha\alpha} = \frac{1}{2\,(1-\Delta_{\mathcal{O}})}\left[\frac{1}{\sqrt{v}}g^{\alpha\lambda}_{2;\ell;0,-1} - \sqrt{u}g^{\alpha\lambda}_{1;\ell;0,1} - \sqrt{v}g^{\alpha\lambda}_{2;\ell;0,1}\right], \tag{4.63}$$

$$\begin{aligned} g_{11}^{\alpha\alpha} = \frac{\sqrt{u}}{2\,(1-\Delta_{\mathcal{O}})}\Big[&-(1-\Delta_3+\Delta_4-2u\partial_u-2v\partial_v)\,g^{\alpha\lambda}_{1;\ell;0,-1} \\ &+(\Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v)\,g^{\alpha\lambda}_{1;\ell;0,1}\Big], \end{aligned} \tag{4.64}$$

$$\begin{aligned} g_{12}^{\alpha\alpha} = \frac{\sqrt{v}}{2\,(1-\Delta_{\mathcal{O}})}\Big[&(1-\Delta_3+\Delta_4-2u\partial_u+2\,(1-v)\,\partial_v)\,g^{\alpha\lambda}_{1;\ell;0,-1} \\ &+(1+\Delta_3-\Delta_4-2u\partial_u)\,g^{\alpha\lambda}_{1;\ell;0,1}\Big], \end{aligned} \tag{4.65}$$

$$\begin{aligned} g_{21}^{\alpha\alpha} = \frac{\sqrt{u}}{2\,(1-\Delta_{\mathcal{O}})}\Big[&-(1-\Delta_3+\Delta_4-2u\partial_u-2v\partial_v)\,g^{\alpha\lambda}_{2;\ell;0,-1} \\ &-(1-\Delta_1+\Delta_2+\Delta_3-\Delta_4+2v\partial_v)\,g^{\alpha\lambda}_{2;\ell;0,1}\Big], \end{aligned} \tag{4.66}$$

$$\begin{aligned} g_{22}^{\alpha\alpha} = \frac{\sqrt{v}}{2\,(1-\Delta_{\mathcal{O}})}\Big[&\left(1-\frac{1}{v}-\Delta_3+\Delta_4-2u\partial_u+2\,(1-v)\,\partial_v\right)g^{\alpha\lambda}_{2;\ell;0,-1} \\ &+(2+\Delta_3-\Delta_4-2u\partial_u)\,g^{\alpha\lambda}_{2;\ell;0,1}\Big]. \end{aligned} \tag{4.67}$$

For $\ell = 0$, this is the entire answer. For $\ell > 0$, we can proceed similarly with the other blocks, obtaining the $\alpha\beta$, $\beta\alpha$, and $\beta\beta$ components given in appendix F. To save space, we have omitted the arguments of the conformal blocks appearing above. For the SVSV blocks (i.e. the expressions on the left-hand-sides above) the arguments are unshifted, $g_p^{rs}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}})$, while for the others we use the previously adopted condensed notation, along with

$$g^{\lambda\alpha}_{i;\ell;P,Q} = g_i^{\lambda\alpha}(u,v;\Delta_1+P,\Delta_2,\Delta_3+Q,\Delta_4;\ell,\Delta_{\mathcal{O}}). \tag{4.68}$$

The combination which occurs in the four-point function is

$$\alpha_{12\mathcal{O}}\alpha_{34\mathcal{O}}g_p^{\alpha\alpha} + \alpha_{12\mathcal{O}}\beta_{34\mathcal{O}}g_p^{\alpha\beta} + \beta_{12\mathcal{O}}\alpha_{34\mathcal{O}}g_p^{\beta\alpha} + \beta_{12\mathcal{O}}\beta_{34\mathcal{O}}g_p^{\beta\beta}. \tag{4.69}$$

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It turns that if we write this combination in terms of scalar conformal blocks it has the remarkably simple form

$$A_1 A_2 \mathcal{D}_p^{--} g_{\ell;-1,-1} + A_1 B_2 \mathcal{D}_p^{-+} g_{\ell;-1,1} + B_1 A_2 \mathcal{D}_p^{+-} g_{\ell;1,-1} + B_1 B_2 \mathcal{D}_p^{++} g_{\ell;1,1}, \tag{4.70}$$

where

$$A_1 = \frac{1}{2\left(\Delta_{\mathcal{O}} - 1\right)} \left(\alpha_{12\mathcal{O}} + \left(\Delta_1 - \Delta_2 - \Delta_{\mathcal{O}} + \ell + 1\right) \frac{\beta_{12\mathcal{O}}}{\ell} \right), \tag{4.71}$$

$$A_2 = \frac{1}{2\left(\Delta_{\mathcal{O}} - 1\right)} \left(\alpha_{34\mathcal{O}} + \left(\Delta_3 - \Delta_4 - \Delta_{\mathcal{O}} + \ell + 1\right) \frac{\beta_{34\mathcal{O}}}{\ell} \right), \tag{4.72}$$

$$B_1 = \frac{1}{2\left(\Delta_{\mathcal{O}} - 1\right)} \left(\alpha_{12\mathcal{O}} + \left(\Delta_1 - \Delta_2 + \Delta_{\mathcal{O}} + \ell - 1\right) \frac{\beta_{12\mathcal{O}}}{\ell} \right), \tag{4.73}$$

$$B_2 = \frac{1}{2\left(\Delta_{\mathcal{O}} - 1\right)} \left(\alpha_{34\mathcal{O}} + \left(\Delta_3 - \Delta_4 + \Delta_{\mathcal{O}} + \ell - 1\right) \frac{\beta_{34\mathcal{O}}}{\ell} \right), \tag{4.74}$$

and $\mathcal{D}_p^{\pm\pm}$ are fairly simple differential operators whose explicit forms are given in a table in appendix H.

We turn next to the evaluation of blocks for exchanged of a mixed symmetry tensor $\mathcal{A}_{a_1 a_2 b_1 \cdots b_k}$ whose representation is labeled by a non-negative integer k ($k = 0$ corresponds to an antisymmetric two-index tensor). The contraction which we need is

$$\left(\gamma_{12\mathcal{A}} m^{(20)}_{ac_1} k^{(012)}_{c_2} k^{(012)}_{d_1} \cdots k^{(012)}_{d_k}\right) \widetilde{\Pi}^{(k)\, c_1 c_2 d_1 \cdots d_k}_{e_1 e_2 f_1 \cdots f_k} \left(\gamma_{34\tilde{\mathcal{A}}} m^{(40)}{}_{b}^{\,e_1} k^{(034)\, e_2} k^{(034)\, f_1} \cdots k^{(034)\, f_k}\right). \tag{4.75}$$

For fixed k, given knowledge of the projector $\widetilde{\Pi}^{(k)}$ as detailed in appendix B.2, we could just expand this contraction by brute force into a sum of monomials in the x_{ij}^2 and x_{0i}^2, with the free indices being carried by $(x_{ij})_a$, in which case we can pull it ouside of the integral, or $(x_{0i})_a$, in which case we can rewrite the corresponding integral as an x_i derivative acting on a scalar integral. Then each term in this collection can be evaluated using the integrals in appendix D.2, and the result could be written in terms of the $\ell = 0$ scalar blocks. However, this approach is impractical for several reasons, most notably that the number of monomials in such an expansion grows exponentially in k. We need a cleaner expression.

Indeed, we can use (B.39), (B.42), and (B.43), as well as other identities from appendices A and B, to rewrite the contraction as

$$\begin{aligned}
&\left(\gamma_{12\mathcal{A}} m^{(20)}_{ac_1} k^{(012)}_{c_2} k^{(012)}_{d_1} \cdots k^{(012)}_{d_k}\right) \widetilde{\Pi}^{(k)\, c_1 c_2 d_1 \cdots d_k}_{e_1 e_2 f_1 \cdots f_k} \left(\gamma_{34\tilde{\mathcal{A}}} m^{(40)}{}_{b}^{\,e_1} k^{(034)\, e_2} k^{(034)\, f_1} \cdots k^{(034)\, f_k}\right) \\
&= \frac{1}{2} \gamma_{12\mathcal{A}} \gamma_{34\tilde{\mathcal{A}}} \sqrt{\frac{x_{02}^2 x_{04}^2 x_{12}^2 x_{34}^2}{x_{01}^2 x_{03}^2}} \left\{ \frac{k+2}{k+1} p_{D,k+1}(t) \frac{\partial^2 t}{\partial x_2^a \partial x_4^b} - \frac{\partial^2}{\partial x_2^a \partial x_4^b} \left[\frac{1}{(k+1)(k+2)} p_{D,k+2}(t) \right.\right. \\
&\left.\left. + \frac{k+2}{(D+2k)(D+2k-2)(D+k-2)} p_{D,k}(t) \right] \right\},
\end{aligned} \tag{4.76}$$

In appendix C we give more details and motivation for how we arrive at this expression.

This is the main formula that will allow us to relate mixed-symmetric conformal blocks to the ones for symmetric-traceless exchange. Notice that derivatives of the polynomials $p(t)$ will always produce terms that appear in the conformal blocks of traceless-symmetric

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operators since, from (B.17), they are related to symmetric contractions where not all of the indices are contracted. Furthermore derivatives of t, i.e.

$$\frac{\partial t}{\partial x_2^a} = \left(\frac{\partial}{\partial x_2^a} k^{(012)\,b}\right) k_b^{(034)} = -\sqrt{\frac{x_{01}^2}{x_{02}^2 x_{12}^2}}\left[\left(m^{20}\cdot k^{(034)}\right)_a + t k_a^{(201)}\right] \tag{4.77}$$

and similar expressions for other x_i[8] produce the tensor structures from three-point functions of operators with spin (see (4.48), (4.49) for example). Therefore if one could write the contractions of more general mixed symmetries in the form (4.76)—i.e. total derivatives of the symmetric contraction $p_{D,k}$ and undifferentiated polynomials times derivatives of t—then the relation of these expressions to those from symmetric exchanges would follow in analogy to our case. We believe that it will be possible to do this in more general situations, but this has not been definitively established. However, to support this conjecture, we present the contraction for $[k+1,1,1]$ in appendix C, which has an analogous form.

From these arguments, we obtain

$$\begin{aligned}
\frac{\partial^2 t}{\partial x_2^a \partial x_4^b} = &\sqrt{\frac{x_{01}^2 x_{03}^2}{x_{02}^2 x_{04}^2 x_{12}^2 x_{34}^2}}\left(m_{ab}^{(24)} - 2\sqrt{\frac{v}{u}} k_a^{(214)} k_b^{(432)}\right) \\
&- \frac{1}{2}\frac{x_{01}^2 x_{03}^2 x_{24}^2 + x_{01}^2 x_{04}^2 x_{23}^2 + x_{02}^2 x_{03}^2 x_{14}^2 + x_{02}^2 x_{04}^2 x_{13}^2}{x_{02}^2 x_{04}^2 x_{12}^2 x_{34}^2} k_a^{(201)} k_b^{(403)} \\
&+ \sqrt{\frac{x_{03}^2 x_{14}^2}{x_{04}^2 x_{12}^2 x_{24}^2 x_{34}^2}} k_a^{(201)} k_b^{(412)} - \sqrt{\frac{x_{03}^2 x_{23}^2}{x_{04}^2 x_{24}^2}} \frac{x_{01}^2 x_{24}^2 + x_{02}^2 x_{14}^2}{x_{02}^2 x_{12}^2 x_{34}^2} k_a^{(201)} k_b^{(432)} \\
&- \sqrt{\frac{x_{01}^2 x_{14}^2}{x_{02}^2 x_{24}^2}} \frac{x_{03}^2 x_{24}^2 + x_{04}^2 x_{23}^2}{x_{04}^2 x_{12}^2 x_{34}^2} k_a^{(214)} k_b^{(403)} + \sqrt{\frac{x_{01}^2 x_{23}^2}{x_{02}^2 x_{12}^2 x_{24}^2 x_{34}^2}} k_a^{(234)} k_b^{(403)},
\end{aligned} \tag{4.78}$$

and also

$$\begin{aligned}
\frac{\partial^2}{\partial x_2^a \partial x_4^b} p_{D,\ell}(t) = \ell^2 &\sqrt{\frac{x_{01}^2 x_{03}^2}{x_{02}^2 x_{04}^2 x_{12}^2 x_{34}^2}}\left(m_{ac_1}^{(20)} + k_a^{(201)} k_{c_1}^{(012)}\right)\left(m^{(40)}{}_b^{\,d_1} + k_b^{(403)} k^{(034)\,d_1}\right) \\
&\times k_{c_2}^{(012)} \cdots k_{c_\ell}^{(012)} \Pi_{d_1\cdots d_\ell}^{(\ell)\,c_1\cdots c_\ell} k^{(034)\,d_2} \cdots k^{(034)\,d_\ell}.
\end{aligned} \tag{4.79}$$

Thus if we define

$$\begin{aligned}
&\left(S^\alpha_{a\,PQ} \circ_\ell S^\alpha_{b\,RS}\right) \\
&= \int d^D x_0 S^\alpha_{a\,c_1\cdots c_\ell}(x_1, x_2, x_0; \Delta_1 + P, \Delta_2 + Q, \Delta_{\mathcal{A}}) S^{\alpha}{}_{b}^{\,c_1\cdots c_\ell}(x_3, x_4, x_0; \Delta_3 + R, \Delta_4 + S, D - \Delta_{\mathcal{A}}),
\end{aligned} \tag{4.80}$$

and similarly for $(S^\alpha_{a\,PQ} \circ_\ell S^\lambda_{RS})$, etc., we can compute (recall that (4.76) appears inside a conformal integral like (4.50))

$$\begin{aligned}
&X_{\Delta_i \gamma_{12\mathcal{A}} \gamma_{34\mathcal{A}}} \widehat{g}_p^{\gamma\gamma}(u,v) t_{ab}^p \\
&= \mathcal{N}_{\mathcal{A}} \gamma_{12\mathcal{A}} \gamma_{34\tilde{\mathcal{A}}} \left\{\frac{1}{2}\frac{k+2}{k+1}\left[\left(m_{ab}^{(24)} - 2\sqrt{\frac{v}{u}} k_a^{(214)} k_b^{(432)}\right)\left(S^\lambda_{00} \circ_{k+1} S^\lambda_{00}\right)\right.\right.
\end{aligned}$$

[8] When there is more than one derivative with respect to the same variable, one has to include extra factors, say proportional to $(x_{12})_a$ in the derivative above, in order to obtain covariant structures (k's and m's).

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$$
\begin{aligned}
&-\frac{1}{2}\frac{1}{\sqrt{x_{12}^2 x_{34}^2}}\Big(x_{24}^2\left(S^\alpha_{a\,-\frac12\,\frac12}\circ_{k+1}S^\alpha_{b\,-\frac12\,\frac12}\right)+x_{23}^2\left(S^\alpha_{a\,-\frac12\,\frac12}\circ_{k+1}S^\alpha_{b\,\frac12\,-\frac12}\right)\\
&+x_{14}^2\left(S^\alpha_{a\,\frac12\,-\frac12}\circ_{k+1}S^\alpha_{b\,-\frac12\,\frac12}\right)+x_{13}^2\left(S^\alpha_{a\,\frac12\,-\frac12}\circ_{k+1}S^\alpha_{b\,\frac12\,-\frac12}\right)\Big)-\sqrt{\frac{x_{14}^2}{x_{24}^2}}k_b^{(412)}\left(S^\alpha_{a\,\frac12\,-\frac12}\circ_{k+1}S^\lambda_{00}\right)\\
&+\sqrt{\frac{x_{23}^2x_{24}^2}{x_{12}^2x_{34}^2}}k_b^{(432)}\left(S^\alpha_{a\,-\frac12\,\frac12}\circ_{k+1}S^\lambda_{00}\right)+x_{14}^2\sqrt{\frac{x_{23}^2}{x_{12}^2x_{24}^2x_{34}^2}}k_b^{(432)}\left(S^\alpha_{a\,\frac12\,-\frac12}\circ_{k+1}S^\lambda_{00}\right)\\
&+\sqrt{\frac{x_{14}^2x_{24}^2}{x_{12}^2x_{34}^2}}k_a^{(214)}\left(S^\lambda_{00}\circ_{k+1}S^\alpha_{b\,-\frac12\,\frac12}\right)+x_{23}^2\sqrt{\frac{x_{14}^2}{x_{12}^2x_{24}^2x_{34}^2}}k_a^{(214)}\left(S^\lambda_{00}\circ_{k+1}S^\alpha_{b\,\frac12\,-\frac12}\right)\\
&-\sqrt{\frac{x_{23}^2}{x_{24}^2}}k_a^{(234)}\left(S^\lambda_{00}\circ_{k+1}S^\alpha_{b\,\frac12\,-\frac12}\right)\Big]-\frac{1}{2}\frac{k+2}{k+1}\Big[(S^\alpha_{a\,00}\circ_{k+2}S^\alpha_{b\,00})-\left(S^\alpha_{a\,00}\circ_{k+2}S^\beta_{b\,00}\right)\\
&-\left(S^\beta_{a\,00}\circ_{k+2}S^\alpha_{b\,00}\right)+\left(S^\beta_{a\,00}\circ_{k+2}S^\beta_{b\,00}\right)\Big]-\frac{1}{2}\frac{k^2\,(k+2)}{(D+2k)\,(D+2k-2)\,(D+k-2)}\\
&\times\Big[(S^\alpha_{a\,00}\circ_k S^\alpha_{b\,00})-\left(S^\alpha_{a\,00}\circ_k S^\beta_{b\,00}\right)-\left(S^\beta_{a\,00}\circ_k S^\alpha_{b\,00}\right)+\left(S^\beta_{a\,00}\circ_k S^\beta_{b\,00}\right)\Big]\Big\}\,.
\end{aligned}
\tag{4.81}
$$

To evaluate these we make use of relations like

$$
\left(S^\lambda_{00}\circ_\ell S^\lambda_{00}\right)=\frac{X_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}}{\mathcal{N}_\mathcal{O}\left(\lambda_{34\widetilde{\mathcal{O}}}/\lambda_{34\mathcal{O}}\right)}\widehat{g}_{\ell;0,0}(u,v)
\tag{4.82}
$$

where $\mathcal{N}_\mathcal{O}$, $\lambda_{34\widetilde{\mathcal{O}}}$, and $\widehat{g}_{\ell;0,0}(u,v)$ are given by (3.12), (3.4), and (4.38) respectively, where we make the substitution $\Delta_\mathcal{O}\to\Delta_\mathcal{A}$ in all three definitions (also we of course substitute $\Delta_1\to\Delta_3$ and $\Delta_2\to\Delta_4$ in the definition of λ). Similarly,

$$
\left(S^\alpha_{a\,PQ}\circ_\ell S^\lambda_{00}\right)=\frac{X_{\Delta_1+P,\Delta_2+Q,\Delta_3,\Delta_4}}{\mathcal{N}_\mathcal{O}\left(\lambda_{34\widetilde{\mathcal{O}}}/\lambda_{34\mathcal{O}}\right)}\left[\widehat{g}^{\alpha\lambda}_{1;\ell;P-Q,0}(u,v)k_a^{(214)}+\widehat{g}^{\alpha\lambda}_{2;\ell;P-Q,0}(u,v)k_a^{(234)}\right].
\tag{4.83}
$$

