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Chiral Supergravity

Melanie Becker[*], **Paul Bruillard**[†] and **Sean Downes**[‡]

George P. and Cynthia W. Mitchell Institute for Fundamental Physics
Texas A&M University, College Station, TX 77843, USA

ABSTRACT: We study the linearized approximation of $\mathcal{N} = 1$ topologically massive supergravity around AdS_3. Linearized gravitino fields are explicitly constructed. For appropriate boundary conditions, the conserved charges demonstrate chiral behavior, so that chiral gravity can be consistently extended to Chiral Supergravity.

KEYWORDS: Supergravity Models.

[*]mbecker@physics.tamu.edu
[†]pjb2357@physics.tamu.edu
[‡]sddownes@physics.tamu.edu

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1. Introduction and Motivation

The immense number of vacua in the String Theory Landscape is surrounded by an even larger number of vacua contained in the Swampland [1]. That is, the set of effective theories that appear valid semiclassically, but are inconsistent quantum theories. One might wonder if, in the gravitational scenery, there could exist a third class of theories: renormalizable and fully consistent quantum theories of gravity that stand independent of String Theory. Candidate theories include $\mathcal{N} = 8$ Supergravity in four dimensions (see [2] for the latest status) and Supergravity in three dimensions. In this paper, we are interested in the latter.

In three dimensions, gravity is highly constrained, suggesting that some theories might be consistent at the quantum level. Chiral Gravity [3], Log Gravity [4–7], and New Massive Gravity [8] may be examples of such theories, although the second is expected to have a non-unitary dual conformal field theory. The first two theories emerged from a critical point in the parameter space of Topologically Massive Gravity (TMG)[1].

TMG was constructed by Deser, Jackiw and Templeton [9]. Though retaining all solutions of Einstein gravity, like the BTZ black hole [10], TMG allows for new solutions. Around an Anti-de Sitter background, it has propagating (massive) gravitons. For generic values in its parameter space, these modes have negative energy. Consequently, the AdS_3 background is unstable.

It was argued in [3] that at the critical or "chiral point," these energies vanish and that the resulting theory is stable. Depending on the asymptotic boundary conditions for the metric two theories emerged: Chiral Gravity and Log Gravity.

The goal of this paper is to construct the supersymmetric extension of Chiral Gravity, while a detailed study of the supersymmetric extension of Log Gravity is work in progress [11]. We focus on the simplest supersymmetric extension of TMG, Topologically Massive Supergravity (TMSG), constructed by Deser and Kay [12] and cosmologically extended by Deser in [13]. As in simple AdS_3 Supergravity [14], Chiral Supergravity might possess a holographic dual. Specifically, a chiral, $\mathcal{N} = 1$ extremal[2] two dimensional superconformal field theory.

There are several reasons which motivated us to construct the supersymmetric extension of Chiral Gravity:

(1) Positive Energy Theorem

The Hamiltonian of supersymmetric theories is expressed as a sum of squares of supercharges, $H = \Sigma\ Q^2$. Naïvely, this would suggest positive energy for TMSG for generic points in parameter space. However, in supersymmetric theories with higher derivative interactions, the total energy does not have to be positive, as became evident from the literature of the late 1970s. At the time, the positive energy theorem of general relativity [16,17] had still not been proven. Supersymmetry seemed an interesting path to follow, and Deser and Teitelboim [18] showed positivity of the total energy in simple Supergravity using $H = \Sigma Q^2$.

Abbott and Deser [19] extended this proof to Supergravity with a cosmological constant. This lead to the idea [20] of using properties of supersymmetric theories to understand bosonic supersymmetrizable theories by setting fermion fields to zero, and culminated in Schoen and Yau's proof of the positive energy theorem [17]. Techniques from Supergravity

[1] However, the sign conventions of Deser et. al. [9] will be disregarded in favor of that of ordinary Einstein gravity.

[2] Extremal SCFT, as defined in [15], have no primaries other than the identity of dimension less than $k^*/2$, with central charge $c = 12k^*$.

then inspired Witten's new proof [16].

It was later realized that supersymmetric higher derivative theories are more complicated. In the $R + R^2$ case, the total energy is not positive definite, due to the presence of ghosts [21].
To illustrate this idea, consider a theory containing two free chiral supermultiplets. The total Lagrangian of this theory is

$$\mathcal{L} = \frac{1}{2} \left(\phi_1^* \Box \phi_1 + i\bar{\lambda}_1 \partial\!\!\!/ \lambda_1 - \phi_2^* \Box \phi_2 - i\bar{\lambda}_2 \partial\!\!\!/ \lambda_2 \right) .$$

In the above equation, the second multiplet describes a ghost. The Hamiltonian and supercharge are a difference of two positive quantities, $H = H_1 - H_2$ and $\mathcal{Q} = \mathcal{Q}_1 - \mathcal{Q}_2$. This implies that $H = \mathrm{tr}\mathcal{Q}^2 = H_1 - H_2$ is not positive even though the theory is supersymmetric. In short, higher derivative supersymmetric theories may contain ghosts. These can lead to negative enregy. The positivity of the energy depends on the concrete model at hand.

It is an important question to ask if a positive energy theorem for TMSG can be derived. This issue has been recently explored in [22, 23] for the non-linear theory where the positivity of the energy could not be shown, but a lower bound was derived. The puzzle of energy positivity was not solved but only transformed into a new question, the type of solutions admitted by the equations of motion. Our own calculations for the theory indicate that at the linearized level, the supersymmetric theory mimics the bosonic theory considered in [3]: For generic points in parameter space, there is a massive gravity supermultiplet with negative energy, and positivity of the total energy is not guaranteed. However, at the chiral point the energy contribution of this multiplet vanishes, and the total energy is positive. Thus, a consistent supersymmetric extension of Chiral Gravity— Chiral Supergravity— exists. The result of our energy calculation matches with the recent work of Andrade and Marlof [24], where it was shown that TMG has ghosts for generic values of the couplings (except at the chiral point).

(2) Uniqueness of String Theory

Having found a supersymmetric generalization of Chiral Gravity one may wonder if it could be embedded in String Theory. We will make some observations regarding the relation to String Theory in our conclusions and will leave the detailed check of whether Chiral Supergravity can be related to String Theory for future work.

(3) Extremal conformal field theories.

Chiral Supergravity may have an interesting dual extremal SCFT description along the lines of [14, 15].

This paper is organized as follows: In Section 2, we describe $\mathcal{N} = 1$ Topologically Massive Supergravity (TMSG). We discuss in some detail the different possibilities of having

$\mathcal{N}=1$ supersymmetry either in the left-moving sector, the right-moving sector, or both. In Section 3, we describe the isometries and supersymmetry properties of the AdS_3 classical background we are interested in. In Section 4, we describe the linearized theory. In Section 4.1, we derive the linearized equations of motion. In Section 4.2, we review the graviton excitations that solve these equations [3] and in Section 4.3, we compute the explicit form of the gravitini excitations, their wave functions, and conformal weights. In Section 5, we calculate the energy and supercharge of TMSG using the Abbott-Deser-Tekin approach [19, 25]. Positivity of the total energy indicates that TMSG is only stable at the chiral point, $\mu\ell = 1$, even though the theory is supersymmetric. We finish in Section 6 with some conclusions, some comments about Log Supergravity and an outlook.

2. $\mathcal{N}=1$ Topologically Massive Supergravity

Shortly after the appearance of TMG [9] in the mid-1980s, Deser and Kay constructed its $\mathcal{N}=(1,0)$ extension, Topologically Massive Supergravity [12] by the addition of a single Majorana gravitino; Deser [13] provided the cosmological extension of the theory. The underlying AdS symmetry group $SO(2,2) \approx SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ is then enhanced to $Osp(1|2;\mathbb{R})_L \times SL(2,\mathbb{R})_R$. It is also possible to include gravitinos in the right moving sector or in both the right and the left moving sector. We will first present the $\mathcal{N}=(1,0)$ theory and discuss the other two cases after that. The action describing the $\mathcal{N}=(1,0)$ theory is

$$\mathcal{S} = \frac{1}{16\pi G}\int d^3x e\mathcal{L} = \frac{1}{16\pi G}\int d^3x e\left[R - 2\Lambda - \frac{1}{2\mu}\epsilon^{\mu\nu\rho}\left(\partial_\mu\omega_\nu^{ab}\omega_{\rho ba} + \frac{2}{3}\omega_{\mu b}^{a}\omega_{\nu c}^{b}\omega_{\rho a}^{c}\right)\right.$$
$$\left. - i\epsilon^{\mu\nu\rho}\bar{\boldsymbol{\psi}}_\mu\left(D_\nu - \frac{1}{2\ell}\gamma_\nu\right)\boldsymbol{\psi}_\rho + \frac{i}{2\mu}\bar{f}^\mu\gamma_\nu\gamma_\mu f^\nu\right] . \tag{2.1}$$

Here, $\Lambda = -\frac{1}{\ell^2}$ is the cosmological constant, the gravitino mass parameter is equal to the reciprocal of the AdS radius ℓ, and G is the three-dimensional Newton's constant. The f^μ appearing in the action is the dual of the gravitino field strength given by

$$f^\mu = \epsilon^{\mu\alpha\beta}D_\alpha\boldsymbol{\psi}_\beta\,; \quad \text{where} \quad D_\alpha\boldsymbol{\psi}_\beta = \partial_\alpha\boldsymbol{\psi}_\beta + \frac{1}{4}\omega_\alpha^{ab}\gamma_{ab}\boldsymbol{\psi}_\beta - \Gamma_{\alpha\beta}^{\lambda}\boldsymbol{\psi}_\lambda\,. \tag{2.2}$$

