MIFPA-10-38

On Global Flipped $SU(5)$ GUTs in F-theory

Yu-Chieh Chung

Department of Physics & Astronomy, Texas A&M University
College Station, TX 77843, USA

`ycchung@physics.tamu.edu`

Abstract

We construct an $SU(4)$ spectral divisor and its factorization of types $(3,1)$ and $(2,2)$ based on the construction proposed in [1]. We calculate the chiral spectra of flipped $SU(5)$ GUTs by using the spectral divisor construction. The results agree with those from the analysis of semi-local spectral covers. Our computations provide evidence for the validity of the spectral divisor construction and suggest that the standard heterotic chirality formulae are applicable to the case of F-theory on an elliptically fibered Calabi-Yau fourfold with no heterotic dual.

1 Introduction

F-theory [2–4] is a twelve-dimensional geometric version of string theory. The construction of F-theory is motivated by the $SL(2,\mathbb{Z})$ symmetry in type IIB string theory. The $SL(2,\mathbb{Z})$ symmetry becomes the geometrical reparametrization symmetry of the torus when the axio-dilaton in type IIB string theory is identified with the complex modulus of a torus. The ten-dimensional background of type IIB string theory is lifted to a twelve-dimensional manifold which admits an elliptic fibration. Due to the monodromy of $SL(2,\mathbb{Z})$, F-theory can be regarded as a non-perturbative completion of type IIB string theory[1]. In F-theory, it was shown [6] that the singularities of elliptic fibers correspond to the gauge groups on the seven-branes. More precisely, the A_n, D_n, and E_n singularities of elliptic fibration correspond to $SU(n+1)$, $SO(2n)$, and E_n gauge groups, respectively. Since F-theory incorporates the exceptional groups, it is believed to be a natural framework for model building. Recently, supersymmetric Grand Unified Theory (GUT) models have been studied extensively in F-theory framework, in particular, the local version of GUT models have been explored in [7–32][2]. The semi-local and global $SU(5)$ GUTs in F-theory have been discussed in [34–56]. For the cases of higher rank GUT groups, global $SO(10)$ GUTs have been studied in [57] and semi-local flipped $SU(5)$ GUTs [58–60] have been constructed in [61]. In this paper we mainly focus on flipped $SU(5)$ GUTs. The purpose of this paper is to promote the semi-local flipped $SU(5)$ models studied in [61] to the global version by using the spectral divisor construction proposed in [1].

In F-theory, semi-local GUT models can be constructed by using spectral cover construction [9, 41]. In particular, one can use $SU(4)$ spectral covers to build flipped $SU(5)$ models [61]. We start with an elliptically fibered Calabi-Yau fourfold Z_4 with a base B_3 which contains a divisor B_2 where Z_4 exhibits an E_8 singularity. To avoid full F-theory on a complicated elliptically fibered Calabi-Yau fourfold, we adopt a bottom-up approach to construct models in the decoupling limit, which lead us to consider a contractible complex surface B_2 inside B_3 such that we can reduce full F-theory on X_4 to an effective eight-dimensional supersymmetric gauge theory on $\mathbb{R}^{3,1} \times B_2$ [7–10]. To achieve the decoupling limit, the surface B_2 has to be a del Pezzo surface [62, 63]. To obtain the gauge group $SU(5)\times U(1)_X$, we unfold

[1]See [5] for a review.

[2]See [33] for a review.

the E_8 singularity into a D_5 singularity corresponding to unbroken $SO(10)$. This unfolding can be encoded in an $SU(4)$ spectral cover. It was shown in [9, 41] that the spectral cover construction naturally encodes the information of the unfolding E_8 singularity and the gauge fluxes. By unfolding an E_8 singularity, we can engineer the singularity of types D_5, D_6, E_6, and E_7 in the Calabi-Yau fourfold Z_4. These operations correspond to the manipulation of the roots of a $SU(4)$ spectral cover. Generally we need to turn on certain fluxes to obtain the chiral spectrum. In F-theory, a natural candidate is the four-form G-flux which consists of three-form fluxes and gauge fluxes. In type IIB theory, these three-form fluxes produce back-reaction in the background geometry. It was shown in [30, 64] that the three-form fluxes induce noncommutative geometric structures and also modify the texture of the Yukawa couplings. An example of noncommutative geometry is a fuzzy space, which has been studied in the context of F-theory in [65]. In this article we shall turn off these three-form fluxes and focus only on the gauge fluxes. The chirality of the matter fields in the representations of $SO(10)$ is determined by the traceless cover fluxes which are $(1,1)$-forms on the spectral covers. To obtain the gauge group $SU(5)\times U(1)_X$, we turn on a line bundle associated with $U(1)_X$ to break $SO(10)$ down to $SU(5)\times U(1)_X$. The spectrum is then determined by the cover fluxes and $U(1)_X$ fluxes. In this paper we shall focus on the $SU(4)$ spectral cover and also consider the factorizations of the spectral cover to construct realistic flipped $SU(5)$ models. For $U(1)_X$ fluxes breaking $SO(10)$ down to $SU(5)\times U(1)_X$ and numerical models, we refer readers to [61] for the details. A brief review of the semi-local $SU(4)$ spectral cover can be found in section 3.1. The analysis of the chiral spectrum under $(3,1)$ and $(2,2)$ factorizations can be found in section 4.1.

The spectral cover construction discussed above is semi-local. To obtain global flipped $SU(5)$ GUT models, we shall use the spectral divisor construction which has recently been proposed in [1]. This construction is motivated by heterotic/F-theory duality [3]. In the heterotic string framework, one can calculate the chirality of matter fields by specifying a line bundle or its twist $\gamma_H^{(4)}$ on an $SU(4)$ spectral cover. It turns out the net chirality $N_{\mathbf{r}}$ of matter field in the representation $\mathbf{r}$ is given by [1, 34]

$$N_{\mathbf{r}}=\int_{\Sigma_{\mathbf{r},H}}\gamma_H^{(4)}, \tag{1.1}$$

[3]For another construction from mirror symmetry and the discussion in a global $U(1)$ gauge symmetry arising from global restrictions of the Tate model, see [51] and references therein.

where $\Sigma_{\mathbf{r},H}$ is the matter curve of representation $\mathbf{r}$. It was shown in [70] that the data of the spectral cover can be encoded in a dP_8 surface. On the other hand, when the Calabi-Yau fourfold admits a global K_3-fibration over B_2, the $K3$ fiber degenerates into two dP_9 surfaces glued together along an elliptic curve in the stable degenerate limit. The elliptic fibration over B_2 becomes the background Calabi-Yau threefold in the dual heterotic string compactification. Moreover, the spectral cover data of E_8 bundles can be encoded in the pair of dP_9 surfaces in F-theory geometry. The subbundles of E_8 correspond to some singularities in dP_9 surfaces. In particular, an unbroken $SO(10)$ gauge group corresponds to a D_5 singularity. It turns out that following heterotic/F-theory duality we can define the dual spectral divisor in F-theory framework, which encodes the data of the spectral cover in heterotic theory [1]. In F-theory, the net chirality formula was proposed to be

$$N_{\mathbf{r}} = \int_{\widehat{\Sigma}_{\mathbf{r}} \cdot p^*_{\mathcal{D}_F^{(4)}} B_2} \gamma_F^{(4)}, \tag{1.2}$$

where $\gamma_F^{(4)}$ is the traceless flux on the $SU(4)$ spectral divisor $\mathcal{D}_F^{(4)}$ and $p^*_{\mathcal{D}_F^{(4)}}$ is the projective map $p^*_{\mathcal{D}_F^{(4)}} : \mathcal{D}_F^{(4)} \to B_3$. It was argued [1] that this formula is intrinsic in the sense that it can be applied to the cases of F-theory compactifications without heterotic duals and that spectral divisor construction can be regarded as a global completion of the semi-local spectral cover construction. The case of an $SU(5)$ spectral divisor has been analyzed in [1]. In this article we shall verify this proposal by comparing the computations of chirality from an semi-local $SU(4)$ spectral cover with that from an $SU(4)$ spectral divisor. It turns out that they agree with each other. Our computation provides evidence to support the validity of the spectral divisor construction. The detailed construction of the $SU(4)$ spectral divisor can be found in section 3.2. We also calculate the chirality under $(3,1)$ and $(2,2)$ factorizations by using the spectral divisor construction. The results can be found in section 4.2.

The organization of the rest of the paper is as follows: in section 2, we first briefly review the $SU(4)$ spectral cover construction and computation of the chiral spectrum in heterotic string compactification on an elliptically fibered Calabi-Yau threefold. We then turn to the del Pezzo surface construction for $SU(4)$ bundles and stable degenerate limits, which are two important ingredients for heterotic/F-theory duality. We construct an $SU(4)$ spectral divisor in F-theory motivated by heterotic/F-theory duality and calculate the chiral spectrum in the end of section 2. In section 3, we consider the cases of F-theory compactifications without heterotic duals. We first briefly review the semi-local $SU(4)$ spectral cover construction and then turn to constructing an $SU(4)$ spectral divisor. In section 4, we study $(3,1)$ and $(2,2)$ factorizations of the $SU(4)$ spectral cover and $SU(4)$ spectral divisor. We also calculate the chirality induced by traceless fluxes and found agreement between these two constructions. We summarize and conclude in section 5.

2 Preliminaries

In this section we shall briefly review the spectral cover construction in heterotic string. In particular, we shall focus on the case of an $SU(4)$ spectral cover. We then give an introduction to heterotic/F-theory duality. In the end of this section, we construct the dual F-theory spectral divisor [1] motivated by heterotic/F-theory duality.

2.1 $SU(4)$ Cover in Heterotic String

The $\mathcal{N}=1$ four-dimensional effective theory of heterotic string compactifications[4] is governed by the data (Z_3, V_1, V_2), where V_1 and V_2 are vector bundles over a six-dimensional manifold Z_3. For simplicity, we only focus on one of the vector bundles, denoted by V whose structure group is G. Supersymmetry requires that Z_3 be a Calabi-Yau threefold and that V admit a connection satisfying the Hermitian Yang-Mills equations [66]

$$F_{ab} = F_{\bar{a}\bar{b}} = 0, \qquad g^{a\bar{b}} F_{a\bar{b}} = 0, \tag{2.1}$$

where g and F are a metric of Z_3 and curvature of the connection, respectively. The unbroken gauge group of the four-dimensional effective theory is then the commutant of G in E_8. To obtain an unbroken $SO(10)$ gauge group, we shall focus on the case of $G = SU(4)$. It is an extremely difficult task to construct solutions of the Hermitian Yang-Mills equations Eq. (2.1) for manifolds of dimension greater than one. However, it was proven in [67–69] that there is a one-to-one correspondence between the solutions of the Hermitian Yang-Mills equations and the construction

[4]Here we focus on the case of $E_8 \times E_8$ heterotic string compactificatuion with vanishing background three-form flux H and with constant dilaton ϕ.

of stable holomorphic vector bundles over the same complex manifold[5]. In other words, one can either attempt to solve the Hermitian Yang-Mills equations or, simply construct the associated stable holomorphic vector bundles. It was shown in [70, 71] that when Z_3 admits an elliptic fibration, stable holomorphic bundles with structure groups $SU(n)$ can be constructed by using spectral covers. In what follows, we briefly review the spectral cover construction and the computation of net chirality of the massless matter fields in a four-dimensional effective theory [34, 70, 72].

Let Z_3 be an elliptically fibered Calabi-Yau threefold $\pi_H : Z_3 \to B_2$ with a section $\sigma_H : B_2 \to Z_3$. Due to the presence of the section σ_H, Z_3 can be described by the Weierstrass model. One can realize Z_3 as a hypersurface of $W\mathbb{P}^2_{2,3,1}$-fibration over B_2 given by

$$y^2 = x^3 + f_4 x u^4 + g_6 u^6, \tag{2.2}$$

where x, y, u are sections of $\mathcal{O}(2\sigma_H) \otimes K_{B_2}^{-2}$, $\mathcal{O}(3\sigma_H) \otimes K_{B_2}^{-3}$, and $\mathcal{O}(\sigma_H)$, respectively, while f_4 and g_6 are sections of $K_{B_2}^{-4}$ and $K_{B_2}^{-6}$, respectively.[6] Note that these sections satisfy the following relation:

$$\sigma_H \cdot (\sigma_H + \pi_H^* c_1) = 0, \tag{2.3}$$

where $c_1 \equiv c_1(B_2)$. At a generic point $b \in B_2$, the fiber $\mathbb{E}_b$ is an elliptic curve. The restriction $V|_{\mathbb{E}_b}$ of the bundle V of rank n to the elliptic curve $\mathbb{E}_b$ is split. Namely, $V|_{\mathbb{E}_b}$ can be decomposed as a direct summand of holomorphic line bundles. The semi-stability of V requires that these line bundles be all of degree zero. Therefore, we can write $V|_{\mathbb{E}_b} = \oplus_{i=1}^n \mathcal{O}_{\mathbb{E}_b}(q_i - e_0)$, where $q_i \in \mathbb{E}_b$ and e_0 is a distinguished point representing the identity element in the group law on $\mathbb{E}_b$. For $SU(n)$ bundles, it is required that $c_1(V) = 0$ which leads to the traceless condition $\sum_{i=1}^n (q_i - e_0) = 0$.

[5]Let E be a holomorphic vector bundle over Z_3 and J_{Z_3} be a Kähler form of Z_3. The slope $\mu(E)$ is defined by $\mu(E) = \frac{\int_{Z_3} c_1(E) \wedge J_{Z_3} \wedge J_{Z_3}}{\mathrm{rk}(E)}$. The vector bundle E is (semi)stable if for every subbundle or subsheaf $\mathcal{E}$ with $\mathrm{rk}(\mathcal{E}) < \mathrm{rk}(E)$, the inequality $\mu(\mathcal{E}) < (\leqslant) \mu(E)$ holds. Assume that $E = \oplus_i^k \mathcal{E}_i$, then E is polystable if each $\mathcal{E}_i$ is a stable bundle with $\mu(\mathcal{E}_1) = ... = \mu(\mathcal{E}_k) = \mu(E)$ [67–69]. The Donaldson-Uhlenbeck-Yau theorem [67–69] states that a (split) irreducible holomorphic bundle E admits a hermitian connection satisfying Eq. (2.1) if and only if E is polystable.

[6]The globally well-defined $W\mathbb{P}^2_{2,3,1}$-fibration can be realized as the total space of the weighted projective bundle $W\mathbb{P}(L^2 \oplus L^3 \oplus \mathcal{O}_{B_2})$. It follows from the condition $c_1(Z_3) = 0$ that $L \cong K_{B_2}^{-1}$, where $K_{B_2}^{-1}$ is the anticanonical bundle of B_2. Let $c_1(\mathcal{O}_P(1)) = \sigma_H$, then the homogeneous coordinates $[x : y : u]$ are sections of $\mathcal{O}(2\sigma_H) \otimes K_{B_2}^{-2}$, $\mathcal{O}(3\sigma_H) \otimes K_{B_2}^{-3}$, and $\mathcal{O}(\sigma_H)$, respectively.

When the point b varies along B_2, $\{q_1.q_2, ..., q_n\}$ spans a n-fold cover over B_2, called $SU(n)$ spectral cover. In particular, the $SU(4)$ spectral cover is given by

$$\mathcal{C}_H^{(4)}: \ a_0u^4 + a_2xu^2 + a_3yu + a_4x^2 = 0, \tag{2.4}$$

with a projection map $p_{\mathcal{C}_H^{(4)}} : \mathcal{C}_H^{(4)} \to B_2$. We denote the homological class $[a_0]$ of the section a_0 by $\pi_H^*\eta$, where $\eta \in H_2(B_2, \mathbb{Z})$ and write the remaining sections as $[a_m] = \pi_H^*(\eta - mc_1)$, where $m = 2, 3, 4$.[7] The sections a_0, a_2, a_3 and a_4 encode the information of deformation of $\mathcal{C}_H^{(4)}$ defined by Eq. (2.4) and can be regarded as complex moduli of the spectral cover. On the other hand, the positions of the points $\{q_1, q_2, q_3, q_4\}$ or the roots of the cover $\mathcal{C}_H^{(4)}$ characterize the deformation of the bundle V. Therefore, $\{a_0, a_2, a_3, a_4\}$ characterize the deformation[8] of V. It follows from Eq. (2.4) that the homological class of $\mathcal{C}_H^{(4)}$ is given by

$$[\mathcal{C}_H^{(4)}] = 4\sigma_H + \pi_H^*\eta. \tag{2.5}$$

An $SU(4)$ bundle can be constructed by specifying a line bundle or its twist $\gamma_H^{(4)}$ which is $(1,1)$-form on $\mathcal{C}_H^{(4)}$. To obtain $SU(4)$ bundles, it is required that $\gamma_H^{(4)}$ satisfies the traceless condition $p_{\mathcal{C}_H^{(4)}*}\gamma_H^{(4)} = 0$. This can be achieved by setting

$$\gamma_H^{(4)} = (4 - p^*_{\mathcal{C}_H^{(4)}} p_{\mathcal{C}_H^{(4)}*})([\mathcal{C}_H^{(4)}] \cdot \sigma_H). \tag{2.6}$$

Turning on an $SU(4)$ bundle over Z_3 breaks E_8 down to $SO(10)$. Under the breaking pattern $E_8 \to SO(10) \times SU(4)$, the adjoint representation of E_8 is decomposed as

$$\begin{aligned} E_8 &\to SO(10) \times SU(4) \\ \mathbf{248} &\to (\mathbf{1},\mathbf{15}) + (\mathbf{45},\mathbf{1}) + (\mathbf{10},\mathbf{6}) + (\mathbf{16},\mathbf{4}) + (\overline{\mathbf{16}},\bar{\mathbf{4}}). \end{aligned} \tag{2.7}$$

The net chirality of matter fields can be calculated by the Atiyah-Singer index theorem or by intersection numbers of matter curves with $\gamma_H^{(4)}$ [34, 70, 72]. Before computing the net chirality, we need to find the homological classes of matter curves. The

[7]Generically, the spectral cover defined by Eq. (2.4) leads to a semistable bundle [70, 71]. A sufficient condition to obtain a holomorphic stable bundle V is that $\mathcal{C}_H^{(4)}$ is irreducible, which can be achieved by imposing the following two conditions: (1) The linear system $|\eta|$ is base-point free in B_2, (2) $\eta - mc_1$ is effective in B_2 [72].

