DISCRIMINANT COAMOEBAS THROUGH HOMOLOGY

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ABSTRACT. Understanding the complement of the coamoeba of a (reduced) A-discriminant is one approach to studying the monodromy of solutions to the corresponding system of A-hypergeometric differential equations. Nilsson and Passare described the structure of the coamoeba and its complement (a zonotope) when the reduced A-discriminant is a function of two variables. Their main result was that the coamoeba and zonotope form a cycle which is equal to the fundamental cycle of the torus, multiplied by the normalized volume of the set A of integer vectors. That proof only worked in dimension two. Here, we use simple ideas from topology to give a new proof of this result in dimension two, one which can be generalized to all dimensions.

INTRODUCTION

A-hypergeometric functions, which are solutions to A-hypergeometric systems of differential equations [4, 5, 12], enjoy two complimentary analytical formulae which together give an approach to studying the monodromy of the solutions [2] at non-resonant parameters. One formula is as explicit power series whose convergence domains in $\mathbf{C}^{N+1}$ have an action of the group $\mathbf{T}^{N+1}$ of phases. These power series form a basis of solutions, with known local monodromy around loops from $\mathbf{T}^{N+1}$. Another formula is as A-hypergeometric Mellin-Barnes integrals [9] evaluated at phases $\theta \in \mathbf{T}^{N+1}$. When the Mellin-Barnes integrals give a basis of solutions, they may be used to glue together the local monodromy groups and determine a subgroup of the monodromy group, which may sometimes be the full monodromy group.

Here, $A \subset \mathbf{Z}^n$ consists of $N+1$ integer vectors that generate $\mathbf{Z}^n$. Considering $\mathbf{Z}^n \subset \mathbf{Z}^{n+1}$ as the vectors with first coordinate 1, we regard A as a collection of $N+1$ vectors in $\mathbf{Z}^{n+1}$. The A-discriminant is a multihomogeneous polynomial in $N+1$ variables with $n+1$ homogeneities corresponding to A. Removing these homogeneities gives the reduced A-discriminant, D_B, which is a hypersurface in $\mathbf{C}^d$ ($d := N-n$) that depends upon a vector configuration $B \subset \mathbf{Z}^d$ Gale dual to A. This reduction corresponds to a homomorphism $\beta : (\mathbf{C}^*)^{N+1} \to (\mathbf{C}^*)^d$ and induces a corresponding map $\mathrm{Arg}(\beta)$ on phases.

The Mellin-Barnes integrals at $\theta \in \mathbf{T}^{N+1}$ give a basis of solutions when $\mathrm{Arg}(\beta)(\theta)$ has a neighborhood in $\mathbf{T}^d$ with the property that no point of D_B has a phase lying in that neighborhood [9]. By results in [6, 11], this means that $\mathrm{Arg}(\beta)(\theta)$ lies in the complement of the closure of the coamoeba $\mathcal{A}_B$ of D_B.

When $d = 2$, the closure of $\mathcal{A}_B$ and its complement were described in [10] as topological chains in $\mathbf{T}^2$ (induced from natural chains in its universal cover $\mathbf{R}^2$, where $\mathbf{T}^2 = (\mathbf{R}/2\pi\mathbf{Z})^2$). The closure of the coamoeba is an explicit chain depending on B. Its edges coincide with the edges of the zonotope Z_B generated by B. The main result of [10] is the following theorem.

Theorem 1. *The sum of the coamoeba chain $\overline{\mathcal{A}_B}$ and the zonotope Z_B forms a two-dimensional cycle in $\mathbf{T}^2$ that is equal to $n!\,\mathrm{vol}(A)$ times the fundamental cycle.*

Here, $n!\,\mathrm{vol}(A)$ is the normalized volume of the convex hull of A, which is the dimension of the space of solutions to the (non-resonant) A-hypergeometric system. The zonotope Z_B gives points in the complement of $\mathcal{A}_B$, by Theorem 1. Its proof in [10] only works when $d = 2$ and it is not clear how to generalize it to $d > 2$. However, any such generalization would be important, for Mellin-Barnes integrals at a set of phases θ where $\mathrm{Arg}(\beta)(\theta)$ are distinct points of Z_B with the same image in $\mathbf{T}^d$ are linearly independent.

We give a proof of Theorem 1 which explains the occurrence of the zonotope and can be generalized to higher dimensions. This proof uses the Horn-Kapranov parametrization of the A-discriminant [7], which implies that the discriminant coamoeba is the image of the coamoeba of a line ℓ_B in $\mathbf{P}^N$ under the map $\mathrm{Arg}(\beta)$. We construct a piecewise linear *zonotope chain* in $\mathbf{T}^N$ (the quotient of $\mathbf{T}^{N+1}$ by the diagonal torus) which is a cone over the boundary of the coamoeba of ℓ_B, and compute the homology class of the sum of the coamoeba and this zonotope chain. This gives a formula for the image of this cycle under $\mathrm{Arg}(\beta)$, which we show is $n!\,\mathrm{vol}(A)$ times the fundamental cycle of $\mathbf{T}^2$. Theorem 1 follows as the map $\mathrm{Arg}(\beta)$ sends the coamoeba of ℓ to the coamoeba $\mathcal{A}_B$ of D_B and sends the zonotope chain to Z_B.