In this expression (as well as in all other expressions defining the theory) the spin connection involves torsion. We work in the second-order formalism and define the functional form of $\omega_\mu^{ab}(e,\boldsymbol{\psi})$ to be precisely that of simple supergravity. This can be determined using the Palatini formalism (see [26]) giving

$$\omega_{\mu ab}(e,\boldsymbol{\psi}) = \omega_{\mu ab}(e) + \kappa_{\mu ab}(e,\boldsymbol{\psi}) \ , \tag{2.3}$$

where

$$\kappa_{\mu ab}(e,\boldsymbol{\psi}) = \frac{i}{4}\left(\bar{\boldsymbol{\psi}}_\mu\gamma_a\boldsymbol{\psi}_b - \bar{\boldsymbol{\psi}}_\mu\gamma_b\boldsymbol{\psi}_a + \bar{\boldsymbol{\psi}}_a\gamma_\mu\boldsymbol{\psi}_b\right) \ . \tag{2.4}$$

We observe that torsion comes from gravitini, and we will refer to the connection $\omega(e,\boldsymbol{\psi})$ as the torsional spin connection as opposed to the standard spin connection $\omega(e)$ defined by the vielbein postulate

$$D_\mu e_\nu^a = \partial_\mu e_\nu^a - e_\rho^a\Gamma_{\mu\nu}^{\rho}(g) + \omega_\mu^{ab}(e)\,e_{\nu b} = 0\,, \tag{2.5}$$

with $\Gamma^{\rho}_{\mu\nu}(g)$ representing the standard Christoffel connection. Note that the torsional spin connection also satisfies the torsional constraint

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^{\rho}_{\mu\nu}\,(g, \boldsymbol{\psi})\, e^a_\rho + \omega^{ab}_\mu\,(g, \boldsymbol{\psi})\, e_{\nu b} = 0 \ , \tag{2.6}$$

which also serves as the definition of the torsional Christoffel connection.

The $\mathcal{N} = (1,0)$ is invariant under the local supersymmetry transformations

$$\delta e^a_\mu = \bar{\epsilon}\gamma^a \boldsymbol{\psi}_\mu \ , \tag{2.7}$$

$$\delta \boldsymbol{\psi}_\mu = 2D_\mu\epsilon - \frac{1}{\ell}\gamma_\mu\epsilon \ , \tag{2.8}$$

where $D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\epsilon$ is the standard covariant derivative of a spinor. The conformal (topological) part of the action is separately invariant under supersymmetry.

The non-linear field equations for the graviton and gravitino following from the action (2.1) are

$$\mathcal{G}_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} + F_{\mu\nu} = 0, \ \text{and} \tag{2.9}$$

$$f^\mu - \frac{1}{2\ell}\gamma^{\mu\nu}\boldsymbol{\psi}_\nu + \frac{1}{\mu}C^\mu = 0 \ . \tag{2.10}$$

In these equations, $\mathcal{G}_{\mu\nu}$ is the cosmologically modified Einstein tensor,

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} \ . \tag{2.11}$$

$C_{\mu\nu}$ is the Cotton tensor,

$$C_{\mu\nu} = \epsilon_\mu{}^{\rho\sigma}\nabla_\rho\left(R_{\sigma\nu} - \frac{1}{4}g_{\sigma\nu}R\right) \ . \tag{2.12}$$

The covariant derivative appearing in this expression is written in terms of the torsionful Christoffel connection. Furthermore, we denote by $F_{\mu\nu}$ the fermionic (up to 6th order in fermions) contribution to the graviton field equation and will give its explicit expression to the relevant order when needed. The supersymmetric partner of the Cotton tensor, the Cottino, is the vector-spinor

$$C^\mu = \frac{1}{2}\gamma^\rho\gamma^{\mu\nu}D_\nu f_\rho - \frac{1}{8}\epsilon^{\lambda\nu\rho}R_{\lambda\nu ab}\left(2\delta^\mu_\rho\gamma^b\boldsymbol{\psi}^a + e^{\mu b}\gamma_\rho\boldsymbol{\psi}^a\right) \ . \tag{2.13}$$

Having the formulas for the $\mathcal{N} = (1,0)$ theory, it is an easy matter to describe the $\mathcal{N} = (0,1)$ theory taking into account that parity relates both theories. A detailed discussion of the action of parity on the $\mathcal{N} = (1,0)$ theory is presented in Appendix B.2. The action of parity effectively reduces to a sign reversal in μ and ℓ, so that all of the previous formulas apply for the $\mathcal{N} = (0,1)$ theory after the corresponding sign changes.

The action for the $\mathcal{N} = (1,1)$ model incorporates two Majorana gravitinos, $\boldsymbol{\psi}_L$, $\boldsymbol{\psi}_R$ with mass terms of opposite sign. Such an action does not seem to have been discussed in the literature, as far as we know. However, for the purpose of studying the linearized theory, we can easily extend the previous formulas to a theory with $\mathcal{N} = (1,1)$ supersymmetry by including an extra gravitino with opposite mass term into the action (2.1). In this case we can apply the Palatini formalism to determine the torsional spin connection to be

$$\omega_\mu^{ab}(e, \boldsymbol{\psi}^R, \boldsymbol{\psi}^L) = \omega_\mu^{ab}(e) + \kappa_\mu^{ab}(\boldsymbol{\psi}^L) + \kappa_\mu^{ab}(\boldsymbol{\psi}^R) \ , \tag{2.14}$$

where $\omega(e)$ and κ are the torsion-free spin connection and the contorsion as previously defined. Eventual interactions between left and right gravitino fields due to this torsional coupling would show up at fourth and higher orders in the perturbative expansion, but they are irrelevant for the linearized theory we are interested in. The supersymmetry transformations for the $\mathcal{N} = (1,1)$ theory are

$$\delta e_\mu^a = \bar{\epsilon}\gamma^a \boldsymbol{\psi}_\mu^L - \bar{\epsilon}\gamma^a \boldsymbol{\psi}_\mu^R \ , \tag{2.15}$$

$$\delta \boldsymbol{\psi}_\mu^L = 2D_\mu \epsilon - \frac{1}{\ell}\gamma_\mu \epsilon \ , \tag{2.16}$$

$$\delta \boldsymbol{\psi}_\mu^R = 2D_\mu \epsilon + \frac{1}{\ell}\gamma_\mu \epsilon \ . \tag{2.17}$$

In Section 5, we calculate the energy and supercharge of the three $\mathcal{N} = 1$ models we just discussed.

3. The Classical Background

Topological massive supergravity has an $\mathcal{N} = 1$ supersymmetric AdS_3 vacuum for which the metric in global coordinates takes the form

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = \ell^2 \left(-\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2\right) \tag{3.1}$$

while the gravitino vanishes. In this section, we describe the form of the (super)symmetry generators that will be used later to calculate the explicit form of the bosonic and fermionic wave functions along the lines of [3].

3.1 Isometries

AdS_3 is maximally symmetric and thus has six Killing vectors, K^μ, which generate the $SO(2,2) \approx SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ group of isometries. When acting on scalars, their generators take the forms [3]:

$$L_0 = K_{(0)}^\mu \partial_\mu = \frac{i}{2}\left(\partial_\tau + \partial_\phi\right) \ , \tag{3.2}$$

$$\bar{L}_0 = \bar{K}_{(0)}^\mu \partial_\mu = \frac{i}{2}\left(\partial_\tau - \partial_\phi\right) \ , \tag{3.3}$$

$$L_{\pm 1} = K_{(\pm 1)}^\mu \partial_\mu = \frac{i}{2} e^{\pm i(\tau+\phi)} \left(\tanh \rho \partial_\tau + \coth \rho \partial_\phi \mp i \partial_\rho\right) \ , \tag{3.4}$$

$$\bar{L}_{\pm 1} = \bar{K}_{(\pm 1)}^\mu \partial_\mu = \frac{i}{2} e^{\pm i(\tau-\phi)} \left(\tanh \rho \partial_\tau - \coth \rho \partial_\phi \pm i \partial_\rho\right) \ , \tag{3.5}$$

where unbarred and barred operators refer to the left- and right-moving algebras respectively. These generators satisfy the conformal algebra

$$[L_0, L_{\pm 1}] = \pm L_{\pm 1} \; ; \qquad [L_1, L_{-1}] = 2L_0 \; , \tag{3.6}$$

and similarly for the right moving operators. In the supersymmetric case, we are interested in this algebra as extended to a super-Virasoro algebra, which we elaborate on in the next Section 3.2.

The conformal algebra in the bulk is enhanced to an infinite dimensional Virasoro algebra on the boundary [27]:

$$[L_n, L_m] = (m-n)\, L_{m+n} + \frac{1}{12} c \left(m^3 - m\right) \delta_{m+n,0} \; , \tag{3.7}$$

and similarly for the barred algebra.

It was shown in [3] that the $SL(2,\mathbb{R})$ algebra can be used to classify the states that satisfy the three-dimensional equations of motion.[3] Gravitons, $|h\rangle$, are described as primary states of this algebra and are labeled by the weights $(h, \bar{h})$

$$L_0|h\rangle = h|h\rangle \; , \quad \bar{L}_0|\bar{h}\rangle = \bar{h}|\bar{h}\rangle \; , \tag{3.8}$$

and satisfy

$$L_n|h\rangle = \bar{L}_n|\bar{h}\rangle = 0 \quad n > 0 \; . \tag{3.9}$$

Equivalently, we can label these states by their energy $E = h + \bar{h}$ and their spin $S = h - \bar{h}$. Unitarity of the representation (e.g. positivity of the norm of all states) imposes constraints on the central change and weight of the primary fields [28, 29]. Unitary representations exist for all values (c, h) with $c \geq 1$ and $h \geq 0$, or equivalently $E \geq |S|$. Representations that saturate this bound are called "massless" and describe non-propagating degrees of freedom. Representations with $E > |S|$ are called "massive" and describe propagating states with helicity S. As will be checked later, primary states in both representation appear in TMSG.