For the other expressions we need to invert the matrix $M_r{}^s$ introduced in (3.7) (substituting $\Delta_\mathcal{O}\to\Delta_\mathcal{A}$, as well as $\Delta_\phi\to\Delta_3+R$ and $\Delta_v\to\Delta_4+S$). Then we have

$$
\left(S^\lambda_{00}\circ_\ell S^\alpha_{a\,RS}\right)=\frac{X_{\Delta_1,\Delta_2,\Delta_3+R,\Delta_4+S}}{\mathcal{N}_\mathcal{O}}\left(M^{-1}\right)_r{}^\alpha\,\widehat{g}^{\lambda r}_{p;\ell;0,R-S}(u,v)t_a^p,
\tag{4.84}
$$

where we sum r over α and β, sum p over 1 and 2, and where $t_a^1=k_a^{(412)}$, $t_a^2=k_a^{(432)}$. Similarly,

$$
\left(S^r_{a\,PQ}\circ_\ell S^s_{b\,RS}\right)=\frac{X_{\Delta_1+P,\Delta_2+Q,\Delta_3+R,\Delta_4+S}}{\mathcal{N}_\mathcal{O}}\left(M^{-1}\right)_t{}^s\,\widehat{g}^{rt}_{p;\ell;P-Q,R-S}(u,v)t^p_{ab}.
\tag{4.85}
$$

The necessary combination of coefficients involves

$$
\begin{aligned}
\mathcal{N}_\mathcal{O}^{-1}M^{-1}&=\pi^{D/2}\frac{\Gamma(\frac{D}{2}-\Delta_\mathcal{O})\Gamma(D-\Delta_\mathcal{O}+\ell-1)}{\Gamma(D-\Delta_\mathcal{O})\Gamma(\Delta_\mathcal{O}+\ell)}\\
&\times\frac{\Gamma(\frac12\left(\Delta_3-\Delta_4+\Delta_\mathcal{O}+\ell+1+R-S\right))\Gamma(\frac12\left(-\Delta_3+\Delta_4+\Delta_\mathcal{O}+\ell-1-R+S\right))}{\Gamma(\frac12\left(D+\Delta_3-\Delta_4-\Delta_\mathcal{O}+\ell-1+R-S\right))\Gamma(\frac12\left(D-\Delta_3+\Delta_4-\Delta_\mathcal{O}+\ell+1-R+S\right))}\\
&\times\begin{pmatrix}\frac{(\Delta_\mathcal{O}-1)(D-\Delta_\mathcal{O}+\ell-1)-(D-\Delta_\mathcal{O}-1)(\Delta_3-\Delta_4+R-S)}{D+\Delta_3-\Delta_4-\Delta_\mathcal{O}+\ell-1+R-S} & 2\Delta_\mathcal{O}-D\\ -\frac{\ell(2\Delta_\mathcal{O}-D)(\Delta_3-\Delta_4+R-S)}{(\Delta_3-\Delta_4+\Delta_\mathcal{O}+\ell-1+R-S)(D+\Delta_3-\Delta_4-\Delta_\mathcal{O}+\ell-1+R-S)} & \frac{(\Delta_\mathcal{O}+\ell-1)(D-\Delta_\mathcal{O}-1)-(\Delta_\mathcal{O}-1)(\Delta_3-\Delta_4+R-S)}{\Delta_3-\Delta_4+\Delta_\mathcal{O}+\ell-1+R-S}\end{pmatrix}
\end{aligned}
\tag{4.86}
$$

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Finally we can plug in these expressions to (4.81) and collect the different tensor structures to obtain

$$
\begin{aligned}
&g_0^{\gamma\gamma}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}})\\
&\quad=\frac{1}{2}\frac{k+2}{k+1}\Bigg[C_1 g_{k+1;0,0}-\frac{1}{2\sqrt{u}}\Big(C_2\left(g^{\alpha\alpha}_{0;k+1;-1,-1}+g^{\alpha\alpha}_{0;k+1;1,-1}\right)+C_3\left(g^{\alpha\beta}_{0;k+1;-1,-1}+g^{\alpha\beta}_{0;k+1;1,-1}\right)\\
&\qquad+C_4\left(vg^{\alpha\alpha}_{0;k+1;-1,1}+g^{\alpha\alpha}_{0;k+1;1,1}\right)+C_5\left(vg^{\alpha\beta}_{0;k+1;-1,1}+g^{\alpha\beta}_{0;k+1;1,1}\right)\Big)\Bigg]\\
&\quad-\frac{1}{2}\frac{k+2}{k+1}\left[C_6\left(g^{\alpha\alpha}_{0;k+2;0,0}-g^{\beta\alpha}_{0;k+2;0,0}\right)+C_7\left(g^{\beta\beta}_{0;k+2;0,0}-g^{\alpha\beta}_{0;k+2;0,0}\right)\right]\\
&\quad-\frac{1}{2}\frac{k^2\,(k+2)}{(D+2k)\,(D+2k-2)\,(D+k-2)}\left[C_8\left(g^{\alpha\alpha}_{0;k;0,0}-g^{\beta\alpha}_{0;k;0,0}\right)+C_9\left(g^{\beta\beta}_{0;k;0,0}-g^{\alpha\beta}_{0;k;0,0}\right)\right],
\end{aligned}
\tag{4.87}
$$

$$
\begin{aligned}
&g_{11}^{\gamma\gamma}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}})\\
&\quad=\frac{1}{2}\frac{k+2}{k+1}\Bigg[-\frac{1}{2\sqrt{u}}\Big(C_2\left(g^{\alpha\alpha}_{11;k+1;-1,-1}+g^{\alpha\alpha}_{11;k+1;1,-1}\right)+C_3\left(g^{\alpha\beta}_{11;k+1;-1,-1}+g^{\alpha\beta}_{11;k+1;1,-1}\right)\\
&\qquad+C_4\left(vg^{\alpha\alpha}_{11;k+1;-1,1}+g^{\alpha\alpha}_{11;k+1;1,1}\right)+C_5\left(vg^{\alpha\beta}_{11;k+1;-1,1}+g^{\alpha\beta}_{11;k+1;1,1}\right)\Big)-C_1 g^{\alpha\lambda}_{1;k+1;1,0}\\
&\quad+\frac{C_2}{\sqrt{u}}g^{\lambda\alpha}_{1;k+1;0,-1}+\frac{C_3}{\sqrt{u}}g^{\lambda\beta}_{1;k+1;0,-1}+\frac{C_4 v}{\sqrt{u}}g^{\lambda\alpha}_{1;k+1;0,1}+\frac{C_5 v}{\sqrt{u}}g^{\lambda\beta}_{1;k+1;0,1}\Bigg]\\
&\quad-\frac{1}{2}\frac{k+2}{k+1}\left[C_6\left(g^{\alpha\alpha}_{11;k+2;0,0}-g^{\beta\alpha}_{11;k+2;0,0}\right)+C_7\left(g^{\beta\beta}_{11;k+2;0,0}-g^{\alpha\beta}_{11;k+2;0,0}\right)\right]\\
&\quad-\frac{1}{2}\frac{k^2\,(k+2)}{(D+2k)\,(D+2k-2)\,(D+k-2)}\left[C_8\left(g^{\alpha\alpha}_{11;k;0,0}-g^{\beta\alpha}_{11;k;0,0}\right)+C_9\left(g^{\beta\beta}_{11;k;0,0}-g^{\alpha\beta}_{11;k;0,0}\right)\right],
\end{aligned}
\tag{4.88}
$$

$$
\begin{aligned}
&g_{12}^{\gamma\gamma}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}})\\
&\quad=\frac{1}{2}\frac{k+2}{k+1}\Bigg[-2\sqrt{\frac{v}{u}}C_1 g_{k+1;0,0}-\frac{1}{2\sqrt{u}}\Big(C_2\left(g^{\alpha\alpha}_{12;k+1;-1,-1}+g^{\alpha\alpha}_{12;k+1;1,-1}\right)\\
&\qquad+C_3\left(g^{\alpha\beta}_{12;k+1;-1,-1}+g^{\alpha\beta}_{12;k+1;1,-1}\right)+C_4\left(vg^{\alpha\alpha}_{12;k+1;-1,1}+g^{\alpha\alpha}_{12;k+1;1,1}\right)\\
&\qquad+C_5\left(vg^{\alpha\beta}_{12;k+1;-1,1}+g^{\alpha\beta}_{12;k+1;1,1}\right)\Big)+C_1\sqrt{\frac{v}{u}}\left(g^{\alpha\lambda}_{1;k+1;-1,0}+g^{\alpha\lambda}_{1;k+1;1,0}\right)+\frac{C_2}{\sqrt{u}}g^{\lambda\alpha}_{2;k+1;0,-1}\\
&\quad+\frac{C_3}{\sqrt{u}}g^{\lambda\beta}_{2;k+1;0,-1}+\frac{C_4 v}{\sqrt{u}}g^{\lambda\alpha}_{2;k+1;0,1}+\frac{C_5 v}{\sqrt{u}}g^{\lambda\beta}_{2;k+1;0,1}\Bigg]\\
&\quad-\frac{1}{2}\frac{k+2}{k+1}\left[C_6\left(g^{\alpha\alpha}_{12;k+2;0,0}-g^{\beta\alpha}_{12;k+2;0,0}\right)+C_7\left(g^{\beta\beta}_{12;k+2;0,0}-g^{\alpha\beta}_{12;k+2;0,0}\right)\right]\\
&\quad-\frac{1}{2}\frac{k^2\,(k+2)}{(D+2k)\,(D+2k-2)\,(D+k-2)}\left[C_8\left(g^{\alpha\alpha}_{12;k;0,0}-g^{\beta\alpha}_{12;k;0,0}\right)+C_9\left(g^{\beta\beta}_{12;k;0,0}-g^{\alpha\beta}_{12;k;0,0}\right)\right],
\end{aligned}
\tag{4.89}
$$

$$
\begin{aligned}
&g_{21}^{\gamma\gamma}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}})\\
&\quad=\frac{1}{2}\frac{k+2}{k+1}\Bigg[-\frac{1}{2\sqrt{u}}\Big(C_2\left(g^{\alpha\alpha}_{21;k+1;-1,-1}+g^{\alpha\alpha}_{21;k+1;1,-1}\right)+C_3\left(g^{\alpha\beta}_{21;k+1;-1,-1}+g^{\alpha\beta}_{21;k+1;1,-1}\right)\\
&\qquad+C_4\left(vg^{\alpha\alpha}_{21;k+1;-1,1}+g^{\alpha\alpha}_{21;k+1;1,1}\right)+C_5\left(vg^{\alpha\beta}_{21;k+1;-1,1}+g^{\alpha\beta}_{21;k+1;1,1}\right)\Big)\\
&\quad-C_1 g^{\alpha\lambda}_{2;k+1;1,0}-C_4\sqrt{v}g^{\lambda\alpha}_{1;k+1;0,1}-C_5\sqrt{v}g^{\lambda\beta}_{1;k+1;0,1}\Bigg]\\
&\quad-\frac{1}{2}\frac{k+2}{k+1}\left[C_6\left(g^{\alpha\alpha}_{21;k+2;0,0}-g^{\beta\alpha}_{21;k+2;0,0}\right)+C_7\left(g^{\beta\beta}_{21;k+2;0,0}-g^{\alpha\beta}_{21;k+2;0,0}\right)\right]\\
&\quad-\frac{1}{2}\frac{k^2\,(k+2)}{(D+2k)\,(D+2k-2)\,(D+k-2)}\left[C_8\left(g^{\alpha\alpha}_{21;k;0,0}-g^{\beta\alpha}_{21;k;0,0}\right)+C_9\left(g^{\beta\beta}_{21;k;0,0}-g^{\alpha\beta}_{21;k;0,0}\right)\right],
\end{aligned}
\tag{4.90}
$$

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$$
\begin{aligned}
&g_{22}^{\gamma\gamma}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}})\\
&= \frac{1}{2}\frac{k+2}{k+1}\Bigg[-\frac{1}{2\sqrt{u}}\Big(C_2\left(g^{\alpha\alpha}_{22;k+1;-1,-1}+g^{\alpha\alpha}_{22;k+1;1,-1}\right)+C_3\left(g^{\alpha\beta}_{22;k+1;-1,-1}+g^{\alpha\beta}_{22;k+1;1,-1}\right)\\
&\qquad +C_4\left(vg^{\alpha\alpha}_{22;k+1;-1,1}+g^{\alpha\alpha}_{22;k+1;1,1}\right)+C_5\left(vg^{\alpha\beta}_{22;k+1;-1,1}+g^{\alpha\beta}_{22;k+1;1,1}\right)\Big)\\
&\qquad +C_1\sqrt{\frac{v}{u}}\left(g^{\alpha\lambda}_{2;k+1;-1,0}+g^{\alpha\lambda}_{2;k+1;1,0}\right)-C_4\sqrt{v}g^{\lambda\alpha}_{2;k+1;0,1}-C_5\sqrt{v}g^{\lambda\beta}_{2;k+1;0,1}\Bigg]\\
&\quad -\frac{1}{2}\frac{k+2}{k+1}\left[C_6\left(g^{\alpha\alpha}_{22;k+2;0,0}-g^{\beta\alpha}_{22;k+2;0,0}\right)+C_7\left(g^{\beta\beta}_{22;k+2;0,0}-g^{\alpha\beta}_{22;k+2;0,0}\right)\right]\\
&\quad -\frac{1}{2}\frac{k^2(k+2)}{(D+2k)(D+2k-2)(D+k-2)}\left[C_8\left(g^{\alpha\alpha}_{22;k;0,0}-g^{\beta\alpha}_{22;k;0,0}\right)+C_9\left(g^{\beta\beta}_{22;k;0,0}-g^{\alpha\beta}_{22;k;0,0}\right)\right].
\end{aligned}
\tag{4.91}
$$

The constants appearing above are written in appendix G. For $k=0$ the k^2 numerator in the last line of each conformal block kills those terms, and so we don't need to worry about the fact that for $k=0$ only $g^{\alpha\alpha}$ is defined. One detail that we do need to worry about is the ℓ-dependent normalization of the scalar conformal blocks, mentioned below equation (4.43). Since the expressions used to compute the mixed symmetry blocks involve adding contributions from scalar conformal blocks of different spins, it is important to use our normalization for the scalar blocks. Otherwise, some of the relative coefficients will be off (e.g. by powers of two relative to another common normalization).[9]

5 Setting up the bootstrap

5.1 General discussion

The picture now is that we are given explicit expressions for the conformal blocks, which depend only on the weights Δ_i and[10] $\Delta_{\mathcal{O}}$, and the SO(D) representations of the five operators in question. The blocks are otherwise theory-independent. Indeed, a CFT is specified by the spectrum of primary operators, i.e. a list of the $\phi_{i\,\bar{a}}$, characterized by their weights Δ_i and representations, and their OPE coefficients $\lambda_{ijk\,r}$ (a finite list of constants enumerated by r for each fixed triple of operators $\phi_{i\,\bar{a}}$, $\phi_{j\,\bar{b}}$, $\phi_{k\,\bar{c}}$). Then from this data we can compute any four-point function by

$$
\langle\phi_{1\,\bar{a}}(x_1)\phi_{2\,\bar{b}}(x_2)\phi_{3\,\bar{c}}(x_3)\phi_{4\,\bar{d}}(x_4)\rangle = X_{\Delta_i}\sum_p\left(\sum_{\mathcal{O}}\sum_{r,s}\lambda_{12\mathcal{O}\,r}\lambda_{34\mathcal{O}\,s}g_p^{rs}(u,v)\right)t^p_{\bar{a}\bar{b}\bar{c}\bar{d}}(x_i),
\tag{5.1}
$$

However, in deriving this expression we made a choice to first perform the OPEs of $\phi_{1\,\bar{a}}$ with $\phi_{2\,\bar{b}}$ and $\phi_{3\,\bar{c}}$ with $\phi_{4\,\bar{d}}$, then evaluating the resulting two-point function. Starting with the same correlation function and the same CFT data, we could have evaluated instead the OPE of $\phi_{1\,\bar{a}}$ with $\phi_{4\,\bar{d}}$ and $\phi_{2\,\bar{b}}$ with $\phi_{3\,\bar{c}}$, or equivalently, we could have performed a $2\leftrightarrow 4$ crossing symmetry exchange before performing our OPEs. This should be an equivalent path to the same four-point correlator, and by comparing the two results we obtain a non-trivial constraint on the defining data of our CFT.

[9] We have checked that our expressions for $k=0,1$ are consistent with the latest version of [46].

[10] In sections discussing very general four-point functions, such as this one, $\mathcal{O}$ will stand for all possible primary exchange operators, of arbitrary SO(D) representations, while in sections discussing particular assignments of representations, such as the following two subsections, $\mathcal{O}$ will refer only to traceless symmetric exchanges. In that case we will also have exchange operators $\mathcal{A}$ of mixed symmetry.