[8]The moduli space of stable $SU(4)$ bundles on $\mathbb{E}_b$ is the projective space $\mathbb{P}^3$. Fitting $\mathbb{P}^3$'s together, we obtain the projective bundle $\mathbb{P}(\mathcal{O}_{B_2} \oplus L^{-2} \otimes L^{-3} \oplus L^{-4})$ over B_2. In general, the moduli space of stable $SU(n)$ bundles is the projective bundle $\mathbb{P}(\mathcal{O}_{B_2} \oplus L^{-2} \otimes L^{-3} \oplus ... \oplus L^{-n})$ [70].

homological class of the matter **16** curve in Z_3 is given by the intersection of $\mathcal{C}_H^{(4)}$ with the zero section

$$[\Sigma_{\mathbf{16},H}] = [\mathcal{C}_H^{(4)}] \cdot \sigma_H. \tag{2.8}$$

The net chirality $N_{\mathbf{16}}$ of the matter **16** can be evaluated by

$$\begin{aligned} N_{\mathbf{16}} &= \int_{\Sigma_{\mathbf{16},H}} \gamma_H^{(4)} \\ &= \gamma_H^{(4)} \cdot [\Sigma_{\mathbf{16},H}] \\ &= -\eta \cdot_{B_2} (\eta - 4c_1). \end{aligned} \tag{2.9}$$

To get the net chirality of matter **10**, we have to resolve the singularity on the associated cover $\mathcal{C}^{(6)}_{\wedge^2 V,H}$ corresponding to the antisymmetric representation **6** in $SU(4)$. It can be done by considering the intersection $\mathcal{C}_H^{(4)} \cap \tau\mathcal{C}_H^{(4)}$, where τ is a $\mathbb{Z}_2$ involution acting on the cover $\mathcal{C}_H^{(4)}$ by $y \to -y$ while keeping x and u untouched. More precisely, the intersection $\mathcal{C}_H^{(4)} \cap \tau\mathcal{C}_H^{(4)}$ is determined by

$$\left\{ \begin{array}{l} a_3 y u = 0 \\ a_0 u^4 + a_2 x u^2 + a_4 x^2 = 0. \end{array} \right. \tag{2.10}$$

The homological class of matter **10** curve in Z_3 can be computed as

$$\begin{aligned} [\Sigma_{\mathbf{10},H}] &= [\mathcal{C}_H^{(4)}] \cdot [\mathcal{C}_H^{(4)}] - [y] \cdot [a_0 u^4] - [u] \cdot [a_4 x^2] \\ &= [\mathcal{C}_H^{(4)}] \cdot \{[\mathcal{C}_H^{(4)}] - 3(\sigma_H + \pi_H^* c_1) - \sigma_H\}. \end{aligned} \tag{2.11}$$

The net chirality $N_{\mathbf{10}}$ can be calculated by the intersection number[9] $\gamma_H \cdot [\Sigma_{\mathbf{10},H}]$

$$\begin{aligned} N_{\mathbf{10}} &= \gamma_H \cdot [\Sigma_{\mathbf{10},H}] \\ &= [\mathcal{C}_H^{(4)}] \cdot [4\sigma_H - \pi_H^*(\eta - 4c_1)] \cdot \{[\mathcal{C}_H^{(4)}] - 3(\sigma_H + \pi_H^* c_1) - \sigma_H\} \\ &= 0. \end{aligned} \tag{2.12}$$

2.1.1 Del Pezzo Surface Construction

In the previous section one can see that the information of the bundle V can be encoded in the spectral cover $\mathcal{C}_H^{(4)}$ and the twist $\gamma_H^{(4)}$. However, the construction can

[9] For the case of $SU(n)$ bundles, $N_{\mathbf{16}} = -\eta \cdot_{B_2} (\eta - nc_1)$ and $N_{\mathbf{10}} = -(n-4)\eta \cdot_{B_2} (\eta - nc_1)$. The factor $(n-4)$ in $N_{\mathbf{10}}$ can be seen from the fact that $\chi(Z_3, \wedge^2 V) = (n-4)\chi(Z_3, V)$ where Z_3 is a Calabi-Yau threefold and V is a vector bundle of rank n with $c_1(V) = 0$.

be translated to another form which involves del Pezzo surfaces and is more suitable for the framework of heterotic/F-theory duality. Before introducing the heterotic/F-theory duality, we briefly review the del Pezzo surface construction for $SU(4)$ bundles. Let S be a del Pezzo surface dP_8 which can be obtained by blowing up eight generic points $p_1, p_2, ..., p_8$ in $\mathbb{P}^2$. The second homology group $H_2(S, \mathbb{Z})$ of S is generated by the basis $\{H, E_1, ..., E_8\}$ with the intersection form given by

$$H \cdot H = 1, \quad H \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{ij}, \quad i, j = 1, 2, ..., 8, \tag{2.13}$$

where H is the pullback of the hyperplane divisor in $\mathbb{P}^2$ and E_i are the exceptional divisors from blow-ups. The anticanonical divisor $-K_S$ of S is given by

$$-K_S = 3H - \sum_{i=1}^{8} E_i. \tag{2.14}$$

The linear system $|-K_S|$ has a base point and general elements $\mathbb{E}$ of $|-K_S|$ are genus one curves.[10] Let us define two subsects of $H_2(S, \mathbb{Z})$ as follows [34, 70]:

$$I_8 = \{l \in H_2(S, \mathbb{Z}) | l \cdot l = -1,\ l \cdot (-K_S) = 1\}, \tag{2.15}$$

$$R_8 = \{C \in H_2(S, \mathbb{Z}) | C \cdot C = -2,\ C \cdot (-K_S) = 0\}. \tag{2.16}$$

Note that I_8 and R_8 are in one-to-one correspondence through $l = C + (-K_S)$ and that the elements in R_8 are in one-to-one correspondence with roots of E_8. The generators of R_8 can be chosen as follows:

$$C_k = E_k - E_{k+1},\ k = 1, 2, ..., 7, \quad C_8 = H - (E_6 + E_7 + E_8). \tag{2.17}$$

The intersection matrix of R_8 is given by $(-C_{E_8})$ where C_{E_8} is the Cartan matrix of E_8. Given $\mathbb{E} \in |-K_S|$, a flat bundle on $\mathbb{E}$ is given by

$$\mathcal{O}_{\mathbb{E}}(C_k|_{\mathbb{E}}) \cong \mathcal{O}_{\mathbb{E}}(q_k - e_0), \tag{2.18}$$

[10]Since $-K_S$ is ample, $H^0(S, \mathcal{O}_{B_2}(-K_S)) \neq 0$. The linear system of $-K_S$ is defined by $|-K_S| = \mathbb{P}H^0(S, \mathcal{O}_S(-K_S))$ and the base point locus is defined by $\bigcap \mathbb{E}_\alpha,\ \mathbb{E}_\alpha \in |-K_S|$. For a del Pezzo surface dP_8, it follows from the Riemann-Roch theorem and Kodaira vanishing theorem that $\dim(|-K_S|) = \chi(S, \mathcal{O}_S(-K_S)) - 1 = (-K_S)^2 = 1$. Since $h^0(S, \mathcal{O}_S(-K_S)) = 2$, we have two homogeneous polynomials of degree one and the base point is the unique common zero $[0:0]$. Moreover, one can show that the linear system $|-3K_S|$ induces a morphism $\Phi_{|-3K_S|} : S \to W\mathbb{P}^3_{2,3,1,1}$. The image of $\Phi_{|-3K_S|}$ is given by Eq. (2.19) [34].

where $C_k \in R_8$ and $q_k \in \mathbb{E}$ given by $l_k \cdot (-K_S)$. Recall that the spectral cover describes a flat bundle on an elliptic fiberation $\pi_H : Z_3 \to B_2$ by specifying a set $\{q_k\}$ for each fiber $\mathbb{E}_b$, $b \in B_2$. Equivalently, one can describe the bundle by starting with embedding an elliptic curve $\mathbb{E}_b$ into a fiber of dP_8-fibration over B_2 $\pi_{W_4} : W_4 \to B_2$ with $\pi_{W_4}|_{Z_3} = \pi_H$. Then the local data $V|_{\mathbb{E}_b}$ of the bundle V can be described by the cycles $\{C_1, C_2, ..., C_n\}$ in R_8 via Eq. (2.18). On the other hand, one can realize a dP_8 surface as a divisor in $W\mathbb{P}^3_{2,3,1,1}$. More precisely, a dP_8 surface in $W\mathbb{P}^3_{2,3,1,1}$ can be described by the Weierstrass model as follows:

$$y^2 = x^3 + \tilde{f}_4(Z_1, Z_2)x + \tilde{g}_6(Z_1, Z_2), \tag{2.19}$$

where $[x : y : Z_1 : Z_2]$ are homogeneous coordinates of $W\mathbb{P}^3_{2,3,1,1}$, $\tilde{f}_4$ and $\tilde{g}_6$ are homogeneous polynomials of degree four and six, respectively. Through this embedding, one can find that the bundle moduli of a flat bundle on $\mathbb{E}$ map to the complex structure moduli of the defining equation Eq. (2.19). For the case of $G = SU(4)$, one can construct the bundle through the spectral cover construction by specifying points $\{q_1, q_2, q_3, q_4\}$ on $\mathbb{E}_b$. The bundle moduli are characterized by the coefficients $\{a_0, a_2, a_3, a_4\}$ of the spectral cover defined by Eq. (2.4). Equivalently, this data can be described by the (-2)-cycles $\{C_1, C_2, C_3, C_4\}$ in dP_8 and their intersection numbers. The intersection of these cycles form the Cartan matrix of $SU(4)$. The complement of the extended Dynkin diagram of $SU(4)$ in E_8 corresponds to the vanishing cycles which leads to a D_5 singularity in dP_8. In other words, the unbroken GUT group $SO(10)$ corresponds to a D_5 singularity in dP_8. Therefore, one can construct an $SO(10)$ GUT group by engineering a D_5 singularity in dP_8. More precisely, one can consider the Weierstrass model

$$y^2 = x^3 + f_4 Z_1^4 x + g_6 Z_1^6 + Z_2 Z_1 (b_0 Z_1^4 + b_2 Z_1^2 x + b_3 Z_1 y + b_4 x^2). \tag{2.20}$$

Note that $Z_2 = 0$ locus is an elliptic curve given by the Weierstrass equation $y^2 = x^3 + f_4 x + g_6$ and that the parenthesis in Eq. (2.20) reduces to the spectral cover $\mathcal{C}_H^{(4)}$ given by Eq. (2.4) when $Z_1 \to u$ with $b_m|_{Z_3} = a_m$. It is clear that in this case the bundle moduli $\{a_0, a_2, a_3, a_4\}$ map to the complex moduli of dP_8 given by Eq. (2.20). The dual F-theory geometry can be described as a dP_9-fibration over B_2, which is obtained by blowing up the base point. The dP_8 construction described above for $SU(n)$ bundles can be realized by a dP_9 surface whose intersection matrix of (-2)-cycles contains the Cartan matrix of E_8 . It can be seen by taking [1, 34]

$$I_8 = \{l \in H_2(dP_9, \mathbb{Z}) | l \cdot l = -1,\ l \cdot (-K_{dP_9}) = 1,\ l \cdot E_9 = 0\}, \tag{2.21}$$

$$R_8 = \{C \in H_2(dP_9, \mathbb{Z}) | C \cdot C = -2,\ C \cdot (-K_{dP_9}) = 0,\ C \cdot E_9 = 0\}, \tag{2.22}$$

where E_9 is an exceptional divisor from the blow-up of the base point. The geometry of a dP_9-fibration can be obtained by taking the stable degenerate limit of a $K3$-fibration on B_2 in F-theory [9, 34, 70, 73, 74]. Through this degenerate limit, we can embed the data of the bundle V into dual F-theory geometry. We shall describe this degenerate limit in the next section.

2.2 Heterotic/F-theory Duality

2.2.1 Stable Degeneration Limit

Let us consider F-theory on an elliptically fibered Calabi-Yau fourfold $\pi_{X_4} : X_4 \to B_3$ with a section $\sigma_{B_3} : B_3 \to X_4$. With the section σ_F, X_4 can be described by the Weierstrass model:

$$y^2 = x^3 + fxu^4 + gu^6. \tag{2.23}$$

The Calabi-Yau condition $c_1(X_4) = 0$ requires that f and g are sections of $K_{B_3}^{-4}$ and $K_{B_3}^{-6}$, respectively.[11] The heterotic/F-theory duality requires that B_3 admits a $\mathbb{P}^1$-fibration over some surface B_2. Let $[Z_1 : Z_2]$ be the homogeneous coordinates of $\mathbb{P}^1$ fiber. Since f and g are the homogeneous polynomials of degree 8 and 12 in terms of $[Z_1 : Z_2]$, respectively, [12] one can expand Eq. (2.23) as

$$y^2 = x^3 + \big(\sum_{i=0}^{8} f_i Z_1^i Z_2^{8-i}\big) xu^4 + \big(\sum_{j=0}^{12} g_j Z_1^j Z_2^{12-j}\big) u^6, \tag{2.24}$$

[11]To see this, we can embed X_4 as a section of a weighted projective bundle over B_3. More precisely, we homogenize Eq. (2.23) to be $y^2 = x^3 + xu^4 + gu^6 \hookrightarrow W\mathbb{P}^2_{2,3,1}$, where f and g are sections of line bundles L^4 and L^4 on B_3, respectively. To obtain a globally well-defined fibration, let $\bar{X}_5$ be the total space of the weighted projective bundle $W\mathbb{P}(L^2 \otimes L^3 \otimes \mathcal{O}_{B_3})$ over B_3 and consider X_4 to be a hypersurface in $\bar{X}_5$. By the adjuction formula [82, 83], we have $c(X_4) = \frac{c(B_3)(1+2r+2\pi^*_{X_4}t)(1+3r+3\pi^*_{X_4}t)(1+r)}{(1+6r+6\pi^*_{X_4}t)}$, where $r \equiv c_1(\mathcal{O}_P(1))$ and $t \equiv c_1(L)$. It follows from the condition $c_1(X_4) = 0$ that $L = K_{B_3}^{-1}$.

[12]Recall that the anticanonical bundle $K_{\mathbb{P}^n}^{-1}$ of n-dimensional complex projective space $\mathbb{P}^n$ is $K_{\mathbb{P}^n}^{-1} = (n+1)H \equiv \mathcal{O}_{\mathbb{P}^n}(n+1)$. So $K_{\mathbb{P}^1}^{-4} = \mathcal{O}_{\mathbb{P}^1}(8)$ and $K_{\mathbb{P}^1}^{-6} = \mathcal{O}_{\mathbb{P}^1}(12)$.

where f_i and g_j are sections of suitable line bundles over B_2. When $Z_1 \to 0$ and set $Z_2 = 1$, Eq. (2.24) becomes [1, 34, 74]

$$y^2 = x^3 + \big(\sum_{i=0}^{4} f_i z_1^i\big) x u^4 + \big(\sum_{j=0}^{6} g_j z_1^j\big) u^6, \tag{2.25}$$

where $z_1 \equiv \frac{Z_1}{Z_2}$. On the other hand, taking $Z_2 \to 0$ and set $Z_1 = 1$, Eq. (2.24) becomes

$$y^2 = x^3 + \big(\sum_{m=0}^{4} f_{m+4} z_2^{4-m}\big) x u^4 + \big(\sum_{l=0}^{6} g_{l+6} z_2^{6-l}\big) u^6, \tag{2.26}$$

where $z_2 \equiv \frac{Z_2}{Z_1}$. These two limits correspond to two dP_9 surfaces[13] glued together along an elliptic curve $\mathbb{E}$ with the Weierstrass equation[14]:

$$y^2 = x^3 + f_4 x u^4 + g_6 u^6. \tag{2.27}$$

This elliptically fibered Calabi-Yau threefold $\pi_H : Z_3 \to B_2$ is the background of heterotic string compactification. Two dP_9 surfaces, Eq. (2.25) and (2.26) encode the data of bundles $E_8 \times E_8$ in the heterotic string. With heterotic/F-theory duality, one can find that constructing an stable $SU(4)$ bundle on an elliptically fibered Z_3 with a base B_2 by using spectral cover construction corresponds to engineering an D_5 singularity in the geometry of dP_9-fibration on B_2 given by Eq. (2.20).

2.2.2 Dual $SU(4)$ spectral Divisor in F-theory

Let Y_4 be a dP_9-fibration over a complex surface B_2 with a projection map $p : Y_4 \to B_2$. Since dP_9 is an elliptic surface, Y_4 can be regarded as an elliptic fibration over a threefold B_3 with a section $\sigma_F : B_3 \to Y_4$ and B_3 admits a $\mathbb{P}^1$-fibration over B_2. The projection map of the elliptic fibration and $\mathbb{P}^1$-fibration are denoted by $\pi_F : Y_4 \to B_3$ and $\varphi : B_3 \to B_2$, respectively. To describe Y_4, we embed the elliptic fiber as a

[13]The hypersurfaces described by Eq. (2.25) and Eq. (2.26) both are homogeneous polynomials of degree six in $W\mathbb{P}^3_{2,3,1,1}$. They are actually dP_8 surfaces. It follows from the adjuction formula that $c_1(S) = x$ and $c_2(S) = 11x^2$, where $x \equiv r + t$. By the Riemann-Roch theorem $12\chi(\mathcal{O}_S) = c_1^2(S) + c_2(S)$, we obtain $x^2 = 1$ and then Euler characteristic $\chi(S) = 11$. For dP_k surfaces, $\chi(dP_k) = 3+k$, which implies that $k = 8$. One can obtain dP_9's by blowing up the point $Z_1 = Z_2 = 0$.

[14]The elliptic curve $\mathbb{E}$ is an effective divisor of the linear system $|-K_S|$. By the adjunction formula, we obtain $2g - 2 = \mathbb{E}(\mathbb{E} + K_S) = 0$, which implies that $\mathbb{E}$ is an elliptic curve.

divisor of $W\mathbb{P}^2_{2,3,1}$ with homogeneous coordinates $[x:y:u]$ and consider the following Weierstrass model:

$$\begin{aligned} y^2 &= x^3 + f_4(Z_1u)^4x + g_6(Z_1u)^6 + Z_2(Z_1u)^{5-n}[b_0(Z_1u)^n + b_2(Z_1u)^{n-2}x \\ &+ b_3(Z_1u)^{n-3}y + ...], \end{aligned} \tag{2.28}$$

where the last term in the bracket is $b_nx^{n/2}$ for n even, or $b_nx^{(n-3)/2}y$ for n odd. Note that x, y, and u are sections of $\mathcal{L}^2$, $\mathcal{L}^3$, and $\mathcal{O}_{B_3}$, respectively, where $\mathcal{L}$ is a line bundle on B_3 and will be determined later. To make $W\mathbb{P}^2_{2,3,1}$-fibration globally well-defined, we consider Y_4 be a divisor in the weighted projective bundle $W\mathbb{P}(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}_{B_3})$. We denote the fiber by $\mathcal{O}_P(1)$. Let $c_1(\mathcal{O}_P(1)) = \sigma_F$ and $c_1(\mathcal{L}) = l$. By using the adjuction formula, we obtain

$$c(Y_4) = c(B_3)\frac{(1+2\sigma_F+2\pi_F^*l)(1+3\sigma_F+3\pi_F^*l)(1+\sigma_F)}{(1+6\sigma_F+6\pi_F^*l)}, \tag{2.29}$$

where c stand for the total Chern class. It follows from Eq. (2.29) that

$$c_1(Y_4) = \pi_F^*c_1(B_3) - \pi_F^*l. \tag{2.30}$$

Let us turn to the geometry of B_3. We take B_3 to be a $\mathbb{P}^1$ bundle over B_2. To be concrete, let $B_3 = \mathbb{P}(\mathcal{O}_{B_2} \oplus \mathcal{M})$ with $c_1(\mathcal{O}_P(1)) = r$ and $c_1(\mathcal{M}) = t$, where $\mathcal{M}$ is a line bundle on B_2. By using the adjuction formula, we have

$$c(B_3) = c(B_2)(1+r)(1+r+\varphi^*t), \tag{2.31}$$

which implies that

$$\left\{ \begin{array}{l} c_1(B_3) = 2r + \varphi^*(c_1+t) \\ c_2(B_3) = \varphi^*c_2 + \varphi^*c_1 \cdot (\varphi^*t + 2r) \\ c_3(B_3) = \varphi^*c_2 \cdot (\varphi^*t + 2r) \end{array} \right. , \tag{2.32}$$

where $c_1 = c_1(B_2)$ and the relation $r \cdot (r + \varphi^*t) = 0$ has been used. On the other hand, it follows from Eq. (2.28) that the heterotic Calabi-Yau threefold Z_3 is given by $Z_2 = 0$ which is a submanifold of Y_4. By using the adjuction formula, we have

$$c(Z_3) = \frac{c(Y_4)}{(1+\pi_F^*r + p^*t)}. \tag{2.33}$$

It follows from the Calabi-Yau condition $c_1(Z_3) = 0$, Eq. (2.30), and Eq. (2.32) that

$$c_1(Y_4) = \pi_F^*r + p^*t, \quad \pi_F^*l = \pi_F^*c_1(B_3) - \pi_F^*r - p^*t = \pi_F^*r + p^*c_1. \tag{2.34}$$