While for A-discriminants, the set A consists of distinct integer vectors and consequently its Gale dual B generates $\mathbf{Z}^2$ and has no two vectors parallel, we establish Theorem 1 in the greater generality of any finite multiset B of integer vectors in $\mathbf{Z}^2$ with sum $\mathbf{0}$ that spans $\mathbf{R}^2$. This generality is useful in our primary application to hypergeometric systems, for example the classical systems of Appell [1] and Lauricella [8] may be expressed as A-hypergeometric systems with repeated vectors in the Gale dual B. In this setting, we replace the reduced A-discriminant by the Horn-Kapranov parametrization given by the vectors B, and study the coamoeba $\mathcal{A}_B$ of the image, which is also written D_B. The normalized volume $n!\,\mathrm{vol}(A)$ of the configuration A is replaced by a quantity d_B that depends upon the vectors in B.

We collect some preliminaries in Section 1. In Section 2 we study the coamoeba of a line in $\mathbf{P}^N$ defined over the real numbers and define its associated zonotope chain. Our main result is a computation of the homology class of the cycle formed by these two chains. In Section 3 we show that under the map $\mathrm{Arg}(\beta)$ the coamoeba and zonotope chains map to the coamoeba $\mathcal{A}_B$ and the zonotope Z_B, and a simple application of the result in Section 2 shows that the homology class of $\overline{\mathcal{A}_B} + Z_B$ is d_B times the fundamental cycle of $\mathbf{T}^2$.

Remark. This approach to reduced A-discriminant coamoebas and their complements was developed during the Winter 2011 semester at the Institut Mittag-Leffler, with the main result obtained in August 2011, along with a sketch of a program to extend it to $d \geq 2$. With the tragic death of Mikael Passare on 15 September 2011, the task of completing this paper

fell to the second author, and the program extending these results is being carried out in collaboration with Mounir Nisse.

1. Coamoebas and cohomology of tori

Throughout N will be an integer strictly greater than 1. Let $\mathbf{P}^N$ be N-dimensional complex projective space, which will always have a preferred set of coordinates $[x_1 : \cdots : x_N : x_{N+1}]$ (up to reordering). Similarly, $\mathbf{C}^N$, $(\mathbf{C}^*)^N$, $\mathbf{R}^N$, and $\mathbf{Z}^N$ are N-tuples of complex numbers, non-zero complex numbers, real numbers, and integers, all with corresponding preferred coordinates. We will write $\mathbf{e}_i$ for the ith basis vector in a corresponding ordered basis.

The argument map $\mathbf{C}^* \ni z = re^{\sqrt{-1}\theta} \mapsto \theta \in \mathbf{T} := \mathbf{R}/2\pi\mathbf{Z}$ induces an argument map $\mathrm{Arg}\colon (\mathbf{C}^*)^N \to \mathbf{T}^N$. To a subvariety $X \subset \mathbf{P}^N$ (or $\mathbf{C}^N$ or $(\mathbf{C}^*)^N$) we associate its *coamoeba* $\mathcal{A}(X) \subset \mathbf{T}^N$ which is the image of $X \cap (\mathbf{C}^*)^N$ under Arg. The closure of the coamoeba $\mathcal{A}(X)$ was studied in [6, 11]. This closure contains $\mathcal{A}(X)$, together with all limits of arguments of unbounded sequences in $X \cap (\mathbf{C}^*)^N$, which constitute the *phase limit set of X, $\mathcal{P}^\infty(X)$*. The main result of [11] (proven when X is a complete intersection in [6]) is that $\mathcal{P}^\infty(X)$ is the union of the coamoebas of all initial degenerations of $X \cap (\mathbf{C}^*)^N$.

Lines in $\mathbf{C}^3$ were studied in [11], and the arguments there imply some basic facts about coamoebas of lines. When $X = \ell \subset \mathbf{C}^N$ is a line which is not parallel to a sum of coordinate directions ($\mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_s}$ for some subset $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, N\}$), its coamoeba is two-dimensional and its phase limit set is a union of at most $N+1$ one-dimensional subtori of $\mathbf{T}^N$, one for each point of ℓ at infinity, whose directions are parallel to sums of coordinate directions. If $\ell' \subset \mathbf{C}^M$ ($M < N$) is the image of ℓ under a coordinate projection, then the coamoeba $\mathcal{A}(\ell')$ is the image of $\mathcal{A}(\ell)$ under the induced projection. If ℓ' is not parallel to a sum of coordinate directions, then the map $\overline{\mathcal{A}(\ell)} \to \overline{\mathcal{A}(\ell')}$ is an injection except for those components of the phase limit set which are collapsed to points.