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Let's recall how this works for four scalars operators ϕ_i. We have

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}(\Delta_3-\Delta_4)}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}(\Delta_1-\Delta_2)} \times \left(x_{12}^2\right)^{-\frac{1}{2}(\Delta_1+\Delta_2)}\left(x_{34}^2\right)^{-\frac{1}{2}(\Delta_3+\Delta_4)}\sum_{\mathcal{O}}\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}g(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}), \quad (5.2)$$

while in the other channel we have

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \left(\frac{x_{12}^2}{x_{13}^2}\right)^{\frac{1}{2}(\Delta_3-\Delta_2)}\left(\frac{x_{24}^2}{x_{12}^2}\right)^{\frac{1}{2}(\Delta_1-\Delta_4)} \times \left(x_{14}^2\right)^{-\frac{1}{2}(\Delta_1+\Delta_4)}\left(x_{23}^2\right)^{-\frac{1}{2}(\Delta_3+\Delta_2)}\sum_{\mathcal{O}}\lambda_{14\mathcal{O}}\lambda_{32\mathcal{O}}g(v,u;\Delta_1,\Delta_4,\Delta_3,\Delta_2;\ell,\Delta_{\mathcal{O}}). \quad (5.3)$$

Comparing the two we learn that

$$0 = \sum_{\mathcal{O}}\Big\{\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}g(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) - \lambda_{14\mathcal{O}}\lambda_{32\mathcal{O}}u^{\frac{1}{2}(\Delta_3+\Delta_4)}v^{-\frac{1}{2}(\Delta_2+\Delta_3)}g(v,u;\Delta_1,\Delta_4,\Delta_3,\Delta_2;\ell,\Delta_{\mathcal{O}})\Big\} \quad (5.4)$$

This constraint is of limited usefulness when the scalars are all distinct. A somewhat better case is when ϕ_2 and ϕ_4 are identical scalars, in which case we get

$$0 = \sum_{\mathcal{O}}\lambda_{12\mathcal{O}}\lambda_{32\mathcal{O}}\left[g_{\ell;0,0}(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_2+\Delta_3)}g_{\ell;0,0}(v,u)\right]. \quad (5.5)$$

In practice, it is not easy to extract information from this form either. Rather, the comparison becomes most powerful (at least in the absence of other information) when the first and third scalars are also identical and the theory is unitary. In this case we get

$$0 = \sum_{\mathcal{O}}\lambda_{12\mathcal{O}}^2\left[g_{\ell;0,0}(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_1+\Delta_2)}g_{\ell;0,0}(v,u)\right]. \quad (5.6)$$

Now the coefficients are all positive (since the λs are real in a unitary theory). For fixed Δ_1 and Δ_2, we can view the quantity in square brackets as a family of functions of u and v, parameterized by ℓ and $\Delta_{\mathcal{O}}$. If we choose a functional F (taking functions of u and v and returning a real number; for instance we can act by an arbitrary differential operator and then evaluate at some choice of fixed u and v), and apply it to the functions in square brackets, then we get a set of real numbers $F_{\ell,\Delta_{\mathcal{O}}}$, and our constraint simply looks like

$$0 = \sum_{\mathcal{O}}\lambda_{12\mathcal{O}}^2 F_{\ell,\Delta_{\mathcal{O}}}. \quad (5.7)$$

A necessary condition for this to have a solution is that the spectrum must include operators $\mathcal{O}_1$ and $\mathcal{O}_2$, both with nonzero OPE coefficients $\lambda_{12\mathcal{O}}$, such that $F_{\ell_1,\Delta_{\mathcal{O}_1}}$ and $F_{\ell_2,\Delta_{\mathcal{O}_2}}$ have opposite sign. The art now is to choose a functional (or better a set of functionals) (given fixed Δ_1 and Δ_2) that splits the space of operators in a useful way. For example,

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in the case of all four scalars being identical, it is possible to choose functionals such that $F_{\ell,\Delta_{\mathcal{O}}} \geq 0$ for $\ell > 0$ and $\Delta_{\mathcal{O}}$ above the unitarity bound ($\Delta_{\mathcal{O}} > D + \ell - 2$), $F_{0,0} = 1$ (this corresponds to the identity operator and excludes a trivial spectrum) and $F_{0,\Delta_{\mathcal{O}}} \geq 0$ for $\Delta_{\mathcal{O}} \geq \Delta_c$, with Δ_c some critical value (that depends on D and Δ_ϕ). In this case, we can conclude that the spectrum must include a scalar operator $\mathcal{O}$ which appears in the $\phi_1\phi_2$ OPE and satisfies $(D-2)/2 \leq \Delta_{\mathcal{O}} \leq \Delta_c$ (the first inequality is the unitarity bound for scalar operators other than the identity).

For three scalars and one vector, there is no configuration which is quite as powerful. For SVSS, we can consider $1 \leftrightarrow 3$ exchange, which leads to

$$\begin{aligned}
0 = \sum_{\mathcal{O}} \Big\{ & \lambda_{34\mathcal{O}} \Big(\Big(\alpha_{12\mathcal{O}} g_1^{\alpha\lambda}(u,v;\Delta_1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) \\
& + \beta_{12\mathcal{O}} g_1^{\beta\lambda}(u,v;\Delta_1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) \Big) k_a^{(214)} + \Big(\alpha_{12\mathcal{O}} g_2^{\alpha\lambda}(u,v;\Delta_1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) \\
& + \beta_{12\mathcal{O}} g_2^{\beta\lambda}(u,v;\Delta_1,\Delta_v,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}) \Big) k_a^{(234)} \Big) \\
& - \lambda_{14\mathcal{O}} u^{\frac{1}{2}(\Delta_3+\Delta_4)} v^{-\frac{1}{2}(\Delta_v+\Delta_3)} \Big(\Big(\alpha_{32\mathcal{O}} g_1^{\alpha\lambda}(v,u;\Delta_3,\Delta_v,\Delta_1,\Delta_4;\ell,\Delta_{\mathcal{O}}) \\
& + \beta_{32\mathcal{O}} g_1^{\beta\lambda}(v,u;\Delta_3,\Delta_v,\Delta_1,\Delta_4;\ell,\Delta_{\mathcal{O}}) \Big) k_a^{(234)} + \Big(\alpha_{32\mathcal{O}} g_2^{\alpha\lambda}(v,u;\Delta_3,\Delta_v,\Delta_1,\Delta_4;\ell,\Delta_{\mathcal{O}}) \\
& + \beta_{32\mathcal{O}} g_2^{\beta\lambda}(v,u;\Delta_3,\Delta_v,\Delta_1,\Delta_4;\ell,\Delta_{\mathcal{O}}) \Big) k_a^{(214)} \Big) \Big\} .
\end{aligned} \tag{5.8}$$

By grouping the two tensor structures, we get two scalar equations. Let's write them out just for the case that ϕ_1 and ϕ_3 are identical. We get

$$\begin{aligned}
0 = \sum_{\mathcal{O}} \lambda_{14\mathcal{O}} \Big\{ & \alpha_{12\mathcal{O}} \Big(g_1^{\alpha\lambda}(u,v) - u^{\frac{1}{2}(\Delta_1+\Delta_4)} v^{-\frac{1}{2}(\Delta_1+\Delta_v)} g_2^{\alpha\lambda}(v,u) \Big) \\
& + \beta_{12\mathcal{O}} \Big(g_1^{\beta\lambda}(u,v) - u^{\frac{1}{2}(\Delta_1+\Delta_4)} v^{-\frac{1}{2}(\Delta_1+\Delta_v)} g_2^{\beta\lambda}(v,u) \Big) \Big\} ,
\end{aligned} \tag{5.9}$$

and a similar equation where we act on the subscript p of

$$g_p^{r\lambda}(u,v) = g_p^{r\lambda}(u,v;\Delta_1,\Delta_v,\Delta_1,\Delta_4;\ell,\Delta_{\mathcal{O}}), \tag{5.10}$$

by exchanging $1 \leftrightarrow 2$.

By considering $2 \leftrightarrow 4$ exchange, we would obtain equations relating a sum over $g_p^{r\lambda}$ blocks of the SVSS correllator with the $g_p^{\lambda r}$ blocks of SSSV, or we could obtain an equation by considering $1 \leftrightarrow 3$ exchange in the SSSV case.

Next we turn to our primary interest in this paper - the case of two scalars and two vectors.

5.2 SVSV case with generic vectors

Finally we consider the case of two scalars, in the 1 and 3 positions, and two vectors, in the 2 and 4 positions. By comparing the four-point function obtained by taking the OPEs of ϕ_1 with v_2 and ϕ_3 with v_4 to the result obtained by taking ϕ_1 with v_4 and ϕ_3 with v_2 (obtained from the former by a $2 \leftrightarrow 4$ exchange), we get

$$\left\{ \sum_{\mathcal{O}} \left[\alpha_{12\mathcal{O}}\alpha_{34\mathcal{O}} g_p^{\alpha\alpha} + \alpha_{12\mathcal{O}}\beta_{34\mathcal{O}} g_p^{\alpha\beta} + \beta_{12\mathcal{O}}\alpha_{34\mathcal{O}} g_p^{\beta\alpha} + \beta_{12\mathcal{O}}\beta_{34\mathcal{O}} g_p^{\beta\beta} \right] + \sum_{\mathcal{A}} \gamma_{12\mathcal{A}}\gamma_{34\mathcal{A}} g_p^{\gamma\gamma} \right\} t_{ab}^p$$

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$$= u^{\frac{1}{2}(\Delta_3+\Delta_4)} v^{-\frac{1}{2}(\Delta_2+\Delta_3)} \Bigg\{ \sum_{\mathcal{O}} \left[\alpha_{14\mathcal{O}}\alpha_{32\mathcal{O}} g_p^{\alpha\alpha} + \alpha_{14\mathcal{O}}\beta_{32\mathcal{O}} g_p^{\alpha\beta} + \beta_{14\mathcal{O}}\alpha_{32\mathcal{O}} g_p^{\beta\alpha} + \beta_{14\mathcal{O}}\beta_{32\mathcal{O}} g_p^{\beta\beta} \right] + \sum_{\mathcal{A}} \gamma_{14\mathcal{A}}\gamma_{32\mathcal{A}} g_p^{\gamma\gamma} \Bigg\} t'^p_{ab}. \quad (5.11)$$

Here we have abbreviated all the blocks on the left-hand side as

$$g_p^{rs} = g_p^{rs}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}}), \quad g_p^{\gamma\gamma} = g_p^{\gamma\gamma}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;k,\Delta_{\mathcal{A}}), \quad (5.12)$$

while on the right-hand side we expand as

$$g_p^{rs} = g_p^{rs}(v,u;\Delta_1,\Delta_4,\Delta_3,\Delta_2;\ell,\Delta_{\mathcal{O}}), \quad g_p^{\gamma\gamma} = g_p^{\gamma\gamma}(v,u;\Delta_1,\Delta_4,\Delta_3,\Delta_2;k,\Delta_{\mathcal{A}}). \quad (5.13)$$

we also recall that we use tensor structures

$$t^0_{ab} = m^{(24)}_{ab},\ t^{11}_{ab} = k^{(214)}_a k^{(412)}_b,\ t^{12}_{ab} = k^{(214)}_a k^{(432)}_b,\ t^{21}_{ab} = k^{(234)}_a k^{(412)}_b,\ t^{22}_{ab} = k^{(234)}_a k^{(432)}_b, \quad (5.14)$$

and the primed tensor structures are obtained by exchanging $2 \leftrightarrow 4$ and $a \leftrightarrow b$,

$$t'^0_{ab} = m^{(24)}_{ab},\ t'^{11}_{ab} = k^{(214)}_a k^{(412)}_b,\ t'^{12}_{ab} = k^{(234)}_a k^{(412)}_b,\ t'^{21}_{ab} = k^{(214)}_a k^{(432)}_b,\ t'^{22}_{ab} = k^{(234)}_a k^{(432)}_b. \quad (5.15)$$

In other words, $t'^0_{ab} = t^0_{ab}$ and $t'^{ij}_{ab} = t^{ji}_{ab}$.

Grouping like tensor structures together now gives us five equations on the underlying data of the CFT. Note that by using (4.70), we can rewrite the symmetric exchange summands in terms of scalar blocks.

As with the case of four-scalars, the equations are much more constraining for the case where we have two identical scalars and two identical vectors. In this case, we saw in section 4.3 that $g^{rs}_{22}(u,v) = g^{sr}_{11}(u,v)$ and $g^{rs}_p = g^{sr}_p$ for the other p. This results in only three independent bootstrap constraints,

$$0 = \sum_{\mathcal{O}} \left[G_0(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_v)} G_0(v,u) \right] + \sum_{\mathcal{A}} \gamma^2_{\phi v\mathcal{A}} \left[g_0^{\gamma\gamma}(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_v)} g_0^{\gamma\gamma}(v,u) \right], \quad (5.16)$$

$$0 = \sum_{\mathcal{O}} \left[G_{11}(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_v)} G_{11}(v,u) \right] + \sum_{\mathcal{A}} \gamma^2_{\phi v\mathcal{A}} \left[g_{11}^{\gamma\gamma}(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_v)} g_{11}^{\gamma\gamma}(v,u) \right], \quad (5.17)$$

and

$$0 = \sum_{\mathcal{O}} \left[G_{12}(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_v)} G_{21}(v,u) \right] + \sum_{\mathcal{A}} \gamma^2_{\phi v\mathcal{A}} \left[g_{12}^{\gamma\gamma}(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+\Delta_v)} g_{21}^{\gamma\gamma}(v,u) \right], \quad (5.18)$$

where we have defined

$$G_p(u,v) = A^2_{\mathcal{O}} \mathcal{D}^{--}_p g_{\ell;-1,-1} + A_{\mathcal{O}} B_{\mathcal{O}} \left(\mathcal{D}^{-+}_p g_{\ell;-1,1} + \mathcal{D}^{+-}_p g_{\ell;1,-1} \right) + B^2_{\mathcal{O}} \mathcal{D}^{++}_p g_{\ell;1,1}, \quad (5.19)$$

with

$$A_{\mathcal{O}} = \frac{1}{2(\Delta_{\mathcal{O}}-1)} \left(\alpha_{\phi v\mathcal{O}} + (\Delta_\phi - \Delta_v - \Delta_{\mathcal{O}} + \ell + 1) \frac{\beta_{\phi v\mathcal{O}}}{\ell} \right), \quad (5.20)$$

$$B_{\mathcal{O}} = \frac{1}{2(\Delta_{\mathcal{O}}-1)} \left(\alpha_{\phi v\mathcal{O}} + (\Delta_\phi - \Delta_v + \Delta_{\mathcal{O}} + \ell - 1) \frac{\beta_{\phi v\mathcal{O}}}{\ell} \right), \quad (5.21)$$

(or $A_{\mathcal{O}} = B_{\mathcal{O}} = \alpha_{\phi v\mathcal{O}}/2(\Delta_{\mathcal{O}}-1)$ for $\ell = 0$) and using the notation of (4.70).

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5.3 SVSV with conserved vectors

The situation is even more tractable in the case that the identical vectors are in fact conserved currents. In this case we have $\Delta_1 = \Delta_3 = \Delta_\phi$ and $\Delta_2 = \Delta_4 = D-1$. There are also restrictions on the data of symmetric operator exchange, i.e. on which operators $\mathcal{O}$ can appear in the $\langle \phi v \mathcal{O} \rangle$ three-point function, which were discussed in section 2.5. Let's split into the cases $\ell = 0$ and $\ell > 0$.

For $\ell = 0$, as reviewed in section 2.5, we can assume that either ϕ is neutral under the symmetry, or that is the real part of a complex scalar operator of charge Q. We'll focus on the latter case, and the former case can be recovered by setting $Q = 0$. Thus, there will be a unique scalar operator $\mathcal{O}$ that can be exchanged, with $\Delta_\mathcal{O} = \Delta_\phi$ and $\alpha_{\phi v \mathcal{O}} = -Q$. In this case we can split this piece out of the bootstrap constraints, much in the same way that the contribution from exchange of the identity operator is typically split off for the case of identical scalars in the SSSS bootstrap, and move it to the right-hand-side of the constraint equations.

For $\ell > 0$ we also have another important result - the relation between $\alpha_{\phi v \mathcal{O}}$ and $\beta_{\phi v \mathcal{O}}$ given in (2.53). Since we could in principle have either $\alpha_{\phi v \mathcal{O}}$ or $\beta_{\phi v \mathcal{O}}$ vanishing, it is more useful to define a non-vanishing constant $c_\mathcal{O}$ related to them by

$$\alpha_{\phi v \mathcal{O}} = (\Delta_\phi - \Delta_\mathcal{O} + D + \ell - 2)\, c_\mathcal{O}, \qquad \beta_{\phi v \mathcal{O}} = (\Delta_\mathcal{O} - \Delta_\phi)\, c_\mathcal{O}, \tag{5.22}$$

which ensures that (2.53) holds. In terms of this, (5.20) and (5.21) become

$$A_\mathcal{O} = -\frac{(\Delta_\mathcal{O} - \Delta_\phi - \ell)\,(\Delta_\mathcal{O} - \Delta_\phi + D + \ell - 2)}{2\ell\,(\Delta_\mathcal{O} - 1)} c_\mathcal{O} =: a_\mathcal{O} c_\mathcal{O}, \tag{5.23}$$

$$B_\mathcal{O} = \left[\frac{\Delta_\mathcal{O} - \Delta_\phi}{\ell} - \frac{(\Delta_\mathcal{O} - \Delta_\phi - \ell)\,(\Delta_\mathcal{O} - \Delta_\phi + D + \ell - 2)}{2\ell\,(\Delta_\mathcal{O} - 1)}\right] c_\mathcal{O} =: b_\mathcal{O} c_\mathcal{O}, \tag{5.24}$$

which defines constants $a_\mathcal{O}$ and $b_\mathcal{O}$ that depend only on the dimensions Δ_ϕ and $\Delta_\mathcal{O}$, and then we have $G_p(u,v) = c_\mathcal{O}^2 \widehat{G}_p(u,v)$, with

$$\widehat{G}_p(u,v) = a_\mathcal{O}^2 \mathcal{D}_p^{--} g_{\ell;-1,-1} + a_\mathcal{O} b_\mathcal{O} \left(\mathcal{D}_p^{-+} g_{\ell;-1,1} + \mathcal{D}_p^{+-} g_{\ell;1,-1}\right) + b_\mathcal{O}^2 \mathcal{D}_p^{++} g_{\ell;1,1}. \tag{5.25}$$

Now we can rewrite the bootstrap constraints as follows,

$$\begin{aligned}
&\sum_{\mathcal{O},\ell>0} c_\mathcal{O}^2 \left[\widehat{G}_0(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi + D - 1)} \widehat{G}_0(v,u)\right] \\
&\qquad + \sum_\mathcal{A} \gamma_{\phi v \mathcal{A}}^2 \left[g_0^{\gamma\gamma}(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi + D - 1)} g_0^{\gamma\gamma}(v,u)\right] \\
&\qquad\qquad = -\frac{Q^2}{4\,(\Delta_\phi - 1)^2} \left[H_0(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi + D - 1)} H_0(v,u)\right],
\end{aligned} \tag{5.26}$$

$$\sum_{\mathcal{O},\ell>0} c_\mathcal{O}^2 \left[\widehat{G}_{11}(u,v) - \left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi + D - 1)} \widehat{G}_{11}(v,u)\right]$$

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$$+\sum_{\mathcal{A}}\gamma^2_{\phi v\mathcal{A}}\left[g^{\gamma\gamma}_{11}(u,v)-\left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+D-1)}g^{\gamma\gamma}_{11}(v,u)\right]$$
$$=-\frac{Q^2}{4\left(\Delta_\phi-1\right)^2}\left[H_{11}(u,v)-\left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+D-1)}H_{11}(v,u)\right], \quad (5.27)$$

$$\sum_{\mathcal{O},\ell>0}c^2_{\mathcal{O}}\left[\widehat{G}_{12}(u,v)-\left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+D-1)}\widehat{G}_{21}(v,u)\right]$$
$$+\sum_{\mathcal{A}}\gamma^2_{\phi v\mathcal{A}}\left[g^{\gamma\gamma}_{12}(u,v)-\left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+D-1)}g^{\gamma\gamma}_{21}(v,u)\right]$$
$$=-\frac{Q^2}{4\left(\Delta_\phi-1\right)^2}\left[H_{12}(u,v)-\left(\frac{u}{v}\right)^{\frac{1}{2}(\Delta_\phi+D-1)}H_{21}(v,u)\right], \quad (5.28)$$

where we have defined

$$H_p(u,v)=\mathcal{D}_p^{--}g_{0;-1,-1}+\mathcal{D}_p^{-+}g_{0;-1,1}+\mathcal{D}_p^{+-}g_{0;1,-1}+\mathcal{D}_p^{++}g_{0;1,1}, \quad (5.29)$$

for the $\ell=0$ exchange, and the scalar blocks in this last expression are evaluated with $\Delta_1=\Delta_3=\Delta_{\mathcal{O}}=\Delta_\phi$ and $\Delta_2=\Delta_4=D-1$. In these equations the left-hand sides are written as sums of functions which depend only on the conformal representations (i.e. only on D, Δ_ϕ, and either $\Delta_{\mathcal{O}}$ and ℓ or $\Delta_{\mathcal{A}}$ and k) but not on the three-point function coefficients, multiplied by manifestly positive real numbers (for unitary theories). This is a situation in which we can profitably use the techniques developed in the scalar bootstrap literature to obtain bounds on the spectrum of our theory. The (numerical) analysis of the consistent space of theories with conserved currents will be carried out in future work.