Therefore, the homological classes of sections appearing in Eq. (2.40) are as follows:

$$[x] = 2(\sigma_F + \pi_F^* r + p^* c_1), \;\; [y] = 3(\sigma_F + \pi_F^* r + p^* c_1), \;\; [u] = \sigma_F, \tag{2.35}$$

$$[Z_1] = \pi_F^* r, \;\; [Z_2] = \pi_F^* r + p^* t, \; [b_m] = p^*[(6-m)c_1 - t], \;\; m = 0,2,3,4. \tag{2.36}$$

Following the proposal in [1], we define the spectral divisor $\mathcal{D}_F^{(n)}$ of Y_4 by

$$\mathcal{D}_F^{(n)} : b_0(Z_1 u)^n + b_2(Z_1 u)^{n-2} x + b_3 (Z_1 u)^{n-3} y + ... = 0, \tag{2.37}$$

where the last term is $b_n x^{n/2}$ for n even, or $b_n x^{(n-3)/2} y$ for n odd. The projection map is denoted by $p_{\mathcal{D}_F^{(n)}} : \mathcal{D}_F^{(n)} \to B_3$. Let $\gamma_F^{(n)}$ be a $(1,1)$ form on $\mathcal{D}_F^{(n)}$. It was proposed in [1] that the net chirality formula for matter in the representation $\mathbf{r}$ can be computed as

$$N_{\mathbf{r}} = [\widehat{\Sigma}_{\mathbf{r}}] \cdot \mathcal{G}_F^{(n)} \cdot p^*_{\mathcal{D}_F^{(n)}} B_2, \tag{2.38}$$

where $[\widehat{\Sigma}_{\mathbf{r}}]$ is the dual matter surface inside $\mathcal{D}_F^{(n)}$ and $\mathcal{G}_F^{(n)}$ is defined by $\gamma_F^{(n)} = [\mathcal{D}_F^{(n)}] \cdot \mathcal{G}_F^{(n)}$ for given $\gamma_F^{(n)}$. For the case of $n = 4$, we have

$$y^2 = x^3 + f_4 (Z_1 u)^4 x + g_6 (Z_1 u)^6 + Z_2 [b_0 (Z_1 u)^5 + b_2 (Z_1 u)^3 x + b_3 (Z_1 u)^2 y + b_4 (Z_1 u) x^2]. \tag{2.39}$$

Note that when $Z_2 = 0$, Eq. (2.39) reduces to Z_3 defined by Eq. (2.2). In this case the spectral divisor is given by

$$\mathcal{D}_F^{(4)} : \; b_0 (Z_1 u)^4 + b_2 (Z_1 u)^2 x + b_3 (Z_1 u) y + b_4 x^2 = 0, \tag{2.40}$$

with a projection map $p_{\mathcal{D}_F^{(4)}} : \mathcal{D}_F^{(4)} \to B_3$. The divisor $\mathcal{D}_F^{(4)}$ can be realized as the union of four exceptional lines of dP_9 comprising a fundamental representation of $SU(4)$ [9,34,75]. With Eq (2.35) and Eq. (2.36), the homological class of $\mathcal{D}_F^{(4)}$ is given by

$$[\mathcal{D}_F^{(4)}] = 4(\sigma_F + \pi_F^* r) + p^*(6c_1 - t). \tag{2.41}$$

The traceless flux $\gamma_F^{(4)}$ can be computed as

$$\begin{aligned} \gamma_F^{(4)} &= (4 - p^*_{\mathcal{D}_F^{(4)}} p_{\mathcal{D}_F^{(4)} *})([\mathcal{D}_F^{(4)}] \cdot \sigma_F) \\ &= [\mathcal{D}_F^{(4)}] \cdot [4\sigma_F - p^*(2c_1 - t)], \end{aligned} \tag{2.42}$$

where the relation $\sigma_F \cdot (\sigma_F + \pi_F^* r + p^* c_1) = 0$ has been used. It follows from Eq. (2.42) that $\mathcal{G}_F^{(4)} = 4\sigma_F - p^*(2c_1 - t)$. To calculate the chiral spectrum, we need to calculate the homological classes of dual matter surfaces. The dual matter surface $\widehat{\Sigma}_{\mathbf{16}}$ sits in the locus of the intersection $\{(Z_1 u) = 0\} \cap \{b_4 = 0\}$ and then its homological class is given by

$$[\widehat{\Sigma}_{\mathbf{16}}] = (\sigma_F + \pi_F^* r) \cdot p^*(2c_1 - t). \tag{2.43}$$

By using the net chirality formula Eq. (2.38), we obtain

$$\begin{aligned} N_{\mathbf{16}} &= [\widehat{\Sigma}_{\mathbf{16}}] \cdot \mathcal{G}_F^{(4)} \cdot \pi_F^* r \\ &= -(6c_1 - t) \cdot_{B_2} (2c_1 - t). \end{aligned} \tag{2.44}$$

On the other hand, the dual matter surface $\widehat{\Sigma}_{\mathbf{10}}$ sits in the locus of $\mathcal{D}_F^{(4)} \cap \tau \mathcal{D}_F^{(4)}$ where τ is a $\mathbb{Z}_2$ involution $y \to -y$ acting on $\mathcal{D}_F^{(4)}$ while keeping x, u, and Z_1 intact. More precisely, the intersection loci of $\mathcal{D}_F^{(4)} \cap \tau \mathcal{D}_F^{(4)}$ are determined by

$$\begin{cases} b_3 (Z_1 u) y = 0 \\ b_0 (Z_1 u)^4 + b_2 (Z_1 u)^2 x + b_4 x^2 = 0. \end{cases} \tag{2.45}$$

It follows from Eq. (2.45) that the homological class of dual matter surface $\widehat{\Sigma}_{\mathbf{10}}$ is

$$\begin{aligned} [\widehat{\Sigma}_{\mathbf{10}}] &= [\mathcal{D}_F^{(4)}] \cdot [\mathcal{D}_F^{(4)}] - [Z_1 u] \cdot [b_4] - [y][b_4 x^2] - 2[x][Z_1] \\ &= (\sigma_F + \pi_F^* r) \cdot p^*(12c_1 - 4t) + p^*(6c_1 - t) \cdot p^*(3c_1 - t). \end{aligned} \tag{2.46}$$

By using Eq. (2.38), the net chirality of matter **10** is

$$\begin{aligned} N_{\mathbf{10}} &= [\widehat{\Sigma}_{\mathbf{10}}] \cdot \mathcal{G}_F^{(4)} \cdot \pi_F^* r \\ &= 0. \end{aligned} \tag{2.47}$$

These results agree with the computations in the dual heterotic string framework by identifying $\mathcal{D}_F^{(4)}|_{Z_3} = \mathcal{C}_H^{(4)}$ and $b_m|_{Z_3} = a_m$, which gives rise to the relation $\eta = 6c_1 - t$. It was argued in [1] that the chirality formula Eq. (2.38) can be applied to the cases of F-theory compactifications without heterotic duals. In section 3, we shall briefly review semi-local $SU(4)$ cover construction [61] and its global completion [1].

3 Global Completion of $SU(4)$ Cover

In this section we shall discuss the case of an F-theory compactification on an elliptically fibered Calabi-Yau fourfold without a heterotic dual. We first briefly review the semi-local $SU(4)$ spectral cover construction studied in [61]. In the second part we construct the $SU(4)$ spectral divisor following the proposal in [1].

3.1 Semi-local $SU(4)$ Cover

Let us consider an elliptically fibered Calabi-Yau fourfold $\pi : Z_4 \to B_3$ with a section $\sigma : B_3 \to Z_4$ and B_2 to be a divisor in B_3 where Z_4 exhibits a D_5 singularity. Generically, Z_4 can be described by the Tate form as follows:

$$y^2 = x^3 + b_4 x^2 z + b_3 y z^2 + b_2 x z^3 + b_0 z^5. \tag{3.1}$$

Let us define $t \equiv -c_1(N_{B_2/B_3})$ and then the homological classes of the sections x, y, z, and b_m can be expressed as

$$[x] = 3(c_1 - t),\ [y] = 2(c_1 - t),\ [z] = -t,\ [b_m] = (6-m)c_1 - t = \eta - mc_1. \tag{3.2}$$

Recall that locally Z_4 can be described by an ALE fibration over B_2. Pick a point $p \in B_2$, the fiber is an ALE space denoted by ALE_p. The ALE space can be constructed by resolving an orbifold $\mathbb{C}^2/\Gamma_{ADE}$, where Γ_{ADE} is a discrete subgroup of $SU(2)$.[15] It was shown that the intersection matrix of the exceptional 2-cycles corresponds to the Cartan matrix of ADE type, which can be described by ADE Dynkin diagrams. Let us take $\alpha_i \in H_2(\text{ALE}_p, \mathbb{Z})$, $i = 1, 2, ..., 8$ to be the roots[16] of E_8 and the extended E_8 Dynkin diagram with roots and Dynkin indices to be shown in Fig 1. Notice that $\alpha_{-\theta}$

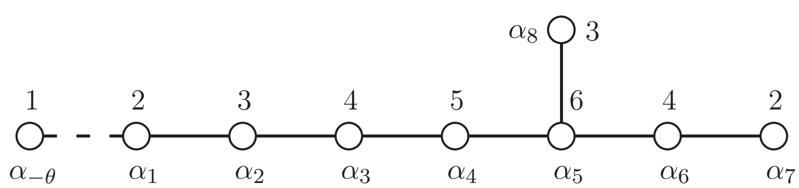

Figure 1: The extended E_8 Dynkin diagram and indices

is the highest root and satisfies the condition $\alpha_{-\theta} + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = 0$. To obtain $SO(10)$, we take the volume of the cycles $\{\alpha_4, \alpha_5, ..., \alpha_8\}$ to be vanishing and then $SU(4)$ is generated by $\{\alpha_1, \alpha_2, \alpha_3\}$. An enhancement to E_6 happens when α_3 or any of its images under the Weyl permutation shrinks to zero size. We define $\{\lambda_1, ..., \lambda_4\}$ to be the periods of these cycles. As described in [10, 41],

[15] For more information, see [76–81].

[16] By abuse of notation, the corresponding exceptional 2-cycles are also denoted by α_i.

theses λ_i are encoded in the coefficients b_m as follows:

$$\begin{cases} \sum_i \lambda_i = \frac{b_1}{b_0} = 0 \\ \sum_{i<j} \lambda_i\lambda_j = \frac{b_2}{b_0} \\ \sum_{i<j<k} \lambda_i\lambda_j\lambda_k = \frac{b_3}{b_0} \\ \prod_l \lambda_l = \frac{b_4}{b_0}. \end{cases} \tag{3.3}$$

Equivalently, $\{\lambda_1, ..., \lambda_4\}$ are the roots of the equation

$$b_0 \prod_{k=1}^{4} (s + \lambda_k) = b_0 s^4 + b_2 s^2 + b_3 s + b_4 = 0. \tag{3.4}$$

When $p \in B_2$ varies along B_2, Eq. (3.4) defines a fourfold cover $\mathcal{C}^{(4)}$ over B_2, the semi-local $SU(4)$ spectral cover. This cover can be described as a section of the canonical bundle $K_{B_2} \to B_2$. When λ_i vanish, $\prod_i \lambda_i = b_4 = 0$ and the corresponding gauge group is enhanced to E_6, which implies that the matter field **16** is localized at the locus $\{b_4 = 0\}$. On the other hand,the matter field **10** corresponds to the anti-symmetric representation **6** of $SU(4)$, associated with a sixfold cover $\mathcal{C}^{(6)}_{\wedge^2 V}$ over B_2. This associated cover $\mathcal{C}^{(6)}_{\wedge^2 V}$ is given by

$$\mathcal{C}^{(6)}_{\wedge^2 V}: \quad b_0^2 \prod_{i<j} (s + \lambda_i + \lambda_j) = b_0^2 s^6 + 2b_0 b_2 s^4 + (b_2^2 - 4b_0 b_4) s^2 - b_3^2 = 0. \tag{3.5}$$

Since matter **10** corresponds to $\lambda_i + \lambda_j = 0, \ i \neq j$, it follows from Eq. (3.5) that $b_3 = 0$, which means that matter **10** is localized at the locus $\{b_3 = 0\}$ as we expected from the D_6 singularity of Eq. (3.1). From the discussion above, we see that spectral cover indeed encodes the information of singularities and gauge group enhancements. Moreover, we can construct a Higgs bundle to calculate the chiral spectrum for matter **16** and **10** by switching on a line bundle on the cover. Let us define X to be the total space of the canonical bundle K_{B_2} over B_2. Note that X is a local Calabi-Yau threefold. but X is non-compact. To obtain a compact space, one can compactify X to the total space $\bar{X}$ of the projective bundle over B_2, *i.e.*

$$\bar{X} = \mathbb{P}(\mathcal{O}_{B_2} \oplus K_{B_2}), \tag{3.6}$$

with a projection map $\pi : \bar{X} \to B_2$, where $\mathcal{O}_{B_2}$ is the trivial bundle over B_2. Notice that $\bar{X}$ is compact but no longer a Calabi-Yau threefold. Let $\mathcal{O}_P(1)$ be a hyperplane section of $\mathbb{P}^1$ fiber and denote its first Chern class by σ_∞. We define the homogeneous coordinates of the fiber by $[U : W]$. Note that $\{U = 0\}$ and $\{W = 0\}$ are sections of $\mathcal{O}_P(1) \otimes K_S$ and $\mathcal{O}_P(1)$, while the class of $\{U = 0\}$ and $\{W = 0\}$ are $\sigma \equiv \sigma_\infty - \pi^* c_1$ and σ_∞, respectively. By the emptiness of intersection of $\{U = 0\}$ and $\{W = 0\}$, we obtain $\sigma \cdot \sigma = -\sigma \cdot \pi^* c_1$. We define the affine coordinate s by $s = U/W$ and then the $SU(4)$ cover given by Eq. (3.4) can be written as

$$\mathcal{C}^{(4)} : \quad b_0 U^4 + b_2 U^2 W^2 + b_3 U W^3 + b_4 W^4 = 0 \tag{3.7}$$

with a projection map $p_{\mathcal{C}^{(4)}} : \mathcal{C}^{(4)} \to B_2$. It is not difficult to see that the homological class $[\mathcal{C}^{(4)}]$ of the cover $\mathcal{C}^{(4)}$ is given by $[\mathcal{C}^{(4)}] = 4\sigma + \pi^*\eta$. We can calculate the matter **16** curve by intersecting $[\mathcal{C}^{(4)}]$ with σ

$$[\mathcal{C}^{(4)}] \cap \sigma = (4\sigma + \pi^*\eta) \cdot \sigma = \sigma \cdot \pi^*(\eta - 4c_1). \tag{3.8}$$

On the other hand, it follows from Eq. (3.5) that the homological class of the cover $\mathcal{C}^{(6)}_{\wedge^2 V}$ is given by

$$[\mathcal{C}^{(6)}_{\wedge^2 V}] = 6\sigma + 2\pi^*\eta. \tag{3.9}$$

However, the cover $\mathcal{C}^{(6)}_{\wedge^2 V}$ is generically singular. To solve this problem, one can consider intersection $\mathcal{C}^{(4)} \cap \tau\mathcal{C}^{(4)}$ and define [72]

$$[D] = [\mathcal{C}^{(4)}] \cap ([\mathcal{C}^{(4)}] - 3\sigma_\infty - \sigma). \tag{3.10}$$

where τ is a $\mathbb{Z}_2$ involution $W \to -W$ acting on the spectral cover $\mathcal{C}^{(4)}$. To obtain chiral spectrum, we turn on a spectral line bundle $\mathcal{L}$ on the cover $\mathcal{C}^{(4)}$. The corresponding Higgs bundle is given by $p_{\mathcal{C}^{(4)}*}\mathcal{L}$. For $SU(n)$ bundles, it is required that $c_1(p_{\mathcal{C}^{(4)}*}\mathcal{L}) = 0$. It follows that

$$p_{\mathcal{C}^{(4)}*} c_1(\mathcal{L}) - \frac{1}{2} p_{\mathcal{C}^{(4)}*} r^{(4)} = 0, \tag{3.11}$$

where $r^{(4)}$ is the ramification divisor given by $r^{(4)} = p_{\mathcal{C}^{(4)}*} c_1 - c_1(\mathcal{C}^{(4)})$. It is convenient to define the cover flux $\gamma^{(4)}$ by

$$c_1(\mathcal{L}) = \lambda \gamma^{(4)} + \frac{1}{2} r^{(4)}, \tag{3.12}$$

where λ is a rational number used to compensate the non-integral class $\frac{1}{2} r^{(4)}$ such that $c_1(\mathcal{L}) \in H_4(\bar{X}, \mathbb{Z})$. The traceless condition $c_1(p_{\mathcal{C}^{(4)}*}\mathcal{L}) = 0$ is then equivalent to the condition $p_{\mathcal{C}^{(4)}*}\gamma^{(4)} = 0$. Up to multiplication of a constant, the only choice of $\gamma^{(4)}$ satisfying the traceless condition is

$$\gamma^{(4)} = (4 - p^*_{\mathcal{C}^{(4)}} p_{\mathcal{C}^{(4)}*})([\mathcal{C}^{(4)}] \cdot \sigma). \tag{3.13}$$

Since the first Chern class of a line bundle must be integral, it follows that λ and $\gamma^{(4)}$ have to obey the following quantization condition

$$\lambda\gamma^{(4)} + \frac{1}{2}[p^*_{\mathcal{C}^{(4)}}c_1 - c_1(\mathcal{C}^{(4)})] \in H_4(\bar{X},\mathbb{Z}). \tag{3.14}$$

With the given cover flux $\gamma^{(4)}$, the net chirality of matter **16** is calculated by [34,41]

$$N_{\mathbf{16}} = ([\mathcal{C}^{(4)}]\cdot\sigma)\cdot\lambda\gamma^{(4)} = -\lambda\eta\cdot(\eta - 4c_1). \tag{3.15}$$

On the other hand, the homological class of matter **10** curve is given by Eq. (3.10). It turns out that the net chirality of matter **10** is computed as [34]

$$N_{\mathbf{10}} = [D]\cdot\gamma^{(4)} = 0. \tag{3.16}$$

One can find that the computations of net chirality agree with those from heterotic spectral cover. Unlike the representation **10** in $SU(5)$ case, the **10** in $SO(10)$ is a real representation. Therefore, it is impossible to engineer a chiral spectrum of **10**'s by using a generic $SU(4)$ spectral cover. From Eq. (3.15) and Eq. (3.16), we obtain an $SO(10)$ model with $-\lambda\eta\cdot(\eta-4c_1)$ copies of matter on the **16** curve and nothing on the **10** curve. The flux does not have many degrees of freedom to tune and the candidate of **10** Higgs is absent. Therefore, we shall consider factorizations of the $SU(4)$ cover $\mathcal{C}^{(4)}$ to enrich the configuration along the line of the $SU(5)$ cover studied in [39,40,44,45]. Before studying the cove factorizations, we shall construct an $SU(4)$ spectral divisor motivated from heterotic/F-theory duality [1] in section 3.2.