It was shown by Brown and Henneaux [27] that for ordinary Einstein theory, the central charges of the left- and right-moving algebra are equal:

$$c_L = c_R = \frac{3\ell}{2G} \; . \tag{3.10}$$

The gravitational Chern-Simons term appearing in TMG deforms these central charges so they are no longer equal [30]

$$c_L = \frac{3\ell}{2G}\left(1 - \frac{1}{\mu\ell}\right), \quad c_R = \frac{3\ell}{2G}\left(1 + \frac{1}{\mu\ell}\right) \; . \tag{3.11}$$

Taking into account that the semiclassical approximation corresponds to large ℓ/G, it is interesting to note that unitarity of the boundary theory demands $|\mu\ell| \geq 1$. For the

[3] Although not all solutions can be obtained in this way, the logarithmic mode is a counterexample.

specific value of the Chern-Simons coupling, $\mu\ell = 1$, it was noticed in [3] that $c_L = 0$ and $c_R = 3\ell/G$. This suggests that right-moving gravitational physics may behave as in pure gravity (albeit with twice the central charge) and that left-moving gravitational physics might be trivial. In other work, it was shown that once particular boundary conditions are imposed, it is indeed possible to obtain a left-moving theory that is trivial [4, 5, 31–33].

3.2 Supersymmetry

We consider again all three cases, i.e. supersymmetry in the left sector, the right sector or both. For the $\mathcal{N} = (1, 0)$ theory the Killing spinor equation takes the form

$$\delta\psi_\mu = 2D_\mu\epsilon - \frac{1}{\ell}\gamma_\mu\epsilon = 0 \ , \tag{3.12}$$

where $\gamma_\mu = e_\mu^a\gamma_a$. There are two Killing spinors that solve this equation:

$$\xi_L = e^{(iu-\rho)/2}\begin{pmatrix} -ie^\rho \\ 1 \end{pmatrix} \quad \text{and} \quad \xi_L^* = e^{-(iu+\rho)/2}\begin{pmatrix} ie^\rho \\ 1 \end{pmatrix} \ , \tag{3.13}$$

where $u = \tau + \phi$. Like the Killing vectors K_1^μ and K_{-1}^μ, these Killing spinors are simply complex conjugates of one another. Killing spinors are associated with the fermionic generators of the super-Virasoro algebra. In general, fermionic fields depending on a compact coordinate (in this case the ϕ coordinate) can be either periodic or anti-periodic under $\phi \to \phi + 2\pi$, corresponding either the Ramond (R) or Neveu-Schwarz (NS) sector algebra. Since the above spinors are anti-periodic, we are interested in the NS sector of the algebra. In this sector, the $\mathcal{N} = (1, 0)$ global subalgebra is generated by

$$L_0, L_{\pm1}, G_{\pm1/2}, \bar{L}_0, \bar{L}_{\pm1}. \tag{3.14}$$

The left-moving superconformal algebra is

$$[L_m, L_n] = (m-n)\,L_{m+n} \ , \quad [L_m, G_r] = \left(\frac{m}{2} - r\right)G_{m+r} \ , \quad \{G_r, G_s\} = 2L_{r+s} \ , \tag{3.15}$$

with $m, n = 0, \pm1$ and $r, s = \pm1/2$. Similarly, as in the bosonic case, we expect this algebra to be enhanced to an infinite dimensional super-Virasoro algebra on the boundary

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{1}{12}c_L\left(m^3 - m\right)\delta_{m+n,0} \ , \tag{3.16}$$

$$\{G_m, G_n\} = 2L_{m+n} + \frac{1}{3}c_L\left(m^2 - \frac{1}{4}\right)\delta_{m+n,0} \ , \tag{3.17}$$

$$[L_m, G_n] = \left(\frac{m}{2} - n\right)G_{m+n} \ . \tag{3.18}$$

The central change of the supersymmetric theory is the same as the central charge of the bosonic theory (see [34]).

In the supersymmetric case graviton and gravitinos are primary fields of the super-Virasoro algebra that satisfy the constraints (3.8), (3.9) and

$$G_r|h\rangle = 0 \ , \quad r > 0 \ . \tag{3.19}$$

Although the explicit form of the fermionic generators will not be needed to obtain the graviton and gravitino wave functions, notice that just as L_m can be expressed in terms of Killing vectors, the fermionic generators, $G_{\pm 1/2}$, can be expressed in terms of Killing spinors. For example, the action of L_1 on a scalar field ϕ is simply the directional derivative

$$L_1\phi = K_1^\mu \partial_\mu \phi \ . \tag{3.20}$$

Similarly, the action of $G_{1/2}$ on ϕ is

$$G_{1/2}\phi = \gamma^\mu \xi_L \partial_\mu \phi \ . \tag{3.21}$$

The $\mathcal{N} = (0,1)$ theory involves an inequivalent representation of the Clifford algebra, see Appendix B.1 for details. A study of the $\mathcal{N} = (0,1)$ yields the resulting Killing spinor equation:

$$D_\mu \epsilon + \frac{1}{2\ell}\gamma_\mu \epsilon = 0 \ . \tag{3.22}$$

There are again two Killing spinors that solve this equation that take the form

$$\xi_R = e^{(iv-\rho)/2}\begin{pmatrix} i \\ e^\rho \end{pmatrix}, \quad \xi_R^* = e^{-(iv+\rho)/2}\begin{pmatrix} -i \\ e^\rho \end{pmatrix} \ . \tag{3.23}$$

Here, $v = \tau - \phi$, and we observe again that both spinors are complex conjugates of one another.

Thus, we see that AdS_3 is a viable supersymmetric background for an $\mathcal{N} = (1,1)$ extension of TMG.

Next, we analyze the stability of the AdS_3 background by considering perturbations around it. We proceed as in [3]: Compute the explicit form of linear graviton and gravitino fields, then compute conserved charges to second order in these perturbations. First, however, we must discuss some generalities.

4. The Perturbative Expansion

Due to the nonlinearity present in Topologically Massive Gravity and the corresponding supergravity theories, one is often forced into a perturbative regime as to make headway. Indeed, the analysis leading to Chiral Gravity and our corresponding work on Topologically Massive Supergravity relies heavily on the methods of perturbation theory. In this section, we will briefly recapitulate the basics of perturbative gravity, and then proceed to determine the perturbative spectrum of the supersymmetric theory.

Expansion in small fluctuations about some fundamental object is central to all variants of perturbation theory. In gravitational theories, the fundamental field is the metric, the starting point of perturbative gravity is to expand in fluctuations about some background metric, $\bar{g}_{\mu\nu}$, which is a known solution of the theory, as follows

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \lambda h_{\mu\nu} + \lambda^2 j_{\mu\nu} + \mathcal{O}\left(\lambda^3\right) \ . \tag{4.1}$$

Here, λ is a small parameter to be used as a book-keeping device. Thus the n^{th} order in perturbation theory is tantamount to $\mathcal{O}\left(\lambda^n\right)$ in the relevant expansions. In supergravity theories there is a second field, the gravitino, which should also be expanded in powers of λ. It is important to note that the gravitino is a fermion and so when performing such an expansion the background term is identically zero:

$$\boldsymbol{\psi}_\mu = \lambda\psi_\mu + \lambda^2\psi_\mu^{(2)} + \mathcal{O}\left(\lambda^3\right) \ . \tag{4.2}$$

Given such perturbative expansions for the fundamental fields of the theory, one constructs perturbative expansions for all objects appearing in the action. In general, given a multi-linear map $M_{\mu\nu\ldots}\left(g,\boldsymbol{\psi}\right)$, its formal perturbative expansion can be written as

$$M_{\mu\nu\ldots}\left(g,\boldsymbol{\psi}\right) = M^{(0)}_{\mu\nu\ldots} + \lambda M^{(1)}_{\mu\nu\ldots} + \lambda^2 M^{(2)}_{\mu\nu\ldots} + \mathcal{O}\left(\lambda^3\right) \ , \tag{4.3}$$

where the functional form of $M^{(n)}_{\mu\nu\ldots}$ is given by

$$M^{(n)}_{\mu\nu\ldots} = \frac{1}{n!}\left.\frac{\partial^n M_{\mu\nu\ldots}\left(g,\boldsymbol{\psi}\right)}{\partial\lambda^n}\right|_{\lambda=0} \ . \tag{4.4}$$

Applying such an expansion to the equations of motion allows one to work order by order in λ and generate the perturbative spectrum of the theory.

4.1 Linearized Supergravity

Consider first the $\mathcal{N}=(1,0)$ theory. In the linearized approximation the field equations of this theory take the form

$$\mathcal{G}^{(1)}_{\mu\nu}\left(h\right) + \frac{1}{\mu}C^{(1)}_{\mu\nu}\left(h\right) = 0 \ , \tag{4.5}$$

and

$$f^{(1)}_\mu\left(\psi\right) - \frac{1}{2\ell}\bar{\gamma}_{\mu\nu}\psi^\nu + \frac{1}{\mu}C^{(1)}_\mu\left(\psi\right) = 0 \ , \tag{4.6}$$

where barred quantities are expressed with respect to the background metric. Here we have introduced the notation

$$\mathcal{G}^{(1)}_{\mu\nu}\left(h\right) = R^{(1)}_{\mu\nu}\left(h\right) - \frac{1}{2}\bar{g}_{\mu\nu}R^{(1)}\left(h\right) - 2\Lambda h_{\mu\nu} \ , \tag{4.7}$$

$$C^{(1)}_{\mu\nu}\left(h\right) = \epsilon^{\alpha\beta}{}_\mu\bar{\nabla}_\alpha\left(R^{(1)}_{\beta\nu}\left(h\right) - \frac{1}{4}\bar{g}_{\beta\nu}R^{(1)}\left(h\right) - 2\Lambda h_{\beta\nu}\right) \ , \tag{4.8}$$

with

$$R^{(1)}(h) = -\bar{\nabla}^2 h + \bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu\nu} - 2\Lambda h \ , \tag{4.9}$$

$$R^{(1)}_{\mu\nu}\left(h\right) = \frac{1}{2}\left(-\bar{\nabla}^2 h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h + \bar{\nabla}^\sigma\bar{\nabla}_\nu h_{\sigma\mu} + \bar{\nabla}^\sigma\bar{\nabla}_\mu h_{\sigma\nu}\right) \ , \tag{4.10}$$

$$f^{(1)}_\mu\left(\psi\right) = \epsilon_{\mu\alpha\beta}\bar{D}^\alpha\psi^\beta \ , \tag{4.11}$$

$$C^{(1)\mu}\left(\psi\right) = \frac{1}{2}\bar{\gamma}^\rho\bar{\gamma}^{\mu\nu}\bar{D}_\mu f^{(1)}_\rho\left(\psi\right) - \frac{1}{4\mu\ell^2}\psi_\mu. \tag{4.12}$$

Note that at linear order the bosonic and fermionic equations completely decouple. This is due to gravitons and gravitini coupling through torsion, which is a second order effect in λ. This allows us to study the graviton wave functions separately from the gravitinos. The form of the linearized bosonic equation of motion is precisely the one found by [3], and so before proceeding to the case of the Rarita-Schwinger field, we will review the metric fluctuations.