6 Conclusions

Our primary goal in this paper was to set up, in explicit detail, the bootstrap equations for a four-point function of two scalars and two vectors (abbreviated $\langle SVSV\rangle$) in a general CFT in arbitrary dimension. To compute this four-point function, one performs OPEs between a scalar and a vector, producing either traceless symmetric tensors $\mathcal{O}$ or mixed symmetry tensors $\mathcal{A}$ (indeed this is the simplest four-point function that includes exchange of a mixed symmetry operator), and evaluates the resulting two-point functions. The bootstrap compares the two different channels for these OPEs. The contribution of a primary operator, either $\mathcal{O}$ or $\mathcal{A}$, and all its descendants to the four-point function is captured by a conformal block g_p^{rs}, where the indices r and s refer to possible tensor structures in either the three-point function $\langle SV\mathcal{O}\rangle$ or $\langle SV\mathcal{A}\rangle$, while p refers to $\langle SVSV\rangle$. For general scalars and vectors, $\langle SV\mathcal{O}\rangle$ has two structures and $\langle SV\mathcal{A}\rangle$ has one, while $\langle SVSV\rangle$ has five structures, so there are really $(2^2+1^2)\times 5=25$ different conformal block functions, and we have computed them all in section 4.4.4. We presented the explicit form of conformal blocks associated to the exchange of symmetric traceless operators $\mathcal{O}$, by applying suitable differential operators to scalar blocks using ideas from [44]. Furthermore we showed that

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the conformal blocks of the mixed operator $\mathcal{A}$ can also be written as differential operators acting on scalar blocks, if we allow them to have shifted spins k,$k+1$,$k+2$. We also found that writing the SVSV blocks in terms of lower spin ones (SVSS,SSSV) makes the expressions simpler. This could potentially be important for making the computation of higher spin blocks (four vectors,four stress tensors) more tractable. For identical scalars and/or identical vectors there are many relations between the blocks which we have spelled out.

Next we set up the bootstrap equations, starting with the most general $\langle SVSV\rangle$ (or $\langle SSSV\rangle$ or $\langle SVSS\rangle$) four-point function, and then specializing to identical scalars and identical vectors. Finally we restrict to the particularly simple case of conserved vectors, for which a unique scalar $\mathcal{O}$ can contribute to $\langle SV\mathcal{O}\rangle$, while for $\mathcal{O}$ with $\ell > 0$ the two possible structures in $\langle SV\mathcal{O}\rangle$ collapse to one. This latter property ensured that the bootstrap equations resemble those of identical scalars with global symmetry [6], i.e. a sum of vector functions with positive coefficients. Thus one can exploit the already developed techniques for bootstrapping identical scalars.[11]

As a check of our results, we have verified numerically that our blocks (as well as the SSSV and SVSS blocks which we construct as intermediate steps) exhibit the correct behavior under exchanges, though we have omitted the details in the present work, preferring to defer all numerical details to a follow up paper. It would be worthwhile to develop further checks on our results, by comparing with cases (such as generalized free CFTs) where the full four-point function can be computed directly. It would also be interesting to understand our results in the context of holography, and particularly to match our results with the bulk geometric quantities studied in [49].

Going forward, there are two natural extensions to this work. The first is to actually apply our formalism to seek, both numerically and analytically, bounds on the data of some general class of CFTs. It would be particularly interesting to derive results for conserved vectors, which could constrain theories with continuous global symmetries and their spectra of scalars charged under the symmetry.

The second direction heading forward is to use the techniques developed in this work to set up the bootstrap for even more complicated four-point functions. The next one to attempt is probably four vectors, either conserved or not, and this case should be tractable by hand. Another possibility would be two scalars and two spin-two tensors, especially for the case where the tensors are conserved (e.g. stress-energy tensors). Finally, the most ambitious goal would be to bootstrap the four-point function of conserved stress-tensors (see for example the discussion in [50]). This is probably not feasible using our current techniques "by hand", but might be possible if we can computerize the necessary steps.

Acknowledgments

D. R. was supported by the grants PHY-1214344 and NSF Focused Research Grant DMS-1159404 and by the George P. and Cynthia W. Mitchell Institute for Fundamental Physics and Astronomy. F. R.-B. was supported by the Mexican Consejo Nacional de Ciencia y

[11] Bootstrapping non-conserved vectors could in principle be done using semi-definite techniques [8, 13, 23].

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Tecnología (CONACyT scholarship 382219). We thank S. El-Showk, S. Hellerman, D. Hofman, H. Osborn, E. Perlmutter, S. Rychkov, and S. Sethi for useful discussions. D. R. would like to thank the Simons Center for Geometry and Physics for hospitality during the final days of preparation of this work.

A Building blocks and identities

In this paper, the physical space is flat $\mathbb{R}^D$ with Euclidean signature. Indices a, b, etc., are raised and lowered with the Kronecker delta δ_{ab}. For two vectors x_i^a and x_j^a, we define

$$x_{ij}^a = x_i^a - x_j^a. \tag{A.1}$$

Two particular structures play a significant role in constructing correlators in a conformal field theory,

$$k_a^{(ijk)} = \frac{x_{ij}^2 \left(x_{ik}\right)_a - x_{ik}^2 \left(x_{ij}\right)_a}{\left(x_{ij}^2 x_{ik}^2 x_{jk}^2\right)^{1/2}}, \tag{A.2}$$

where x_i, x_j, and x_k are assumed to be distinct points, and

$$m_{ab}^{(ij)} = \delta_{ab} - \frac{2}{x_{ij}^2} \left(x_{ij}\right)_a \left(x_{ij}\right)_b, \tag{A.3}$$

where again x_i and x_j are distinct.

From the fact that $x_{ij} + x_{jk} = x_{ik}$, we can show that

$$k_a^{(ik\ell)} = -\sqrt{\frac{x_{i\ell}^2 x_{jk}^2}{x_{ij}^2 x_{k\ell}^2}} k_a^{(ijk)} + \sqrt{\frac{x_{ik}^2 x_{j\ell}^2}{x_{ij}^2 x_{k\ell}^2}} k_a^{(ij\ell)} \tag{A.4}$$

Using the basic identity that

$$x_{ij} \cdot x_{k\ell} = \frac{1}{2}\left(-x_{ik}^2 + x_{i\ell}^2 + x_{jk}^2 - x_{j\ell}^2\right), \tag{A.5}$$

we can prove identities

$$k^{(ijk)} \cdot k^{(i\ell m)} = \frac{1}{2}\left(x_{ij}^2 x_{ik}^2 x_{i\ell}^2 x_{im}^2 x_{jk}^2 x_{\ell m}^2\right)^{-1/2} \left(-x_{ij}^2 x_{i\ell}^2 x_{km}^2 + x_{ij}^2 x_{im}^2 x_{k\ell}^2 + x_{ik}^2 x_{i\ell}^2 x_{jm}^2 - x_{ik}^2 x_{im}^2 x_{j\ell}^2\right), \tag{A.6}$$

$$m_{ab}^{(ij)} k^{(jk\ell)\,b} = -\sqrt{\frac{x_{ik}^2 x_{j\ell}^2}{x_{ij}^2 x_{k\ell}^2}} k_a^{(ijk)} + \sqrt{\frac{x_{i\ell}^2 x_{jk}^2}{x_{ij}^2 x_{k\ell}^2}} k_a^{(ij\ell)}, \tag{A.7}$$

and

$$\delta^{cd} m_{ac}^{(ik)} m_{db}^{(kj)} = m_{ab}^{(ij)} - 2 k_a^{(ijk)} k_b^{(jik)}. \tag{A.8}$$

As special cases of these fomulae, we have

$$\left(k^{(ijk)}\right)^2 = 1, \qquad m_{ab}^{(ij)} k^{(jik)\,b} = k_a^{(ijk)}, \qquad \delta^{cd} m_{ac}^{(ij)} m_{db}^{(ji)} = \delta_{ab}. \tag{A.9}$$

One more useful identity is

$$\frac{\partial}{\partial x_k^a} k_b^{(ijk)} = -\sqrt{\frac{x_{ij}^2}{x_{ik}^2 x_{jk}^2}} \left(m_{ab}^{(ki)} + k_a^{(kij)} k_b^{(ijk)}\right). \tag{A.10}$$

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B Lorentz representation projectors

We will be grouping tensor operators by their representations under SO(D). There is a large body of work on irreducible representations of SO(D) (for instance see the nice discussion in [46] and references therein), but we really don't need the full power of this theory for the current work.

Consider a tensor with n indices. It must transform as a sub-representation of the tensor product $\mathbf{D}^{\otimes n}$ of n copies of the D-dimensional vector representation. To distinguish the different irreducible representations I which appear in the decomposition of $\mathbf{D}^{\otimes n}$, we can use projectors, $\Pi^{I\,a_1\cdots a_n}_{b_1\cdots b_n}$. Being projectors, these must satisfy

$$\Pi^{I\,a_1\cdots a_n}_{c_1\cdots c_n}\Pi^{J\,c_1\cdots c_n}_{b_1\cdots b_n} = \delta^{IJ}\Pi^{I\,a_1\cdots a_n}_{b_1\cdots b_n}. \tag{B.1}$$

The projectors are built exclusively with Kronecker deltas $\delta^{a_i}_{b_j}$, $\delta^{a_ia_j}$, or $\delta_{b_ib_j}$.

Below, we will need the projectors for the totally symmetric traceless representation of spin ℓ (i.e. with ℓ indices), and also for a mixed symmetry representation with $k+2$ indices which we will describe below.

B.1 Totally symmetric

Consider first the projector onto totally symmetric traceless representations, $\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}$. By the symmetries of the problem, it must have the form

$$\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell} = A_0\delta^{(a_1}_{(b_1}\cdots\delta^{a_\ell)}_{b_\ell)} + \sum_{i=1}^{\lfloor\ell/2\rfloor} A_i\delta^{(a_1a_2}\cdots\delta^{a_{2i-1}a_{2i}}\delta_{(b_1b_2}\cdots\delta_{b_{2i-1}b_{2i}}\delta^{a_{2i+1}}_{b_{2i+1}}\cdots\delta^{a_\ell)}_{b_\ell)}, \tag{B.2}$$

where the A_i are constants. For $\ell\geq 2$, taking the trace with $\delta^{b_{\ell-1}b_\ell}$ we get

$$\begin{aligned}
&A_0\delta^{(a_1}_{(b_1}\cdots\delta^{a_{\ell-2}}_{b_{\ell-2})}\delta^{a_{\ell-1}a_\ell)}\\
&\quad+\sum_{i=1}^{\lfloor\ell/2\rfloor} A_i\left\{\frac{(\ell-2i)\,(\ell-1-2i)}{\ell\,(\ell-1)}\delta^{(a_1a_2}\cdots\delta^{a_{2i+1}a_{2i+2}}\delta_{(b_1b_2}\cdots\delta_{b_{2i-1}b_{2i}}\delta^{a_{2i+3}}_{b_{2i+1}}\cdots\delta^{a_\ell)}_{b_{\ell-2})}\right.\\
&\qquad\left.+\frac{2i\,(D+2\ell-2i-2)}{\ell\,(\ell-1)}\delta^{(a_1a_2}\cdots\delta^{a_{2i-1}a_{2i}}\delta_{(b_1b_2}\cdots\delta_{b_{2i-3}b_{2i-2}}\delta^{a_{2i+1}}_{b_{2i-1}}\cdots\delta^{a_\ell)}_{b_{\ell-2})}\right\}.
\end{aligned} \tag{B.3}$$

Thus tracelessness requires

$$A_i = -\frac{(\ell+2-2i)\,(\ell+1-2i)}{2i\,(D+2\ell-2-2i)}A_{i-1}, \qquad \text{for } 1\leq i\leq\lfloor\ell/2\rfloor, \tag{B.4}$$

or

$$A_i = (-1)^i\,\frac{\ell!\Gamma(\frac{D}{2}+\ell-i-1)}{2^{2i}\,(\ell-2i)!i!\Gamma(\frac{D}{2}+\ell-1)}A_0. \tag{B.5}$$

Finally, we can fix A_0 by the condition that $\Pi^2=\Pi$, i.e.

$$\Pi^{(\ell)\,a_1\cdots a_\ell}_{c_1\cdots c_\ell}\Pi^{(\ell)\,c_1\cdots c_\ell}_{b_1\cdots b_\ell} = \Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}. \tag{B.6}$$

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In fact we only need to check the leading terms, not the subleading traceless terms, because the latter can't contribute to the former when we square. Then since

$$\delta^{(a_1}_{(c_1}\cdots\delta^{a_\ell)}_{c_\ell)}\delta^{(c_1}_{(b_1}\cdots\delta^{c_\ell)}_{b_\ell)}=\delta^{(a_1}_{(b_1}\cdots\delta^{a_\ell)}_{b_\ell)}, \tag{B.7}$$

we require $A_0^2=A_0$, and hence we should take $A_0=1$, and we can write

$$\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}=\sum_{i=0}^{\lfloor\ell/2\rfloor}\left(-\frac{1}{4}\right)^i\frac{\ell!\,\Gamma(\frac{D}{2}+\ell-i-1)}{i!\,(\ell-2i)!\,\Gamma(\frac{D}{2}+\ell-1)}\times\delta^{(a_1a_2}\cdots\delta^{a_{2i-1}a_{2i}}\delta_{(b_1b_2}\cdots\delta_{b_{2i-1}b_{2i}}\delta^{a_{2i+1}}_{b_{2i+1}}\cdots\delta^{a_\ell)}_{b_\ell)}. \tag{B.8}$$

These projectors obey certain recursion relations. With the explicit expressions for coefficients above, one can show that

$$\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}=\delta^{(a_1}_{(b_1}\Pi^{(\ell-1)\,a_2\cdots a_\ell)}_{b_2\cdots b_\ell)}-\frac{(D+\ell-4)\,(\ell-1)}{(D+2\ell-6)\,(D+2\ell-4)}\delta^{(a_1a_2}\delta_{(b_1b_2}\Pi^{(\ell-2)\,a_3\cdots a_\ell)}_{b_3\cdots b_\ell)}. \tag{B.9}$$

Now we can define polynomials $p_{D,\ell}(t)$ by

$$X_{a_1}\cdots X_{a_\ell}\Pi^{(\ell)\,a_1\cdots a_\ell}_{b_1\cdots b_\ell}Y^{b_1}\cdots Y^{b_\ell}=\left(X^2Y^2\right)^{\ell/2}p_{D,\ell}(t),\qquad t=\frac{X\cdot Y}{\sqrt{X^2Y^2}}. \tag{B.10}$$

Explicitly, using (B.8), we have

$$p_{D,\ell}(t)=\sum_{i=0}^{\lfloor\ell/2\rfloor}\left(-\frac{1}{4}\right)^i\frac{\ell!\Gamma(\frac{D}{2}+\ell-i-1)}{i!(\ell-2i)!\Gamma(\frac{D}{2}+\ell-1)}t^{\ell-2i}. \tag{B.11}$$

These are related to the more familiar Gegenbauer polynomials by

$$p_{D,\ell}(t)=\frac{\ell!\Gamma(\frac{D}{2}-1)}{2^\ell\Gamma(\frac{D}{2}+\ell-1)}C_\ell^{(\frac{D}{2}-1)}(t). \tag{B.12}$$

They obey a simple differential identity,

$$p'_{D,\ell}(t)=\ell p_{D+2,\ell-1}(t), \tag{B.13}$$

and also

$$p_{D+2,\ell}(t)=p_{D,\ell}(t)+\frac{\ell\,(\ell-1)}{(D+2\ell-2)\,(D+2\ell-4)}p_{D+2,\ell-2}(t). \tag{B.14}$$

We can also prove a recursion relation for fixed D from (B.9),

$$p_{D,\ell}(t)=tp_{D,\ell-1}(t)-\frac{(D+\ell-4)\,(\ell-1)}{(D+2\ell-4)\,(D+2\ell-6)}p_{D,\ell-2}(t). \tag{B.15}$$

The first few of these polynomials are

$$p_0=1,\qquad p_1=t,\qquad p_2=t^2-\frac{1}{D},\qquad p_3=t^3-\frac{3}{D+2}t,$$

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$$p_4 = t^4 - \frac{6}{D+4}t^2 + \frac{3}{(D+2)(D+4)}, \quad p_5 = t^5 - \frac{10}{D+6}t^3 + \frac{15}{(D+4)(D+6)}t. \tag{B.16}$$

We will also need the result of the following partial contractions of $\Pi^{(\ell)}$,

$$\begin{aligned} X_{c_1}\cdots X_{c_{\ell-1}}\Pi^{(\ell)\,ac_1\cdots c_{\ell-1}}_{bd_1\cdots d_{\ell-1}}Y^{d_1}\cdots Y^{d_{\ell-1}} &= \frac{1}{\ell^2}\frac{\partial}{\partial X_a}\frac{\partial}{\partial Y^b}\left[\left(X^2Y^2\right)^{\ell/2}p_\ell(t)\right] \\ &= \frac{1}{\ell^2}\left(X^2Y^2\right)^{\frac{\ell-1}{2}}\left[\delta^a_b\partial_t + \left(\frac{X^aX_b}{X^2}+\frac{Y^aY_b}{Y^2}\right)\left((\ell-1)\,\partial_t - t\partial_t^2\right)\right. \\ &\quad \left. + \frac{X^aY_b}{\sqrt{X^2Y^2}}\left(\ell^2-(2\ell-1)\,t\partial_t+t^2\partial_t^2\right) + \frac{Y^aX_b}{\sqrt{X^2Y^2}}\partial_t^2\right]p_\ell(t) \\ &= \frac{1}{\ell}\left(X^2Y^2\right)^{\frac{\ell-1}{2}}\left[\delta^a_b p_{D+2,\ell-1}(t) + (\ell-1)\left(\frac{X^aX_b}{X^2}+\frac{Y^aY_b}{Y^2}\right)\left(p_{D+2,\ell-1}(t) - tp_{D+4,\ell-2}(t)\right)\right. \\ &\quad + \frac{X^aY_b}{\sqrt{X^2Y^2}}\left(\ell p_{D,\ell}(t) - (2\ell-1)\,tp_{D+2,\ell-1}(t) + (\ell-1)\,t^2p_{D+4,\ell-2}(t)\right) \\ &\quad \left. + (\ell-1)\,\frac{Y^aX_b}{\sqrt{X^2Y^2}}p_{D+4,\ell-2}(t)\right]. \end{aligned} \tag{B.17}$$

B.2 Mixed symmetry

Now we would like to find projectors onto the mixed symmetry representations that we need for the scalar-vector bootstrap. Recall that these tensors are antisymmetric in their first two indices, totally symmetric in their remaining k indices, they vanish when antisymmetrized over any three indices (this condition is trivial unless the three are the first two indices plus one more), and are completely traceless. We will write the corresponding projectors $\widetilde{\Pi}^{(k)\,a_1a_2b_1\cdots b_k}_{c_1c_2d_1\cdots d_k}$ with tildes to distinguish from the totally symmetric case considered above.