The integral cohomology of the compact torus $\mathbf{T}^N$ is the exterior algebra $\wedge^* \mathbf{Z}^N$. Under the natural identification of homology with the linear dual of cohomology (which is again $\wedge^* \mathbf{Z}^N$), we will write $\mathbf{e}_i$ for the fundamental 1-cycle $[\mathbf{T}_i]$ of the coordinate circle $\mathbf{T}_i := 0^{i-1} \times \mathbf{T} \times 0^{N-i}$ and $\mathbf{e}_i \wedge \mathbf{e}_j$ is the fundamental cycle $[\mathbf{T}_{i,j}]$ of the coordinate 2-torus $\mathbf{T}_{i,j} \simeq \mathbf{T}^2$ in the directions i and j with the implied orientation. Given a continuous map $\rho\colon \mathbf{T}^N \to \mathbf{T}^2$, the induced map in homology is $\rho_*\colon H_*(\mathbf{T}^N, \mathbf{Z}) \to H_*(\mathbf{T}^2, \mathbf{Z})$ where $\rho_*(\mathbf{e}_i) = [\rho(\mathbf{T}_i)]$, where we interpret $[\rho(\mathbf{T}_i)]$ as a cycle—the set of points in $\rho(\mathbf{T}_i)$ over which ρ has degree n will appear in $[\rho(\mathbf{T}_i)]$ with coefficient n. By the identification of $H_*(\mathbf{T}^N, \mathbf{Z})$ with $\wedge^* \mathbf{Z}^N$, such a map is determined by its action on $H_1(\mathbf{T}^N, \mathbf{Z})$, where it is an integer linear map $\mathbf{Z}^N \to \mathbf{Z}^2$.

2. The coamoeba and zonotope chains of a real line

We study the coamoeba $\mathcal{A}(\ell)$ of a line ℓ in $\mathbf{P}^N$ defined by real equations. Its closure $\overline{\mathcal{A}(\ell)}$ is a two-dimensional chain in $\mathbf{T}^N$ whose boundary consists of at most $N+1$ one-dimensional subtori parallel to sums of coordinate directions. We describe a piecewise linear two-dimensional chain—the *zonotope chain* of ℓ—which has the same boundary as the

coamoeba, but with opposite orientation. The union of the coamoeba and the zonotope chain forms a cycle whose homology class we compute.

The line ℓ has a parametrization

$$\Phi \,:\, \mathbf{P}^1 \ni z \,\longmapsto\, [b_1(z) \,:\, b_2(z) \,:\, \cdots \,:\, b_{N+1}(z)] \,\in\, \mathbf{P}^N \,,$$

where $b_1, \ldots, b_{N+1}$ are real linear forms with zeroes $\xi_1, \ldots, \xi_{N+1} \in \mathbf{RP}^1$. The formulation and statement of our results about the coamoeba of ℓ will be with respect to particular orderings of the forms b_i, which we now describe.

Definition 2.1. Suppose that these zeroes are in a weakly increasing cyclic order on $\mathbf{RP}^1$,

$$(2.1) \hspace{4cm} \xi_1 \,\leq\, \xi_2 \,\leq\, \cdots \,\leq\, \xi_{N+1} \,.$$

Next, identify $\mathbf{P}^1 \smallsetminus \{\xi_{N+1}\}$ with $\mathbf{C}$, so that ξ_{N+1} is the point ∞ at infinity, and suppose that the distinct zeroes are

$$(2.2) \hspace{4cm} \zeta_1 \,<\, \zeta_2 \,<\, \cdots \,<\, \zeta_M \,<\, \zeta_{M+1} \,=\, \infty \,.$$

(Note that $M \leq N$.) Let $\mathbf{R} = \mathbf{RP}^1 \smallsetminus \{\infty\}$ and consider the forms b_i as affine functions on $\mathbf{R}$. Fix a scaling of these functions so that $b_{N+1} = 1$. On the interval $(-\infty, \zeta_1)$ the sign of each function b_i is constant. Define $\mathrm{sgn}_i \in \{\pm 1\}$ to be this sign.

By (2.1) and (2.2), there exist numbers $1 = m_1 < \cdots < m_{M+1} < m_{M+2} = N{+}2$ such that $b_i(\zeta_j) = 0$ if and only if $i \in [m_j, m_{j+1})$. We further suppose that on each of these intervals $[m_j, m_{j+1})$ the signs sgn_i are weakly ordered. Specifically, there are integers $n_1, \ldots, n_{M+1}$ with $m_j < n_j \leq m_{j+1}$ such that one of the following holds

$$(2.3) \hspace{1cm} \mathrm{sgn}_{m_j} = \mathrm{sgn}_{m_j+1} = \cdots = \mathrm{sgn}_{n_j-1} = -1 \;\; < \;\; 1 = \mathrm{sgn}_{n_j} = \cdots = \mathrm{sgn}_{m_{j+1}-1} \,, \hspace{1cm} \text{or}$$

$$(2.4) \hspace{1cm} \mathrm{sgn}_{m_j} = \mathrm{sgn}_{m_j+1} = \cdots = \mathrm{sgn}_{n_j-1} = 1 \;\; > \;\; -1 = \mathrm{sgn}_{n_j} = \cdots = \mathrm{sgn}_{m_{j+1}-1} \,,$$

for $j = 1, \ldots, M{+}1$. If $n_j = m_{j+1}$, then all the signs are the same; otherwise both signs occur. Since $b_{N+1} = 1$, either (2.3) occurs with $n_{M+1} \leq N{+}1$ or (2.4) occurs with $n_{M+1} = N{+}1$. ◰