3.2 $SU(4)$ Spectral Cover Divisor

Recall that Z_4 is an elliptically fibered Calabi-Yau fourfold $\pi : Z_4 \to B_3$ with a section $\sigma : B_3 \to Z_4$. In general, Z_4 can be described by the Weierstrass model[17]

$$y^2 = x^3 + fxu^4 + gu^6, \tag{3.17}$$

where f and g are sections of $K_{B_3}^{-4}$ and $K_{B_3}^{-6}$, respectively. We now consider the case that Z_4 exhibits a D_5 singularity along a divisor B_2 inside B_3. We define z to be a

[17]Recall that Z_4 can be embedded as a hypersurface of $W\mathbb{P}^3_{2,3,1}$ -fibration over B_3. It follows from the Calabi-Yau condition $c_1(Z_4) = 0$ that x, y, and u are sections of $\mathcal{O}_{B_3}(2\sigma)\otimes K_{B_3}^{-2}$, $\mathcal{O}_{B_3}(3\sigma)\otimes K_{B_3}^{-3}$, and $\mathcal{O}_{B_3}(\sigma)$, respectively.

section of the normal bundle N_{B_2/B_3} of B_2 in B_3. Locally we can expand f and g in the Weierstrass model Eq. (3.17) in terms of z. With suitable choice of variables, we obtain

$$y^2 = x^3 + u(zu)[b_0(zu)^4 + b_2(zu)^2 x + b_3(zu)y + b_4 x^2] + \mathcal{O}(z;u), \tag{3.18}$$

where $\mathcal{O}(z;u)$ stands for the higher order terms of z for each fixed order in u. Following the proposal in [1], we define the $SU(4)$ spectral divisor as

$$\mathcal{D}^{(4)}:\ b_0(zu)^4 + b_2(zu)^2 x + b_3(zu)y + b_4 x^2 = 0 \tag{3.19}$$

with a projection map $p_{\mathcal{D}^{(4)}} : \mathcal{D}^{(4)} \to B_2$. Note that local behavior of $\mathcal{D}^{(4)}$ is the same as the union of the exceptional lines described by Eq. (2.40) and that the homological classes of x, y, u, z, and b_m in Eq. (3.19) are

$$[x] = 2[\sigma + \pi^* c_1(B_3)],\ \ [y] = 3[\sigma + \pi^* c_1(B_3)],\ \ [u] = \sigma,\ \ [z] = \pi^* B_2 \tag{3.20}$$

$$[b_m] = (6-m)\pi^* c_1(B_3) - (5-m)\pi^* B_2, \quad m = 0, 2, 3, 4. \tag{3.21}$$

The homological class of the divisor $\mathcal{D}^{(4)}$ is then given by

$$[\mathcal{D}^{(4)}] = 4\sigma + \pi^*[6c_1(B_3) - B_2]. \tag{3.22}$$

In this case the dual matter **16** surface $\widehat{\Sigma}_{\mathbf{16}}$ is determined by the locus of $\{(zu) = 0\} \cap \{b_4 = 0\}$ with homological class

$$[\widehat{\Sigma}_{\mathbf{16}}] = (\sigma + \pi^* B_2) \cdot \pi^*[2c_1(B_3) - B_2]. \tag{3.23}$$

On the other hand, the dual matter **10** surface sits inside the locus of the intersection $\mathcal{D}^{(4)} \cap \tau\mathcal{D}^{(4)}$, where τ is a $\mathbb{Z}_2$ involution acting on the cover by $y \to -y$ while keeping x, u, and z invariant. More precisely, the intersection $\mathcal{D}^{(4)} \cap \tau\mathcal{D}^{(4)}$ is given by

$$\begin{cases} b_3(zu)y = 0 \\ b_0(zu)^4 + b_2(zu)^2 x + b_4 x^2 = 0. \end{cases} \tag{3.24}$$

We can compute the homological class $[\widehat{\Sigma}_{\mathbf{10}}]$ of dual matter **10** surface as

$$\begin{aligned} [\widehat{\Sigma}_{\mathbf{10}}] &= [\mathcal{D}^{(4)}] \cdot [\mathcal{D}^{(4)}] - [zu] \cdot [b_4] - [y] \cdot [b_4 x^2] - 2[x] \cdot [z] \\ &= \sigma \cdot \pi^*[12c_1(B_3) - 8B_2] + \pi^*[3c_1(B_3) - 2B_2] \cdot \pi^*[6c_1(B_3) - B_2]. \end{aligned} \tag{3.25}$$

To obtain chiral spectrum, we turn on a spectral line bundle $\mathcal{N}$ over $\mathcal{D}^{(4)}$. The corresponding Higgs bundle is given by $E = p_{\mathcal{D}^{(4)}*}\mathcal{N}$. For $SU(n)$ bundles, it is required that $c_1(E) = 0$. It follows that

$$c_1(p_{\mathcal{D}^{(4)}*}\mathcal{N}) = p_{\mathcal{D}^{(4)}*} c_1(\mathcal{N}) - \frac{1}{2} p_{\mathcal{D}^{(4)}*} \widehat{r}^{(4)} = 0, \tag{3.26}$$

where $\widehat{r}^{(4)}$ is the ramification divisor given by $\widehat{r}^{(4)} = p_{\mathcal{D}^{(4)}*}c_1(B_3) - c_1(\mathcal{D}^{(4)})$. It is convenient to define the flux $\widehat{\gamma}^{(4)}$ by

$$c_1(\mathcal{N}) = \lambda\widehat{\gamma}^{(4)} + \frac{1}{2}\widehat{r}^{(4)}, \tag{3.27}$$

where λ is a rational number used to compensate the non-integral class $\frac{1}{2}\widehat{r}^{(4)}$ such that $c_1(\mathcal{N}) \in H_2(\mathcal{D}^{(4)}, \mathbb{Z})$. The traceless condition $c_1(p_{\mathcal{D}^{(4)}*}\mathcal{N}) = 0$ is then equivalent to the condition $p_{\mathcal{D}^{(4)}*}\widehat{\gamma}^{(4)} = 0$. Up to multiplication of a constant, the only choice of $\widehat{\gamma}^{(4)}$ satisfying the traceless condition is

$$\widehat{\gamma}^{(4)} = (4 - p^*_{\mathcal{D}^{(4)}}p_{\mathcal{D}^{(4)}*})([\mathcal{D}^{(4)}] \cdot \sigma). \tag{3.28}$$

Since the first Chern class of a line bundle must be integral, it follows that λ and $\widehat{\gamma}^{(4)}$ have to obey the following quantization condition

$$\lambda\widehat{\gamma}^{(4)} + \frac{1}{2}[p^*_{\mathcal{D}^{(4)}}c_1(B_3) - c_1(\mathcal{D}^{(4)})] \in H_2(\mathcal{D}^{(4)}, \mathbb{Z}). \tag{3.29}$$

In the case of $SU(4)$ spectral divisor, the traceless flux $\widehat{\gamma}^{(4)}$ is given by

$$\widehat{\gamma}^{(4)} = (4 - p^*_{\mathcal{D}^{(4)}}p_{\mathcal{D}^{(4)}*})([\mathcal{D}^{(4)}] \cdot \sigma) = [\mathcal{D}^{(4)}] \cdot \{4\sigma - \pi^*[2c_1(B_3) - B_2]\}. \tag{3.30}$$

It follows from Eq. (3.30) and the definition $\widehat{\gamma}^{(4)} = [\mathcal{D}^{(4)}] \cdot \mathcal{G}^{(4)}$ that $\mathcal{G}^{(4)} = 4\sigma - \pi^*[2c_1(B_3) - B_2]$. With the given cover flux $\widehat{\gamma}^{(4)}$, the net chirality of matter **16** and **10** are respectively given by

$$\begin{aligned} N_{\mathbf{16}} &= [\widehat{\Sigma}_{\mathbf{16}}] \cdot \mathcal{G}^{(4)} \cdot \pi^* B_2 \\ &= -(6c_1 - t) \cdot_{B_2} (2c_1 - t), \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} N_{\mathbf{10}} &= [\widehat{\Sigma}_{\mathbf{10}}] \cdot \mathcal{G}^{(4)} \cdot \pi^* B_2 \\ &= 0, \end{aligned} \tag{3.32}$$

where the fact that $B_2|_{B_2} = -t$ and $c_1(B_3)|_{B_2} = c_1 - t$ has been used. We found agreement between net chirality from semi-local spectral cover and from spectral divisor construction.

4 Chirality

In this section we consider flipped $SU(5)$ GUTs in F-theory. As mentioned in section 1, the construction contains two steps. The first step is to break E_8 down to $SO(10)$

by using $SU(4)$ spectral covers. The second step is to turn on $U(1)_X$ fluxes to break $SO(10)$ down to $SU(5) \times U(1)_X$. In what follows we shall focus on the first step, namely breaking E_8 down to $SO(10)$ by using a semi-local $SU(4)$ spectral cover and its global completion, $SU(4)$ spectral divisors. We also analyze the chiral spectra induced by the fluxes. For the analysis of $U(1)_X$ fluxes and numerical models, we refer readers to [61] for the details. We first briefly review $(3,1)$ and $(2,2)$ factorizations of the semi-local $SU(4)$ spectral cover and induced chirality. Then we construct the factorized $SU(4)$ spectral divisor for each factorization and calculate the chirality induced by the fluxes.

4.1 Semi-local $SU(4)$ Spectral Cover

4.1.1 Constraints

Before computing the chiral spectra, we take a moment to analyze the constraints for the cover fluxes. Let us consider the case of the cover factorization $\mathcal{C}^{(n)} \to \mathcal{C}^{(l)} \times \mathcal{C}^{(m)}$. To obtain well-defined cover fluxes and maintain supersymmetry, we impose the following constraints [40]:

$$c_1(p_{\mathcal{C}^{(l)}*}\mathcal{L}^{(l)}) + c_1(p_{\mathcal{C}^{(m)}*}\mathcal{L}^{(m)}) = 0, \tag{4.1}$$

$$c_1(\mathcal{L}^{(k)}) \in H_2(\mathcal{C}^{(k)}, \mathbb{Z}), \quad k = l, m, \tag{4.2}$$

$$[c_1(p_{\mathcal{C}^{(l)}*}\mathcal{L}^{(l)}) - c_1(p_{\mathcal{C}^{(m)}*}\mathcal{L}^{(m)})] \cdot_{B_2} [\omega] = 0, \tag{4.3}$$

where $p_{\mathcal{C}^{(k)}}$ denotes the projection map $p_{\mathcal{C}^{(k)}} : \mathcal{C}^{(k)} \to B_2$, $\mathcal{L}^{(k)}$ is a line bundle over $\mathcal{C}^{(k)}$ and $[\omega]$ is an ample divisor dual to a Kähler form of B_2. The first constraint Eq. (4.1) is the traceless condition for the induced Higgs bundle[18]. The second constraint Eq. (4.2) requires that the first Chern class of a well-defined line bundle $\mathcal{L}^{(k)}$ must be integral. The third constraint states that the 2-cycle $[c_1(p_{\mathcal{C}^{(l)}*}\mathcal{L}^{(l)}) - c_1(p_{\mathcal{C}^{(m)}*}\mathcal{L}^{(m)})]$ in B_2 is supersymmetic. Note that Eq. (4.1) can be expressed as

$$p_{\mathcal{C}^{(l)}*}c_1(\mathcal{L}^{(l)}) - \frac{1}{2}p_{\mathcal{C}^{(l)}*}r^{(l)} + p_{\mathcal{C}^{(m)}*}c_1(\mathcal{L}^{(m)}) - \frac{1}{2}p_{\mathcal{C}^{(m)}*}r^{(m)} = 0, \tag{4.4}$$

[18]We may think of Eq. (4.2) as the traceless condition of an $SU(4)$ bundle V_4 over B_2 split into $V_3 \oplus L$ with $V_3 = p_{a*}\mathcal{L}^{(a)}$ and $L = p_{b*}\mathcal{L}^{(b)}$. Therefore, the traceless condition of V_4 can be expressed by $c_1(V_4) = c_1(p_{a*}\mathcal{L}^{(a)}) + c_1(p_{b*}\mathcal{L}^{(b)}) = 0$.

where $r^{(l)}$ and $r^{(m)}$ are the ramification divisors for the maps $p_{\mathcal{C}^{(l)}}$ and $p_{\mathcal{C}^{(m)}}$, respectively. Recall that the ramification divisor $r^{(k)}$ is defined by

$$r^{(k)} = p^*_{\mathcal{C}^{(k)}} c_1 - c_1(\mathcal{C}^{(k)}), \quad k = l, m. \tag{4.5}$$

It is convenient to define cover fluxes $\gamma^{(k)}$ as

$$c_1(\mathcal{L}^{(k)}) = \gamma^{(k)} + \frac{1}{2} r^{(k)}, \quad k = l, m. \tag{4.6}$$

With Eq. (4.6), the traceless condition Eq. (4.1) can be expressed as $p_{\mathcal{C}^{(l)}*}\gamma^{(l)} + p_{\mathcal{C}^{(m)}*}\gamma^{(m)} = 0$. By using Eq. (4.5) and Eq. (4.6), we can recast the quantization condition Eq. (4.2) by $\gamma^{(k)} + \frac{1}{2}[p^*_{\mathcal{C}^{(k)}} c_1 - c_1(\mathcal{C}^{(k)})] \in H_2(\mathcal{C}^{(k)}, \mathbb{Z}), \quad k = l, m$. It follows from Eq. (4.1) that the condition Eq. (4.3) can be reduced to $p_{\mathcal{C}^{(k)}*}\gamma^{(k)} \cdot_{B_2} [\omega] = 0$. We summarize the constraints for the cover fluxes $\gamma^{(k)}$ as follows:

$$p_{\mathcal{C}^{(l)}*}\gamma^{(l)} + p_{\mathcal{C}^{(m)}*}\gamma^{(m)} = 0, \tag{4.7}$$

$$\gamma^{(k)} + \frac{1}{2}[p^*_{\mathcal{C}^{(k)}} c_1 - c_1(\mathcal{C}^{(k)})] \in H_2(\mathcal{C}^{(k)}, \mathbb{Z}), \quad k = l, m, \tag{4.8}$$

$$p_{\mathcal{C}^{(k)}*}\gamma^{(k)} \cdot_{B_2} [\omega] = 0, \quad k = l, m. \tag{4.9}$$

In the next section, we shall calculate the homological classes of matter curves for $(3, 1)$ and $(2, 2)$ factorizations. We also compute the chirality induced by the restriction of the fluxes to each matter curve.

4.1.2 $(3, 1)$ Factorization

We consider the $(3, 1)$ factorization, $\mathcal{C}^{(4)} \to \mathcal{C}^{(a)} \times \mathcal{C}^{(b)}$ corresponding to the factorization of Eq. (3.7) as follows:

$$\mathcal{C}^{(a)} \times \mathcal{C}^{(b)}: \quad (a_0 U^3 + a_1 U^2 W + a_2 U W^2 + a_3 W^3)(d_0 U + d_1 W) = 0. \tag{4.10}$$

By comparing with Eq. (3.7), we can obtain the following decomposition:

$$b_0 = a_0 d_0, \quad b_1 = a_1 d_0 + a_0 d_1 = 0, \quad b_2 = a_2 d_0 + a_1 d_1, \quad b_3 = a_3 d_0 + a_2 d_1, \quad b_4 = a_3 d_1. \tag{4.11}$$

We denote the classes $[d_1]$ by $\pi^*\xi_1$ and then write

$$[d_0] = \pi^*(c_1 + \xi_1), \qquad [a_k] = \pi^*[\eta - (k+1)c_1 - \xi_1], \quad k = 0, 1, 2, 3. \tag{4.12}$$

To solve the traceless condition $b_1 = 0$, we use ansatz $a_0 = \alpha d_0$ and $a_1 = -\alpha d_1$ where $[\alpha] = \pi^*(\eta - 2c_1 - 2\xi_1)$. It is easy to see that the homological classes of $\mathcal{C}^{(a)}$ and $\mathcal{C}^{(b)}$ in $\bar{X}$ are

$$[\mathcal{C}^{(a)}] = 3\sigma + \pi^*(\eta - c_1 - \xi_1), \quad [\mathcal{C}^{(b)}] = \sigma + \pi^*(c_1 + \xi_1). \tag{4.13}$$

To obtain the **10** curves, we follow the method proposed in [39, 40, 44, 72] to calculate the intersection $\mathcal{C}^{(4)} \cap \tau\mathcal{C}^{(4)}$, where τ is the $\mathbb{Z}_2$ involution $\tau : W \to -W$ acting on the spectral cover. Since the calculation is straightforward, we omit the detailed calculation here and only summarize the results in Table 1[19][20].

	$[\mathcal{C}^{(b)(b)}]$	$2[\mathcal{C}^{(a)(b)}]$	$[\mathcal{C}^{(a)(a)}]$
16	$\sigma \cdot \pi^*\xi_1$	-	$\sigma \cdot \pi^*(\eta - 4c_1 - \xi_1)$
10	-	$2[\sigma + \pi^*(c_1 + \xi_1)]$ $\cdot \pi^*(\eta - 3c_1 - \xi_1) + 2\sigma \cdot \pi^*\xi_1$	$[2\sigma + \pi^*(\eta - 2c_1 - \xi_1)]$ $\cdot \pi^*(\eta - 3c_1 - \xi_1) + 2(\sigma + \pi^* c_1) \cdot \pi^*\xi_1$
∞	$\sigma_\infty \cdot \pi^*(c_1 + \xi_1)$	$4\sigma_\infty \cdot \pi^*(c_1 + \xi_1)$	$\sigma_\infty \cdot \pi^*(\eta - c_1 - \xi_1)$ $+2\sigma_\infty \cdot \pi^*(\eta - 2c_1 - 2\xi_1)$

Table 1: Matter curves for the factorization $\mathcal{C}^{(4)} = \mathcal{C}^{(a)} \times \mathcal{C}^{(b)}$.

It follows from Table 1 that the homological classes of **16** curves are

$$[\Sigma_{\mathbf{16}^{(a)}}] = \sigma \cdot \pi^*(\eta - 4c_1 - \xi_1) \tag{4.14}$$

$$[\Sigma_{\mathbf{16}^{(b)}}] = \sigma \cdot \pi^*\xi_1 \tag{4.15}$$

and that the homological classes of $[\Sigma_{\mathbf{10}^{(a)(a)}}]$ and $[\Sigma_{\mathbf{10}^{(a)(b)}}]$ are[21]

$$[\Sigma_{\mathbf{10}^{(a)(a)}}] = [2\sigma + \pi^*(\eta - 2c_1 - \xi_1)] \cdot \ \pi^*(\eta - 3c_1 - \xi_1) + 2(\sigma + \pi^* c_1) \cdot \pi^*\xi_1 \tag{4.16}$$

$$[\Sigma_{\mathbf{10}^{(a)(b)}}] = [\sigma + \pi^*(c_1 + \xi_1)] \cdot \ \pi^*(\eta - 3c_1 - \xi_1) + \sigma \cdot \pi^*\xi_1. \tag{4.17}$$

[19]To simplify notations, we denote $\mathcal{C}^{(k)} \cap \tau\mathcal{C}^{(l)}$ by $\mathcal{C}^{(k)(l)}$ and notice that $[\mathcal{C}^{(k)(l)}] = [\mathcal{C}^{(l)(k)}]$.

[20]To avoid a singularity of non-Kodaira type, we impose the condition $\xi_1 \cdot_{B_2} (c_1 + \xi_1) = 0$. Therefore, $[\Sigma_{\mathbf{10}^{(b)(b)}}] = \pi^*\xi_1 \cdot \pi^*(c_1 + \xi_1)$=0.