For $k=0$, the only index structure compatible with antisymmetry is

$$\widetilde{\Pi}^{(0)\,a_1a_2}_{c_1c_2} = A_0\left(\delta^{a_1}_{c_1}\delta^{a_2}_{c_2} - \delta^{a_1}_{c_2}\delta^{a_2}_{c_1}\right). \tag{B.18}$$

Imposing $\widetilde{\Pi}^2 = \widetilde{\Pi}$ then implies $A_0 = 1/2$.

For $k=1$, there are three terms compatible with the antisymmetry in the a_i and the c_i,

$$\begin{aligned} \widetilde{\Pi}^{(1)\,a_1a_2b}_{c_1c_2d} &= A_0\left(\delta^{a_1}_{c_1}\delta^{a_2}_{c_2} - \delta^{a_1}_{c_2}\delta^{a_2}_{c_1}\right)\delta^b_d + B_0\left(\delta^{a_1}_{c_1}\delta^{a_2}_d\delta^b_{c_2} - \delta^{a_1}_{c_2}\delta^{a_2}_d\delta^b_{c_1} - \delta^{a_1}_d\delta^{a_2}_{c_1}\delta^b_{c_2} + \delta^{a_1}_d\delta^{a_2}_{c_2}\delta^b_{c_1}\right) \\ &\quad + C_1\left(\delta^{a_1b}\delta_{c_1d}\delta^{a_2}_{c_2} - \delta^{a_1b}\delta_{c_2d}\delta^{a_2}_{c_1} - \delta^{a_2b}\delta_{c_1d}\delta^{a_1}_{c_2} + \delta^{a_2b}\delta_{c_2d}\delta^{a_1}_{c_1}\right). \end{aligned} \tag{B.19}$$

Demanding that this vanish on antisymetrizing $[a_1a_2b]$ leads to the constraint

$$2A_0 - 4B_0 = 0, \tag{B.20}$$

while demanding that it vanishes when we trace with δ^{c_2d} gives

$$A_0 + B_0 + (D-1)\,C_1 = 0. \tag{B.21}$$

Finally, demanding that $\widetilde{\Pi}^2 = \widetilde{\Pi}$ requires

$$2A_0^2 + 4B_0^2 = A_0, \quad 4A_0B_0 - 2B_0^2 = B_0, \quad \text{and} \quad 4A_0C_1 + 4B_0C_1 + 2\,(D-1)\,C_1^2 = C_1. \tag{B.22}$$

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The unique non-vanishing solution to these constraints is that

$$A_0 = \frac{1}{3}, \qquad B_0 = \frac{1}{6}, \qquad C_1 = -\frac{1}{2\,(D-1)}, \tag{B.23}$$

so

$$\begin{aligned}\widetilde{\Pi}^{(1)\,a_1a_2b}_{c_1c_2d} = {}& \frac{1}{3}\left(\delta^{a_1}_{c_1}\delta^{a_2}_{c_2} - \delta^{a_1}_{c_2}\delta^{a_2}_{c_1}\right)\delta^{b}_{d} + \frac{1}{6}\left(\delta^{a_1}_{c_1}\delta^{a_2}_{d}\delta^{b}_{c_2} - \delta^{a_1}_{c_2}\delta^{a_2}_{d}\delta^{b}_{c_1} - \delta^{a_1}_{d}\delta^{a_2}_{c_1}\delta^{b}_{c_2} + \delta^{a_1}_{d}\delta^{a_2}_{c_2}\delta^{b}_{c_1}\right) \\ & - \frac{1}{2\,(D-1)}\left(\delta^{a_1b}\delta_{c_1d}\delta^{a_2}_{c_2} - \delta^{a_1b}\delta_{c_2d}\delta^{a_2}_{c_1} - \delta^{a_2b}\delta_{c_1d}\delta^{a_1}_{c_2} + \delta^{a_2b}\delta_{c_2d}\delta^{a_1}_{c_1}\right). \end{aligned} \tag{B.24}$$

For $k > 1$, the following structure is the most general consistent with antisymmetry of the a_i and c_i, symmetry of the b_i and d_i, and symmetry between upper and lower indices,

$$\begin{aligned}&\widetilde{\Pi}^{(k)\,a_1a_2b_1\cdots b_k}_{c_1c_2d_1\cdots d_k} \\ &= \sum_{i=0}^{\lfloor k/2\rfloor} A_i\left(\delta^{a_1}_{c_1}\delta^{a_2}_{c_2} - \delta^{a_1}_{c_2}\delta^{a_2}_{c_1}\right)\delta^{(b_1b_2}\cdots\delta^{b_{2i-1}b_{2i}}\delta_{(d_1d_2}\cdots\delta_{d_{2i-1}d_{2i}}\delta^{b_{2i+1}}_{d_{2i+1}}\cdots\delta^{b_k)}_{d_k)} \\ &\quad + \sum_{i=0}^{\lfloor (k-1)/2\rfloor} B_i\left(\delta^{a_1}_{c_1}\delta^{a_2}_{(d_1}\delta^{(b_1}_{|c_2|} - \delta^{a_1}_{c_2}\delta^{a_2}_{(d_1}\delta^{(b_1}_{|c_1|} - \delta^{a_1}_{(d_1}\delta^{a_2}_{|c_1}\delta^{(b_1}_{c_2|} + \delta^{a_1}_{(d_1}\delta^{a_2}_{|c_2}\delta^{(b_1}_{c_1|}\right) \\ &\qquad \times\delta^{b_2b_3}\cdots\delta^{b_{2i}b_{2i+1}}\delta_{d_2d_3}\cdots\delta_{d_{2i}d_{2i+1}}\delta^{b_{2i+2}}_{d_{2i+2}}\cdots\delta^{b_k)}_{d_k)} \\ &\quad + \sum_{i=1}^{\lfloor (k+1)/2\rfloor} C_i\left(\delta^{a_1(b_1}\delta_{c_1(d_1}\delta^{|a_2|}_{|c_2|} - \delta^{a_1(b_1}\delta_{c_2(d_1}\delta^{|a_2|}_{|c_1|} - \delta^{a_2(b_1}\delta_{c_1(d_1}\delta^{|a_1|}_{|c_2|} + \delta^{a_2(b_1}\delta_{c_2(d_1}\delta^{|a_1|}_{|c_1|}\right) \\ &\qquad \times\delta^{b_2b_3}\cdots\delta^{b_{2i-2}b_{2i-1}}\delta_{d_2d_3}\cdots\delta_{d_{2i-2}d_{2i-1}}\delta^{b_{2i}}_{d_{2i}}\cdots\delta^{b_k)}_{d_k)} \\ &\quad + \sum_{i=1}^{\lfloor k/2\rfloor} D_i\left(\delta^{a_1(b_1}\delta_{c_1(d_1}\delta^{|a_2|}_{d_2}\delta^{b_2}_{|c_2|} - \delta^{a_1(b_1}\delta_{c_2(d_1}\delta^{|a_2|}_{d_2}\delta^{b_2}_{|c_1|} - \delta^{a_2(b_1}\delta_{c_1(d_1}\delta^{|a_1|}_{d_2}\delta^{b_2}_{|c_2|}\right. \\ &\qquad \left.+\delta^{a_2(b_1}\delta_{c_2(d_1}\delta^{|a_1|}_{d_2}\delta^{b_2}_{|c_1|}\right)\delta^{b_3b_4}\cdots\delta^{b_{2i-1}b_{2i}}\delta_{d_3d_4}\cdots\delta_{d_{2i-1}d_{2i}}\delta^{b_{2i+1}}_{d_{2i+1}}\cdots\delta^{b_k)}_{d_k)} \\ &\quad + \sum_{i=1}^{\lfloor k/2\rfloor} E_i\left(\delta^{a_1(b_1}\delta_{(d_1d_2}\delta^{|a_2|}_{|c_1|}\delta^{b_2}_{c_2|} - \delta^{a_1(b_1}\delta_{(d_1d_2}\delta^{|a_2|}_{c_2}\delta^{b_2}_{c_1|} - \delta^{a_2(b_1}\delta_{(d_1d_2}\delta^{|a_1|}_{|c_1}\delta^{b_2}_{c_2|}\right. \\ &\qquad +\delta^{a_2(b_1}\delta_{(d_1d_2}\delta^{|a_1|}_{|c_2}\delta^{b_2}_{c_1|} + \delta^{(b_1b_2}\delta_{c_1(d_1}\delta^{|a_1}_{|c_2|}\delta^{a_2|}_{d_2} - \delta^{(b_1b_2}\delta_{c_2(d_1}\delta^{|a_1}_{|c_1|}\delta^{a_2|}_{d_2} - \delta^{(b_1b_2}\delta_{c_1(d_1}\delta^{|a_1}_{d_2}\delta^{a_2|}_{|c_2|} \\ &\qquad \left.+\delta^{(b_1b_2}\delta_{c_2(d_1}\delta^{|a_1}_{d_2}\delta^{a_2|}_{|c_1|}\right)\delta^{b_3b_4}\cdots\delta^{b_{2i-1}b_{2i}}\delta_{d_3d_4}\cdots\delta_{d_{2i-1}d_{2i}}\delta^{b_{2i+1}}_{d_{2i+1}}\cdots\delta^{b_k)}_{d_k)}. \end{aligned} \tag{B.25}$$

Demanding this vanish when we antisymmetrize over $[a_1a_2b_1]$, when we trace with $\delta^{d_{k-1}d_k}$, and when we trace with $\delta^{c_2d_k}$ fixes everything up to one constant A_0 which can then be fixed by the condition that $\widetilde{\Pi}^2 = \widetilde{\Pi}$. The result is that

$$B_i = \frac{k-2i}{2}A_i, \tag{B.26}$$

$$C_i = -\frac{k-2i+2}{D+k-2}\left[\frac{i\,(D+k-1)}{D+2k-2i} + \frac{1}{2}\right]A_{i-1}, \tag{B.27}$$

$$D_i = \frac{i\,(D+2k)}{D+k-2}A_i, \tag{B.28}$$

$$E_i = -iA_i, \tag{B.29}$$

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while the A_i are given by

$$A_0 = \frac{1}{k+2}, \tag{B.30}$$

and the recursion

$$A_i = -\frac{(k-2i+1)\,(k-2i+2)}{2i\,(D+2k-2i)} A_{i-1}, \tag{B.31}$$

solved by

$$A_i = \left(-\frac{1}{4}\right)^i \frac{k!\,\Gamma(\frac{D}{2}+k-i)}{(k+2)\,i!\,(k-2i)!\,\Gamma(\frac{D}{2}+k)}. \tag{B.32}$$

For $D \leq 4$, the story so far is not quite complete.

In $D = 2$, these mixed symmetry tensors labeled by k are equivalent to spin-k symmetric traceless tensors, with the map

$$\mathcal{A}_{a_1 a_2 b_1 \cdots b_k} = \epsilon_{a_1 a_2} \mathcal{O}_{b_1 \cdots b_k}, \qquad \mathcal{O}_{a_1 \cdots a_k} = \frac{1}{2} \epsilon^{b_1 b_2} \mathcal{A}_{b_1 b_2 a_1 \cdots a_k}. \tag{B.33}$$

In $D = 3$ similarly, there is an isomorphism between mixed symmetry labeled by k and traceless symmetric of spin $k+1$, via

$$\mathcal{A}_{a_1 a_2 b_1 \cdots b_k} = \epsilon_{a_1 a_2}{}^c \mathcal{O}_{b_1 \cdots b_k c} + \frac{k}{2} \left(\epsilon_{a_1 (b_1}{}^c \mathcal{O}_{|a_2| b_2 \cdots b_k) c} - \epsilon_{a_2 (b_1}{}^c \mathcal{O}_{|a_1| b_2 \cdots b_k) c} \right), \tag{B.34}$$

and

$$\mathcal{O}_{a_1 \cdots a_{k+1}} = \frac{1}{k+2} \epsilon_{(a_1}{}^{c_1 c_2} \mathcal{A}_{|c_1 c_2 | a_2 \cdots a_{k+1})}. \tag{B.35}$$

Finally, in $D = 4$ we don't have to worry about any isomorphisms of this sort, but we instead need to recognize that our mixed symmetry representations are in fact reducible. To split the two pieces apart, we can define

$$\Pi^{(\pm)\, a_1 a_2}_{b_1 b_2} = \frac{1}{4} \left(\delta^{a_1}_{b_1} \delta^{a_2}_{b_2} - \delta^{a_1}_{b_2} \delta^{a_2}_{b_1} \pm \epsilon^{a_1 a_2}{}_{b_1 b_2} \right), \tag{B.36}$$

and then define

$$\widetilde{\Pi}^{(k\pm)\, a_1 a_2 b_1 \cdots b_k}_{c_1 c_2 d_1 \cdots d_k} = \Pi^{(\pm)\, a_1 a_2}_{e_1 e_2} \Pi^{(\pm)\, f_1 f_2}_{c_1 c_2} \widetilde{\Pi}^{(k)\, e_1 e_2 b_1 \cdots b_k}_{f_1 f_2 d_1 \cdots d_k}. \tag{B.37}$$

As in the symmetric case, we will need to consider the result of contracting these projectors with vectors X and Y, so we consider the expression

$$X_{c_1} \cdots X_{c_{k+1}} \widetilde{\Pi}^{(k)\, a c_1 \cdots c_{k+1}}_{b d_1 \cdots d_{k+1}} Y^{d_1} \cdots Y^{d_{k+1}}. \tag{B.38}$$

The free indices a and b can only be carried by a Kronecker delta δ^a_b or by the vectors X^a and Y^a. Moreover, the expression must be symmetric under simultaneous interchange of X with Y and a with b, and it must be identically zero when we contract with X_a or with Y^b. These conditions imply that it must have the form

$$\begin{aligned} &X_{c_1} \cdots X_{c_{k+1}} \widetilde{\Pi}^{(k)\, a c_1 \cdots c_{k+1}}_{b d_1 \cdots d_{k+1}} Y^{d_1} \cdots Y^{d_{k+1}} \\ &= \left(X^2 Y^2\right)^{\frac{k+1}{2}} \left[\left(-\delta^a_b + \frac{X^a X_b}{X^2} + \frac{Y^a Y_b}{Y^2} - \frac{X^a Y_b}{\sqrt{X^2 Y^2}} t \right) f_{k-1}(t) + \left(-\delta^a_b t + \frac{Y^a X_b}{\sqrt{X^2 Y^2}} \right) g_k(t) \right], \end{aligned} \tag{B.39}$$

for some polynomials $f_{k-1}(t)$ and $g_k(t)$ of degree $k-1$ and k respectively, with $t = X \cdot Y/\sqrt{X^2Y^2}$ as before. These polynomials can be determined by explicit contraction of (B.25) and use of the solutions for coefficients determined above. The result is

$$f_{k-1}(t) = \frac{1}{2}\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\left(-\frac{1}{4}\right)^i \frac{k!\Gamma(\frac{D}{2}+k-i)}{i!\,(k-2i-1)!\,(D+k-2)\,\Gamma(\frac{D}{2}+k)}t^{k-2i-1}, \tag{B.40}$$

and

$$g_k(t) = -\frac{1}{2}\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\left(-\frac{1}{4}\right)^i \frac{k!\,(D+k-2i-2)\,\Gamma(\frac{D}{2}+k-i)}{i!\,(k-2i)!\,(D+k-2)\,\Gamma(\frac{D}{2}+k)}t^{k-2i}. \tag{B.41}$$

Actually, these can be recast in terms of the polynomials $p_{D,\ell}(t)$ which we defined in the symmetric case (and which are related to the usual Gegenbauer polynomials),

$$f_{k-1}(t) = \frac{1}{2\,(k+1)\,(D+k-2)}p''_{D,k+1}(t) = \frac{k}{2\,(D+k-2)}p_{D+4,k-1}(t), \tag{B.42}$$

$$\begin{aligned} g_k(t) &= -\frac{1}{2\,(k+1)\,(D+k-2)}\left((D-2)\,p'_{D,k+1}(t) + tp''_{D,k+1}(t)\right) \\ &= -\frac{1}{2\,(D+k-2)}\left((D-2)\,p_{D+2,k}(t) + ktp_{D+4,k-1}(t)\right), \end{aligned} \tag{B.43}$$

where a prime denotes the derivative with respect to the argument t.