4.2 Linearized Gravitons

The graviton field equation becomes much simpler if a particular gauge is chosen. It was shown in [3] that a convenient gauge choice is the divergence free-gauge. At the linear level it reads

$$\bar{\nabla}_\mu \left(h^{\mu\nu} + \bar{g}^{\mu\nu} h\right) = 0 \; . \tag{4.13}$$

Combining this with the bosonic field equation yields the traceless condition $h = 0$, so that the linearized bosonic field can be chosen to be divergence free and traceless:

$$h = 0, \quad \bar{\nabla}_\mu h^{\mu\nu} = 0 \; . \tag{4.14}$$

It was further shown in [3] that implementation of this gauge condition reduces the linearized bosonic equation of motion to

$$\left(\bar{\nabla}^2 + \frac{2}{\ell^2}\right)\left(h_{\mu\nu} + \frac{1}{\mu}\epsilon_\mu{}^{\rho\sigma}\bar{\nabla}_\rho h_{\sigma\nu}\right) = 0 \; . \tag{4.15}$$

This equation can be written in terms of the SL$(2,\mathbb{R})$ Casimir, $L^2 + \bar{L}^2 = -\frac{\ell^2}{2}\nabla^2$, which motivated [3] to use the SL$(2,\mathbb{R})$ algebra to find the solutions to the equation of motion. Gravitons are described by harmonic functions on AdS_3, which take the form

$$h_{\mu\nu} = e^{-i(E\tau + S\phi)} M_{\mu\nu}\left(\rho\right) \; , \tag{4.16}$$

where $M_{\mu\nu}$ is a symmetric two-index tensor depending only on ρ whose explicit form will be calculated. The spin is determined by the gauge choice for $h_{\mu\nu}$ and is found to be $S = \pm 2$. Furthermore, $M_{\mu\nu}\left(\rho\right)$ can be determined through the application of the primary field constraints

$$L_1 h_{\mu\nu} = \bar{L}_1 h_{\mu\nu} = 0 \; . \tag{4.17}$$

These constraints can be rearranged into the more convenient form

$$\left(L_1 \pm \bar{L}_1\right) h_{\mu\nu} = 0 \; . \tag{4.18}$$

The generators, L_n, $\bar{L}_n$ are taken to be Lie derivatives along the Killing vector fields. In particular, when acting on a two-index tensor, the Lie derivative along a vector field $K_{(n)}$ takes the form [35]:

$$L_n h_{\mu\nu} = K^\lambda_{(n)}\left(\bar{\nabla}_\lambda h_{\mu\nu}\right) + \left(\bar{\nabla}^\lambda K_{(n)\mu}\right) h_{\lambda\nu} + \left(\bar{\nabla}^\lambda K_{(n)\nu}\right) h_{\mu\lambda} \; . \tag{4.19}$$

The f fields in these equations are taken to be the Killing vectors $K^{\mu}_{(n)}$, $\bar{K}^{\mu}_{(n)}$ defined in (3.2)-(3.5). With the lower sign of (4.18), the tensor $M_{\mu\nu}$ can be determined to be

$$M_{\mu\nu}(\rho) = f(\rho) \begin{pmatrix} 1 & \frac{S}{2} & ia \\ \frac{S}{2} & 1 & \frac{iSa}{2} \\ ia & \frac{iSa}{2} & -a^2 \end{pmatrix} , \tag{4.20}$$

with

$$a = \frac{1}{\sinh\rho\cosh\rho}. \tag{4.21}$$

Taking the upper sign of (4.18) allows one to determine the matrix prefactor, $f(\rho)$. In particular, one arrives at the differential equation

$$\partial_\rho f(\rho) + \frac{E\sinh^2\rho - 2\cosh^2\rho}{\sinh\rho\cosh\rho} f(\rho) = 0 , \tag{4.22}$$

which admits the solution

$$f(\rho) = \frac{\sinh^2\rho}{\cosh^E\rho} . \tag{4.23}$$

Combining these results yields the graviton wave function up to overall normalization [3]:

$$h_{\mu\nu} = N_b e^{-i(E\tau+S\phi)} \frac{\sinh^2\rho}{\cosh^E\rho} \begin{pmatrix} 1 & \frac{S}{2} & ia \\ \frac{S}{2} & 1 & \frac{iSa}{2} \\ ia & \frac{iSa}{2} & -a^2 \end{pmatrix} . \tag{4.24}$$

The energy value E can be determined by inserting this solution into the linearized equation of motion. Upon restricting to normalizable modes, the weights or energy and spin can be fixed to

$$(E, S) = (2, \pm 2) \quad \text{or} \quad (1 \pm \mu\ell, \pm 2) . \tag{4.25}$$

In [3], $(E, S) = (2, 2)$ is referred to as the left-moving graviton, $(E, S) = (2, -2)$ is the right-moving graviton, and $(E, S) = (1 + \mu\ell, 2)$ is the massive graviton. The final case, $(E, S) = (1 - \mu\ell, -2)$, is not considered a solution to the theory since the wave function is non-normalizable. At the chiral point, $\mu\ell = 1$, the wave function of the massive mode coincides with the one of the left-moving graviton. It was argued in [3] that for suitable boundary conditions this left moving wave function can be gauged away so that the theory becomes chiral, with only a right-moving degree of freedom.

4.3 Linearized Gravitini

Deriving the gravitino wave functions at the linear level proceeds in a similar fashion. Consider again first the $\mathcal{N} = (1, 0)$ theory. The left-moving gravitino wave functions are vector-spinors on AdS_3

$$\psi_\mu = e^{-i(E\tau+S\phi)} \zeta_\mu(\rho) , \tag{4.26}$$

where ζ_μ with $\mu = 0, 1, 2$ is a two-component spinor depending only on ρ. As with the gravitons, a suitably chosen gauge simplifies the equations of motion.

It is understood that the Rarita-Schwinger field carries its own gauge freedom. Specifically, equivalent physical states are obtained by $\psi_\mu \to \psi_\mu + (D_\mu \pm \frac{1}{2\ell}\gamma_\mu)\kappa$, where κ is some spinor field. This gauge freedom allows one to fix

$$\bar{\gamma}^\mu \psi_\mu = 0\ . \tag{4.27}$$

This is the natural choice for the superpartner of the traceless graviton. In fact, applying a supersymmetry transformation to the linearized graviton trace-free gauge condition yields the gamma-traceless condition (4.27). Expanding this condition yields the relationship

$$\psi_2 = \bar{\gamma}_1 \psi_0 - \bar{\gamma}_0 \psi_1\ , \tag{4.28}$$

which can be used to determine ψ_2 once ψ_0 and ψ_1 are known. To determine ψ_0 and ψ_1 it is sufficient to apply the lowest-weight/primary-field conditions

$$\left(L_1 \pm \bar{L}_1\right)\psi_\mu = 0\ , \tag{4.29}$$

where, as in the bosonic case, the L_1, $\bar{L}_1$ operators are Lie derivatives along the Killing vector fields K_1^μ, $\bar{K}_1^\mu$ acting on a vector-spinor. Specifically, they are given by [35]:

$$L_n \psi_\mu = K^\lambda_{(n)} \bar{D}_\lambda \psi_\mu + \frac{1}{2}(\bar{\nabla}_\alpha K_{(n)\beta})\bar{\gamma}^{\alpha\beta}\psi_\mu + \left(\bar{\nabla}_\mu K^\lambda_{(n)}\right)\psi_\lambda\ . \tag{4.30}$$

However, when $K^\lambda_{(n)}$, $\bar{K}^\lambda_{(n)}$ are Killing vectors, one can apply the AdS_3 algebra to reduce these expressions to

$$L_n \psi_\mu = K^\lambda_{(n)}(\bar{D}_\lambda - \frac{1}{2\ell}\bar{\gamma}_\lambda)\psi_\mu + \left(\bar{\nabla}_\mu K^\lambda_{(n)}\right)\psi_\lambda \tag{4.31}$$

and

$$\bar{L}_n \psi_\mu = \bar{K}^\lambda_{(n)}(\bar{D}_\lambda + \frac{1}{2\ell}\bar{\gamma}_\lambda)\psi_\mu + \left(\bar{\nabla}_\mu \bar{K}^\lambda_{(n)}\right)\psi_\lambda\ . \tag{4.32}$$

Choosing the minus sign in (4.29), one finds that $S = \frac{3}{2}$ and

$$\psi_\mu = N_f e^{-iE\tau - iS\phi} F_\mu(\rho) \begin{pmatrix} i \\ e^\rho \end{pmatrix}\ . \tag{4.33}$$