[21]It follows from Eqs. (4.16) and (4.17) that $[\Sigma_{\mathbf{10}^{(a)(a)}}]$ and $[\Sigma_{\mathbf{10}^{(a)(b)}}]$ correspond to the same matter curve in B_2 with homological class $\eta - 3c_1$. In other words, $\Sigma_{\mathbf{10}^{(a)(a)}}$ and $\Sigma_{\mathbf{10}^{(a)(b)}}$ both are lifts of the same curve in B_2. The **10** matter curve inside the cover $\mathcal{C}^{(4)}$ is actually 4-sheeted cover of the corresponding matter curve in B_2. A nice description of the cover structure for the **10** curve can be found in [34].

For the $(3,1)$ factorization, the ramification divisors for the spectral covers $\mathcal{C}^{(a)}$ and $\mathcal{C}^{(b)}$ are given by

$$\begin{aligned} r^{(a)} &= [\mathcal{C}^{(a)}]\cdot[\sigma+\pi^*(\eta-2c_1-\xi_1)] \\ r^{(b)} &= [\mathcal{C}^{(b)}]\cdot(-\sigma+\pi^*\xi_1), \end{aligned} \tag{4.18}$$

respectively. We define traceless fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$ by

$$\begin{aligned} \gamma_0^{(a)} &= (3-p^*_{\mathcal{C}^{(a)}}p_{\mathcal{C}^{(a)}*})\gamma^{(a)} = [\mathcal{C}^{(a)}]\cdot[3\sigma-\pi^*(\eta-4c_1-\xi_1)] \\ \gamma_0^{(b)} &= (1-p^*_{\mathcal{C}^{(b)}}p_{\mathcal{C}^{(b)}*})\gamma^{(b)} = [\mathcal{C}^{(b)}]\cdot(\sigma-\pi^*\xi_1)\,, \end{aligned} \tag{4.19}$$

where $\gamma^{(a)}$ and $\gamma^{(b)}$ are non-traceless fluxes and defined by

$$\gamma^{(a)} = [\mathcal{C}^{(a)}]\cdot\sigma,\quad \gamma^{(b)} = [\mathcal{C}^{(b)}]\cdot\sigma. \tag{4.20}$$

Then we can calculate the restriction of fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$ to each matter curve. We omit the calculation here and only summarize the results in Table 2. We also can

	$\gamma_0^{(b)}$	$\gamma_0^{(a)}$
$\mathbf{16}^{(b)}$	$-\xi_1\cdot_{B_2}(c_1+\xi_1)$	0
$\mathbf{16}^{(a)}$	0	$-(\eta-c_1-\xi_1)\cdot_{B_2}(\eta-4c_1-\xi_1)$
$\mathbf{10}^{(a)(b)}$	0	$-(\eta-3c_1-3\xi_1)\cdot_{B_2}(\eta-4c_1-\xi_1)$
$\mathbf{10}^{(a)(a)}$	0	$(\eta-3c_1-3\xi_1)\cdot_{B_2}(\eta-4c_1-\xi_1)$

Table 2: Chirality induced by the fluxes $\gamma_0^{(a)}$ and $\gamma_0^{(b)}$.

define additional fluxes $\delta^{(a)}$ and $\delta^{(b)}$ by

$$\begin{aligned} \delta^{(a)} &= (1-p^*_{\mathcal{C}^{(b)}}p_{\mathcal{C}^{(a)}*})\gamma^{(a)} = [\mathcal{C}^{(a)}]\cdot\sigma-[\mathcal{C}^{(b)}]\cdot\pi^*(\eta-4c_1-\xi_1) \\ \delta^{(b)} &= (3-p^*_{\mathcal{C}^{(a)}}p_{\mathcal{C}^{(b)}*})\gamma^{(b)} = [\mathcal{C}^{(b)}]\cdot 3\sigma-[\mathcal{C}^{(a)}]\cdot\pi^*\xi_1. \end{aligned} \tag{4.21}$$

Another flux we can include is [40]

$$\rho^{(3,1)} = (3p^*_{\mathcal{C}^{(b)}}-p^*_{\mathcal{C}^{(a)}})\rho, \tag{4.22}$$

where $\rho\in H_2(B_2,\mathbb{R})$. We summarize the restriction of fluxes $\delta^{(a)}$, $\delta^{(b)}$ and $\rho^{(3,1)}$ to each matter curve in Table 3.

With Eqs. (4.19), (4.21), and (4.22), we define the universal cover flux Γ to be [40]

$$\Gamma = k_a\gamma_0^{(a)}+k_b\gamma_0^{(b)}+m_a\delta^{(a)}+m_b\delta^{(b)}+\rho^{(3,1)} \equiv \Gamma^{(a)}+\Gamma^{(b)}, \tag{4.23}$$

	$\delta^{(b)}$	$\delta^{(a)}$	$\rho^{(3,1)}$
$\mathbf{16}^{(b)}$	$-3c_1 \cdot_{B_2} \xi_1$	$-\xi_1 \cdot_{B_2} (\eta - 4c_1 - \xi_1)$	$3\rho \cdot_{B_2} \xi_1$
$\mathbf{16}^{(a)}$	$-\xi_1 \cdot_{B_2} (\eta - 4c_1 - \xi_1)$	$-c_1 \cdot_{B_2} (\eta - 4c_1 - \xi_1)$	$-\rho \cdot_{B_2} (\eta - 4c_1 - \xi_1)$
$\mathbf{10}^{(a)(b)}$	$\xi_1 \cdot_{B_2} (2\eta - 9c_1 - 3\xi_1)$	$-(\eta - 3c_1 - \xi_1) \cdot_{B_2} (\eta - 4c_1 - \xi_1)$	$2\rho \cdot_{B_2} (\eta - 3c_1)$
$\mathbf{10}^{(a)(a)}$	$-2\xi_1 \cdot_{B_2} (\eta - 3c_1)$	$(\eta - 3c_1 - \xi_1) \cdot_{B_2} (\eta - 4c_1 - \xi_1)$	$-2\rho \cdot_{B_2} (\eta - 3c_1)$

Table 3: Chirality induced by the fluxes $\delta^{(a)}$, $\delta^{(b)}$, and $\rho^{(3,1)}$.

where $\Gamma^{(a)}$ and $\Gamma^{(b)}$ are defined by

$$\Gamma^{(a)} = [\mathcal{C}^{(a)}] \cdot [(3k_a + m_a)\sigma - \pi^*(k_a(\eta - 4c_1 - \xi_1) + m_b\xi_1 + \rho)]\,, \tag{4.24}$$

$$\Gamma^{(b)} = [\mathcal{C}^{(b)}] \cdot [(k_b + 3m_b)\sigma - \pi^*(k_b\xi_1 + m_a(\eta - 4c_1 - \xi_1) - 3\rho)]\,. \tag{4.25}$$

Note that

$$p_{\mathcal{C}^{(a)}*}\Gamma^{(a)} = -3m_b\xi_1 + m_a(\eta - 4c_1 - \xi_1) - 3\rho, \tag{4.26}$$

$$p_{\mathcal{C}^{(b)}*}\Gamma^{(b)} = 3m_b\xi_1 - m_a(\eta - 4c_1 - \xi_1) + 3\rho. \tag{4.27}$$

Clearly, $\Gamma^{(a)}$ and $\Gamma^{(b)}$ obey the traceless condition $p_{\mathcal{C}^{(a)}*}\Gamma^{(a)} + p_{\mathcal{C}^{(b)}*}\Gamma^{(b)} = 0$. Besides, the quantization condition in this case becomes

$$(3k_a + m_a + \frac{1}{2})\sigma - \pi^*[k_a(\eta - 4c_1 - \xi_1) + m_b\xi_1 + \rho - \frac{1}{2}(\eta - 2c_1 - \xi_1)] \in H_4(\bar{X}, \mathbb{Z}), \tag{4.28}$$

$$(k_b + 3m_b - \frac{1}{2})\sigma - \pi^*[k_b\xi_1 + m_a(\eta - 4c_1 - \xi_1) - 3\rho - \frac{1}{2}\xi_1] \in H_4(\bar{X}, \mathbb{Z}). \tag{4.29}$$

The supersymmetry condition is given by

$$[3m_b\xi_1 - m_a(\eta - 4c_1 - \xi_1) + 3\rho] \cdot_{B_2} [\omega] = 0. \tag{4.30}$$

4.1.3 $(2,2)$ Factorization

In the case of the (2,2) factorization, the cover is split as $\mathcal{C}^{(4)} \to \mathcal{C}^{(d_1)} \times \mathcal{C}^{(d_2)}$. We then can factorize Eq. (3.7) into the following form:

$$\mathcal{C}^{(d_1)} \times \mathcal{C}^{(d_2)}: \quad (e_0U^2 + e_1UW + e_2W^2)(f_0U^2 + f_1UW + f_2W^2) = 0 \tag{4.31}$$

By comparing the coefficients with Eq. (3.7), we obtain

$$b_0 = e_0f_0, \;\; b_1 = e_0f_1 + e_1f_0 = 0, \;\; b_2 = e_0f_2 + e_1f_1 + e_2f_0, \;\; b_3 = e_1f_2 + e_2f_1, \;\; b_4 = e_2f_2. \tag{4.32}$$

By denoting the homological class of f_2 by $\pi^*\xi_2$, the classes of other sections can be written as

$$[f_1] = \pi^*(c_1 + \xi_2), \quad [f_0] = \pi^*(2c_1 + \xi_2), \quad [e_m] = \pi^*[\eta - (m+2)c_1 - \xi_2], \quad m = 0, 1, 2. \tag{4.33}$$

To solve the traceless condition $b_1 = 0$, we impose the condition $e_0 = \beta f_0$ and $e_1 = -\beta f_1$ where $[\beta] = \pi^*(\eta - 4c_1 - 2\xi_2)$. In this case, the homological classes of $\mathcal{C}^{(d_1)}$ and $\mathcal{C}^{(d_2)}$ are given by

$$[\mathcal{C}^{(d_1)}] = 2\sigma + \pi^*(\eta - 2c_1 - \xi_2), \quad [\mathcal{C}^{(d_2)}] = 2\sigma + \pi^*(2c_1 + \xi_2). \tag{4.34}$$

To find the **10** curves, we again follow the method proposed in [39, 40, 44, 72] to calculate the intersection $\mathcal{C}^{(4)} \cap \tau\mathcal{C}^{(4)}$. We omit the detailed calculation here and only summarize the results in Table 4.

	$[\mathcal{C}^{(d_2)(d_2)}]$	$2[\mathcal{C}^{(d_1)(d_2)}]$	$[\mathcal{C}^{(d_1)(d_1)}]$
16	$\sigma \cdot \pi^*\xi_2$	-	$\sigma \cdot \pi^*(\eta - 4c_1 - \xi_2)$
10	$[2\sigma + \pi^*(2c_1 + \xi_2)] \cdot \pi^*(c_1 + \xi_2)$	$2[2\sigma + \pi^*(2c_1 + \xi_2)] \cdot \pi^*(\eta - 4c_1 - \xi_2)$	$\pi^*(\eta - 3c_1 - \xi_2) \cdot \pi^*(\eta - 4c_1 - \xi_2) + 2(\sigma + \pi^*c_1) \cdot \pi^*(c_1 + \xi_2)$
∞	$\sigma_\infty \cdot \pi^*(2c_1 + \xi_2)$	$4\sigma_\infty \cdot \pi^*(2c_1 + \xi_2)$	$\sigma_\infty \cdot \pi^*(\eta - 2c_1 - \xi_2) + 2\sigma_\infty \cdot \pi^*(\eta - 4c_1 - 2\xi_2)$

Table 4: Matter curves for the factorization $\mathcal{C}^{(4)} = \mathcal{C}^{(d_1)} \times \mathcal{C}^{(d_2)}$.

It follows from Table 4 that the homological classes of the factorized **16** curves are

$$[\Sigma_{\mathbf{16}^{(d_1)}}] = \sigma \cdot \pi^*(\eta - 4c_1 - \xi_2), \tag{4.35}$$
$$[\Sigma_{\mathbf{16}^{(d_2)}}] = \sigma \cdot \pi^*\xi_2, \tag{4.36}$$

and that the homological classes of the factorized **10** curves are[22]

$$[\Sigma_{\mathbf{10}^{(d_1)(d_1)}}] = 2(\sigma + \pi^*c_1) \cdot \pi^*(c_1 + \xi_2) + \pi^*(\eta - 3c_1 - \xi_2) \cdot \pi^*(\eta - 4c_1 - \xi_2), \tag{4.37}$$
$$[\Sigma_{\mathbf{10}^{(d_1)(d_2)}}] = [2\sigma + \pi^*(2c_1 + \xi_2)] \cdot \pi^*(\eta - 4c_1 - \xi_2), \tag{4.38}$$
$$[\Sigma_{\mathbf{10}^{(d_2)(d_2)}}] = [2\sigma + \pi^*(2c_1 + \xi_2)] \cdot \pi^*(c_1 + \xi_2). \tag{4.39}$$

[22] It follows from Eqs. (4.37)-(4.39) that $[\Sigma_{\mathbf{10}^{(d_1)(d_1)}}]$ and $[\Sigma_{\mathbf{10}^{(d_2)(d_2)}}]$ correspond to the same curve with class $c_1 + \xi_2$ in B_2, and $[\Sigma_{\mathbf{10}^{(d_1)(d_2)}}]|_\sigma = 2(\eta - 4c_1 - \xi_2)$ in B_2.

In the $(2,2)$ factorization, the ramification divisors $r^{(d_1)}$ and $r^{(d_2)}$ for the covers $\mathcal{C}^{(d_1)}$ and $\mathcal{C}^{(d_2)}$ are given by

$$\begin{aligned} r^{(d_1)} &= [\mathcal{C}^{(d_1)}]\cdot\pi^*(\eta-3c_1-\xi_2), \\ r^{(d_2)} &= [\mathcal{C}^{(d_2)}]\cdot\pi^*(c_1+\xi_2), \end{aligned} \tag{4.40}$$

respectively. We then define traceless cover fluxes $\gamma_0^{(d_1)}$ and $\gamma_0^{(d_2)}$ by

$$\begin{aligned} \gamma_0^{(d_1)} &= (2-p^*_{\mathcal{C}^{(d_1)}}p_{\mathcal{C}^{(d_1)}*})\gamma^{(d_1)} = [\mathcal{C}^{(d_1)}]\cdot[2\sigma-\pi^*(\eta-4c_1-\xi_2)]\,, \\ \gamma_0^{(d_2)} &= (2-p^*_{\mathcal{C}^{(d_2)}}p_{\mathcal{C}^{(d_2)}*})\gamma^{(d_2)} = [\mathcal{C}^{(d_2)}]\cdot(2\sigma-\pi^*\xi_2)\,, \end{aligned} \tag{4.41}$$

where $\gamma^{(d_1)}$ and $\gamma^{(d_21)}$ are non-traceless fluxes and defined by

$$\gamma^{(d_1)} = [\mathcal{C}^{(d_1)}]\cdot\sigma, \quad \gamma^{(d_2)} = [\mathcal{C}^{(d_2)}]\cdot\sigma. \tag{4.42}$$

We summarize the restriction of the fluxes to each factorized curve in Table 5. We

	$\gamma_0^{(d_2)}$	$\gamma_0^{(d_1)}$
$\mathbf{16}^{(d_2)}$	$-\xi_2\cdot_{B_2}(2c_1+\xi_2)$	0
$\mathbf{16}^{(d_1)}$	0	$-(\eta-2c_1-\xi_2)\cdot_{B_2}(\eta-4c_1-\xi_2)$
$\mathbf{10}^{(d_2)(d_2)}$	0	0
$\mathbf{10}^{(d_1)(d_2)}$	0	$-2(\eta-4c_1-2\xi_2)\cdot_{B_2}(\eta-4c_1-\xi_2)$
$\mathbf{10}^{(d_1)(d_1)}$	0	$2(\eta-4c_1-2\xi_2)\cdot_{B_2}(\eta-4c_1-\xi_2)$

Table 5: Chirality induced by the fluxes $\gamma_0^{(d_1)}$ and $\gamma_0^{(d_2)}$.

also can define two fluxes

$$\begin{aligned} \delta^{(d_1)} &= (2-p^*_{\mathcal{C}^{(d_2)}}p_{\mathcal{C}^{(d_1)}*})\gamma^{(d_1)} = [\mathcal{C}^{(d_1)}]\cdot 2\sigma-[\mathcal{C}^{(d_2)}]\cdot\pi^*(\eta-4c_1-\xi_2), \\ \delta^{(d_2)} &= (2-p^*_{\mathcal{C}^{(d_1)}}p_{\mathcal{C}^{(d_2)}*})\gamma^{(d_2)} = [\mathcal{C}^{(d_2)}]\cdot 2\sigma-[\mathcal{C}^{(d_1)}]\cdot\pi^*\xi_2. \end{aligned} \tag{4.43}$$

Another flux we can include is [40]

$$\rho^{(2,2)} = (p^*_{\mathcal{C}^{(d_2)}}-p^*_{\mathcal{C}^{(d_1)}})\rho, \tag{4.44}$$

where $\rho\in H_2(B_2,\mathbb{R})$. We summarize the restriction of the fluxes $\delta^{(d_1)}$, $\delta^{(d_2)}$, and $\rho^{(2,2)}$ to each factorized curve in Table 6.