C Mixed symmetric contractions

In general one expects that the contraction of the projector $\Pi^{[\lambda]}$ associated to some Young symmetry λ is given by

$$X_{f_1}\cdots X_{f_k}\Pi^{[\lambda]e_1\cdots e_n f_1\cdots f_k}_{g_1\cdots g_n h_1\cdots h_k}Y^{h_1}\cdots Y^{h_k} = \sum_i T_i(X,Y)^{e_1\cdots e_n}_{g_1\cdots g_n}\mathcal{P}_i(t), \tag{C.1}$$

where T_i are tensor structures made out of combinations of X, Y, and the Kronecker delta, and $\mathcal{P}$ polynomials on $t \equiv \frac{X\cdot Y}{\sqrt{X^2Y^2}}$. In the previous section we showed this, explicitly, for $[k+1,1]$. More generally, from the work of [51–53], one can understand this expression as the result of a particular differential operator (say in X) acting on the symmetric contraction of λ_1 indices

$$X_{f_1}\cdots X_{f_k}\Pi^{[\lambda]e_1\cdots e_n f_1\cdots f_k}_{g_1\cdots g_n h_1\cdots h_k}Y^{h_1}\cdots Y^{h_k} = \mathcal{D}^{[\lambda]e_1\cdots e_n}_{g_1\cdots g_n}(X)H_{\lambda_1}(X\cdot Y)^{\lambda_1}, \tag{C.2}$$

where

$$H_{\lambda_1}(X\cdot Y)^{\lambda_1} = X_{f_1}\cdots X_{f_{\lambda_1}}\Pi^{[\lambda_1]f_1\cdots f_{\lambda_1}}_{h_1\cdots h_{\lambda_1}}Y^{h_1}\cdots Y^{h_{\lambda_1}} = (X^2Y^2)^{\lambda_1/2}p_{D,\lambda_1}(t), \tag{C.3}$$

and λ_1 is the length of the top row of the Young pattern $[\lambda]$ (in our case this is $k+1$). In the context of conformal blocks, the extra indices e_j are contracted with $m^{(10)}$, $m^{(20)}$, and the indices g_j with $m^{(30)}$, $m^{(40)}$. Furthermore, $X \equiv k^{(012)}$, $Y \equiv k^{(034)}$ with $X^2 = Y^2 = 1$. Thus a generic contraction with T_i

$$m^{10}\cdots m^{10}m^{20}\cdots m^{20}\cdot T_i(k^{(012)},k^{(034)})\cdot m^{(30)}\cdots m^{(30)}m^{(40)}\cdots m^{(40)}, \tag{C.4}$$

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can include combinations of 1- and 2-index elements

$$m^{(i0)}_{ab} k^{(012)\,b}, \quad m^{(i0)}_{ab} k^{(034)\,b}, \quad m^{(i0)}_{ab} m^{(0j)\,b}{}_{c}, \tag{C.5}$$

with $i, j = 1, 2, 3, 4$. The result presented in this paper suggests that one can write the contractions (C.4) as derivatives of t only. For example, for (B.39) we have

$$\begin{aligned} m^{(20)}_{ae} T^{e}_{1\,c} m^{(04)\,c}{}_{b} &= m^{(20)}_{ae} (t\delta^{e}_{c} - k^{(012)}_{c} k^{(034)\,e}) m^{(04)\,c}{}_{b} \\ &= \sqrt{\frac{x^2_{02} x^2_{04} x^2_{12} x^2_{34}}{x^2_{01} x^2_{03}}} \left(t \frac{\partial^2 t}{\partial x^a_2 \partial x^b_4} - \frac{\partial t}{\partial x^a_2} \frac{\partial t}{\partial x^b_4} \right). \end{aligned} \tag{C.6}$$

$$\begin{aligned} & m^{(20)}_{ae} T^{e}_{2\,c} m^{(04)\,c}{}_{b} \\ & = m^{(20)}_{ae} \left((t^2-1)\delta^{e}_{c} + k^{(012)}_{c} k^{(012)\,e} + k^{(034)}_{c} k^{(034)\,e} - t(k^{(012)}_{c} k^{(034)\,e} + k^{(034)}_{c} k^{(012)\,e}) \right) m^{(04)\,c}{}_{b} \\ &= \sqrt{\frac{x^2_{02} x^2_{04} x^2_{12} x^2_{34}}{x^2_{01} x^2_{03}}} \left((t^2-1) \frac{\partial^2 t}{\partial x^a_2 \partial x^b_4} - t \frac{\partial t}{\partial x^a_2} \frac{\partial t}{\partial x^b_4} \right), \end{aligned} \tag{C.7}$$

where we have extracted T_1 and T_2 by rewriting (B.39) as

$$- T^{a}_{1\,b} \, (t f_{k-1} + g_k) + T^{a}_{2\,b} f_{k-1}(t), \tag{C.8}$$

and we picked the particular combinations because f_{k-1} and $t f_{k-1} + g_k$ are just constant multiples of $p_{D+4,k-1}$ and $p_{D+2,k}$ respectively. These results rely on the fact that

$$\frac{\partial k^{(0ij)}_{c}}{\partial x^a_j} = - \sqrt{\frac{x^2_{0i}}{x^2_{0j} x^2_{ij}}} \left(m^{(0j)}_{ac} + k^{(j0i)}_{a} k^{(0ij)}_{c} \right), \tag{C.9}$$

and the key observation is that the particular combinations of δ, $k^{(012)}$, $k^{(034)}$, that appear in T_i, are such that the terms $k^{(j0i)}_{a}$ cancel out, leaving only the terms $m^{(0j)}$ that we want. This leads to

$$\begin{aligned} & m^{(20)}_{ac} k^{(012)}_{d_1} \cdots k^{(012)}_{d_{k+1}} \widetilde{\Pi}^{(k)\,c d_1 \cdots d_{k+1}}_{e f_1 \cdots f_{k+1}} m^{(04)\,e}{}_{b} k^{(034)\,f_1} \cdots k^{(034)\,f_{k+1}} \\ & \quad = \sqrt{\frac{x^2_{02} x^2_{04} x^2_{12} x^2_{34}}{x^2_{01} x^2_{03}}} \frac{1}{2(D+k-2)} \left(k \left((t^2-1) \frac{\partial^2 t}{\partial x^a_2 \partial x^b_4} - t \frac{\partial t}{\partial x^a_2} \frac{\partial t}{\partial x^b_4} \right) p_{D+4,k-1} \right. \\ & \qquad \left. + (D-2) \left(t \frac{\partial^2 t}{\partial x^a_2 \partial x^b_4} - \frac{\partial t}{\partial x^a_2} \frac{\partial t}{\partial x^b_4} \right) p_{D+2,k} \right). \end{aligned} \tag{C.10}$$

Equation (4.76) then follows from the chain rule and simple Gegenbauer identities listed in appendix B.

As an extra result, we present the contraction under the projector $\Pi^{[k+1,1,1]}$ associated to the Young pattern $[k+1, 1, 1]$. Using techniques from [51–53] one obtains

$$\mathcal{P}^{[k+1,1,1]e_1 e_2}_{c_1 c_2} \equiv X_{f_1} \cdots X_{f_{k+1}} \Pi^{[k+1,1,1]e_1 e_2 f_1 \cdots f_{k+1}}_{c_1 c_2 d_1 \cdots d_{k+1}} Y^{d_1} \cdots Y^{d_{k+1}} \tag{C.11}$$

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$$\propto \frac{(k+1)(X^2Y^2)^{\frac{k+1}{2}}}{(k+3)(D+k-3)}\delta_{c_1}^{[d}\delta_{c_2}^{f]}\delta_{[g}^{e_1}\delta_{h]}^{e_2}\left((D-3)\delta_d^g\left[t\delta_f^h-2\frac{X_fY^h}{\sqrt{X^2Y^2}}\right]p_{D+2,k}(t)\right.$$
$$\left.+2k\left[\delta_d^g\left(\frac{(t^2-1)\delta_f^h}{2}+\frac{X_fX^h}{X^2}-\frac{t\left(X_fY^h+Y_fX^h\right)}{\sqrt{X^2Y^2}}+\frac{Y_fY^h}{Y^2}\right)-\frac{X_dY_fX^gY^h}{X^2Y^2}\right]p_{D+4,k-1}\right).$$

Thus from the previous discussion one finds that

$$m_{ae_1}^{(10)}m_{be_2}^{(20)}\mathcal{P}_{c_1c_2}^{[k+1,1,1]e_1e_2}m_c^{(30)\,c_1}m_d^{(40)\,c_2}$$
$$=\frac{x_{02}^2x_{04}^2x_{12}^2x_{34}^2}{x_{01}^2x_{03}^2}m_{ab'}^{(12)}m_{cd'}^{(34)}\left\{\frac{\partial^2t}{\partial x_2^b\partial x_4^{[d|}}\frac{\partial^2t}{\partial x_2^{b'}\partial x_4^{|d']}}p_{D,k+1}-2\frac{\partial^2t}{\partial x_2^{[b|}\partial x_4^{[d|}}\frac{\partial^2}{\partial x_2^{|b']}\partial x_4^{|d']}}\right.$$
$$\left.\times\left[\frac{1}{(k+2)(k+3)}p_{D,k+2}+\frac{(k+1)}{(D+2k)(D+2k-2)(D+k-3)}p_{D,k}\right]\right\},\quad\text{(C.12)}$$

where in the second term, the square bracket notation is indicating that b (d) is antisymmetrized with b' (d').

This is one of the two new contractions that appear in the conformal blocks for the $[k+1,1,1]$ exchange in $\langle VVVV\rangle$. Those conformal blocks are left for future work.

D Integrals

Much of the material in this appendix follows [45].

The basic building block for our integrals is

$$\int d^Dx_0\left(\sum_i a_ix_{0i}^2\right)^{-D}=\frac{2^{1-D}\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})}\left(\sum_{i<j}a_ia_jx_{ij}^2\right)^{-D/2},\quad\text{(D.1)}$$

along with the Feynman-Schwinger trick which uses the identity

$$\frac{1}{\prod_{i=1}^n X_i^{c_i}}=\frac{\Gamma(\sum_{i=1}^n c_i)}{\prod_{j=1}^n\Gamma(c_j)}\left(\prod_{k=2}^n\int_0^\infty d\mu_k\,\mu^{c_k-1}\right)\frac{1}{(X_1+\sum_{\ell=2}^n\mu_\ell X_\ell)^{\sum_{m=1}^n c_m}}.\quad\text{(D.2)}$$

D.1 Three-point integrals

Suppose $\alpha+\beta+\gamma=D$. Then the integral

$$I_{\alpha,\beta,\gamma}(x_1,x_2,x_3)=\int\frac{d^Dx_0}{\left(x_{01}^2\right)^\alpha\left(x_{02}^2\right)^\beta\left(x_{03}^2\right)^\gamma},\quad\text{(D.3)}$$

will be a conformal scalar of weight α, β, and γ under conformal transformations of x_1, x_2, and x_3 respectively. To evaluate the integral, we first use (D.2) and then (D.1) to write

$$I_{\alpha,\beta,\gamma}(x_1,x_2,x_3)=\frac{\Gamma(D)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int d^Dx_0\int_0^\infty ds\,s^{\beta-1}\int_0^\infty dt\,t^{\gamma-1}\frac{1}{\left(x_{01}^2+sx_{02}^2+tx_{03}^2\right)^D}$$
$$=\frac{2^{1-D}\pi^{\frac{D+1}{2}}\Gamma(D)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\frac{D+1}{2})}\int_0^\infty ds\,s^{\beta-1}\int_0^\infty dt\,t^{\gamma-1}\left(sx_{12}^2+tx_{13}^2+stx_{23}^2\right)^{-D/2}.\quad\text{(D.4)}$$

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To perform the remaining integrals, we recall one of the representations of the beta function

$$\int_0^\infty du \frac{u^{x-1}}{(1+u)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{D.5}$$

Then

$$\begin{aligned}
&I_{\alpha,\beta,\gamma}(x_1,x_2,x_3)\\
&= \frac{2^{1-D}\pi^{\frac{D+1}{2}}\Gamma(D)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\frac{D+1}{2})}\int_0^\infty ds\, s^{\beta-1}\left(sx_{12}^2\right)^{-D/2}\int_0^\infty dt\, t^{\gamma-1}\left(1+t\left(\frac{x_{13}^2+sx_{23}^2}{sx_{12}^2}\right)\right)^{-D/2}\\
&= \frac{2^{1-D}\pi^{\frac{D+1}{2}}\Gamma(D)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\frac{D+1}{2})}\left(x_{12}^2\right)^{\gamma-\frac{D}{2}}\int_0^\infty ds\, s^{\beta+\gamma-\frac{D}{2}-1}\left(x_{13}^2+sx_{23}^2\right)^{-\gamma}\int_0^\infty du\, u^{\gamma-1}(1+u)^{-D/2}\\
&= \frac{2^{1-D}\pi^{\frac{D+1}{2}}\Gamma(D)\Gamma(\frac{D}{2}-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\frac{D+1}{2})\Gamma(\frac{D}{2})}\left(x_{12}^2\right)^{\gamma-\frac{D}{2}}\left(x_{13}^2\right)^{-\gamma}\int_0^\infty ds\, s^{\beta+\gamma-\frac{D}{2}-1}\left(1+s\frac{x_{23}^2}{x_{13}^2}\right)^{-\gamma}\\
&= \frac{\pi^{D/2}\Gamma(\frac{D}{2}-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\left(x_{12}^2\right)^{\gamma-\frac{D}{2}}\left(x_{13}^2\right)^{\beta-\frac{D}{2}}\left(x_{23}^2\right)^{\frac{D}{2}-\beta-\gamma}\int_0^\infty dv\, v^{\beta+\gamma-\frac{D}{2}-1}(1+v)^{-\gamma}\\
&= \frac{\pi^{D/2}\Gamma(\frac{D}{2}-\gamma)\Gamma(\beta+\gamma-\frac{D}{2})\Gamma(\frac{D}{2}-\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\left(x_{12}^2\right)^{\gamma-\frac{D}{2}}\left(x_{13}^2\right)^{\beta-\frac{D}{2}}\left(x_{23}^2\right)^{\frac{D}{2}-\beta-\gamma}\\
&= \pi^{D/2}\frac{\Gamma(\frac{D}{2}-\alpha)\Gamma(\frac{D}{2}-\beta)\Gamma(\frac{D}{2}-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\left(x_{12}^2\right)^{\gamma-\frac{D}{2}}\left(x_{13}^2\right)^{\beta-\frac{D}{2}}\left(x_{23}^2\right)^{\alpha-\frac{D}{2}},
\end{aligned} \tag{D.6}$$

where we have also made use of the duplication formula for the gamma function, which in this case tells us

$$\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D+1}{2}\right) = 2^{1-D}\sqrt{\pi}\Gamma(D). \tag{D.7}$$

Similarly, we will need to evaluate

$$I_{\alpha\beta\gamma;a_1\cdots a_n}(x_1,x_2,x_3) = \Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n}\int\frac{d^Dx_0}{\left(x_{01}^2\right)^\alpha\left(x_{02}^2\right)^\beta\left(x_{03}^2\right)^\gamma}k^{(302)}_{b_1}\cdots k^{(302)}_{b_n}, \tag{D.8}$$

which for $\alpha+\beta+\gamma = D$ will be a conformal scalar of weight α (β) under conformal transformations of x_1 (x_2), and a traceless symmetric tensor of conformal weight γ under transformations of x_3. We compute by doing a binomial expansion of the $k^{(302)}$'s,

$$\begin{aligned}
&I_{\alpha\beta\gamma;a_1\cdots a_n}(x_1,x_2,x_3) = \Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n}\sum_{k=0}^n\frac{n!}{k!\,(n-k)!}(-1)^k\left(x_{23}^2\right)^{\frac{n-2k}{2}}(x_{23})_{b_1}\cdots(x_{23})_{b_k}\\
&\quad\times\int\frac{d^Dx_0}{\left(x_{01}^2\right)^\alpha\left(x_{02}^2\right)^{\beta+\frac{n}{2}}\left(x_{03}^2\right)^{\gamma+\frac{n-2k}{2}}}(x_{03})_{b_{k+1}}\cdots(x_{03})_{b_n}\\
&= \Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n}\sum_{k=0}^n\frac{n!}{k!\,(n-k)!}(-1)^k\left(x_{23}^2\right)^{\frac{n-2k}{2}}(x_{23})_{b_1}\cdots(x_{23})_{b_k}\\
&\quad\times\frac{\Gamma(\gamma-\frac{n}{2})}{2^{n-k}\Gamma(\gamma+\frac{n}{2}-k)}\frac{\partial}{\partial x_3^{b_{k+1}}}\cdots\frac{\partial}{\partial x_3^{b_n}}I_{\alpha,\beta+\frac{n}{2},\gamma-\frac{n}{2}}(x_1,x_2,x_3)\\
&= \pi^{D/2}\Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n}\sum_{k=0}^n\sum_{m=0}^{n-k}\frac{n!}{k!m!\,(n-k-m)!}(-1)^k\\
&\quad\times\frac{\Gamma(\frac{D}{2}+n-m-k-\alpha)\Gamma(\frac{D}{2}-\frac{n}{2}+m-\beta)\Gamma(\frac{D}{2}+\frac{n}{2}-\gamma)}{\Gamma(\alpha)\Gamma(\beta+\frac{n}{2})\Gamma(\gamma+\frac{n}{2}-k)}
\end{aligned}$$

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$$
\begin{aligned}
&\times \left(x_{12}^2\right)^{\gamma-\frac{n}{2}-\frac{D}{2}} \left(x_{13}^2\right)^{\beta+\frac{n}{2}-m-\frac{D}{2}} \left(x_{23}^2\right)^{\alpha-\frac{n}{2}+m-\frac{D}{2}} (x_{13})_{b_1}\cdots(x_{13})_{b_m}(x_{23})_{b_{m+1}}\cdots(x_{23})_{b_n} \\
&= \pi^{D/2}\Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n} \sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!}(-1)^{n-m} \frac{\Gamma(\frac{D}{2}-\alpha)\Gamma(\frac{D}{2}+\frac{n}{2}-\beta)\Gamma(\frac{D}{2}+\frac{n}{2}-\gamma)}{\Gamma(\alpha)\Gamma(\beta+\frac{n}{2})\Gamma(\gamma+\frac{n}{2})} \\
&\times \left(x_{12}^2\right)^{\gamma-\frac{n}{2}-\frac{D}{2}} \left(x_{13}^2\right)^{\beta+\frac{n}{2}-m-\frac{D}{2}} \left(x_{23}^2\right)^{\alpha-\frac{n}{2}+m-\frac{D}{2}} (x_{13})_{b_1}\cdots(x_{13})_{b_m}(x_{23})_{b_{m+1}}\cdots(x_{23})_{b_n} \\
&= \pi^{D/2}\Pi^{(n)\,b_1\cdots b_n}_{a_1\cdots a_n} \frac{\Gamma(\frac{D}{2}-\alpha)\Gamma(\frac{D}{2}+\frac{n}{2}-\beta)\Gamma(\frac{D}{2}+\frac{n}{2}-\gamma)}{\Gamma(\alpha)\Gamma(\beta+\frac{n}{2})\Gamma(\gamma+\frac{n}{2})} \\
&\times \left(x_{12}^2\right)^{\gamma-\frac{D}{2}} \left(x_{13}^2\right)^{\beta-\frac{D}{2}} \left(x_{23}^2\right)^{\alpha-\frac{D}{2}} k_{b_1}^{(312)}\cdots k_{b_n}^{(312)},
\end{aligned}
\tag{D.9}
$$

where we used the identity

$$
\sum_{k=0}^{N} \frac{N!}{k!\,(N-k)!}(-1)^k \frac{\Gamma(x-k)}{\Gamma(y-k)} = (-1)^N \frac{\Gamma(x-N)\Gamma(y-x+N)}{\Gamma(y)\Gamma(y-x)},
\tag{D.10}
$$

with $N = n-m$, $x = \frac{D}{2}+n-m-\alpha$, and $y = \gamma+\frac{n}{2}$.