The point $\mathrm{Arg}(b_1(z), \ldots, b_N(z)) \in \mathbf{T}^N$ is constant for z in each interval of $\mathbf{R}^1 \smallsetminus \{\zeta_1, \ldots, \zeta_M\}$. Let $p_1 := (\arg(\mathrm{sgn}_i) \mid i = 1, \ldots, N)$ be the point coming from the interval $(-\infty, \zeta_1)$, and for each $j = 1, \ldots, M$, let p_{j+1} be the point coming from the interval (ζ_j, ζ_{j+1}). These $M{+}1$ points $p_1, \ldots, p_{M+1}$ of $\mathbf{T}^N$ are the vertices of the coamoeba $\mathcal{A}(\ell)$ of ℓ.

To understand the rest of the coamoeba, note that when $M \geq 2$ the map $\mathrm{Arg} \circ \Phi$ is injective on $\mathbf{P}^1 \smallsetminus \mathbf{RP}^1$ (see [11, § 2]). (When $M = 1$, ℓ is parallel to a sum of coordinate directions and $\mathcal{A}(\ell)$ is a translate of the corresponding one-dimensional subtorus of $\mathbf{T}^N$.) It suffices to consider the image of the upper half plane, as the image of the lower half plane is obtained by multiplying by -1 (induced by complex conjugation). For the upper half plane, consider $\mathrm{Arg} \circ \Phi(z)$ for z lying on a contour C as shown in Figure 1 that contains semicircles of radius ϵ centered at each root ζ_j and a semicircle of radius $1/\epsilon$ centered at 0, but otherwise lies along the real axis, for ϵ a sufficiently small positive number.

As z moves along C, $\mathrm{Arg} \circ \Phi(z)$ takes on values $p_1, \ldots, p_{M+1}$, for $z \in C \cap \mathbf{R}$. On the semicircular arc around ζ_j, it traces a curve from p_j to p_{j+1} in which nearly every component
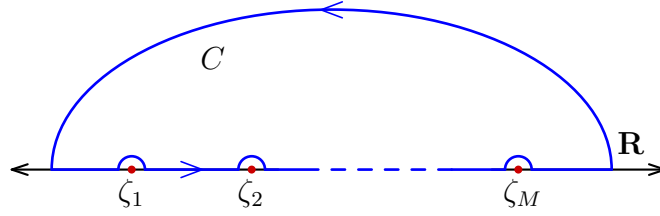
FIGURE 1. Contour in upper half plane

is constant, except for those i where $b_i(\zeta_j) = 0$, each of which decreases by π. In the limit as $\epsilon \to 0$, this becomes the line segment between p_j and p_{j+1} with direction $-\mathbf{f}_j$, where

$$\mathbf{f}_j := \sum_{i:\, b_i(\xi_j)=0} \mathbf{e}_i = \sum_{i=m_j}^{m_{j+1}-1} \mathbf{e}_i \, ,$$

and where we set $\mathbf{e}_{N+1} := -(\mathbf{e}_1 + \cdots + \mathbf{e}_N)$. This is because we are really working in the torus for $\mathbf{P}^N$, which is the quotient $\mathbf{T}^{N+1}/\Delta(\mathbf{T})$ of $\mathbf{T}^{N+1}$ modulo the diagonal torus, and $\mathbf{e}_i \in \mathbf{T}^{N+1}/\Delta(\mathbf{T})$ is the image of the standard basis element in $\mathbf{T}^{N+1}$. Thus $\mathbf{e}_1 + \cdots + \mathbf{e}_{N+1} = 0$.

Along the arc near infinity, $\mathrm{Arg} \circ \Phi(z)$ approaches the line segment between p_{M+1} and p_1 which has direction $-\mathbf{f}_{M+1}$, where

$$(2.5) \qquad \mathbf{f}_{M+1} = -\sum_{i:\, b_i(\infty)\neq 0} \mathbf{e}_j = -(\mathbf{f}_1 + \cdots + \mathbf{f}_M) \, .$$

This polygonal path connecting $p_1, \ldots, p_{M+1}$ in cyclic order forms the boundary of the image of the upper half plane under $\mathrm{Arg} \circ \Phi$, which is a two-dimensional membrane in $\mathbf{T}^N$.