Again we conclude the universal cover flux to be

$$\Gamma = k_{d_1}\gamma_0^{(d_1)}+k_{d_2}\gamma_0^{(d_2)}+m_{d_1}\delta^{(d_1)}+m_{d_2}\delta^{(d_2)}+\rho^{(2,2)} = \Gamma^{(d_1)}+\Gamma^{(d_2)}, \tag{4.45}$$

	$\delta^{(d_2)}$	$\delta^{(d_1)}$	$\rho^{(2,2)}$
$\mathbf{16}^{(d_2)}$	$-2c_1 \cdot_{B_2} \xi_2$	$-\xi_2 \cdot_{B_2} (\eta - 4c_1 - \xi_2)$	$\rho \cdot_{B_2} \xi_2$
$\mathbf{16}^{(d_1)}$	$-\xi_2 \cdot_{B_2} (\eta - 4c_1 - \xi_2)$	$-2c_1 \cdot_{B_2} (\eta - 4c_1 - \xi_2)$	$-\rho \cdot_{B_2} (\eta - 4c_1 - \xi_2)$
$\mathbf{10}^{(d_2)(d_2)}$	$2\xi_2 \cdot_{B_2} (c_1 + \xi_2)$	$-2(c_1 + \xi_2) \cdot_{B_2} (\eta - 4c_1 - \xi_2)$	$2\rho \cdot_{B_2} (c_1 + \xi_2)$
$\mathbf{10}^{(d_1)(d_2)}$	0	$-2(\eta - 4c_1 - 2\xi_2) \cdot_{B_2} (\eta - 4c_1 - \xi_2)$	0
$\mathbf{10}^{(d_1)(d_1)}$	$-2\xi_2 \cdot_{B_2} (c_1 + \xi_2)$	$2(\eta - 3c_1 - \xi_2) \cdot_{B_2} (\eta - 4c_1 - \xi_2)$	$-2\rho \cdot_{B_2} (c_1 + \xi_2)$

Table 6: Chirality induced by the fluxes $\delta^{(d_1)}$, $\delta^{(d_2)}$, and $\rho^{(2,2)}$.

where

$$\begin{aligned} \Gamma^{(d_1)} &= [\mathcal{C}^{(d_1)}] \cdot \{2(k_{d_1} + m_{d_1})\sigma - \pi^*[k_{d_1}(\eta - 4c_1 - \xi_2) + m_{d_2}\xi_2 + \rho]\}\,, \\ \Gamma^{(d_2)} &= [\mathcal{C}^{(d_2)}] \cdot \{2(k_{d_2} + m_{d_2})\sigma - \pi^*[k_{d_2}\xi_2 + m_{d_1}(\eta - 4c_1 - \xi_2) - \rho]\}\,. \end{aligned} \tag{4.46}$$

Note that

$$p_{\mathcal{C}^{(d_1)}*}\Gamma^{(d_1)} = -2m_{d_2}\xi_2 + 2m_{d_1}(\eta - 4c_1 - \xi_2) - 2\rho, \tag{4.47}$$

$$p_{\mathcal{C}^{(d_2)}*}\Gamma^{(d_2)} = 2m_{d_2}\xi_2 - 2m_{d_1}(\eta - 4c_1 - \xi_2) + 2\rho. \tag{4.48}$$

It is easy to see that $\Gamma^{(d_1)}$ and $\Gamma^{(d_2)}$ satisfy the traceless condition $p_{\mathcal{C}^{(d_1)}*}\Gamma^{(d_1)} + p_{\mathcal{C}^{(d_2)}*}\Gamma^{(d_2)} = 0$. In addition, the quantization condition in this case becomes

$$2(k_{d_1} + m_{d_1})\sigma - \pi^*[k_{d_1}(\eta - 4c_1 - \xi_2) + m_{d_2}\xi_2 + \rho - \frac{1}{2}(\eta - 3c_1 - \xi_2)] \in H_4(\bar{X}, \mathbb{Z}), \tag{4.49}$$

$$2(k_{d_2} + m_{d_2})\sigma - \pi^*[k_{d_2}\xi_2 + m_{d_1}(\eta - 4c_1 - \xi_2) - \rho - \frac{1}{2}(c_1 + \xi_2)] \in H_4(\bar{X}, \mathbb{Z}). \tag{4.50}$$

The supersymmetry condition is then given by

$$[2m_{d_2}\xi_2 - 2m_{d_1}(\eta - 4c_1 - \xi_2) + 2\rho] \cdot_{B_2} [\omega] = 0. \tag{4.51}$$

4.2 Global $SU(4)$ Spectral Divisor

4.2.1 Constraints

Similar to the analysis in the last section, we analyze the constraints for the fluxes of the spectral divisors. It was argued in [1] that these constraints could be consistent with that for the semi-local cover fluxes. Let us consider the case of the cover factorization $\mathcal{D}^{(n)} \to \mathcal{D}^{(l)} \times \mathcal{D}^{(m)}$. To obtain well-defined cover fluxes and maintain supersymmetry, we impose the following constraints [1]:

$$c_1(p_{\mathcal{D}^{(l)}*}\mathcal{N}^{(l)}) + c_1(p_{\mathcal{D}^{(m)}*}\mathcal{N}^{(m)}) = 0, \tag{4.52}$$

$$c_1(\mathcal{N}^{(k)}) \in H_2(\mathcal{D}^{(k)}, \mathbb{Z}), \quad k = l, m, \tag{4.53}$$

where $p_{\mathcal{D}^{(k)}}$ denotes the projection map $p_{\mathcal{D}^{(k)}} : \mathcal{D}^{(k)} \to B_3$, $\mathcal{N}^{(k)}$ is a line bundle over $\mathcal{C}^{(k)}$. The first constraint, Eq. (4.52) is the traceless condition for the induced Higgs bundle. The second constraint, Eq. (4.53) requires that the first Chern class of a well-defined line bundle $\mathcal{N}^{(k)}$ must be integral. Note that Eq. (4.52) can be expressed as

$$p_{\mathcal{D}^{(l)}*}c_1(\mathcal{N}^{(l)}) - \frac{1}{2}p_{\mathcal{D}^{(l)}*}\widehat{r}^{(l)} + p_{\mathcal{D}^{(m)}*}c_1(\mathcal{N}^{(m)}) - \frac{1}{2}p_{\mathcal{D}^{(m)}*}\widehat{r}^{(m)} = 0, \tag{4.54}$$

where $\widehat{r}^{(l)}$ and $\widehat{r}^{(m)}$ are the ramification divisors for the maps $p_{\mathcal{D}^{(l)}}$ and $p_{\mathcal{D}^{(m)}}$, respectively. Recall that the ramification divisor $\widehat{r}^{(k)}$ is defined by

$$\widehat{r}^{(k)} = p^*_{\mathcal{D}^{(k)}}c_1(B_3) - c_1(\mathcal{D}^{(k)}), \quad k = l, m. \tag{4.55}$$

It is convenient to define fluxes $\widehat{\gamma}^{(k)}$ as

$$c_1(\mathcal{N}^{(k)}) = \widehat{\gamma}^{(k)} + \frac{1}{2}\widehat{r}^{(k)}, \quad k = l, m. \tag{4.56}$$

With Eq. (4.56), the traceless condition Eq. (4.52) can be expressed as $p_{\mathcal{D}^{(l)}*}\widehat{\gamma}^{(l)} + p_{\mathcal{D}^{(m)}*}\widehat{\gamma}^{(m)} = 0$. By using Eq. (4.55) and Eq. (4.56), we can recast the quantization condition Eq. (4.53) by $\widehat{\gamma}^{(k)} + \frac{1}{2}[p^*_{\mathcal{D}^{(k)}}c_1(B_3) - c_1(\mathcal{D}^{(k)})] \in H_2(\mathcal{D}^{(k)}, \mathbb{Z}), \quad k = l, m$. We summarize the constraints for the fluxes $\widehat{\gamma}^{(k)}$ as follows:

$$p_{\mathcal{D}^{(l)}*}\widehat{\gamma}^{(l)} + p_{\mathcal{D}^{(m)}*}\widehat{\gamma}^{(m)} = 0 \tag{4.57}$$

$$\widehat{\gamma}^{(k)} + \frac{1}{2}[p^*_{\mathcal{D}^{(k)}}c_1(B_3) - c_1(\mathcal{D}^{(k)})] \in H_2(\mathcal{D}^{(k)}, \mathbb{Z}), \quad k = l, m. \tag{4.58}$$

In the next section, we shall calculate the homological classes of the dual matter surfaces for $(3,1)$ and $(2,2)$ factorizations. We also compute the chirality induced by the restriction of the fluxes to each dual matter surface.

4.2.2 $(3,1)$ Factorization

It will be convenient to define $x = \zeta^2$ and $y = \zeta^3$ where ζ is a section of $\mathcal{O}_{B_3}(\sigma) \otimes K^{-1}_{B_3}$. Then the $SU(4)$ spectral divisor defined by Eq. (3.19) can be written as

$$\mathcal{D}^{(4)} : \quad b_0(zu)^4 + b_2(zu)^2\zeta^2 + b_3(zu)\zeta^3 + b_4\zeta^4 = 0. \tag{4.59}$$

We now consider the $(3,1)$ factorization $\mathcal{D}^{(4)} \to \mathcal{D}^{(a)} \times \mathcal{D}^{(b)}$ corresponding to the factorization of Eq. (4.59)

$$\mathcal{D}^{(a)} \times \mathcal{D}^{(b)}: \quad [\widetilde{a}_0(zu)^3 + \widetilde{a}_1(zu)^2\zeta + \widetilde{a}_2(zu)\zeta^2 + \widetilde{a}_3\zeta^3][\widetilde{d}_0(zu) + \widetilde{d}_1\zeta] = 0, \tag{4.60}$$

with projection maps $p_{\mathcal{D}^{(a)}} : \mathcal{D}^{(a)} \to B_3$ and $p_{\mathcal{D}^{(b)}} : \mathcal{D}^{(b)} \to B_3$. By comparing with Eq. (4.59), we can obtain the following relations:

$$b_0 = \widetilde{a}_0\widetilde{d}_0, \quad b_1 = \widetilde{a}_1\widetilde{d}_0 + \widetilde{a}_0\widetilde{d}_1 = 0, \quad b_2 = \widetilde{a}_2\widetilde{d}_0 + \widetilde{a}_1\widetilde{d}_1, \quad b_3 = \widetilde{a}_3\widetilde{d}_0 + \widetilde{a}_2\widetilde{d}_1, \quad b_4 = \widetilde{a}_3\widetilde{d}_1. \tag{4.61}$$

We denote the homological class of $[\widetilde{d}_1]$ by $\pi^*\widehat{\xi}_1$ and then write

$$[\widetilde{d}_0] = \pi^*[c_1(B_3) - B_2 + \widehat{\xi}_1], \quad [\widetilde{a}_k] = \pi^*[(5-m)c_1(B_3) - (4-m)B_2 - \widehat{\xi}_1], \;\; m = 0,1,2,3. \tag{4.62}$$

It is easy to see that the homological classes of $\mathcal{D}^{(a)}$ and $\mathcal{D}^{(b)}$ are given by

$$[\mathcal{D}^{(a)}] = 3\sigma + \pi^*[5c_1(B_3) - B_2 - \widehat{\xi}_1], \quad [\mathcal{D}^{(b)}] = \sigma + \pi^*[c_1(B_3) + \widehat{\xi}_1]. \tag{4.63}$$

Note that the unfactorized dual matter **16** surface sits inside the locus of $\{(zu) = 0\} \cap \{b_4 = 0\}$. Due to the factorization in Eq. (4.60), the factorized dual matter **16** surfaces sit inside the loci $\{(zu) = 0\} \cap \{\widetilde{a}_3 = 0\}$ and $\{(zu) = 0\} \cap \{\widetilde{d}_1 = 0\}$. The homological class of dual matter surfaces $\widehat{\Sigma}_{\mathbf{16}^{(a)}}$ and $\widehat{\Sigma}_{\mathbf{16}^{(b)}}$ are given by

$$[\widehat{\Sigma}_{\mathbf{16}^{(a)}}] = (\sigma + \pi^*B_2) \cdot \pi^*[2c_1(B_3) - B_2 - \widehat{\xi}_1], \quad [\widehat{\Sigma}_{\mathbf{16}^{(b)}}] = (\sigma + \pi^*B_2) \cdot \pi^*\widehat{\xi}_1. \tag{4.64}$$

To obtain dual matter surface $\widehat{\Sigma}_{\mathbf{10}}$'s, we calculate the intersection $\mathcal{D}^{(4)} \cap \tau\mathcal{D}^{(4)}$, where τ is a $\mathbb{Z}_2$ involution $\zeta \to -\zeta$ acting on $\mathcal{D}^{(4)}$ [39, 40, 44, 72]. Under $(3,1)$ factorization $\mathcal{D}^{(4)} \to \mathcal{D}^{(a)} \times \mathcal{D}^{(b)}$, the intersection $\mathcal{D}^{(4)} \cap \tau\mathcal{D}^{(4)}$ can be decomposed into several components $\mathcal{D}^{(a)} \cap \tau\mathcal{D}^{(a)}$, $\mathcal{D}^{(a)} \cap \tau\mathcal{D}^{(b)}$, and $\mathcal{D}^{(b)} \cap \tau\mathcal{D}^{(b)}$. We first consider the case of $\mathcal{D}^{(a)} \cap \tau\mathcal{D}^{(a)}$. This intersection is determined by

$$\begin{cases} (zu)[\widetilde{a}_0(zu)^2 + \widetilde{a}_2\zeta^2] = 0 \\ \zeta[\widetilde{a}_1(zu)^2 + \widetilde{a}_3\zeta^2] = 0. \end{cases} \tag{4.65}$$

To solve the constraint $b_1 = \widetilde{a}_1\widetilde{d}_0 + \widetilde{a}_0\widetilde{d}_1 = 0$, we use ansatz $\widetilde{a}_0 = \widetilde{\alpha}\widetilde{d}_0$ and $\widetilde{a}_1 = -\widetilde{\alpha}\widetilde{d}_1$ where the homological class of $\widetilde{\alpha}$ is $[\widetilde{\alpha}] = \pi^*[4c_1(B_3) - 3B_2 - 2\widehat{\xi}_1]$. By using the ansatz, we obtain

$$\begin{cases} (zu)[\widetilde{\alpha}\widetilde{d}_0(zu)^2 + \widetilde{a}_2\zeta^2] = 0 \\ \zeta[-\widetilde{\alpha}\widetilde{d}_1(zu)^2 + \widetilde{a}_3\zeta^2] = 0. \end{cases} \tag{4.66}$$

It follows from Eq. (4.66) that the homological class of dual matter surface $\widehat{\Sigma}_{\mathbf{10}^{(a)(a)}}$ is given by

$$\begin{aligned}[\widehat{\Sigma}_{\mathbf{10}^{(a)(a)}}] &= [\mathcal{D}^{(a)}]\cdot[\mathcal{D}^{(a)}] - [\zeta]\cdot[\widetilde{a}_0] - [zu]\cdot[\widetilde{a}_3] - 9[\zeta]\cdot[zu] - 2[\zeta]\cdot[\widetilde{\alpha}] \\ &= \{2\sigma + \pi^*[4c_1(B_3) - B_2 - \widehat{\xi}_1]\}\cdot\pi^*[3c_1(B_3) - 2B_2 - \widehat{\xi}_1] \\ &+ 2[\sigma + \pi^* c_1(B_3)]\cdot\pi^*\widehat{\xi}_1. \end{aligned} \tag{4.67}$$

Next we calculate the intersection $\mathcal{D}^{(a)} \cap \tau\mathcal{D}^{(b)}$ which is given by

$$\begin{cases} \widetilde{a}_0(zu)^3 + \widetilde{a}_1(zu)^2\zeta + \widetilde{a}_2(zu)\zeta^2 + \widetilde{a}_3\zeta^3 = 0 \\ \widetilde{d}_0(zu) - \widetilde{d}_1\zeta = 0. \end{cases} \tag{4.68}$$

By using the ansatz, we can rewrite Eq. (4.68) as

$$\begin{cases} \zeta^2[\widetilde{a}_2(zu) + \widetilde{a}_3\zeta] = 0 \\ \widetilde{d}_0(zu) - \widetilde{d}_1\zeta = 0. \end{cases} \tag{4.69}$$

It follows from Eq. (4.69) that the homological class of dual matter surface $\widehat{\Sigma}_{\mathbf{10}^{(a)(b)}}$ is

$$\begin{aligned}[\widehat{\Sigma}_{\mathbf{10}^{(a)(b)}}] &= [\mathcal{D}^{(a)}]\cdot[\mathcal{D}^{(b)}] - 2[\zeta]\cdot[\widetilde{d}_0] - 3[\zeta]\cdot[zu] \\ &= \{\sigma + \pi^*[c_1(B_3) + \widehat{\xi}_1]\}\cdot\pi^*[3c_1(B_3) - 2B_2 - \widehat{\xi}_1] \\ &+ (\sigma + \pi^* B_2)\cdot\pi^*\widehat{\xi}_1. \end{aligned} \tag{4.70}$$

Let us turn to the case of $\mathcal{D}^{(b)} \cap \tau\mathcal{D}^{(b)}$ which is determined by

$$\begin{cases} \widetilde{d}_0(zu) = 0 \\ \widetilde{d}_1\zeta = 0. \end{cases} \tag{4.71}$$

Then the homological class of dual matter surface $\widehat{\Sigma}_{\mathbf{10}^{(b)(b)}}$ is given by

$$\begin{aligned}[\widehat{\Sigma}_{\mathbf{10}^{(b)(b)}}] &= [\mathcal{D}^{(b)}]\cdot[\mathcal{D}^{(b)}] - [\zeta]\cdot[\widetilde{d}_0] - [zu]\cdot[\widetilde{d}_1] - [\zeta]\cdot[zu] \\ &= \pi^*[c_1(B_3) - B_2 + \widehat{\xi}_1]\cdot\pi^*\widehat{\xi}_1. \end{aligned} \tag{4.72}$$

We summarize the homological classes of dual matter **16** and **10** surfaces in Table 7[23].

In $(3,1)$ factorization, the ramification divisors for $\mathcal{D}^{(a)}$ and $\mathcal{D}^{(b)}$ are given by

$$\begin{aligned} \widehat{r}^{(a)} &= [\mathcal{D}^{(a)}]\cdot\{\sigma + \pi^*[4c_1(B_3) - 2B_2 - \widehat{\xi}_1]\}, \\ \widehat{r}^{(b)} &= [\mathcal{D}^{(b)}]\cdot[-\sigma - \pi^*(B_2 - \widehat{\xi}_1)], \end{aligned} \tag{4.73}$$

[23]In the case of $\mathbf{10}^{(b)(b)}$, we impose the condition $\pi^*[c_1(B_3) - B_2 + \widehat{\xi}_1]\cdot\pi^*\widehat{\xi}_1 = 0$ to avoid the appearance of a singularity.

Field	Homological Class
$\mathbf{16}^{(b)}$	$(\sigma+\pi^*B_2)\cdot\pi^*\widehat{\xi}_1$
$\mathbf{16}^{(a)}$	$(\sigma+\pi^*B_2)\cdot\pi^*[2c_1(B_3)-B_2-\widehat{\xi}_1]$
$\mathbf{10}^{(b)(b)}$	-
$\mathbf{10}^{(a)(b)}$	$\{\sigma+\pi^*[c_1(B_3)+\widehat{\xi}_1]\}\cdot\pi^*[3c_1(B_3)-2B_2-\widehat{\xi}_1]$ $+(\sigma+\pi^*B_2)\cdot\pi^*\widehat{\xi}_1$
$\mathbf{10}^{(a)(a)}$	$\{2\sigma+\pi^*[4c_1(B_3)-B_2-\widehat{\xi}_1]\}\cdot\pi^*[3c_1(B_3)-2B_2-\widehat{\xi}_1]$ $+2[\sigma+\pi^*c_1(B_3)]\cdot\pi^*\widehat{\xi}_1$

Table 7: Dual matter surfaces for the factorization $\mathcal{D}^{(4)}=\mathcal{D}^{(a)}\times\mathcal{D}^{(b)}$.

respectively. We define traceless fluxes $\widehat{\gamma}_0^{(a)}$ and $\widehat{\gamma}_0^{(b)}$ by

$$\begin{aligned}\widehat{\gamma}_0^{(a)}&=(3-p^*_{\mathcal{D}^{(a)}}p_{\mathcal{D}^{(a)}*})\widehat{\gamma}^{(a)}=[\mathcal{D}^{(a)}]\cdot\{3\sigma-\pi^*[2c_1(B_3)-B_2-\widehat{\xi}_1]\},\\ \widehat{\gamma}_0^{(b)}&=(1-p^*_{\mathcal{D}^{(b)}}p_{\mathcal{D}^{(b)}*})\widehat{\gamma}^{(b)}=[\mathcal{D}^{(b)}]\cdot(\sigma-\pi^*\widehat{\xi}_1),\end{aligned}\tag{4.74}$$

where $\widehat{\gamma}^{(a)}$ and $\widehat{\gamma}^{(b)}$ are non-traceless fluxes and defined by

$$\widehat{\gamma}^{(a)}=[\mathcal{D}^{(a)}]\cdot\sigma,\quad \widehat{\gamma}^{(b)}=[\mathcal{D}^{(b)}]\cdot\sigma.\tag{4.75}$$

Following the formula in section 3.2, the net chirality of matter in the representation $\mathbf{r}$ induced by the flux $\mathcal{G}$ is

$$N_{\mathbf{r}}=[\widehat{\Sigma}_{\mathbf{r}}]\cdot\mathcal{G}\cdot\pi^*B_2,\tag{4.76}$$

where $[\widehat{\Sigma}_{\mathbf{r}}]$ is the homological class of dual surface for matter in the representation $\mathbf{r}$. By using Eq. (4.76) and $\widehat{\xi}_1|_{B_2}=\xi_1$, we can calculate the restriction of fluxes $\widehat{\gamma}_0^{(a)}$ and $\widehat{\gamma}_0^{(b)}$ to each dual matter surface. We omit the calculation here and only summarize the results in Table 8.