We will also need one more result along these lines,

$$
\begin{aligned}
I_{\alpha,\beta,\gamma;a;b_1\cdots b_n} &= \Pi^{(n)\,c_1\cdots c_n}_{b_1\cdots b_n} \int \frac{d^Dx_0}{\left(x_{01}^2\right)^{\alpha}\left(x_{02}^2\right)^{\beta}\left(x_{03}^2\right)^{\gamma}} k_a^{(203)} k_{c_1}^{(302)}\cdots k_{c_n}^{(302)} \\
&= \Pi^{(n)\,c_1\cdots c_n}_{b_1\cdots b_n} \left\{ \frac{\sqrt{x_{23}^2}}{2\beta+n-1}\frac{\partial}{\partial x_2^a} I_{\alpha,\beta-\frac{1}{2},\gamma+\frac{1}{2};c_1\cdots c_n} + \frac{2\beta-1}{2\beta+n-1}\frac{(x_{23})_a}{\sqrt{x_{23}^2}} I_{\alpha,\beta-\frac{1}{2},\gamma+\frac{1}{2};c_1\cdots c_n} \right. \\
&\quad \left. + \frac{n m^{(23)}_{ac_1}}{2\beta+n-1} I_{\alpha,\beta,\gamma;c_2\cdots c_n} \right\} \\
&= \pi^{D/2}\Pi^{(n)\,c_1\cdots c_n}_{b_1\cdots b_n} \frac{\Gamma(\frac{D}{2}-\alpha)\Gamma(\frac{D}{2}+\frac{n-1}{2}-\beta)\Gamma(\frac{D}{2}+\frac{n-1}{2}-\gamma)}{\Gamma(\alpha)\Gamma(\beta+\frac{n+1}{2})\Gamma(\gamma+\frac{n+1}{2})} \left(x_{12}^2\right)^{\gamma-\frac{D}{2}} \left(x_{13}^2\right)^{\beta-\frac{D}{2}} \left(x_{23}^2\right)^{\alpha-\frac{D}{2}} \\
&\quad \times \left[\left(\frac{D}{2}+\frac{n-1}{2}-\beta\right)\left(\frac{D}{2}+\frac{n-1}{2}-\gamma\right) k_a^{(213)} k_{c_1}^{(312)} + \frac{n}{2}\left(\frac{D}{2}-\alpha\right) m^{(23)}_{ac_1} \right] k_{c_2}^{(312)}\cdots k_{c_n}^{(312)}.
\end{aligned}
\tag{D.11}
$$

In this case we made use of (A.10).

D.2 Four-point integrals

As with the previous section, we start with integrals of the form

$$
I_{\alpha,\beta,\gamma,\delta}(x_1,x_2,x_3,x_4) = \int \frac{d^Dx_0}{\left(x_{01}^2\right)^{\alpha}\left(x_{02}^2\right)^{\beta}\left(x_{03}^2\right)^{\gamma}\left(x_{04}^2\right)^{\delta}},
\tag{D.12}
$$

where $\alpha+\beta+\gamma+\delta = D$. Using (D.1) and (D.2) we can show

$$
\begin{aligned}
I_{\alpha,\beta,\gamma,\delta} = \frac{2^{1-D}\pi^{\frac{D+1}{2}}\Gamma(D)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)\Gamma(\frac{D+1}{2})} \int_0^\infty ds\, s^{\beta-1}\int_0^\infty dt\, t^{\gamma-1}\int_0^\infty dq\, q^{\delta-1} \\
\times \left(sx_{12}^2+tx_{13}^2+qx_{14}^2+stx_{23}^2+sqx_{24}^2+tqx_{34}^2\right)^{-D/2}.
\end{aligned}
\tag{D.13}
$$

After a change of variables we can do one of the three integrals, giving us a result

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$$I_{\alpha,\beta,\gamma,\delta} = \pi^{D/2}\frac{\Gamma(\frac{D}{2}-\delta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\left(x_{14}^2\right)^{-\alpha}\left(x_{23}^2\right)^{\delta-\frac{D}{2}}\left(x_{24}^2\right)^{\frac{D}{2}-\beta-\delta}\left(x_{34}^2\right)^{\frac{D}{2}-\gamma-\delta} \times \widehat{f}_{\alpha,\beta,\gamma,\delta}(uv^{-1},v^{-1}), \quad \text{(D.14)}$$

where we have defined

$$\widehat{f}_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = \int_0^\infty ds\, s^{\beta-1}\int_0^\infty dt\, t^{\gamma-1}\left(sz_1+tz_2+st\right)^{\frac{\delta-\alpha-\beta-\gamma}{2}}(1+s+t)^{-\delta}, \quad \text{(D.15)}$$

and u and v are the usual invariant cross-ratios defined in (2.27).

As explained in section 4.4.1, the monodromy projection requires us to keep only the terms in $\widehat{f}_{\alpha,\beta,\gamma,\delta}(z_1,z_2)$ which are invariant under $z_1 \to e^{4\pi i}z_1$. In [45] it is shown how to do this very elegantly using contour deformation arguments, with the result that the invariant pieces are given precisely by

$$\begin{aligned} f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) &= \widehat{f}_{\alpha,\beta,\gamma,\delta}(z_1,z_2)\Big|_{\text{monodromy}-\text{invariant}} \quad \text{(D.16)}\\ &= \frac{\sin(\pi\delta)}{\sin(\frac{\pi}{2}(\gamma+\delta-\alpha-\beta))}\int_0^\infty ds\, s^{\beta-1}\int_{s+1}^\infty dt\, t^{\gamma-1}\left(st+tz_2-sz_1\right)^{\frac{\delta-\alpha-\beta-\gamma}{2}}(t-s-1)^{-\delta}. \end{aligned}$$

The function $f_{\alpha,\beta,\gamma,\delta}(z_1,z_2)$ obeys several easily verified identities (also $\widehat{f}$ obeys the same identities),

$$\frac{\partial}{\partial z_1}f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = \frac{\delta-\alpha-\beta-\gamma}{2}f_{\alpha+1,\beta+1,\gamma,\delta}(z_1,z_2), \quad \text{(D.17)}$$

$$\frac{\partial}{\partial z_2}f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = \frac{\delta-\alpha-\beta-\gamma}{2}f_{\alpha+1,\beta,\gamma+1,\delta}(z_1,z_2), \quad \text{(D.18)}$$

as well as

$$f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = f_{\alpha+1,\beta,\gamma,\delta+1}(z_1,z_2) + f_{\alpha,\beta+1,\gamma,\delta+1}(z_1,z_2) + f_{\alpha,\beta,\gamma+1,\delta+1}(z_1,z_2), \quad \text{(D.19)}$$

and

$$f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = f_{\alpha,\beta+1,\gamma+1,\delta}(z_1,z_2) + z_1 f_{\alpha+1,\beta+1,\gamma,\delta}(z_1,z_2) + z_2 f_{\alpha+1,\beta,\gamma+1,\delta}(z_1,z_2). \quad \text{(D.20)}$$

When $\alpha+\beta+\gamma+\delta$ is an even integer, which we will call $2h$ (so $h = D/2$ in the four-point integral above, and this would be valid in even dimensions), then $f_{\alpha,\beta,\gamma,\delta}$ can actually be evaluated explicitly in terms of hypergeometric functions. First we change from z_1 and z_2 to a complex variable x related by

$$z_1 = \frac{x\bar{x}}{(1-x)(1-\bar{x})}, \qquad z_2 = \frac{1}{(1-x)(1-\bar{x})}, \quad \text{(D.21)}$$

and then it can be shown [41, 45] that

$$\begin{aligned} f_{\alpha,\beta,\gamma,\delta}(z_1,z_2) = {}& \frac{\Gamma(\alpha)\Gamma(1-h+\beta)\Gamma(1-\delta)\Gamma(h-\gamma)\Gamma(\gamma+\delta-h)}{\Gamma(\delta)\Gamma(h-\delta)\Gamma(1+h-\gamma-\delta)}\left((1-x)(1-\bar{x})\right)^{h-\delta} \\ &\times\left(\frac{1}{x-\bar{x}}\left(x\partial_x-\bar{x}\partial_{\bar{x}}\right)\right)^{h-1}\left[{}_2F_1(1-h+\beta,1-\delta,1+h-\gamma-\delta;x)\right. \\ &\left.\times{}_2F_1(1-h+\beta,1-\delta,1+h-\gamma-\delta;\bar{x})\right]. \quad \text{(D.22)} \end{aligned}$$

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E Mixing matrices and normalization factors

For the case of two scalars and a symmetric traceless tensor, inserting (3.1) into (3.2) leads to

$$\left\langle \phi_1(x_1)\phi_2(x_2)\widetilde{\mathcal{O}}_{a_1\cdots a_\ell}(x_3)\right\rangle = \Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}\int \frac{d^Dx_0}{(x_{03}^2)^{D-\Delta_\mathcal{O}}} m^{(03)}{}_{b_1}^{\ c_1}\cdots m^{(03)}{}_{b_\ell}^{\ c_\ell} \tag{E.1}$$
$$\times\left(\lambda_\mathcal{O}\,(x_{01}^2)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_\mathcal{O})}\,(x_{02}^2)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_\mathcal{O})}\,(x_{12}^2)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_\mathcal{O})}\,\Pi^{(\ell)\,d_1\cdots d_\ell}_{c_1\cdots c_\ell}k^{(012)}_{d_1}\cdots k^{(012)}_{d_\ell}\right).$$

Since (as reviewed in appendix A) $m^{(03)}{}_a^{\ c}m^{(03)}_{bc} = \delta_{ab}$, and since $\Pi^{(\ell)}$ removes traces, it follows that

$$\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}m^{(03)}{}_{b_1}^{\ c_1}\cdots m^{(03)}{}_{b_\ell}^{\ c_\ell}\Pi^{(\ell)\,d_1\cdots d_\ell}_{c_1\cdots c_\ell}k^{(012)}_{d_1}\cdots k^{(012)}_{d_\ell} = \Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}y_{b_1}\cdots y_{b_\ell}, \tag{E.2}$$

where

$$y_a = m^{(03)}_{ab}k^{(012)\,b} = \left(\sqrt{\frac{x_{01}^2x_{23}^2}{x_{03}^2x_{12}^2}} - \frac{x_{02}^2x_{13}^2}{\sqrt{x_{01}^2x_{03}^2x_{12}^2x_{23}^2}}\right)k_a^{(302)} + \sqrt{\frac{x_{02}^2x_{13}^2}{x_{01}^2x_{23}^2}}k_a^{(312)}. \tag{E.3}$$

Expanding in a trinomial expansion, we then obtain

$$\begin{aligned}
&\left\langle \phi_1(x_1)\phi_2(x_2)\widetilde{\mathcal{O}}_{a_1\cdots a_\ell}(x_3)\right\rangle\\
&= \lambda_\mathcal{O}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}\sum_{k=0}^{\ell}\sum_{m=0}^{\ell-k}\frac{\ell!}{k!m!\,(\ell-k-m)!}(-1)^m\\
&\quad\times(x_{12}^2)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_\mathcal{O}-k-m)}(x_{13}^2)^{\frac{1}{2}(\ell-k+m)}(x_{23}^2)^{k-\frac{\ell}{2}}k^{(312)}_{b_1}\cdots k^{(312)}_{b_{\ell-k-m}}\\
&\quad\times I_{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_\mathcal{O}+\ell-2k),\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_\mathcal{O}-\ell+k-m),D-\Delta_\mathcal{O}+\frac{k+m}{2};b_{\ell-k-m+1}\cdots b_\ell}(x_1,x_2,x_3)\\
&= \pi^{D/2}\lambda_\mathcal{O}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}k^{(312)}_{b_1}\cdots k^{(312)}_{b_\ell}\sum_{k=0}^{\ell}\sum_{m=0}^{\ell-k}\frac{\ell!}{k!m!\,(\ell-k-m)!}(-1)^m\\
&\quad\times\frac{\Gamma(\frac{1}{2}(D-\Delta_1+\Delta_2-\Delta_\mathcal{O}-\ell)+k)\Gamma(\frac{1}{2}(D+\Delta_1-\Delta_2-\Delta_\mathcal{O}+\ell)+m)\Gamma(\Delta_\mathcal{O}-\frac{D}{2})}{\Gamma(\frac{1}{2}(\Delta_1-\Delta_2+\Delta_\mathcal{O}+\ell)-k)\Gamma(\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_\mathcal{O}-\ell)+k)\Gamma(D-\Delta_\mathcal{O}+k+m)}\\
&\quad\times(x_{12}^2)^{\frac{1}{2}(D-\Delta_1-\Delta_2-\Delta_\mathcal{O})}(x_{13}^2)^{\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_\mathcal{O}-D)}(x_{23}^2)^{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_\mathcal{O}-D)}\\
&= \pi^{D/2}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}k^{(312)}_{b_1}\cdots k^{(312)}_{b_\ell}(x_{12}^2)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_{\tilde{\mathcal{O}}})}(x_{13}^2)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_{\tilde{\mathcal{O}}})}(x_{23}^2)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_{\tilde{\mathcal{O}}})}\\
&\quad\times\frac{\Gamma(\Delta_\mathcal{O}-\frac{D}{2})\Gamma(\Delta_\mathcal{O}+\ell-1)}{\Gamma(\Delta_\mathcal{O}-1)\Gamma(D-\Delta_\mathcal{O}+\ell)}\\
&\quad\times\frac{\Gamma(\frac{1}{2}(D+\Delta_1-\Delta_2-\Delta_\mathcal{O}+\ell))\Gamma(\frac{1}{2}(D-\Delta_1+\Delta_2-\Delta_\mathcal{O}+\ell))}{\Gamma(\frac{1}{2}(\Delta_1-\Delta_2+\Delta_\mathcal{O}+\ell))\Gamma(\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_\mathcal{O}+\ell))}\lambda_\mathcal{O},
\end{aligned} \tag{E.4}$$

where we use the notation and results for integrals defined in appendix D.1, and we evaluated the sums, first over m and then over k, using the identities

$$\sum_{k=0}^{N}\frac{N!}{k!\,(N-k)!}(-1)^k\frac{\Gamma(x+k)}{\Gamma(y+k)} = \frac{\Gamma(x)\Gamma(y-x+N)}{\Gamma(y+N)\Gamma(y-x)}, \tag{E.5}$$

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which is equivalent to (D.10), and

$$\sum_{k=0}^{N}\frac{N!}{k!\,(N-k)!}\frac{1}{\Gamma(x+k)\Gamma(y-k)}=\frac{\Gamma(x+y+N-1)}{\Gamma(x+N)\Gamma(y)\Gamma(x+y-1)}. \tag{E.6}$$

Thus comparing with (3.3) one can read off (3.4).

Now for a scalar, a vector, and a traceless symmetric tensor we have

$$\begin{aligned}
&\left\langle\phi(x_1)v_a(x_2)\widetilde{\mathcal{O}}_{b_1\cdots b_\ell}(x_3)\right\rangle \\
&=\Pi^{(\ell)\,c_1\cdots c_\ell}_{b_1\cdots b_\ell}\int\frac{d^Dx_0}{(x_{03}^2)^{D-\Delta_\mathcal{O}}}m^{(03)}{}_{c_1}^{\ d_1}\cdots m^{(03)}{}_{c_\ell}^{\ d_\ell}\left(x_{01}^2\right)^{\frac{1}{2}(-\Delta_\phi+\Delta_v-\Delta_\mathcal{O})}\left(x_{02}^2\right)^{\frac{1}{2}(\Delta_\phi-\Delta_v-\Delta_\mathcal{O})} \\
&\quad\times\left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_\phi-\Delta_v+\Delta_\mathcal{O})}\Pi^{(\ell)\,e_1\cdots e_\ell}_{d_1\cdots d_\ell}\left[-\alpha_\mathcal{O}k_a^{(201)}k_{e_1}^{(012)}+\beta_\mathcal{O}m_{ae_1}^{(20)}\right]k_{e_2}^{(012)}\cdots k_{e_\ell}^{(012)} \\
&=\Pi^{(\ell)\,c_1\cdots c_\ell}_{b_1\cdots b_\ell}\left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_\phi-\Delta_v+\Delta_\mathcal{O})}\int d^Dx_0\left(x_{01}^2\right)^{\frac{1}{2}(-\Delta_\phi+\Delta_v-\Delta_\mathcal{O})}\left(x_{02}^2\right)^{\frac{1}{2}(\Delta_\phi-\Delta_v-\Delta_\mathcal{O})}\left(x_{03}^2\right)^{\Delta_\mathcal{O}-D} \\
&\quad\times\left[-\alpha_\mathcal{O}\left(\sqrt{\frac{x_{03}^2x_{12}^2}{x_{01}^2x_{23}^2}}k_a^{(203)}-\sqrt{\frac{x_{02}^2x_{13}^2}{x_{01}^2x_{23}^2}}k_a^{(213)}\right)y_{c_1}+\beta_\mathcal{O}\left(m_{ac_1}^{(23)}-2k_a^{(203)}k_{c_1}^{(302)}\right)\right]y_{c_2}\cdots y_{c_\ell},
\end{aligned} \tag{E.7}$$

using identities from appendix A.

We then proceed as before, performing trinomial expansions on the y_a's, perform the integrals using the results of appendix D.1, and the identities (E.5) and (E.6). This results in (3.5) and (3.6).

Related to these integration techniques is the determination of the normalization factor $\mathcal{N}_\mathcal{O}$ that appears in the shadow projector $P_\mathcal{O}$. As discussed in the main text, this is fixed by requiring

$$\begin{aligned}
&\langle\varphi_1(x_1)\varphi_2(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle=\langle\mathcal{O}_{a_1\cdots a_\ell}(x_3)P_\mathcal{O}\varphi_1(x_1)\varphi_2(x_2)\rangle \\
&=\mathcal{N}_\mathcal{O}\int d^Dx_0\,\langle\mathcal{O}_{a_1\cdots a_\ell}(x_3)\mathcal{O}_{b_1\cdots b_\ell}(x_0)\rangle\left\langle\widetilde{\mathcal{O}}^{b_1\cdots b_\ell}(x_0)\varphi_1(x_1)\varphi_2(x_2)\right\rangle \\
&=\mathcal{N}_\mathcal{O}\lambda_{12\widetilde{\mathcal{O}}}\Pi^{(\ell)\,c_1\cdots c_\ell}_{b_1\cdots b_\ell}\Pi^{(\ell)\,d_1\cdots d_\ell}_{a_1\cdots a_\ell}\int d^Dx_0\left(x_{12}^2\right)^{\frac{1}{2}(D-\Delta_1-\Delta_2-\Delta_\mathcal{O})}\left(x_{01}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_\mathcal{O}-D)} \\
&\quad\times\left(x_{02}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_\mathcal{O}-D)}k_{c_1}^{(012)}\cdots k_{c_\ell}^{(012)}\left(x_{03}^2\right)^{-\Delta_\mathcal{O}}m^{(03)}{}_{d_1}^{\ b_1}\cdots m^{(03)}{}_{d_\ell}^{\ b_\ell} \\
&=\mathcal{N}_\mathcal{O}\lambda_{12\widetilde{\mathcal{O}}}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}\left(x_{12}^2\right)^{\frac{1}{2}(D-\Delta_1-\Delta_2-\Delta_\mathcal{O})}\int d^Dx_0\left(x_{01}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_\mathcal{O}-D)} \\
&\quad\times\left(x_{02}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2+\Delta_\mathcal{O}-D)}\left(x_{03}^2\right)^{-\Delta_\mathcal{O}}y_{b_1}\cdots y_{b_\ell} \\
&=\mathcal{N}_\mathcal{O}\lambda_{12\widetilde{\mathcal{O}}}\Pi^{(\ell)\,b_1\cdots b_\ell}_{a_1\cdots a_\ell}\left(x_{12}^2\right)^{\frac{1}{2}(-\Delta_1-\Delta_2+\Delta_\mathcal{O})}\left(x_{13}^2\right)^{\frac{1}{2}(-\Delta_1+\Delta_2-\Delta_\mathcal{O})}\left(x_{23}^2\right)^{\frac{1}{2}(\Delta_1-\Delta_2-\Delta_\mathcal{O})} \\
&\quad\times k_{b_1}^{(312)}\cdots k_{b_\ell}^{(312)}\frac{\Gamma(\frac{D}{2}-\Delta_\mathcal{O})}{\Gamma(D-\Delta_\mathcal{O}-1)}\frac{\Gamma(D-\Delta_\mathcal{O}+\ell-1)}{\Gamma(\Delta_\mathcal{O}+\ell)} \\
&\quad\times\frac{\Gamma(\frac{1}{2}(\Delta_1-\Delta_2+\Delta_\mathcal{O}+\ell))}{\Gamma(\frac{1}{2}(D+\Delta_1-\Delta_2-\Delta_\mathcal{O}+\ell))}\frac{\Gamma(\frac{1}{2}(-\Delta_1+\Delta_2+\Delta_\mathcal{O}+\ell))}{\Gamma(\frac{1}{2}(D-\Delta_1+\Delta_2-\Delta_\mathcal{O}+\ell))} \\
&=\langle\varphi_1(x_1)\varphi_2(x_2)\mathcal{O}_{a_1\cdots a_\ell}(x_3)\rangle\,\mathcal{N}_\mathcal{O}\pi^D \\
&\quad\times\frac{\Gamma(\Delta_\mathcal{O}-\frac{D}{2})\Gamma(\frac{D}{2}-\Delta_\mathcal{O})}{(\Delta_\mathcal{O}+\ell-1)\,(D-\Delta_\mathcal{O}+\ell-1)\,\Gamma(\Delta_\mathcal{O}-1)\Gamma(D-\Delta_\mathcal{O}-1)},
\end{aligned} \tag{E.8}$$

where we read off (3.12).