Here,

$$F_0(\rho) = F_1(\rho) = F(\rho)\,, \quad F_2(\rho) = \frac{iF(\rho)}{\sinh\rho\cosh\rho}\ , \tag{4.34}$$

and N_f is some overall normalization. $F(\rho)$ can now be fixed by choosing the positive sign in (4.29), which leads to the differential equation

$$\partial_\rho \psi_1 + (E\tanh\rho - \coth\rho)\psi_1 + \frac{i\bar{\gamma}_1}{2\cosh\rho}\psi_1 = 0 \tag{4.35}$$

with solution

$$F(\rho) = \frac{e^{-\rho/2}\sinh\rho}{\cosh^{E+1/2}\rho}\ . \tag{4.36}$$

Notice that since ψ_μ are harmonic functions on AdS_3 they satisfy the Dirac equation

$$\left(\not{D} - \frac{(E-1)}{\ell} \right) \psi_\mu = 0 \, . \tag{4.37}$$

After fixing the gamma-traceless gauge, the linearized fermionic equation reduces to

$$\left(\not{D} - \frac{1}{2\ell} \right) \psi_\mu - \frac{1}{\mu} \left(\not{D}^2 - \frac{1}{4\ell^2} \right) \psi_\mu = 0 \, , \tag{4.38}$$

where the Feynman slash notation has been adopted so that $\not{D} = \bar{\gamma}^\mu D_\mu$. Inserting the linearized modes into this equation yields

$$\frac{E-1}{\ell} \psi_\mu - \frac{1}{2\ell} \psi_\mu - \frac{1}{\mu} \left(\frac{(E-1)^2}{\ell^2} - \frac{1}{4\ell^2} \right) \psi_\mu = 0 \, , \tag{4.39}$$

which fixes the value of E. The energy and spin of the left-moving gravitini are

$$(E, S) = \left(\frac{3}{2}, \frac{3}{2} \right) \quad \text{or} \quad \left(\frac{1}{2} + \mu\ell, \frac{3}{2} \right) \, . \tag{4.40}$$

The former is clearly a solution to simple supergravity and hence satisfies the requisite Dirac equation with appropriate mass. It corresponds to the left moving gravitino, the supersymmetric partner of the left moving graviton. The second mode corresponds to the so-called fermionic "massive" propagating degree of freedom. As in the bosonic theory, we observe that at the chiral point $\mu\ell = 1$ the wave functions of the massless and massive gravitino coincide. As will be shown in the next section, this chiral behavior will extend to the conserved charges, hence Topologically Massive Supergravity preserves the chiral structure found in [3].

Note that in the classical supergravity analysis, it is understood that fermions are Grassmann-valued Majorana spinors. Thus, the physical wave functions are the real (or, alternatively, imaginary) parts of ψ_μ and there are implicit Grassmann-valued numbers associated with all fermion spinor components. The "physical" temporal component of ψ_μ may be written as

$$Re\,(\psi_0) = \frac{Re\,(N_f)\, e^{-\rho/2} \sinh\rho}{\cosh^{E+1/2}\rho} \begin{pmatrix} \sin\left(E\tau + S\phi\right) \theta_1 \\ e^\rho \cos\left(E\tau + S\phi\right) \theta_2 \end{pmatrix} \, , \tag{4.41}$$

where $\theta_i \theta_j = -\theta_j \theta_i$, and similarly for the other components of ψ_μ.

The previous analysis can be carried over for the $\mathcal{N} = (0, 1)$ theory, by taking the corresponding sign changes in μ and ℓ into account. Given the ansatz (4.26) and the primary field constraint (4.29), it is straightforward to show that the spin is fixed to $S = -\frac{3}{2}$ and that right-moving gravitino fields are given by

$$\psi_\mu = N_f e^{-iE\tau - iS\phi} F_\mu(\rho) \begin{pmatrix} -ie^\rho \\ 1 \end{pmatrix} \, , \tag{4.42}$$

where

$$F_0(\rho) = -F_1(\rho) = F(\rho), \quad F_2(\rho) = \frac{iF(\rho)}{\sinh\rho\cosh\rho} , \tag{4.43}$$

and $F(\rho)$ is given by the same expression as before. Note the sign change for F_1. The equations of motion then fix the energy and spin to be

$$(E, S) = (\frac{3}{2}, -\frac{3}{2}) \quad \text{or} \quad (\frac{1}{2} + \mu\ell, -\frac{3}{2}) . \tag{4.44}$$

These states correspond to a right-moving gravitino and a massive gravitino that propagates in the bulk. The $\mathcal{N} = (1,1)$ theory will contain three gravitinos with conformal weights given by

$$(E, S) = \left(\frac{3}{2}, \pm\frac{3}{2}\right) \quad \text{and} \quad \left(\frac{1}{2} + \mu\ell, \frac{3}{2}\right) . \tag{4.45}$$

5. Energy and Supercharge in TMSG

In this section the stability under perturbations of Topologically Massive Supergravity is analyzed. This is done through the study of conserved charges as defined by Abbott and Deser [19]. The energy was previously calculated for the bosonic model [3] through Hamiltonian methods, where it was shown that for generic values of μ, the energies of some of the modes are negative, indicating an instability in the theory. However, at the chiral point, the energies for all linear perturbations are positive semi-definite, and so the theory defined on an AdS_3 background is stable against metric perturbations.

We first compute the energy of individual modes and show that TMSG on an AdS_3 background is unstable in general. Stability is restored at the chiral point $\mu\ell = 1$. One might have hoped to make use of supersymmetry to arrive at a positive energy theorem. As may be inferred, one is not able to show positivity. As explained in the introduction, this failure can be traced back to the existence of ghosts in higher derivative theories. Indeed, the recent calculation of [24] shows that TMG has ghosts for $\mu\ell \neq 1$.

Once the energy is computed, we extend the analysis to determine the supercharge. All conserved charges of the supersymmetric theory exhibit chiral behavior allowing us to show that the theory is stable against perturbations. Thus, one can speak of Chiral Supergravity.

5.1 Bosonic Charges: Energy

In their work [19] Abbott and Deser noted that in the presence of a non-vanishing cosmological constant, the conventional definition of gravitational energy fails and they subsequently determined a modified definition. Their key observation was that for a non-vanishing cosmological constant, the space-time is not asymptotically flat. This leads to the failure of conservation for the conventional charges. To rectify the situation, these authors constructed conserved charges for a background that satisfies Einstein's equations with arbitrary cosmological constant

$$\mathcal{G}^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = 0 . \tag{5.1}$$

To construct the charges, the metric is divided into two parts: a background value $\bar{g}_{\mu\nu}$ and a deviation $h_{\mu\nu}$, which does not have to be small, but needs to vanish fast enough at infinity

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \ . \tag{5.2}$$

Insertion of this expansion into the Einstein equations allows for the definition of the energy momentum pseudo-tensor. This is done by partitioning the equation of motion into three pieces: a piece dependent only on the background, a piece linear in metric fluctuation, and a final term containing all terms quadratic or higher order in $h_{\mu\nu}$. Since the background is Einsteinian the zeroth order contribution vanishes immediately. One then takes the terms nonlinear in metric fluctuations to define the energy momentum pseudo-tensor $T_{\mu\nu}$, so that the equation of motion reads

$$\mathcal{G}^{(1)}_{\mu\nu} = T_{\mu\nu} \tag{5.3}$$

where

$$\mathcal{G}^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} R^{(1)} - \Lambda h_{\mu\nu} \ . \tag{5.4}$$

Here, taking the superscript (n) to denote the expansion of the relevant object containing terms only of order n in $h_{\mu\nu}$. It is straightforward to show that the energy momentum pseudo-tensor satisfies the background Bianchi identity

$$\bar{\nabla}_\mu T^{\mu\nu} = 0 \ . \tag{5.5}$$

One can obtain conserved currents by contracting the energy momentum pseudo-tensor with a Killing vector ξ_ν, [19]:

$$\partial_\mu \left(T^{\mu\nu} \xi_\nu \right) = 0 \ . \tag{5.6}$$

When the Killing vector is taken to be time like, this defines the gravitational energy-momentum density, thus the gravitational energy

$$E\left(\xi_\nu\right) \equiv \frac{1}{8\pi G} \int e d^3 x T^{0\nu} \xi_\nu \ . \tag{5.7}$$

A similar analysis can be carried out for higher-derivative gravities (see [36] [25]). They later applied their analysis to TMG . In this case, the equations are modified by the presence of the Cotton tensor, and the background is now taken to be a solution of the vacuum equations of TMG. Insertion of the expansion yields

$$\mathcal{G}^{(1)\mu\nu} + \frac{1}{\mu} C^{(1)\mu\nu} = T^{\mu\nu} \ . \tag{5.8}$$

Here, $\mathcal{G}^{(1)\mu\nu}$ is the linearized Einstein tensor, $C^{(1)\mu\nu}$ is the Cotton tensor with only terms linear in metric fluctuations retained and the stress energy pseudo-tensor is the collection of all terms quadratic or higher order in $h_{\mu\nu}$. The expressions for the linearized Einstein and Cotton tensors were given in Section 4. To obtain the explicit form of the energy momentum pseudo-tensor of the $\mathcal{N} = (1,0)$ theory, we apply the perturbation expansion of Section 4

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \lambda h_{\mu\nu} + \lambda^2 j_{\mu\nu} + \mathcal{O}\left(\lambda^3\right) \ , \tag{5.9}$$

and

$$\boldsymbol{\psi}_\mu = \lambda\psi_\mu + \lambda^2\psi_\mu^{(2)} + \mathcal{O}\left(\lambda^3\right) \ . \tag{5.10}$$

We remind the reader that the gravitino expansion starts at $\mathcal{O}(\lambda)$ due to the absence of background fermions. Applying these expansions to the graviton field equation (2.9) of TMSG and working to $\mathcal{O}\left(\lambda^2\right)$, one finds

$$G_{\mu\nu}^{(1)}(j) + \frac{1}{\mu}C_{\mu\nu}^{(1)}(j) = -G_{\mu\nu}^{(2)}(h,\psi) - \frac{1}{\mu}C_{\mu\nu}^{(2)}(h,\psi) - F_{\mu\nu}^{(2)}(\psi) = T_{\mu\nu}^{(2)} \ , \tag{5.11}$$

where the functional forms for $\mathcal{G}_{\mu\nu}^{(n)}$ and $C_{\mu\nu}^{(n)}$ can be determined by the general procedure (4.3) and are given explicitly in Appendix A. Given these explicit forms, one can obtain the Abbott-Deser-Tekin gravitational energy to $\mathcal{O}\left(\lambda^2\right)$ in the fashion discussed above. To do so, one identifies the time-like Killing vector as

$$\xi^\mu = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} \ . \tag{5.12}$$