The boundary of the image of the lower half plane is also a piecewise linear path connecting $p_1, \ldots, p_{M+1}$ in cyclic order, but the edge directions are $\mathbf{f}_1, \ldots, \mathbf{f}_{M+1}$.

Example 2.2. Let $N = 3$ and suppose that the affine functions b_i are z, $1-2z$, $z-2$, and 1. Then $M = N$, $\xi_i = \zeta_i$, $\zeta_1 = 0$, $\zeta_1 = 1/2$, $\zeta_2 = 2$, and $\mathbf{f}_i = \mathbf{e}_i$. The vertices of $\mathcal{A}(\ell)$ are

$$p_1 = (\pi, 0, \pi), \quad p_2 = (0, 0, \pi), \quad p_3 = (0, -\pi, \pi), \quad \text{and} \quad p_4 = (0, -\pi, 0) \, .$$

Figure 2 shows two views of $\mathcal{A}(\ell)$ in the fundamental domain $[-\pi, \pi]^3 \subset \mathbf{R}^3$ of $\mathbf{T}^3$, where the opposite faces of the cube are identified to form $\mathbf{T}^3$.

Example 2.3. We consider three examples when $N = 3$ in which the affine functions have repeated zeroes. For the first, suppose that the affine functions b_i are $-1-z, -1-z, 2z$, and 2. These have zeroes $-1 \leq -1 < 0 < \infty$ and the vertices of the coamoeba $\mathcal{A}(\ell)$ are

$$(0, 0, \pi), \quad (-\pi, -\pi, \pi), \quad \text{and} \quad (-\pi, -\pi, 0) \, .$$

So $\mathcal{A}(\ell)$ consists of two triangles with edges parallel to $\mathbf{e}_1+\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$. It lies in the plane $\theta_1 = \theta_2$.
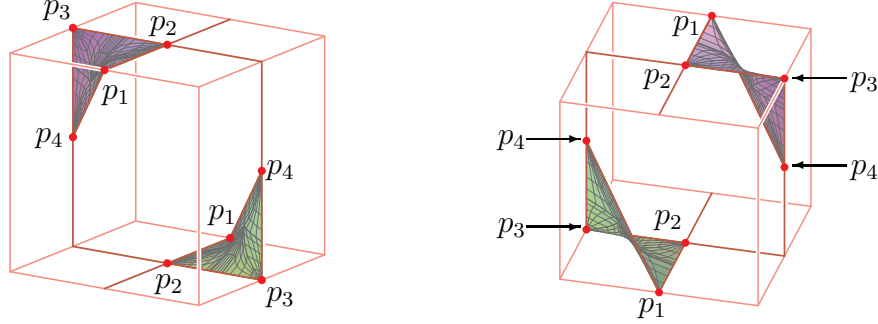
FIGURE 2. Two views of $\mathcal{A}(\ell)$

For a second example, suppose that the affine functions b_i are $\frac{1}{2}+z$, $\frac{1}{2}-z$, -2, and 1. These have zeroes $-1, 1, \infty$, and ∞. The vertices of the coamoeba $\mathcal{A}(\ell)$ are

$$(\pi, 0, \pi), \quad (0, 0, \pi), \quad \text{and} \quad (0, -\pi, \pi).$$

So $\mathcal{A}(\ell)$ consists of two triangles with edges parallel to $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_1+\mathbf{e}_2$. It lies in the plane $\theta_3 = \pi$.

Finally, suppose that the affine functions b_i are $-z, 1-z, 2z-2$, and 1. These have zeroes $0, 1, 1$, and ∞. The vertices of the coamoeba $\mathcal{A}(\ell)$ are

$$(0, 0, \pi), \quad (-\pi, 0, \pi), \quad \text{and} \quad (-\pi, -\pi, 0).$$

So $\mathcal{A}(\ell)$ consists of two triangles with edges parallel to $\mathbf{e}_1$, $\mathbf{e}_2+\mathbf{e}_3$, and $\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$. It lies in the plane $\theta_3 = \theta_2 + \pi$. We display all three coamoebas in Figure 4. ▨

The *coamoeba chain* $\overline{\mathcal{A}(\ell)}$ of ℓ is the closure of the coamoeba of ℓ in which the image of each half plane (under $\mathrm{Arg} \circ \Phi(\cdot)$) is oriented so that its boundary is an oriented polygonal path connecting $p_1, \ldots, p_{M+1}, p_1$. On the upper half plane this agrees with the orientation induced by the parametrization $\mathbf{P}^1 \smallsetminus \underline{\mathbf{RP}^1} \to \mathcal{A}(\ell)$, but it has the opposite orientation on the lower half plane. The boundary of $\overline{\mathcal{A}(\ell)}$ consists of $M+1$ circles in which p_j and p_{j+1} are antipodal points on the jth circle and both semicircles (each is the boundary of the image of a half plane) are oriented to point from p_j to p_{j+1}. This coamoeba chain is not a closed chain, as it has nonempty oriented boundary, but there is a natural zonotope chain $Z(\ell)$ such that $\overline{\mathcal{A}(\ell)} + Z(\ell)$ is closed.