	$\widehat{\gamma}_0^{(b)}$	$\widehat{\gamma}_0^{(a)}$
$\mathbf{16}^{(b)}$	$-\xi_1\cdot_{B_2}(c_1+\xi_1)$	0
$\mathbf{16}^{(a)}$	0	$-(5c_1-t-\xi_1)\cdot_{B_2}(2c_1-t-\xi_1)$
$\mathbf{10}^{(a)(b)}$	0	$-(3c_1-t-3\xi_1)\cdot_{B_2}(2c_1-t-\xi_1)$
$\mathbf{10}^{(a)(a)}$	0	$(3c_1-t-3\xi_1)\cdot_{B_2}(2c_1-t-\xi_1)$

Table 8: Chirality induce by the fluxes $\widehat{\gamma}_0^{(a)}$ and $\widehat{\gamma}_0^{(b)}$.

We also can define additional fluxes $\widehat{\delta}^{(a)}$ and $\widehat{\delta}^{(b)}$ by

$$\begin{aligned}\widehat{\delta}^{(a)} &= (1 - p^*_{\mathcal{D}^{(b)}} p_{\mathcal{D}^{(a)}*})\widehat{\gamma}^{(a)} = [\mathcal{D}^{(a)}] \cdot \sigma - [\mathcal{D}^{(b)}] \cdot \pi^*[2c_1(B_3) - B_2 - \widehat{\xi}_1],\\ \widehat{\delta}^{(b)} &= (3 - p^*_{\mathcal{D}^{(a)}} p_{\mathcal{D}^{(b)}*})\widehat{\gamma}^{(b)} = [\mathcal{D}^{(b)}] \cdot 3\sigma - [\mathcal{D}^{(a)}] \cdot \pi^*\widehat{\xi}_1. \end{aligned} \tag{4.77}$$

Another flux we can include is [40]

$$\widehat{\rho}^{(3,1)} = (3p^*_{\mathcal{D}^{(b)}} - p^*_{\mathcal{D}^{(a)}})\widehat{\rho}, \tag{4.78}$$

where $\widehat{\rho} \in H_2(B_3, \mathbb{R})$ with $\widehat{\rho}|_{B_2} = \rho$. We summarize the restriction of fluxes $\widehat{\delta}^{(a)}$, $\widehat{\delta}^{(b)}$ and $\widehat{\rho}^{(3,1)}$ to each matter curve in Table 9.

	$\widehat{\delta}^{(b)}$	$\widehat{\delta}^{(a)}$	$\widehat{\rho}^{(3,1)}$
$\mathbf{16}^{(b)}$	$-3c_1 \cdot_{B_2} \xi_1$	$-\xi_1 \cdot_{B_2} (2c_1 - t - \xi_1)$	$3\rho \cdot_{B_2} \xi_1$
$\mathbf{16}^{(a)}$	$-\xi_1 \cdot_{B_2} (2c_1 - t - \xi_1)$	$-c_1 \cdot_{B_2} (2c_1 - t - \xi_1)$	$-\rho \cdot_{B_2} (2c_1 - t - \xi_1)$
$\mathbf{10}^{(a)(b)}$	$\xi_1 \cdot_{B_2} (3c_1 - 2t - 3\xi_1)$	$-(3c_1 - t - \xi_1) \cdot_{B_2} (2c_1 - t - \xi_1)$	$2\rho \cdot_{B_2} (3c_1 - t)$
$\mathbf{10}^{(a)(a)}$	$-2\xi_1 \cdot_{B_2} (3c_1 - t)$	$(3c_1 - t - \xi_1) \cdot_{B_2} (2c_1 - t - \xi_1)$	$-2\rho \cdot_{B_2} (3c_1 - t)$

Table 9: Chirality induce by the fluxes $\widehat{\delta}^{(a)}$, $\widehat{\delta}^{(b)}$, and $\widehat{\rho}^{(3,1)}$.

With Eq. (4.74), (4.77), and (4.78), we define the universal flux $\widehat{\Gamma}$ to be [40]

$$\widehat{\Gamma} = \widetilde{k}_a \widehat{\gamma}_0^{(a)} + \widetilde{k}_b \widehat{\gamma}_0^{(b)} + \widetilde{m}_a \widehat{\delta}^{(a)} + \widetilde{m}_b \widehat{\delta}^{(b)} + \widehat{\rho} \equiv \widehat{\Gamma}^{(a)} + \widehat{\Gamma}^{(b)}, \tag{4.79}$$

where $\widehat{\Gamma}^{(a)}$ and $\widehat{\Gamma}^{(b)}$ are defined by

$$\widehat{\Gamma}^{(a)} = [\mathcal{D}^{(a)}] \cdot \{(3\widetilde{k}_a + \widetilde{m}_a)\sigma + \pi^*[2\widetilde{k}_a c_1(B_3) - (4\widetilde{k}_a + \widetilde{m}_a)B_2 + (\widetilde{m}_b - \widetilde{k}_a)\widehat{\xi}_1 + \widehat{\rho}]\}, \tag{4.80}$$

$$\widehat{\Gamma}^{(b)} = [\mathcal{D}^{(b)}] \cdot \{(\widetilde{k}_b + 3\widetilde{m}_b)\sigma - \pi^*[2\widetilde{m}_a c_1(B_3) - (\widetilde{k}_b + 4\widetilde{m}_b)B_2 + (\widetilde{k}_b - \widetilde{m}_b)\widehat{\xi}_1 - 3\widehat{\rho}]\}. \tag{4.81}$$

Note that

$$p_{\mathcal{D}^{(a)}*}\widehat{\Gamma}^{(a)} = 2\widetilde{m}_a c_1(B_3) - \widetilde{m}_a B_2 - (3\widetilde{m}_b + \widetilde{m}_a)\widehat{\xi}_1 - 3\widehat{\rho}, \tag{4.82}$$

$$p_{\mathcal{D}^{(b)}*}\widehat{\Gamma}^{(b)} = -2\widetilde{m}_a c_1(B_3) + \widetilde{m}_a B_2 + (3\widetilde{m}_b + \widetilde{m}_a)\widehat{\xi}_1 + 3\widehat{\rho}. \tag{4.83}$$

Clearly, $\widehat{\Gamma}^{(a)}$ and $\widehat{\Gamma}^{(b)}$ obey the traceless condition $p_{\mathcal{D}^{(a)}*}\widehat{\Gamma}^{(a)} + p_{\mathcal{D}^{(b)}*}\widehat{\Gamma}^{(b)} = 0$. In this case the quantization conditions are

$$\{(3\widetilde{k}_a + \widetilde{m}_a + \frac{1}{2})\sigma + \pi^*[(2\widetilde{k}_a - 1)c_1(B_3) - (4\widetilde{k}_a + \widetilde{m}_a - 1)B_2 + (\widetilde{m}_b - \widetilde{k}_a + \frac{1}{2})\widehat{\xi}_1 + \widehat{\rho}]\} \in H_4(Z_4, \mathbb{Z}), \tag{4.84}$$

$$\{(\widetilde{k}_b + 3\widetilde{m}_b - \frac{1}{2})\sigma - \pi^*[2\widetilde{m}_a c_1(B_3) - (\widetilde{k}_b + 4\widetilde{m}_b - \frac{1}{2})B_2 + (\widetilde{k}_b - \widetilde{m}_b - \frac{1}{2})\widehat{\xi}_1 - 3\widehat{\rho}]\} \in H_4(Z_4, \mathbb{Z}). \tag{4.85}$$

4.2.3 (2, 2) Factorization

In the (2,2) factorization $\mathcal{D}^{(4)} \to \mathcal{D}^{(d_1)} \times \mathcal{D}^{(d_2)}$, the divisor $\mathcal{D}^{(4)}$ splits into two components $\mathcal{D}^{(d_1)}$ and $\mathcal{D}^{(d_2)}$. We then factorize Eq. (4.59) into the following form:

$$\mathcal{D}^{(d_1)} \times \mathcal{D}^{(d_2)}: \quad [\widetilde{e}_0(zu)^2 + \widetilde{e}_1(zu)\zeta + \widetilde{e}_2\zeta^2][\widetilde{f}_0(zu)^2 + \widetilde{f}_1(zu)\zeta + \widetilde{f}_2\zeta^2] = 0. \tag{4.86}$$

with projection maps $p_{\mathcal{D}^{(d_1)}} : \mathcal{D}^{(d_1)} \to B_3$ and $p_{\mathcal{D}^{(d_1)}} : \mathcal{D}^{(d_2)} \to B_3$. By comparing the coefficients with Eq. (4.59), we obtain the following relations:

$$b_0 = \widetilde{e}_0\widetilde{f}_0, \;\; b_1 = \widetilde{e}_0\widetilde{f}_1 + \widetilde{e}_1\widetilde{f}_0 = 0, \;\; b_2 = \widetilde{e}_0\widetilde{f}_2 + \widetilde{e}_1\widetilde{f}_1 + \widetilde{e}_2\widetilde{f}_0, \;\; b_3 = \widetilde{e}_1\widetilde{f}_2 + \widetilde{e}_2\widetilde{f}_1, \;\; b_4 = \widetilde{e}_2\widetilde{f}_2. \tag{4.87}$$

By denoting the homological class of $\widetilde{f}_2$ by $\pi^*\widehat{\xi}_2$, the homological classes of other sections can be written as

$$[\widetilde{f}_k] = \pi^*\{(2-k)[c_1(B_3) - B_2] + \widehat{\xi}_2\}, \;\; k = 0, 1, \tag{4.88}$$

$$[\widetilde{e}_m] = \pi^*[(m-3)B_2 - (m-4)c_1(B_3) - \widehat{\xi}_2], \;\; m = 0, 1, 2. \tag{4.89}$$

In this case, the homological classes of $\mathcal{D}^{(d_1)}$ and $\mathcal{D}^{(d_2)}$ are given by

$$[\mathcal{D}^{(d_1)}] = 2\sigma + \pi^*[4c_1(B_3) - B_2 - \widehat{\xi}_2], \quad [\mathcal{D}^{(d_2)}] = 2\sigma + \pi^*[2c_1(B_3) + \widehat{\xi}_2]. \tag{4.90}$$

With Eq. (4.87), the dual matter **16** surfaces sit inside the loci $\{(zu) = 0\} \cap \{\widetilde{e}_2 = 0\}$ and $\{(zu) = 0\} \cap \{\widetilde{f}_2 = 0\}$. The homological classes of dual matter surfaces $\widehat{\Sigma}_{\mathbf{16}^{(d_1)}}$ and $\widehat{\Sigma}_{\mathbf{16}^{(d_2)}}$ are given by

$$[\widehat{\Sigma}_{\mathbf{16}^{(d_1)}}] = (\sigma + \pi^*B_2)\cdot\pi^*[2c_1(B_3) - B_2 - \widehat{\xi}_2], \quad [\widehat{\Sigma}_{\mathbf{16}^{(d_2)}}] = (\sigma + \pi^*B_2)\cdot\pi^*\widehat{\xi}_2, \tag{4.91}$$

respectively. We can obtain the homological classes of dual matter surfaces $\widehat{\Sigma}_{\mathbf{10}}$'s by calculating the intersection $\mathcal{D}^{(4)} \cap \tau\mathcal{D}^{(4)}$, where τ is a $\mathbb{Z}_2$ involution $\zeta \to -\zeta$ [39, 40, 44, 72]. Under (2, 2) factorization $\mathcal{D}^{(4)} \to \mathcal{D}^{(d_1)} \times \mathcal{D}^{(d_2)}$, $\mathcal{D}^{(4)} \cap \tau\mathcal{D}^{(4)}$ can be decomposed into several components $\mathcal{D}^{(d_1)} \cap \tau\mathcal{D}^{(d_1)}$, $\mathcal{D}^{(d_1)} \cap \tau\mathcal{D}^{(d_2)}$, and $\mathcal{D}^{(d_2)} \cap \tau\mathcal{D}^{(d_2)}$. For the case of $\mathcal{D}^{(d_1)} \cap \tau\mathcal{D}^{(d_1)}$, this intersection is determined by

$$\begin{cases} \widetilde{e}_0(zu)^2 + \widetilde{e}_2\zeta^2 = 0 \\ \widetilde{e}_1(zu)\zeta = 0. \end{cases} \tag{4.92}$$

To solve the constraint $b_1 = \widetilde{e}_0\widetilde{f}_1 + \widetilde{e}_1\widetilde{f}_0 = 0$, we use ansatz $\widetilde{e}_0 = \widetilde{\beta}\widetilde{f}_0$ and $\widetilde{e}_1 = -\widetilde{\beta}\widetilde{f}_1$, where $[\widetilde{\beta}] = \pi^*[2c_1(B_3) - B_2 - 2\widehat{\xi}_2]$. With the ansatz, Eq. (4.92) can be written as

$$\begin{cases} \widetilde{\beta}\widetilde{f}_0(zu)^2 + \widetilde{e}_2\zeta^2 = 0 \\ \widetilde{\beta}\widetilde{f}_1(zu)\zeta = 0. \end{cases} \tag{4.93}$$

It follows from Eq. (4.93) that the homological class of dual matter surface $\widehat{\Sigma}_{\mathbf{10}^{(d_1)(d_1)}}$ can be computed as

$$\begin{aligned}
[\widehat{\Sigma}_{\mathbf{10}^{(d_1)(d_1)}}] &= [\mathcal{D}^{(d_1)}]\cdot[\mathcal{D}^{(d_1)}]-[\zeta]\cdot[\widetilde{e}_0]-[zu]\cdot[\widetilde{e}_2]-4[\zeta]\cdot[zu]-2[\zeta]\cdot[\beta] \\
&= 2[\sigma+\pi^*c_1(B_3)]\cdot\pi^*[c_1(B_3)-B_2+\widehat{\xi}_2] \\
&+ \pi^*[3c_1(B_3)-2B_2-\widehat{\xi}_2]\cdot\pi^*[2c_1(B_3)-B_2-\widehat{\xi}_2]. \qquad (4.94)
\end{aligned}$$

Next we calculate the intersection $\mathcal{D}^{(d_1)}\cap\tau\mathcal{D}^{(d_2)}$ which given by

$$\begin{cases} \widetilde{e}_0(zu)^2+\widetilde{e}_1(zu)\zeta+\widetilde{e}_2\zeta^2=0 \\ \widetilde{f}_0(zu)^2-\widetilde{f}_1(zu)\zeta+\widetilde{f}_2\zeta^2=0. \end{cases} \qquad (4.95)$$

By using the ansatz, we can recast Eq. (4.95) as

$$\begin{cases} \zeta^2[-\beta\widetilde{f}_2+\widetilde{e}_2]=0 \\ \widetilde{f}_0(zu)^2-\widetilde{f}_1(zu)\zeta+\widetilde{f}_2\zeta^2=0. \end{cases} \qquad (4.96)$$

Then the homological class of dual matter surface $\widehat{\Sigma}_{\mathbf{10}^{(d_1)(d_2)}}$ is given by

$$\begin{aligned}
[\widehat{\Sigma}_{\mathbf{10}^{(d_1)(d_2)}}] &= [\mathcal{D}^{(d_1)}]\cdot[\mathcal{D}^{(d_2)}]-2[\zeta]\cdot[\widetilde{f}_0]-4[\zeta]\cdot[zu] \\
&= \{2\sigma+\pi^*[2c_1(B_3)+\widehat{\xi}_2]\}\cdot\pi^*[2c_1(B_3)-B_2-\widehat{\xi}_2]. \qquad (4.97)
\end{aligned}$$

Let us turn to the case of $\mathcal{D}^{(d_2)}\cap\tau\mathcal{D}^{(d_2)}$. This intersection is described by

$$\begin{cases} \widetilde{f}_0(zu)^2+\widetilde{f}_2\zeta^2=0 \\ \widetilde{f}_1(zu)\zeta=0. \end{cases} \qquad (4.98)$$

It follows from Eq. (4.98) that the homological class of dual matter surface $\widehat{\Sigma}_{\mathbf{10}^{(d_2)(d_2)}}$ is calculated as

$$\begin{aligned}
[\widehat{\Sigma}_{\mathbf{10}^{(d_2)(d_2)}}] &= [\mathcal{D}^{(d_2)}]\cdot[\mathcal{D}^{(d_2)}]-[\zeta]\cdot[\widetilde{f}_0]-[zu]\cdot[\widetilde{f}_2]-4[\zeta]\cdot[zu] \\
&= \{2\sigma+\pi^*[2c_1(B_3)+\widehat{\xi}_2]\}\cdot\pi^*[c_1(B_3)-B_2+\widehat{\xi}_2]. \qquad (4.99)
\end{aligned}$$

We summarize the homological classes of dual matter **16** and **10** surfaces in Table 10.

We can calculate the ramification divisors for the $(2,2)$ factorization and obtain

$$\begin{aligned}
\widehat{r}^{(d_1)} &= [\mathcal{D}^{(d_1)}]\cdot\pi^*[3c_1(B_3)-2B_2-\widehat{\xi}_2], \\
\widehat{r}^{(d_2)} &= [\mathcal{D}^{(d_2)}]\cdot\pi^*[c_1(B_3)-B_2+\widehat{\xi}_2], \qquad (4.100)
\end{aligned}$$

Field	Homological Class
$\mathbf{16}^{(d_2)}$	$(\sigma + \pi^* B_2) \cdot \pi^* \widehat{\xi}_2$
$\mathbf{16}^{(d_1)}$	$(\sigma + \pi^* B_2) \cdot \pi^*[2c_1(B_3) - B_2 - \widehat{\xi}_2]$
$\mathbf{10}^{(d_2)(d_2)}$	$\{2\sigma + \pi^*[2c_1(B_3) + \widehat{\xi}_2]\} \cdot \pi^*[c_1(B_3) - B_2 + \widehat{\xi}_2]$
$\mathbf{10}^{(d_1)(d_2)}$	$\{2\sigma + \pi^*[2c_1(B_3) + \widehat{\xi}_2]\} \cdot \pi^*[2c_1(B_3) - B_2 - \widehat{\xi}_2]$
$\mathbf{10}^{(d_1)(d_1)}$	$2[\sigma + \pi^* c_1(B_3)] \cdot \pi^*[c_1(B_3) - B_2 + \widehat{\xi}_2]$ $+\pi^*[3c_1(B_3) - 2B_2 - \widehat{\xi}_2] \cdot \pi^*[2c_1(B_3) - B_2 - \widehat{\xi}_2]$

Table 10: Dual matter surfaces for the factorization $\mathcal{D}^{(4)} = \mathcal{D}^{(d_1)} \times \mathcal{D}^{(d_2)}$.

where $\widehat{r}^{(d_1)}$ and $\widehat{r}^{(d_2)}$ are the ramification divisors for the cover $\mathcal{D}^{(d_1)}$ and $\mathcal{D}^{(d_2)}$, respectively. We then define traceless cover fluxes $\widehat{\gamma}_0^{(d_1)}$ and $\widehat{\gamma}_0^{(d_2)}$ by

$$\begin{aligned} \widehat{\gamma}_0^{(d_1)} &= (2 - p^*_{\mathcal{D}^{(d_1)}} p_{\mathcal{D}^{(d_1)}*})\widehat{\gamma}^{(d_1)} = [\mathcal{D}^{(d_1)}] \cdot \{2\sigma - \pi^*[2c_1(B_3) - 3B_2 - \widehat{\xi}_2]\}, \\ \widehat{\gamma}_0^{(d_2)} &= (2 - p^*_{\mathcal{D}^{(d_2)}} p_{\mathcal{D}^{(d_2)}*})\widehat{\gamma}^{(d_2)} = [\mathcal{D}^{(d_2)}] \cdot [2\sigma + \pi^*(2B_2 - \widehat{\xi}_2)], \end{aligned} \tag{4.101}$$

where $\widehat{\gamma}^{(d_1)}$ and $\widehat{\gamma}^{(d_2)}$ are non-traceless fluxes and defined by

$$\widehat{\gamma}^{(d_1)} = [\mathcal{D}^{(d_1)}] \cdot \sigma, \quad \widehat{\gamma}^{(d_2)} = [\mathcal{D}^{(d_2)}] \cdot \sigma. \tag{4.102}$$