The bosonic contributions to the energy determined in this fashion exactly agree with the results of [3], the Hamiltonian approach was used. Next, to quadratic order, the fermionic contribution to the energy is given by

$$E_F = \frac{1}{8\pi G\ell}\left(1 - \frac{1}{2\mu\ell}\left(1 + 4\left(1-E\right)^2\right)\right)\int ed^3x\epsilon^{0\mu\nu}S_{\nu\mu0}^{(2)}(\psi) \ , \tag{5.13}$$

where we have introduced the torsion, $S_{\mu\nu}{}^\rho$, given by the antisymmetric part of the torsional Christoffel connection [26] and explicitly by

$$S_{\mu\nu}{}^\rho = \frac{1}{4}\bar{\boldsymbol{\psi}}_\mu\gamma^\rho\boldsymbol{\psi}_\nu \ . \tag{5.14}$$

As usual, the (2) superscript in (5.13) denotes expansion to second order in perturbation theory. Recalling the energy and spin values for the left, right, and massive modes (4.25) we find the total energies are given by

$$\begin{aligned} E_L = &\frac{1}{8\pi G\ell}\left(1 - \frac{1}{\mu\ell}\right)\int ed^3x\epsilon^{0\mu\nu}S_{\nu\mu0}^{(2)}(\psi_L) \\ &+ \frac{1}{32\pi G}\left(-1 + \frac{1}{\mu\ell}\right)\int ed^3x\bar{\nabla}^0 h_L^{\mu\nu}\dot{h}_{L\mu\nu} \ , \end{aligned} \tag{5.15}$$

$$E_R = \frac{1}{32\pi G}\left(-1 - \frac{1}{\mu\ell}\right)\int ed^3x\bar{\nabla}^0 h_R^{\mu\nu}\dot{h}_{R\mu\nu} \ , \tag{5.16}$$

$$\begin{aligned} E_M = &\frac{1}{8\pi G\ell^2}\left(1-\mu\ell\right)\left(2\mu\ell-1\right)\int ed^3x\epsilon^{0\mu\nu}S_{\nu\mu0}^{(2)}(\psi_M) \\ &+ \frac{1}{64\pi G}\left(\mu^2\ell^2 - 1\right)\int ed^3x\epsilon_\beta^{0\mu}h_M^{\beta\nu}\dot{h}_{M\mu\nu} \ . \end{aligned} \tag{5.17}$$

Taking into account that the $\mathcal{N}=(1,0)$ and $\mathcal{N}=(0,1)$ theories are related by parity, it is easy to see that in the $\mathcal{N}=(0,1)$ theory the energies take the form

$$E_L = \frac{1}{32\pi G}\left(-1-\frac{1}{\mu\ell}\right)\int ed^3x\bar{\nabla}^0 h_L^{\mu\nu}\dot{h}_{L\mu\nu}\ , \tag{5.18}$$

$$\begin{aligned} E_R = &\frac{1}{8\pi G\ell}\left(1-\frac{1}{\mu\ell}\right)\int ed^3x\epsilon^{0\mu\nu}S^{(2)}_{\nu\mu 0}\left(\psi_R\right) \\ &+\frac{1}{32\pi G}\left(-1+\frac{1}{\mu\ell}\right)\int ed^3x\bar{\nabla}^0 h_R^{\mu\nu}\dot{h}_{R\mu\nu}\ , \end{aligned} \tag{5.19}$$

$$\begin{aligned} E_M = &\frac{1}{8\pi G\ell^2}\left(1-\mu\ell\right)\left(2\mu\ell-1\right)\int ed^3x\epsilon^{0\mu\nu}S^{(2)}_{\nu\mu 0}\left(\psi_M\right) \\ &+\frac{1}{64\pi G}\left(\mu^2\ell^2-1\right)\int ed^3x\epsilon^{0\mu}_{\beta}h_M^{\beta\nu}\dot{h}_{M\mu\nu}\ . \end{aligned} \tag{5.20}$$

At the linear level the corresponding expressions for the energies of the $\mathcal{N}=(1,1)$ theory are given by

$$E_L = -\left(1-\frac{1}{\mu\ell}\right)E_{B,L}+\left(1-\frac{1}{\mu\ell}\right)E_{F,L}\ , \tag{5.21}$$

$$E_R = -\left(1+\frac{1}{\mu\ell}\right)E_{B,R}-\left(1-\frac{1}{\mu\ell}\right)E_{F,R}\ , \tag{5.22}$$

$$E_M = \left(\mu^2\ell^2-1\right)E_{B,M}+\left(1-\mu\ell\right)\left(2\mu\ell-1\right)E_{F,M}\ , \tag{5.23}$$

where $E_{F,i}$ is as defined in (5.13) with the μ dependence factored out, and $E_{B,i}$ are the bosonic energies as given in [3].

The evaluation of the fermionic energy indicates that fermions do not contribute to quadratic order, though we expect them to contribute to the next order in perturbation theory. By the same reasoning as in the bosonic model [3], we conclude that for generic values of the coupling constants, the energy is not positive semi-definite as one may naively expect for a supersymmetric theory. As elaborated in the introduction, it is known that when there are ghosts in the theory, their contribution to the energy will be negative, thereby spoiling the usual positivity arguments. Indeed, in concurrent work, Andrade and Marolf showed that TMG at general values of the coupling contains ghosts [24]. Similarly, as in the bosonic model, the energies of the $\mathcal{N}=(1,1)$ model become positive semi-definite at the chiral point $\mu\ell=1$ such that

$$E_L = E_M = 0, \quad E_R = 2E_B > 0\ , \tag{5.24}$$

in agreement with the disappearance of ghosts found in [24].

5.2 Fermionic Charges: Supercharge

Using the a perturbative expansion of the gravitino equation of motion, we can calculate the supercharge to order λ^2 so that

$$Q = \frac{1}{8\pi G}\int ed^3xJ_0\ . \tag{5.25}$$

Here J_0 is the temporal component of the supercurrent J_μ, which is the gravitino field equation contracted with the appropriate Killing spinor. We begin by considering the $\mathcal{N}=(1,0)$ theory. As with the energy, we can calculate Q to second order. Schematically, the gravitino field equation takes the form

$$F_\mu = \lambda F_\mu^{(1)}(\psi) + \lambda^2 \left(F_\mu^{(1)}(\psi^{(2)}) + F_\mu^{(2)}(\psi) \right) + \mathcal{O}(\lambda^3) \;, \tag{5.26}$$

with $F^{(1)\mu}$ given in (2.10). $F^{(2)\mu}$ requires expanding the gravitino field equation to order λ^2, and extracting the ψ dependent terms. The result is

$$F^{(2)\mu}(\psi) = \epsilon^{\mu\nu\rho}\frac{1}{4}\omega_\nu^{(1)mn}(h)\bar{\gamma}_{mn}\psi_\rho + \frac{1}{2\ell}h^{\mu\nu}\psi_\nu - \frac{i}{8\mu}\epsilon^{\sigma\nu\rho}\Big(R_{\sigma\nu}^{(1)\;ab}(2\delta_\rho^\mu\gamma_b\psi_a + e_b^\mu\bar{\gamma}_\rho\psi_a) + \frac{1}{\ell^2}(h_\nu^\mu\bar{\gamma}_\rho\psi_\sigma - \delta_\nu^\mu h_{\rho\lambda}\bar{\gamma}^\lambda\psi_\sigma + \delta_\nu^\mu h_{\sigma\lambda}\bar{\gamma}_\rho\psi^\lambda)\Big) - \frac{i}{\mu}\left(\frac{1}{2\ell}R^{(2)\mu} - \frac{1}{4\ell^2}h^{\mu\nu}\psi_\nu\right) \;, \tag{5.27}$$

where

$$R^{(2)\mu}(\psi) = \frac{1}{4}\epsilon^{\mu\nu\rho}\omega_\nu^{(1)nm}(h)\bar{\gamma}_{mn}\psi_\rho \;, \tag{5.28}$$

and

$$R_{\mu\nu ab}^{(1)} = (R_{\mu\nu\alpha\beta}e_a^\alpha e_b^\beta)^{(1)} \;. \tag{5.29}$$

After plugging in the first-order modes, we find the $\mathcal{N}=(1,0)$ supercharges (to order λ^2)

$$Q_L = \left(1 - \frac{1}{\mu\ell}\right) Q_{\omega,L} + \left(1 - \frac{1}{\mu\ell}\right) Q_{h,L} \tag{5.30}$$

and

$$Q_M = \left(1 - \frac{1}{\mu\ell}\right) Q_{\omega,M} + \left(1 - \frac{1}{\mu\ell}\right) Q_{h,M} \;. \tag{5.31}$$

At the chiral point, $\mu\ell = 1$, we have

$$Q_L = 0 \quad \text{and} \quad Q_M = 0 \;. \tag{5.32}$$

In the above formulas, we defined

$$Q_\omega\left(\xi,\psi,h\right) = \frac{1}{32\pi G}\int e d^3x \bar{\xi}\left(\epsilon^{0\nu\rho}\omega_\nu^{(1)mn}(h)\bar{\gamma}_{mn}\psi_\rho\right) \;, \tag{5.33}$$

$$Q_h\left(\xi,\psi,h\right) = \frac{1}{32\pi G\ell}\int e d^3x \bar{\xi}\epsilon^{0\nu\rho}h_{\nu\lambda}\bar{\gamma}^\lambda\psi_\rho \;, \tag{5.34}$$

$$Q_{\omega,i} = Q_\omega\left(\xi_i,\psi_i,h_i\right), \tag{5.35}$$

$$Q_{h,i} = Q_h\left(\xi_i,\psi_i,h_i\right) \;, \tag{5.36}$$

where the subindex labels individual modes.