Intuitively, $Z(\ell)$ is the cone over the boundary of $\overline{\mathcal{A}(\ell)}$ with vertex the origin $\mathbf{0} := (0, \ldots, 0)$. Unfortunately, there is no notion of a cone in $\mathbf{T}^N$ and the zonotope chain may be more than just this cone. We instead define a chain in $\mathbf{R}^N$ as the cone over an oriented polygon $P(\ell)$ with vertex the origin and set $Z(\ell)$ to be the image of this chain in $\mathbf{T}^N$.

Definition 2.4. Recall that the affine functions $b_1, \ldots, b_N, b_{N+1} = 1$ are ordered in the following way. Their zeroes are $\zeta_1 < \cdots < \zeta_M < \zeta_{M+1} = \infty$ and there are integers $1 = m_1 < \cdots < m_{M+1} \leq N+1$ and $n_1, \ldots, n_{M+1}$ with $m_j < n_j \leq m_{j+1}$ such that one of (2.3) or (2.4) holds, where sgn_i is the sign of b_i on $(-\infty, \zeta_1)$.

We had defined $\mathbf{f}_j := \sum_{i=m_j}^{m_{j+1}-1} \mathbf{e}_i$. We will need the following vectors

$$\mathbf{g}_j := \sum_{i=m_j}^{n_j-1} \mathbf{e}_i \qquad \text{and} \qquad \mathbf{h}_j := \sum_{i=m_j}^{m_{j+1}-1} \mathrm{sgn}_i\, \mathbf{e}_i = \mathrm{sgn}_{m_j}(2\mathbf{g}_j - \mathbf{f}_j)\ .$$

We first define a sequence of points $\widetilde{p}_1, \widetilde{p}_1', \ldots, \widetilde{p}_{2M+2}, \widetilde{p}_{2M+2}' \in (\pi\mathbf{Z})^N$ with the property that $\widetilde{p}_i, \widetilde{p}_i', \widetilde{p}_{M+1+i}$, and $\widetilde{p}_{M+1+i}'$ all map to $p_i \in \mathbf{T}^N$. To begin, set $\widetilde{p}_1$ to be the unique point in $\{0, \pi\}^N \subset \mathbf{R}^N$ which maps to $p_1 \in \mathbf{T}^N$,

$$(2.6) \qquad\qquad \widetilde{p}_{1,i} = \arg(\mathrm{sgn}_i) = \begin{cases} \pi & \text{if } \mathrm{sgn}_i = -1 \\ 0 & \text{if } \mathrm{sgn}_i = 1 \end{cases}.$$

For each $j = 1, \ldots, M{+}1$, set $\widetilde{p}_{j+1} := \widetilde{p}_j + \pi\mathbf{h}_j$. Since $\mathbf{h}_j = \mathrm{sgn}_{m_j}(2\mathbf{g}_j - \mathbf{f}_j)$, we have that $\widetilde{p}_{j+1}$ maps to p_{j+1}, as $p_{j+1} = p_j - \pi\mathbf{f}_j \mod (2\pi\mathbf{Z})^N$. For the remainder of the points, if $n_j < m_{j+1}$, so that both signs occur, set $\widetilde{p}_j' := \widetilde{p}_j + 2\pi\,\mathrm{sgn}_{m_j}\mathbf{g}_j$, and otherwise set $\widetilde{p}_j' := \widetilde{p}_j$. Observe that $\widetilde{p}_j'$ maps to p_j and that in every case, $\widetilde{p}_{j+1} = \widetilde{p}_j' - \pi\,\mathrm{sgn}_{m_j}\mathbf{f}_j$.

We claim that $\widetilde{p}_{M+2} = -\widetilde{p}_1$. Since $\widetilde{p}_{M+2} = \widetilde{p}_1 + \pi(\mathbf{h}_1 + \cdots + \mathbf{h}_{M+1})$, we need to show that $\pi(\mathbf{h}_1 + \cdots + \mathbf{h}_{M+1}) = -2\widetilde{p}_1$. By definition,

$$\mathbf{h}_1 + \cdots + \mathbf{h}_{M+1} = \sum_{i=1}^{N+1} \mathrm{sgn}_i\, \mathbf{e}_i\ .$$

We have $\mathrm{sgn}_{N+1} = 1$ as $b_{N+1} = 1$. Since we defined $\mathbf{e}_{N+1}$ to be $-(\mathbf{e}_1 + \cdots + \mathbf{e}_N)$, we see that

$$\mathbf{h}_1 + \cdots + \mathbf{h}_{M+1} = \sum_{i=1}^{N} (\mathrm{sgn}_i - 1)\mathbf{e}_i\ .$$

The ith component of this sum is -2 if $\mathrm{sgn}_i = -1$ and 0 if $\mathrm{sgn}_i = 1$. Since $\widetilde{p}_{1,i} = \arg(\mathrm{sgn}_i)$, this proves the claim.