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F $\alpha\beta$, $\beta\alpha$, and $\beta\beta$ components of the $\langle SVSV\rangle$ blocks

Here we write the additional conformal block components that appear for $\ell > 0$. These are expressed in condensed notation where the blocks on the l.h.s., as well as the $\alpha\alpha$ blocks on the right-hand-sides have unshifted arguments, $g_p^{rs}(u,v;\Delta_1,\Delta_2,\Delta_3,\Delta_4;\ell,\Delta_{\mathcal{O}})$. The others follow the conventions in the main text

$$\begin{aligned}
g_0^{\alpha\beta} &= \frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_0^{\alpha\alpha}+\frac{1}{\ell}\left[\sqrt{u}g_{1;\ell;0,1}^{\alpha\lambda}+\sqrt{v}g_{2;\ell;0,1}^{\alpha\lambda}\right],\\
g_{11}^{\alpha\beta} &= \frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{11}^{\alpha\alpha}-\frac{1}{\ell}\sqrt{u}\left(\Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v\right)g_{1;\ell;0,1}^{\alpha\lambda},\\
g_{12}^{\alpha\beta} &= \frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{12}^{\alpha\alpha}-\frac{1}{\ell}\sqrt{v}\left(\Delta_3-\Delta_4+1-2u\partial_u\right)g_{1;\ell;0,1}^{\alpha\lambda},\\
g_{21}^{\alpha\beta} &= \frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{21}^{\alpha\alpha}-\frac{1}{\ell}\sqrt{u}\left(\Delta_1-\Delta_2-\Delta_3+\Delta_4-1-2v\partial_v\right)g_{2;\ell;0,1}^{\alpha\lambda},\\
g_{22}^{\alpha\beta} &= \frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{22}^{\alpha\alpha}-\frac{1}{\ell}\sqrt{v}\left(\Delta_3-\Delta_4+2-2u\partial_u\right)g_{2;\ell;0,1}^{\alpha\lambda},
\end{aligned}\tag{F.1}$$

$$\begin{aligned}
g_0^{\beta\alpha} &= \frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_0^{\alpha\alpha}+\frac{1}{\ell}\left[g_{1;\ell;1,0}^{\lambda\alpha}+\sqrt{\frac{u}{v}}g_{2;\ell;1,0}^{\lambda\alpha}\right],\\
g_{11}^{\beta\alpha} &= \frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{11}^{\alpha\alpha}-\frac{1}{\ell}\left(\Delta_1-\Delta_2+2-2u\partial_u\right)g_{1;\ell;1,0}^{\lambda\alpha},\\
g_{12}^{\beta\alpha} &= \frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{12}^{\alpha\alpha}-\frac{1}{\ell}\left(\Delta_1-\Delta_2+1-2u\partial_u\right)g_{2;\ell;1,0}^{\lambda\alpha},\\
g_{21}^{\beta\alpha} &= \frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{21}^{\alpha\alpha}+\frac{1}{\ell}\sqrt{\frac{u}{v}}2v\partial_v g_{1;\ell;1,0}^{\lambda\alpha},\\
g_{22}^{\beta\alpha} &= \frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell}g_{22}^{\alpha\alpha}-\frac{1}{\ell}\sqrt{\frac{u}{v}}\left(1-2v\partial_v\right)g_{2;\ell;1,0}^{\lambda\alpha},
\end{aligned}\tag{F.2}$$

$$\begin{aligned}
g_0^{\beta\beta} &= \frac{\left(\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1\right)\left(\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1\right)}{\ell^2}g_0^{\alpha\alpha}\\
&\quad+\frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\left[\sqrt{u}g_{1;\ell;0,1}^{\alpha\lambda}+\sqrt{v}g_{2;\ell;0,1}^{\alpha\lambda}\right]\\
&\quad+\frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\left[g_{1;\ell;1,0}^{\lambda\alpha}+\sqrt{\frac{u}{v}}g_{2;\ell;1,0}^{\lambda\alpha}\right]\\
&\quad-\frac{1}{\ell^2}\sqrt{u}\left(\Delta_1-\Delta_2+1-2u\partial_u-2v\partial_v\right)g_{\ell;1,1},\\
g_{11}^{\beta\beta} &= \frac{\left(\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1\right)\left(\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1\right)}{\ell^2}g_{11}^{\alpha\alpha}\\
&\quad-\frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{u}\left(\Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v\right)g_{1;\ell;0,1}^{\alpha\lambda}\\
&\quad-\frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\left(\Delta_1-\Delta_2+2-2u\partial_u\right)g_{1;\ell;1,0}^{\lambda\alpha}\\
&\quad+\frac{1}{\ell^2}\sqrt{u}\left(\Delta_1-\Delta_2+1-2u\partial_u\right)\left(\Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v\right)g_{\ell;1,1},\\
g_{12}^{\beta\beta} &= \frac{\left(\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1\right)\left(\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1\right)}{\ell^2}g_{12}^{\alpha\alpha}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{v}\,(\Delta_3-\Delta_4+1-2u\partial_u)\,g^{\alpha\lambda}_{1;\ell;0,1}\\
&-\frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\,(\Delta_1-\Delta_2+1-2u\partial_u)\,g^{\lambda\alpha}_{2;\ell;1,0}\\
&+\frac{1}{\ell^2}\sqrt{v}\,(\Delta_1-\Delta_2+1-2u\partial_u)\,(\Delta_3-\Delta_4+1-2u\partial_u)\,g_{\ell;1,1},
\end{aligned}
\tag{F.3}
$$

$$
\begin{aligned}
g^{\beta\beta}_{21} =\ & \frac{(\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1)\,(\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1)}{\ell^2}g^{\alpha\alpha}_{21}\\
&-\frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{u}\,(\Delta_1-\Delta_2-\Delta_3+\Delta_4-1-2v\partial_v)\,g^{\alpha\lambda}_{2;\ell;0,1}\\
&+\frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{\frac{u}{v}}2v\partial_v g^{\lambda\alpha}_{1;\ell;1,0}\\
&-\frac{1}{\ell^2}\frac{u}{\sqrt{v}}2v\partial_v\,(\Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v)\,g_{\ell;1,1},\\
g^{\beta\beta}_{22} =\ & \frac{(\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1)\,(\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1)}{\ell^2}g^{\alpha\alpha}_{22}\\
&-\frac{\Delta_1-\Delta_2-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{v}\,(\Delta_3-\Delta_4+2-2u\partial_u)\,g^{\alpha\lambda}_{2;\ell;0,1}\\
&-\frac{\Delta_3-\Delta_4-\Delta_{\mathcal{O}}+\ell+1}{\ell^2}\sqrt{\frac{u}{v}}\,(1-2v\partial_v)\,g^{\lambda\alpha}_{2;\ell;1,0}\\
&-\frac{1}{\ell^2}\sqrt{u}2v\partial_v\,(\Delta_3-\Delta_4+1-2u\partial_u)\,g_{\ell;1,1},
\end{aligned}
$$

G Mixed symmetric constants

The constants appearing in the mixed-symmetric conformal blocks are defined by

$$
C_1=\frac{\mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}}{\left(\mathcal{N}_{\mathcal{O}}\lambda_{34\widetilde{\mathcal{O}}}/\lambda_{34\mathcal{O}}\right)_{k+1\,00}}=\frac{D-\Delta_{\mathcal{A}}-1}{D-\Delta_{\mathcal{A}}-2},
\tag{G.1}
$$

$$
\begin{aligned}
C_2=&\,\mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(M^{-1}\right)_{\alpha}{}^{\alpha}\right)_{k+1\,-\frac{1}{2}\,\frac{1}{2}}\\
&=\frac{(\Delta_{\mathcal{A}}-1)\,(D-\Delta_{\mathcal{A}}+k)-(D-\Delta_{\mathcal{A}}-1)\,(\Delta_3-\Delta_4-1)}{(D-\Delta_{\mathcal{A}}-2)\,(D-\Delta_3+\Delta_4-\Delta_{\mathcal{A}}+k+1)},
\end{aligned}
\tag{G.2}
$$

$$
\begin{aligned}
C_3=&\,\mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(M^{-1}\right)_{\beta}{}^{\alpha}\right)_{k+1\,-\frac{1}{2}\,\frac{1}{2}}\\
&=\frac{(k+1)\,(D-2\Delta_{\mathcal{A}})\,(\Delta_3-\Delta_4-1)}{(D-\Delta_{\mathcal{A}}-2)\,(D-\Delta_3+\Delta_4-\Delta_{\mathcal{A}}+k+1)\,(\Delta_3-\Delta_4+\Delta_{\mathcal{A}}+k-1)}
\end{aligned}
\tag{G.3}
$$

$$
\begin{aligned}
C_4=&\,\mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(M^{-1}\right)_{\alpha}{}^{\alpha}\right)_{k+1\,\frac{1}{2}\,-\frac{1}{2}}\\
=&\,\frac{(\Delta_3-\Delta_4+\Delta_{\mathcal{A}}+k+1)\,((\Delta_{\mathcal{A}}-1)\,(D-\Delta_{\mathcal{A}}+k)-(D-\Delta_{\mathcal{A}}-1)\,(\Delta_3-\Delta_4+1))}{(D-\Delta_{\mathcal{A}}-2)\,(D+\Delta_3-\Delta_4-\Delta_{\mathcal{A}}+k+1)\,(-\Delta_3+\Delta_4+\Delta_{\mathcal{A}}+k-1)},
\end{aligned}
\tag{G.4}
$$

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$$C_5 = \mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(M^{-1}\right)_{\beta}{}^{\alpha}\right)_{k+1\,\frac{1}{2}\,-\frac{1}{2}}$$
$$= \frac{(k+1)\,(D-2\Delta_{\mathcal{A}})\,(\Delta_3-\Delta_4+1)}{(D-\Delta_{\mathcal{A}}-2)\,(D+\Delta_3-\Delta_4-\Delta_{\mathcal{A}}+k+1)\,(-\Delta_3+\Delta_4+\Delta_{\mathcal{A}}+k-1)}, \tag{G.5}$$

$$C_6 = \mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(\left(M^{-1}\right)_{\alpha}{}^{\alpha}-\left(M^{-1}\right)_{\alpha}{}^{\beta}\right)\right)_{k+2\,00}$$
$$= \frac{(D-\Delta_{\mathcal{A}}+k)\,(\Delta_3-\Delta_4+\Delta_{\mathcal{A}}+k+1)}{(D-\Delta_{\mathcal{A}}-2)\,(\Delta_{\mathcal{A}}+k+1)} \times \frac{(D-\Delta_{\mathcal{A}}-1)\,(D-\Delta_{\mathcal{A}}+k+1)-(\Delta_{\mathcal{A}}-1)\,(\Delta_3-\Delta_4)}{(D+\Delta_3-\Delta_4-\Delta_{\mathcal{A}}+k+1)\,(D-\Delta_3+\Delta_4-\Delta_{\mathcal{A}}+k+1)}, \tag{G.6}$$

$$C_7 = \mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(\left(M^{-1}\right)_{\beta}{}^{\beta}-\left(M^{-1}\right)_{\beta}{}^{\alpha}\right)\right)_{k+2\,00} = \frac{D-\Delta_{\mathcal{A}}+k}{D-\Delta_{\mathcal{A}}-2}$$
$$\times \frac{(D-\Delta_{\mathcal{A}}-1)\,(\Delta_{\mathcal{A}}+k+1)\,(D-\Delta_{\mathcal{A}}+k+1)-(\Delta_{\mathcal{A}}-1)\,(\Delta_3-\Delta_4)^2}{(\Delta_{\mathcal{A}}+k+1)\,(D+\Delta_3-\Delta_4-\Delta_{\mathcal{A}}+k+1)\,(D-\Delta_3+\Delta_4-\Delta_{\mathcal{A}}+k+1)}, \tag{G.7}$$

$$C_8 = \mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(\left(M^{-1}\right)_{\alpha}{}^{\alpha}-\left(M^{-1}\right)_{\alpha}{}^{\beta}\right)\right)_{k\,00}$$
$$= \frac{(\Delta_{\mathcal{A}}+k)\,((D-\Delta_{\mathcal{A}}-1)\,(D-\Delta_{\mathcal{A}}+k-1)-(\Delta_{\mathcal{A}}-1)\,(\Delta_3-\Delta_4))}{(D-\Delta_{\mathcal{A}}-2)\,(D-\Delta_{\mathcal{A}}+k-1)\,(-\Delta_3+\Delta_4+\Delta_{\mathcal{A}}+k-1)}, \tag{G.8}$$

$$C_9 = \mathcal{N}_{\mathcal{A}}\gamma_{34\widetilde{\mathcal{A}}}/\gamma_{34\mathcal{A}}\left(\mathcal{N}_{\mathcal{O}}^{-1}\left(\left(M^{-1}\right)_{\beta}{}^{\beta}-\left(M^{-1}\right)_{\beta}{}^{\alpha}\right)\right)_{k\,00}$$
$$= \frac{(\Delta_{\mathcal{A}}+k)\left((D-\Delta_{\mathcal{A}}-1)\,(\Delta_{\mathcal{A}}+k-1)\,(D-\Delta_{\mathcal{A}}+k-1)-(\Delta_{\mathcal{A}}-1)\,(\Delta_3-\Delta_4)^2\right)}{(D-\Delta_{\mathcal{A}}-2)\,(D-\Delta_{\mathcal{A}}+k-1)\,(\Delta_3-\Delta_4+\Delta_{\mathcal{A}}+k-1)\,(-\Delta_3+\Delta_4+\Delta_{\mathcal{A}}+k-1)}. \tag{G.9}$$

In computing these constants we have used notation where a subscript on a quantity in parentheses, $(f)_{k'\,P\,Q}$ means that we should evaluate f (which is given in terms of three-point function data) for external particles of weights Δ_3+P and Δ_4+Q, and an exchange operator of spin $\ell=k'$ and dimension $\Delta_{\mathcal{O}}=\Delta_{\mathcal{A}}$.

H Operators appearing in symmetric exchange blocks

Define

$$\delta_1 = \Delta_3-\Delta_4+2u\partial_u+2v\partial_v, \qquad \delta_2 = \Delta_1-\Delta_2-\Delta_3+\Delta_4-2v\partial_v,$$
$$\delta_3 = 2v\partial_v, \qquad \delta_4 = \Delta_1-\Delta_2-2u\partial_u-2v\partial_v. \tag{H.1}$$

Then the operators which appear in the expression (4.70) are

$$\mathcal{D}_0^{--} = \sqrt{u}\,(\delta_1-1)\,, \quad \mathcal{D}_0^{-+} = \sqrt{u}\,(\delta_2-2)\,, \quad \mathcal{D}_0^{+-} = -\frac{\sqrt{u}}{v}\delta_3, \quad \mathcal{D}_0^{++} = -\sqrt{u}\,(\delta_4+1)\,, \tag{H.2}$$

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$$\mathcal{D}_{11}^{--} = -\sqrt{u}\,(\delta_2 + v\,(\delta_1+1))\,(\delta_1-1)\,, \qquad \mathcal{D}_{11}^{-+} = -\sqrt{u}\,(\delta_2 + v\,(\delta_1+1))\,(\delta_2-2)\,,$$
$$\mathcal{D}_{11}^{+-} = \sqrt{u}\,(\delta_1-1)\,(\delta_3+\delta_4+1)\,, \qquad \mathcal{D}_{11}^{++} = \sqrt{u}\delta_2\,(\delta_3+\delta_4+1)\,, \tag{H.3}$$

$$\begin{aligned}
\mathcal{D}_{12}^{--} &= -\frac{1}{\sqrt{v}}\,(\delta_3 - v\,(\delta_1-1))\,(\delta_2 + v\,(\delta_1-1))\,,\\
\mathcal{D}_{12}^{-+} &= \sqrt{v}\,(\delta_2 - 2 + v\,(\delta_1+1))\,(\delta_2-\delta_4-1)\,,\\
\mathcal{D}_{12}^{+-} &= \frac{1}{\sqrt{v}}\,(\delta_3 - v\,(\delta_1-1))\,(\delta_3+\delta_4+1)\,, \qquad \mathcal{D}_{12}^{++} = -\sqrt{v}\,(\delta_2-\delta_4-1)\,(\delta_3+\delta_4+1)\,,
\end{aligned} \tag{H.4}$$

$$\begin{aligned}
\mathcal{D}_{21}^{--} &= u\sqrt{v}\,(\delta_1-1)\,(\delta_1+1)\,, & \mathcal{D}_{21}^{-+} &= u\sqrt{v}\,(\delta_1+1)\,(\delta_2-2)\,,\\
\mathcal{D}_{21}^{+-} &= -\frac{u}{\sqrt{v}}\,(\delta_1-1)\,\delta_3, & \mathcal{D}_{21}^{++} &= -\frac{u}{\sqrt{v}}\delta_2\delta_3,
\end{aligned} \tag{H.5}$$

$$\begin{aligned}
\mathcal{D}_{22}^{--} &= \sqrt{u}\,(\delta_3 - v\,(\delta_1+1))\,(\delta_1-1)\,, & \mathcal{D}_{22}^{-+} &= -\sqrt{u}v\,(\delta_1+1)\,(\delta_2-\delta_4-1)\,,\\
\mathcal{D}_{22}^{+-} &= -\frac{\sqrt{u}}{v}\,(\delta_3 - 2 - v\,(\delta_1-1))\,\delta_3, & \mathcal{D}_{22}^{++} &= \sqrt{u}\,(\delta_2-\delta_4-1)\,\delta_3.
\end{aligned} \tag{H.6}$$

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