Applying a parity transformation leads to the supercharges of the $\mathcal{N}=(0,1)$ theory:

$$Q_R = \left(1 - \frac{1}{\mu\ell}\right) Q_{\omega,R} - \left(1 - \frac{1}{\mu\ell}\right) Q_{h,R} \;, \tag{5.37}$$

and

$$Q_M = \left(1 - \frac{1}{\mu\ell}\right) Q_{\omega,M} - \left(1 - \frac{1}{\mu\ell}\right) Q_{h,M} \ . \tag{5.38}$$

At the chiral point, $\mu\ell = 1$, we have

$$Q_M = 0 \quad \text{and} \quad Q_R = 0 \ . \tag{5.39}$$

Similarly, at the linear level $\mathcal{N} = (1,1)$, charges are found to be

$$Q_L = \left(1 - \frac{1}{\mu\ell}\right) Q_{\omega,L} + \left(1 - \frac{1}{\mu\ell}\right) Q_{h,L} \ , \tag{5.40}$$

$$Q_R = \left(1 + \frac{1}{\mu\ell}\right) Q_{\omega,R} - \left(1 + \frac{1}{\mu\ell}\right) Q_{h,R} \ , \tag{5.41}$$

$$Q_M = \left(1 - \frac{1}{\mu\ell}\right) Q_{\omega,M} + \left(1 - \frac{1}{\mu\ell}\right) Q_{h,M} \ . \tag{5.42}$$

At the chiral point only the right moving supercharge is non-vanishing

$$Q_L = 0 \ , \quad Q_R = 2Q_{\omega,R} - 2Q_{h,R} \ , \quad \text{and} \quad Q_M = 0 \ . \tag{5.43}$$

Therefore, the fermionic charges share the bosonic charges' chiral behavior.

6. Conclusion

In this paper we have constructed Chiral Supergravity, the $\mathcal{N} = 1$ supersymmetric extension of Chiral Gravity. The theory has been studied in a perturbative regime around the AdS_3 background. The wave functions have been constructed, and the conserved charges were computed to second order. These charges picked out a distinguished point in parameter space, $\mu\ell = 1$, at which the theory acquires a chiral nature in a fashion similar to its bosonic counterpart [3].

Although the positivity of energy could not be proven, at the chiral point, it was shown that AdS_3 is stable against perturbations. Thus, Chiral Supergravity is a consistent theory at the classical level.

Several important questions remain: Given the existence of a classical supersymmetric extension of Chiral Gravity, it would be interesting to examine if the quantum theory is consistent. Also, given the calculation of the partition function [5], as well as the recent claim about renormalizability of TMG [37], it seems likely that Chiral Supergravity is a consistent theory of gravity even at the quantum level. Some more work on the supersymmetric theory needs to be done to confirm this.

If Chiral Supergravity is consistent at the quantum level, it would be of value to see whether or not it is derivable from String Theory. Some work in this direction was done by Gupta and Sen [38]. They found a consistent truncation of higher dimensional Supergravity with matter fields to pure three-dimensional, cosmological Supergravity with a gravitational Chern-Simons term. Their truncation involves a scale hierachy, and the Chern-Simons coupling is assumed to be smaller than the (anti) de Sitter radius. In Chiral Supergravity, unfortunately, these paramteres are equal. Given that Chiral Supergravity is three-dimensional, an alternative way to find a relation to String Theory might be to map Chiral Supergravity to a string sigma model.

It would also be interesting to compute the partition function for Chiral Supergravity. Due to the recent work of Maloney, Song and Strominger [5], one may anticipate this partition function to correspond to the chiral part of the $\mathcal{N} = 1$ partition function calculated in [39].

In light of the recent work of Gaberdiel et. al. [15], where difficulties in constructing $\mathcal{N} = 2$ extremal conformal field theories with large central charge were reported, it would be interesting to construct an $\mathcal{N} = 2$ version of TMG. Kaura and Sahoo had one attempt [40], but clearly more work needs to be done.

Though we focused on the supersymmetric version of Chiral Gravity, there are indications for the existence of a supersymmetric version of Log Gravity. As in the bosonic case [4, 41], there is an additional solution to the gravitino equation of motion at the chiral point

$$\psi_\mu = y(\tau, \rho)\psi_\mu,$$

with $y = -i\tau - \ln\cosh\rho$. The fermionic boundary terms of the action are needed to verify that this mode obeys the variational principle and better specify the boundary conditions. This would allow for a detailed study of Log Supergravity and a possible dual logarithmic superconformal field theory. Fermionic boundary conditions have been discussed in [42] and references therein. We hope to address some of these questions in the future.

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A. Appendix: Notation and Conventions

Throughout the course of this paper we have worked with an index structure such that flat coordinates are labeled by Latin indices, $a,\ b, \ldots = 0,\ 1,\ 2$ and curved coordinates are labeled by Greek indices, $\mu,\ \nu, \ldots = 0,\ 1,\ 2$. Moreover, the manifold parameterized by the flat coordinates is endowed with a Minkowski metric η of signature $(-,+,+)$. The background curved space is taken to be AdS_3 in global coordinates:

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = \ell^2 \left(-\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2\right) ,$$

and the Lorentzian theory is considered so that only ϕ is cyclic. In such a geometry, one can speak of a conformal boundary understood to lie at $\rho = \infty$. Moreover, we chose a space-time orientation such that

$$\epsilon_{012} = -\epsilon^{012} = +1 , \quad \epsilon^{\mu\alpha\beta}\epsilon_{\mu\lambda\sigma} = -\delta^{\alpha\beta}_{\lambda\sigma} , \quad \text{and} \quad \epsilon^{\mu\alpha\beta}\epsilon_{\mu\alpha\sigma} = -2\delta^{\beta}_{\sigma} .$$

The theories under consideration all involve spinors which are taken to lie in the Majorana representation. Moreover all fermions are implicitly Grassmann-valued spinors. Taking χ and ϕ to be Grassmann-valued Majorana spinors, we have the following useful identities:

$$\bar{\chi}\phi = \bar{\phi}\chi , \quad \bar{\chi}\gamma_\mu\phi = -\bar{\phi}\gamma_\mu\chi , \quad \bar{\chi}\gamma_\mu\gamma_\nu\phi = \bar{\phi}\gamma_\nu\gamma_\mu\chi ,$$

where $\gamma_\mu = e^a_\mu \gamma_a$ are three-dimensional gamma matrices and bar denotes the Dirac adjoint $\bar{\chi} = \chi^\dagger\gamma_0$, with $\chi^\dagger = (\chi^*)^T$.

The curvatures and connections are given by

$$\omega^{ab}_\mu = e^a_\nu \partial_\mu e^{\nu b} + e^a_\nu e^{\sigma b}\Gamma^\nu_{\mu\sigma} , \quad \Gamma^\lambda_{(\mu\nu)} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - g_\rho g_{\mu\nu}\right) ,$$
$$R = e^\mu_a e^\nu_b R^{ab}_{\mu\nu}(\omega) , \quad \text{and} \quad R_{\mu\nu}{}^{ab} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{ac}_\mu \omega^b_{\nu c} - \omega^{ac}_\nu \omega^b_{\mu c} .$$

Covariant derivatives acting on spinors ϵ, vector-spinors ψ_μ, and tensors are

$$D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\omega^{ab}_\mu \gamma_{ab}\epsilon , \quad D_\alpha \psi_\beta = \partial_\alpha \psi_\beta + \frac{1}{4}\omega^{ab}_\alpha \gamma_{ab}\psi_\beta - \Gamma^\lambda_{\alpha\beta}\psi_\lambda , \quad \text{and}$$
$$\nabla_\mu X_{\nu\rho} = \partial_\mu X_{\nu\rho} - \Gamma^\sigma_{\mu\nu} X_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} X_{\nu\sigma} .$$

Upon a perturbative expansion of the metric (4.1) and the Rarita-Schwinger field (4.2), we find the tensors of the $\mathcal{N}=(1,0)$ theory:

$$
\begin{aligned}
R^{(1)}_{\mu\nu\alpha\beta} &= \frac{1}{2\ell^2}(h_{\mu\alpha}\bar{g}_{\nu\beta}) - h_{\nu\beta}\bar{g}_{\mu\alpha}) \ , \\
\bar{R}_{\mu\nu}{}^{ab} &= \Lambda\left(e^a_\mu e^b_\nu - e^b_\mu e^a_\nu\right) \ , \\
\bar{R}_{\mu\nu} &= 2\Lambda\bar{g}_{\mu\nu} \ , \\
\bar{R} &= 6\Lambda, \\
R^{(1)} &= -\bar{\nabla}^2 h + \bar{\nabla}_\mu\bar{\nabla}_\nu h^{\mu\nu} - 2\Lambda h \ , \\
\mathcal{G}^{(1)}_{\mu\nu}(h) &= R^{(1)}_{\mu\nu}(h) - \frac{1}{2}\bar{g}_{\mu\nu}R^{(1)}(h) - \Lambda h_{\mu\nu} \ , \\
\mathcal{G}^{(2)}_{\mu\nu}(h,\psi) &= \mathcal{G}^{(2)}_{\mu\nu}(h) + \bar{\nabla}_\rho\kappa^{(2)\rho}_{\mu\nu}(\psi) \ , \\
\mathcal{G}^{(2)}_{\mu\nu}(h) &= R^{(2)}_{\mu\nu}(h) - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}(h) \ , \\
R^{(2)}_{\mu\nu}(h) &= \frac{1}{4}\bar{\nabla}_\mu h_{\alpha\beta}\bar{\nabla}_\nu h^{\alpha\beta} + \bar{\nabla}_{[\beta}h_{\alpha]\mu}\bar{\nabla}^\beta h^\alpha{}_\nu + h^{\alpha\beta}\left(\bar{\nabla}_\alpha\left(\bar{\nabla}_{[\beta}h_{\mu]\nu}\right) + \bar{\nabla}_\nu\left(\bar{\nabla}_{[\mu}h_{\alpha]\beta}\right)\right) \ , \\
R^{(2)}(h) &= \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}(h) - \frac{3}{\ell^2}h^{\mu\nu}h_{\mu\nu} \ , \\
C^{(2)}_{\mu\nu}(h) &= \frac{1}{2}\left[\epsilon_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha\mathcal{G}^{(2)}_{\beta\nu}(h) + h_{\mu\lambda}\epsilon^{\lambda\alpha\beta}\mathcal{G}^{(2)}_{\beta\nu} - \frac{h}{2}\epsilon_\mu{}^{\alpha\beta}\bar{\nabla}\mathcal{G}^{(1)}_{\beta\nu}(h) - \epsilon_\mu{}^{\alpha\beta}\Gamma^{(1)\lambda}_{\nu\alpha}\mathcal{G}^{(1)}_{\beta\lambda}(h) + (\mu\leftrightarrow\nu)\right] \ , \\
\Gamma^{(1)\lambda}_{\mu\nu}(h) &= \frac{1}{2}\left(\bar{\nabla}_\mu h^\lambda_\nu + \bar{\nabla}_\nu h^\lambda_\mu - \bar{\nabla}^\lambda h_{\mu\nu}\right) \ , \\
\Gamma^{(2)\lambda}_{\mu\nu}(h) &= -\frac{1}{2}h^{\lambda\rho}\left(\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}\right) \ , \\
C^{(2)}_{\mu\nu}(h,\psi) &= C^{(2)}_{\mu\nu}(h) + \epsilon_\mu{}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\rho\kappa^{(2)\rho}_{\beta\nu}(\psi) - \frac{i}{8}\Lambda\epsilon_\mu{}^{\alpha\beta}\bar{\psi}_\alpha\bar{\gamma}_\nu\psi_\beta \ , \\
F^{(2)}_{\mu\nu}(\psi) &= \frac{i}{4\ell\bar{e}}\epsilon^{\sigma\lambda\rho}\bar{g}_{\lambda\mu}\bar{\psi}_\sigma\bar{\gamma}_\nu\psi_\rho - \frac{i}{8\mu\ell^2\bar{e}}\epsilon^{\rho\sigma\tau}\bar{g}_{\mu\tau}\bar{\psi}_\rho\bar{\gamma}_\nu\psi_\sigma \ , \\
&\quad - \frac{i}{2\mu\ell^2\bar{e}}(1-E)^2\left(\bar{g}_{\mu\nu}\epsilon^{\rho\sigma\tau}\bar{\psi}_\sigma\bar{\gamma}_\tau\psi_\rho - 2\bar{g}_{\mu\sigma}\epsilon^{\sigma\rho\tau}\bar{\psi}_\rho\bar{\gamma}_\tau\psi_\nu\right) \ , \\
\kappa^{(2)\rho}_{\mu\nu}(\psi) &= \frac{i}{4}\left(\bar{\psi}_\mu\bar{\gamma}_\nu\psi^\rho - \bar{\psi}_\mu\bar{\gamma}^\rho\psi_\nu + \bar{\psi}_\nu\bar{\gamma}_\mu\psi^\rho\right) \ .
\end{aligned}
$$

The vielbein decomposes as

$$
e^a_\mu = \bar{e}^a_\mu + \lambda e^{(1)a}_\mu + \mathcal{O}\left(\lambda^2\right) \ .
$$

Employing the relationship between the metric and the vielbein along with our linearized metric solution we find

$$
\begin{aligned}
e^{(1)0}_\tau &= -\frac{h_{00}}{2\cosh\rho} \ , & e^{(1)0}_\phi &= -\frac{h_{01}}{2\cosh\rho} \ , & e^{(1)0}_\rho &= -\frac{h_{02}}{2\cosh\rho} \ , \\
e^{(1)1}_\tau &= \frac{h_{01}}{2\sinh\rho} \ , & e^{(1)1}_\phi &= \frac{h_{11}}{2\sinh\rho} \ , & e^{(1)1}_\rho &= \frac{h_{12}}{2\sinh\rho} \ , \\
e^{(1)2}_\tau &= \frac{1}{2}h_{02} \ , & e^{(1)2}_\phi &= \frac{1}{2}h_{12} \ , \quad \text{and} & e^{(1)2}_\rho &= \frac{1}{2}h_{22} \ .
\end{aligned}
$$

The linearized spin connection in terms of the vielbeins is

$$
\begin{aligned}
\omega^{(1)}_{\mu ab} = &-\frac{1}{2}e^{(1)\nu}_a(\partial_\mu \bar{e}_{b\nu} - \partial_\nu \bar{e}_{\mu b}) + \frac{1}{2}\bar{e}^\nu_a(\partial_\mu e^{(1)}_{b\nu} - \partial_\nu e^{(1)}_{\mu b}) - \frac{1}{2}\bar{e}^\alpha_a \bar{e}^\beta_b \bar{e}^c_\mu(\partial_\alpha e^{(1)}_{\beta c}) \\
&- \frac{1}{2}\bar{e}^\alpha_a \bar{e}^\beta_b e^{(1)c}_\mu(\partial_\alpha \bar{e}_{\beta c}) + \frac{1}{2}e^{(1)\alpha}_a \bar{e}^\beta_b \bar{e}^c_\mu(\partial_\alpha \bar{e}_{\beta c}) + \frac{1}{2}\bar{e}^\alpha_a e^{(1)\beta}_b \bar{e}^c_\mu(\partial_\alpha \bar{e}_{\beta c}) - (a \leftrightarrow b) \, .
\end{aligned}
$$

It is useful to note

$$
\left[\bar{\nabla}_\sigma, \bar{\nabla}_\mu\right] h^\sigma_\mu = \bar{R}^\sigma{}_{\lambda\sigma\mu} h^\lambda_\nu - \bar{R}^\lambda{}_{\nu\sigma\mu} h^\sigma_\lambda = 3\Lambda h_{\mu\nu} - \Lambda h \bar{g}_{\mu\nu}
$$

and

$$
\left[D_\mu, D_\nu\right] \psi_a = R_{\mu\nu ab}\psi^b + \frac{1}{4}R_{\mu\nu bc}\gamma^{bc}\psi_a \, .
$$

At the linear level, the fermionic field strength of the $\mathcal{N} = (1, 0)$ theory satisfies

$$
f^{(1)}_\mu = \frac{1}{\ell}\left(E - 1\right)\psi_\mu \, .
$$

B. Clifford Algebras, Spinor Representations, and Discrete Symmetries

In TMG, a parity transformation effectively takes $\mu \to -\mu$. When dealing with fermions in TMSG, we must also consider the operator $\mathcal{P}$ acting on fermions.

B.1 Clifford Algebra and Spinor Representation

The structure of the gravitino fields descends from defining spinors on the global group of isometries, $SO(2, 2)$. This is the isometry group of $\mathbb{R}^{2+2}$ of which AdS_3 is a hypersurface. The Clifford algebra of gamma matrices is

$$
\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \, . \tag{B.1}
$$

Here the metric is $\eta = \mathrm{diag}(-1, -1, 1, 1)$ and $A = -, 0, 1, 2$. The "$-$" index represents the additional direction in $\mathbb{R}^{2+2}$, so that the gamma matrices explicitly read:

$$
\Gamma_A = \begin{pmatrix} 0 & \gamma_A \\ \hat{\gamma}_A & 0 \end{pmatrix} \, ,
$$

with

$$
\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \, , \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, , \quad \text{and} \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, ,
$$

$\gamma_- = \hat{\gamma}_- = \mathbb{I}$, and $\hat{\gamma}_m = -\gamma_m$, for $m = 0, 1, 2$. The hypersurface in $\mathbb{R}^{2+2}$ is a curved space. We employ the dreibein e^m_μ to define γ_μ on this curved space:

$$
\gamma_\mu = e^m_\mu \gamma_m \, .
$$

Three-dimensional gamma matrices satisfy

$$
\{\gamma_\mu \gamma_\nu\} = 2g_{\mu\nu} \, ,
$$

and in addition

$$[\gamma_\mu, \gamma_\nu] = \epsilon_{\mu\nu\rho}\gamma^\rho \ , \quad [\hat{\gamma}_\mu, \hat{\gamma}_\nu] = -\epsilon_{\mu\nu\rho}\hat{\gamma}^\rho \ ,$$

where again, with curved indices $\epsilon_{012} = \ell \sinh\rho \cosh\rho$. We have used the notation

$$\gamma^{\mu\nu} = \frac{1}{2!}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \ , \quad \text{and} \quad \gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = \epsilon^{\mu\nu\rho}.$$

Four component "Dirac" spinors are decomposed into two component Weyl spinors. These Weyl spinors have the correct dimensionality for spinor fields in three dimensions. Fermions are taken to be Majorana spinors and have anticommuting components.

B.2 Discrete Symmetries

The Dirac notation is convenient for discussing discrete symmetries. We use the block diagonal form from equation (B.1). If we define a Dirac spinor on AdS_3 by

$$\Psi = \begin{pmatrix} \psi^R \\ \psi^L \end{pmatrix} \ ,$$

then the parity operator acts on this spinor as

$$\mathcal{P}\Psi = i\Gamma_1\Psi \ .$$

In Weyl language, we have

$$\mathcal{P}\psi^R_\mu = -i\gamma_1\psi^R_\mu = \psi^L_\mu$$

and

$$\mathcal{P}\psi^L_\mu = i\gamma_1\psi^L_\mu = \psi^R_\mu \ .$$

It is also useful to define the charge conjugation operator $\mathcal{C}$ using Dirac notation. In this case:

$$\mathcal{C} = \begin{pmatrix} \hat{\gamma}_0 & 0 \\ 0 & \gamma_0 \end{pmatrix} \ .$$

In Weyl language, we have

$$\begin{aligned} \mathcal{C}\psi^R_\mu &= -\gamma_0\psi^R_\mu \ , \\ \mathcal{C}\psi^L_\mu &= \gamma_0\psi^L_\mu \ . \end{aligned}$$

The Majorana condition is then

$$\bar{\Psi} = \Psi^T\mathcal{C} \ .$$

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