Finally, for each $M{+}2 \le j \le 2M{+}2$, set

$$\widetilde{p}_j := -\widetilde{p}_{j-(M+1)} \qquad \text{and} \qquad \widetilde{p}_j' := -\widetilde{p}_{j-(M+1)}\,,$$

and let $P(\ell)$ be the cyclically oriented path obtained by connecting

$$\widetilde{p}_{2M+2}'\,,\ \widetilde{p}_{2M+2}\,,\ \widetilde{p}_{2M+1}'\,,\ \widetilde{p}_{2M+1}\,,\ \ldots,\widetilde{p}_2'\,,\ \widetilde{p}_2\,,\ \widetilde{p}_1'\,,\ \widetilde{p}_1$$

in cyclic order. The cone over $P(\ell)$ with vertex the origin is the union of possibly degenerate triangles of the form

$$\mathrm{conv}(\mathbf{0}, \widetilde{p}_{i+1}, \widetilde{p}_i') \qquad \text{and} \qquad \mathrm{conv}(\mathbf{0}, \widetilde{p}_i', \widetilde{p}_i) \qquad \text{for} \qquad i = 2M{+}2, \ldots, 2, 1\,,$$

where $\widetilde{p}_{2M+3} := \widetilde{p}_1$. Each triangle is oriented so its three vertices occur in positive order along its boundary. If a point $\widetilde{p}_i$ or $\widetilde{p}_i'$ is $\mathbf{0}$, then the triangles involving it degenerate into line segments, as do triangles $\mathrm{conv}(\mathbf{0}, \widetilde{p}_i', \widetilde{p}_i)$ when $\widetilde{p}_i' = \widetilde{p}_i$. Let $\widetilde{Z(\ell)}$ be the union of these oriented triangles, which is a chain in $\mathbf{R}^N$. Define the *zonotope chain* $Z(\ell)$ to be the image in $\mathbf{T}^N$ of $\widetilde{Z(\ell)}$.

Example 2.5. Figure 3 shows two views of the zonotope chain with the coamoeba chain of Figure 2. Now consider the zonotope chains for the three lines of Example 2.3. When ℓ is
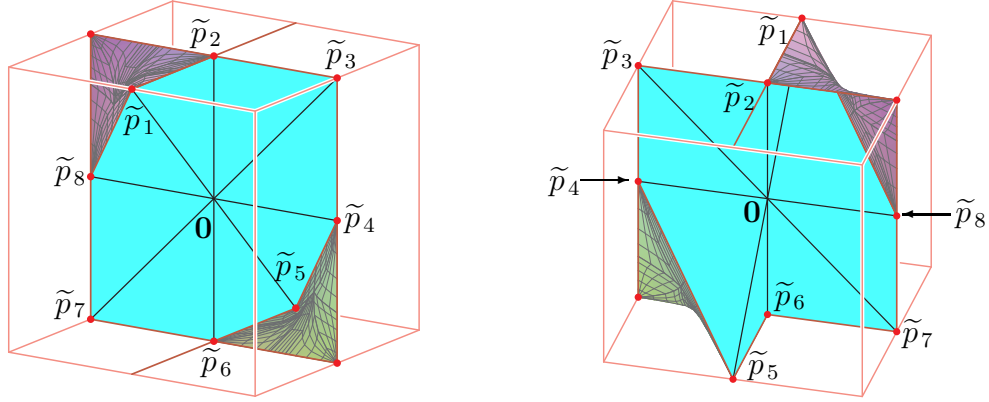

FIGURE 3. Two views of the coamoeba and zonotope chains

defined by $z \mapsto [-1 - z, -1 - z, 2z, 2]$, the points $\widetilde{p}_1, \ldots, \widetilde{p}_6'$ (omitting repeated points) are

$$(0, 0, \pi), \ (\pi, \pi, \pi), \ (\pi, \pi, 0), \ (0, 0, -\pi), \ (-\pi, -\pi, -\pi), \quad \text{and} \quad (-\pi, -\pi, 0).$$

We display the coamoeba chain and the zonotope chain of ℓ at the left of Figure 4.
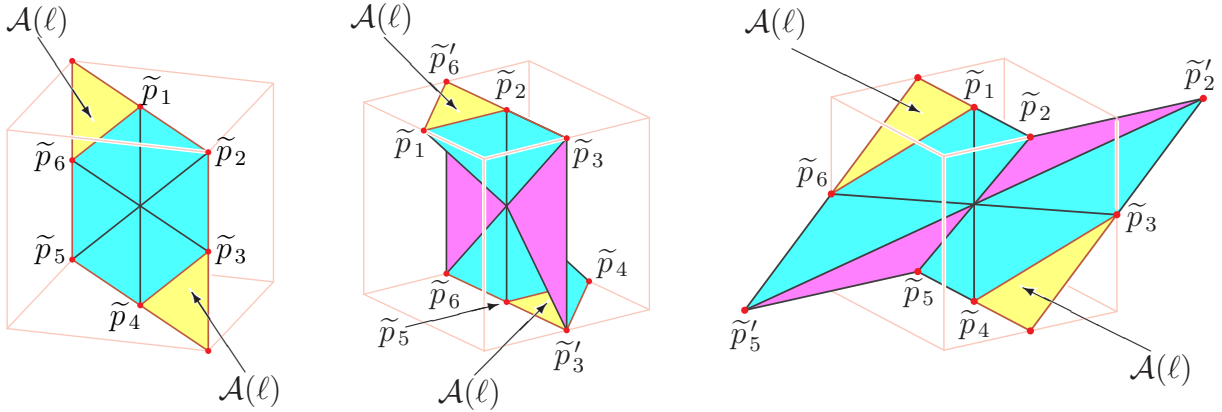