We summarize the restriction of the fluxes to each factorized curve in Table 11. We

	$\widehat{\gamma}_0^{(d_2)}$	$\widehat{\gamma}_0^{(d_1)}$
$\mathbf{16}^{(d_2)}$	$-\xi_2 \cdot_{B_2} (2c_1 + \xi_2)$	0
$\mathbf{16}^{(d_1)}$	0	$-(4c_1 - t - \xi_2) \cdot_{B_2} (2c_1 - t - \xi_2)$
$\mathbf{10}^{(d_2)(d_2)}$	0	0
$\mathbf{10}^{(d_1)(d_2)}$	0	$-2(2c_1 - t - 2\xi_2) \cdot_{B_2} (2c_1 - t - \xi_2)$
$\mathbf{10}^{(d_1)(d_1)}$	0	$2(2c_1 - t - 2\xi_2) \cdot_{B_2} (2c_1 - t - \xi_2)$

Table 11: Chirality induced by the fluxes $\widehat{\gamma}_0^{(d_1)}$ and $\widehat{\gamma}_0^{(d_2)}$.

also can define two fluxes

$$\begin{aligned} \widehat{\delta}^{(d_1)} &= (2 - p^*_{\mathcal{D}^{(d_2)}} p_{\mathcal{D}^{(d_1)}*})\widehat{\gamma}^{(d_1)} = [\mathcal{D}^{(d_1)}] \cdot 2\sigma - [\mathcal{C}^{(d_2)}] \cdot \pi^*[2c_1(B_3) - B_2 - \widehat{\xi}_2], \\ \widehat{\delta}^{(d_2)} &= (2 - p^*_{\mathcal{D}^{(d_1)}} p_{\mathcal{D}^{(d_2)}*})\widehat{\gamma}^{(d_2)} = [\mathcal{D}^{(d_2)}] \cdot 2\sigma - [\mathcal{C}^{(d_1)}] \cdot \pi^* \widehat{\xi}_2. \end{aligned} \tag{4.103}$$

Another flux we can include is [40]

$$\widehat{\rho}^{(2,2)} = (p^*_{\mathcal{D}^{(d_2)}} - p^*_{\mathcal{D}^{(d_1)}})\widehat{\rho}, \tag{4.104}$$

	$\widehat{\delta}^{(d_2)}$	$\widehat{\delta}^{(d_1)}$	$\widehat{\rho}^{(2,2)}$
$\mathbf{16}^{(d_2)}$	$-2c_1 \cdot_{B_2} \xi_2$	$-\xi_2 \cdot_{B_2} (2c_1 - t - \xi_2)$	$\rho \cdot_{B_2} \xi_2$
$\mathbf{16}^{(d_1)}$	$-\xi_2 \cdot_{B_2} (2c_1 - t - \xi_2)$	$-2c_1 \cdot_{B_2} (2c_1 - t - \xi_2)$	$-\rho \cdot_{B_2} (2c_1 - t - \xi_2)$
$\mathbf{10}^{(d_2)(d_2)}$	$2\xi_2 \cdot_{B_2} (c_1 + \xi_2)$	$-2(c_1 + \xi_2) \cdot_{B_2} (2c_1 - t - \xi_2)$	$2\rho \cdot_{B_2} (c_1 + \xi_2)$
$\mathbf{10}^{(d_1)(d_2)}$	0	$-2(2c_1 - t - 2\xi_2) \cdot_{B_2} (2c_1 - t - \xi_2)$	0
$\mathbf{10}^{(d_1)(d_1)}$	$-2\xi_2 \cdot_{B_2} (c_1 + \xi_2)$	$2(3c_1 - t - \xi_2) \cdot_{B_2} (2c_1 - t - \xi_2)$	$-2\rho \cdot_{B_2} (c_1 + \xi_2)$

Table 12: Chirality induced by the fluxes $\widehat{\delta}^{(d_1)}$, $\widehat{\delta}^{(d_2)}$, and $\widehat{\rho}^{(2,2)}$.

where $\widehat{\rho} \in H_2(B_3, \mathbb{R})$ with $\widehat{\rho}|_{B_2} = \rho$. We summarize the restriction of the fluxes $\widehat{\delta}^{(d_1)}$, $\widehat{\delta}^{(d_2)}$, and $\widehat{\rho}$ to each factorized curve in Table 12.

Again we set the universal flux to be

$$\widehat{\Gamma} = \widetilde{k}_{d_1}\widehat{\gamma}_0^{(d_1)} + \widehat{k}_{d_2}\widehat{\gamma}_0^{(d_2)} + \widehat{m}_{d_1}\widehat{\delta}^{(d_1)} + \widehat{m}_{d_2}\widehat{\delta}^{(d_2)} + \widehat{\rho} = \widehat{\Gamma}^{(d_1)} + \widehat{\Gamma}^{(d_2)}, \tag{4.105}$$

where

$$\widehat{\Gamma}^{(d_1)} = [\mathcal{D}^{(d_1)}]\cdot\Big\{2(\widetilde{k}_{d_1} + \widetilde{m}_{d_1})\sigma - \pi^*[2\widetilde{k}_{d_1}c_1(B_3) - (3\widetilde{k}_{d_1} + 2\widetilde{m}_{d_1})B_2 + (\widetilde{m}_{d_2} - \widetilde{k}_{d_1})\widehat{\xi}_2 + \widehat{\rho}]\Big\}, \tag{4.106}$$

$$\widehat{\Gamma}^{(d_2)} = [\mathcal{D}^{(d_2)}]\cdot\Big\{2(\widetilde{k}_{d_2} + \widetilde{m}_{d_2})\sigma - \pi^*[2\widetilde{m}_{d_1}c_1(B_3) - (2\widetilde{k}_{d_2} + 3\widetilde{m}_{d_2})B_2 + (\widetilde{k}_{d_2} - \widetilde{m}_{d_1})\widehat{\xi}_2 - \widehat{\rho}]\Big\}. \tag{4.107}$$

Note that

$$p_{\mathcal{D}^{(d_1)}*}\widehat{\Gamma}^{(d_1)} = 4\widetilde{m}_1 d_1 c_1(B_3) - 2\widetilde{m}_{d_1}B_2 - 2(\widetilde{m}_{d_2} + \widetilde{m}_{d_1})\widehat{\xi}_2 - 2\widehat{\rho}, \tag{4.108}$$

$$p_{\mathcal{D}^{(d_2)}*}\widehat{\Gamma}^{(d_2)} = -4\widetilde{m}_1 d_1 c_1(B_3) + 2\widetilde{m}_{d_1}B_2 + 2(\widetilde{m}_{d_2} - \widetilde{m}_{d_1})\widehat{\xi}_2 + 2\widehat{\rho}. \tag{4.109}$$

It is easy to see that $\widehat{\Gamma}^{(d_1)}$ and $\widehat{\Gamma}^{(d_2)}$ satisfy the traceless condition $p_{\mathcal{D}^{(d_1)}*}\widehat{\Gamma}^{(d_1)} + p_{\mathcal{D}^{(d_2)}*}\widehat{\Gamma}^{(d_2)} = 0$. In this case the quantization conditions are given by

$$\{2(\widetilde{k}_{d_1}+\widetilde{m}_{d_1})\sigma-\pi^*[2(\widetilde{k}_{d_1}-\frac{3}{2})c_1(B_3)-(5\widetilde{k}_{d_1}+4\widetilde{m}_{d_1}+1)B_2+(\widetilde{m}_{d_2}-\widetilde{k}_{d_1}+\frac{1}{2})\widehat{\xi}_2+\widehat{\rho}]\} \in H_4(Z_4,\mathbb{Z}), \tag{4.110}$$

$$\{2(\widetilde{k}_{d_2}+\widetilde{m}_{d_2})\sigma-\pi^*[2\widetilde{m}_{d_1}c_1(B_3)-(4\widetilde{k}_{d_2}+5\widetilde{m}_{d_2}+1)B_2+(\widetilde{k}_{d_2}-\widetilde{m}_{d_1}-\frac{1}{2})\widehat{\xi}_2-\widehat{\rho}]\} \in H_4(Z_4,\mathbb{Z}). \tag{4.111}$$

5 Conclusions

In this paper we construct an $SU(4)$ spectral divisor of F-theory compactified on an elliptically fibered Calabi-Yau fourfold by using heterotic/F-theory duality. We also explicitly calculate the net chirality of matter fields **16** and **10** by using the net chirality formula Eq. (1.2). We then found agreement between the computations in F-theory framework and in dual heterotic string. It was argued in [1] that the net chirality formula does not depend on heterotic/F-theory duality and would be intrinsic to F-theory. Therefore, this formula would be applicable to the cases of F-theory compactifications without heterotic duals and the spectral divisors can be regarded as the global completion of semi-local spectral covers. To verify the validity of the net chirality formula, we construct an $SU(4)$ spectral divisor in F-theory geometry with no heterotic dual. By using this spectral divisor and net chirality formula Eq. (1.2), we calculate the net chirality of matter fields **16** and **10**. It turns out that the computations agree with the analysis of the semi-local $SU(4)$ spectral cover.

To obtain realistic models, we also consider $(3,1)$ and $(2,2)$ factorizations of the $SU(4)$ spectral divisor. The explicit computation of chiral spectra shows that the net chirality formula can be applied to the factorized spectral divisors. By comparing with the spectra calculated by using semi-local spectral covers, we again found agreement between the computation in factored spectral divisors and in factored spectral divisors. Our computations provide an example for the validity of the spectral divisor construction and net chirality formula. In heterotic compactifications, the net chirality formula can be recast as an index on a Calabi-Yau threefold. More precisely, it can be expressed as an integral of the third Chern class of a stable holomorphic vector bundle on the Calabi-Yau threefold. It would be interesting to lift the net chirality formula in F-theory framework to an index on a Calabi-Yau fourfold. The structure of the net chirality formula should shed light on the geometry of F-theory compactification and the nature of heterotic/F-theory duality.

Acknowledgments

I gratefully acknowledge hospitality and support from the Summer School on Mathematical String Theory at Virginia Tech and PiTP 2010 Summer Program at the

Institute for Advanced Study. This work is supported in part by the NSF under grant PHY-0555575 and by Texas A&M University.

References

[1] J. Marsano, N. Saulina and S. Schafer-Nameki, arXiv:1006.0483 [hep-th].

[2] C. Vafa, Nucl. Phys. B **469**, 403 (1996) [arXiv:hep-th/9602022].

[3] D. R. Morrison and C. Vafa, Nucl. Phys. B **473**, 74 (1996) [arXiv:hep-th/9602114].

[4] D. R. Morrison and C. Vafa, Nucl. Phys. B **476**, 437 (1996) [arXiv:hep-th/9603161].

[5] F. Denef, arXiv:0803.1194 [hep-th].

[6] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, Nucl. Phys. B **481**, 215 (1996) [arXiv:hep-th/9605200].

[7] C. Beasley, J. J. Heckman and C. Vafa, JHEP **0901**, 058 (2009) [arXiv:0802.3391 [hep-th]].

[8] C. Beasley, J. J. Heckman and C. Vafa, arXiv:0806.0102 [hep-th].

[9] R. Donagi and M. Wijnholt, arXiv:0802.2969 [hep-th].

[10] R. Donagi and M. Wijnholt, arXiv:0808.2223 [hep-th].

[11] J. J. Heckman and C. Vafa, arXiv:0809.1098 [hep-th].

[12] J. J. Heckman and C. Vafa, arXiv:0809.3452 [hep-ph].

[13] J. J. Heckman and C. Vafa, arXiv:0811.2417 [hep-th].

[14] J. J. Heckman, G. L. Kane, J. Shao and C. Vafa, arXiv:0903.3609 [hep-ph].

[15] V. Bouchard, J. J. Heckman, J. Seo and C. Vafa, arXiv:0904.1419 [hep-ph].

[16] J. J. Heckman and C. Vafa, arXiv:0904.3101 [hep-th].

[17] R. Blumenhagen, arXiv:0812.0248 [hep-th].

[18] J. L. Bourjaily, arXiv:0901.3785 [hep-th].

[19] J. L. Bourjaily, arXiv:0905.0142 [hep-th].

[20] A. Font and L. E. Ibanez, arXiv:0811.2157 [hep-th].

[21] A. Font and L. E. Ibanez, JHEP **0909**, 036 (2009) [arXiv:0907.4895 [hep-th]].

[22] L. Randall and D. Simmons-Duffin, arXiv:0904.1584 [hep-ph].

[23] J. Jiang, T. Li, D. V. Nanopoulos and D. Xie, arXiv:0811.2807 [hep-th].

[24] J. Jiang, T. Li, D. V. Nanopoulos and D. Xie, arXiv:0905.3394 [hep-th].

[25] T. Li, arXiv:0905.4563 [hep-th].

[26] C.-M. Chen and Y.-C. Chung, Nucl. Phys. B **824**, 273 (2010) [arXiv:0903.3009 [hep-th]].

[27] J. J. Heckman, A. Tavanfar and C. Vafa, arXiv:0906.0581 [hep-th].

[28] J. P. Conlon and E. Palti, arXiv:0907.1362 [hep-th].

[29] J. P. Conlon and E. Palti, arXiv:0910.2413 [hep-th].

[30] S. Cecotti, M. C. N. Cheng, J. J. Heckman and C. Vafa, arXiv:0910.0477 [hep-th].

[31] Y. C. Chung, JHEP **1003**, 006 (2010) [arXiv:0911.0427 [hep-th]].

[32] S. F. King, G. K. Leontaris and G. G. Ross, arXiv:1005.1025 [hep-ph].

[33] J. J. Heckman, arXiv:1001.0577 [hep-th].

[34] H. Hayashi, R. Tatar, Y. Toda, T. Watari and M. Yamazaki, Nucl. Phys. B **806**, 224 (2009) [arXiv:0805.1057 [hep-th]].

[35] A. Collinucci, arXiv:0812.0175 [hep-th].

[36] H. Hayashi, T. Kawano, R. Tatar and T. Watari, arXiv:0901.4941 [hep-th].

[37] B. Andreas and G. Curio, arXiv:0902.4143 [hep-th].

[38] J. Marsano, N. Saulina and S. Schafer-Nameki, JHEP **0908**, 030 (2009) [arXiv:0904.3932 [hep-th]].

[39] J. Marsano, N. Saulina and S. Schafer-Nameki, JHEP **0908**, 046 (2009) [arXiv:0906.4672 [hep-th]].

[40] J. Marsano, N. Saulina and S. Schafer-Nameki, JHEP **1004**, 095 (2010) [arXiv:0912.0272 [hep-th]].

[41] R. Donagi and M. Wijnholt, arXiv:0904.1218 [hep-th].

[42] R. Tatar, Y. Tsuchiya and T. Watari, Nucl. Phys. B **823**, 1 (2009) [arXiv:0905.2289 [hep-th]].

[43] A. Collinucci, arXiv:0906.0003 [hep-th].

[44] R. Blumenhagen, T. W. Grimm, B. Jurke and T. Weigand, JHEP **0909**, 053 (2009) [arXiv:0906.0013 [hep-th]].

[45] R. Blumenhagen, T. W. Grimm, B. Jurke and T. Weigand, arXiv:0908.1784 [hep-th].

[46] H. Hayashi, T. Kawano, Y. Tsuchiya and T. Watari, arXiv:0910.2762 [hep-th].

[47] T. W. Grimm, S. Krause and T. Weigand, arXiv:0912.3524 [hep-th].

[48] H. Hayashi, T. Kawano, Y. Tsuchiya and T. Watari, arXiv:1004.3870 [hep-th].

[49] R. Blumenhagen, A. Collinucci and B. Jurke, arXiv:1002.1894 [hep-th].

[50] M. Cvetic, I. Garcia-Etxebarria and J. Halverson, arXiv:1003.5337 [hep-th].

[51] T. W. Grimm and T. Weigand, arXiv:1006.0226 [hep-th].

[52] K. S. Choi, arXiv:1007.3843 [hep-th].

[53] K. S. Choi and T. Kobayashi, arXiv:1003.2126 [hep-th].

[54] E. Dudas and E. Palti, arXiv:1007.1297 [hep-ph].

[55] E. Dudas and E. Palti, JHEP **1001**, 127 (2010) [arXiv:0912.0853 [hep-th]].

[56] J. J. Heckman and C. Vafa, arXiv:1006.5459 [hep-th].

[57] C. M. Chen, J. Knapp, M. Kreuzer and C. Mayrhofer, arXiv:1005.5735 [hep-th].

[58] S. M. Barr, Phys. Lett. B **112**, 219 (1982).

[59] J. P. Derendinger, J. E. Kim and D. V. Nanopoulos, Phys. Lett. B **139**, 170 (1984).

[60] I. Antoniadis, J. R. Ellis, J. S. Hagelin and D. V. Nanopoulos, Phys. Lett. B **194**, 231 (1987).

[61] C. M. Chen and Y. C. Chung, arXiv:1005.5728 [hep-th].

[62] Demazure, Pinkham et Teissier, "Seminaire sur les Singularities des Surfaces," Ecole Polytechnique, 1976-1977.

[63] Y. I. Manin, Cubic forms: Algebra, geometry, arithmetic. North-Holland Publishing Co., Amsterdam, second ed., 1986. Translated from the Russian by M. Hazewinkel.

[64] F. Marchesano and L. Martucci, Phys. Rev. Lett. **104**, 231601 (2010) [arXiv:0910.5496 [hep-th]].

[65] J. J. Heckman and H. Verlinde, arXiv:1005.3033 [hep-th].

[66] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory" Cambridge University Press 1987.

[67] S. Donaldson, Proc. London Math. Soc. 50 1 (1985).

[68] K. Uhlenbeck and S.-T. Yau, Comm. Pure App. Math. **39**, 257 (1986).

[69] K. Uhlenbeck and S.-T. Yau, Comm. Pure App. Math. **42**, 703 (1986).

[70] R. Friedman, J. Morgan and E. Witten, Commun. Math. Phys. **187**, 679 (1997) [arXiv:hep-th/9701162].

[71] R. Friedman, J. Morgan and E. Witten, J. Alg. Geom. **279**, 8 (1999) [alg-geom/9709029].

[72] R. Donagi, Y. H. He, B. A. Ovrut and R. Reinbacher, JHEP **0412**, 054 (2004) [arXiv:hep-th/0405014].

[73] P. S. Aspinwall, Adv. Theor. Math. Phys. **1**, 127 (1998) [arXiv:hep-th/9707014].

[74] R. Donagi, arXiv:hep-th/9802093.

[75] G. Curio and R. Y. Donagi, Nucl. Phys. B **518**, 603 (1998) [arXiv:hep-th/9801057].

[76] M. R. Douglas and G. W. Moore, arXiv:hep-th/9603167.

[77] Eguchi, Gilkey, and Hanson, Phys. Rep. **66**, 214 (1980).

[78] P. Kronheimer, J. Diff. Geom. **29**, 665 (1989).

[79] P. Kronheimer, J. Diff. Geom. **29**, 685 (1989).

[80] N. Hitchin, Math. Proc. Camb. Phil. Soc. **85**, 465 (1979).

[81] N. Hitchin, Sem. Bourbaki, Asterisque **206**, 137 (1992).

[82] R. Hartshorne, "Algebraic geometry," New York : Springer-Verlag, 1977.

[83] P. Griffith and J. Harris, "Principles of Algebraic Geometry," Wiley NY 1994.