FIGURE 4. Coamoeba and zonotope chains

When ℓ is defined by $z \mapsto [\frac{1}{2} + z, \frac{1}{2} - z, -2, 1]$, the points $\widetilde{p}_1, \ldots, \widetilde{p}_6'$ are

$$\widetilde{p}_1 = (\pi, 0, \pi), \ \widetilde{p}_2 = (0, 0, \pi), \ \widetilde{p}_3 = (0, \pi, \pi), \ \widetilde{p}_3' = (0, \pi, -\pi),$$
$$\widetilde{p}_4 = (-\pi, 0, -\pi), \ \widetilde{p}_5 = (0, 0, -\pi), \ \widetilde{p}_6 = (0, -\pi, -\pi), \ \widetilde{p}_6' = (0, -\pi, \pi).$$

We display the coamoeba and zonotope chains of ℓ in the middle of Figure 4.

When ℓ is defined by $z \mapsto [-z, 1-z, 2z-2, 1]$, the points $\widetilde{p}_1, \ldots, \widetilde{p}_6'$ are

$$\widetilde{p}_1 = (0,0,\pi), \ \widetilde{p}_2 = (\pi,0,\pi), \ \widetilde{p}_2' = (\pi,2\pi,\pi), \ \widetilde{p}_3 = (\pi,\pi,0),$$
$$\widetilde{p}_4 = (0,0,-\pi), \ \widetilde{p}_5 = (-\pi,0,-\pi), \ \widetilde{p}_5' = (-\pi,-2\pi,-\pi), \ \widetilde{p}_6 = (-\pi,-\pi,0).$$

We display the coamoeba and zonotope chains of ℓ on the right of Figure 4.

We state the main result of this section.

Theorem 2.6. *The sum,* $\overline{\mathcal{A}(\ell)} + Z(\ell)$, *of the coamoeba chain and the zonotope chain forms a cycle in* $\mathbf{T}^N$ *whose homology class is*

$$\left[\overline{\mathcal{A}(\ell)} + Z(\ell)\right] = \sum_{\substack{1 \leq i < j \leq N \\ (\widetilde{p}_{1,i}, \widetilde{p}_{1,j}) = (0,\pi)}} \mathbf{e}_i \wedge \mathbf{e}_j.$$

Example 2.7. For the line of Example 2.2, $\widetilde{p}_1 = (\pi, 0, \pi)$, and the only entries $i < j$ with 0 at i and π at j are $i = 2$ and $j = 3$, and so

$$\left[\overline{\mathcal{A}(\ell)} + Z(\ell)\right] = \mathbf{e}_2 \wedge \mathbf{e}_3.$$

For the first line of Example 2.3, $\widetilde{p}_1 = (0,0,\pi)$, and so

$$\left[\overline{\mathcal{A}(\ell)} + Z(\ell)\right] = \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_3.$$

For the second line of Example 2.3, $\widetilde{p}_1 = (\pi, 0, \pi)$, so that $\left[\overline{\mathcal{A}(\ell)} + Z(\ell)\right] = \mathbf{e}_2 \wedge \mathbf{e}_3$. For the third line of Example 2.3, $\widetilde{p}_1 = (0,0,\pi)$, and $\left[\overline{\mathcal{A}(\ell)} + Z(\ell)\right] = \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_3$. These homology classes are apparent from Figures 3 and 4.
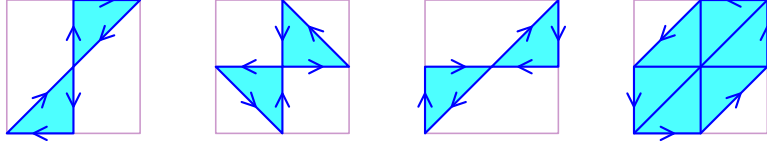
Example 2.8. Our proof of Theorem 2.6 rests on the case of $N = 2$. Suppose first that $M = 2$. Up to positive rescaling and translation in the domain $\mathbf{RP}^1$, there are four lines.

$$[z : z{-}1 : 1] \qquad [z : 1{-}z : 1] \qquad [-z : 1{-}z : 1] \qquad [-z : z{-}1 : 1]$$
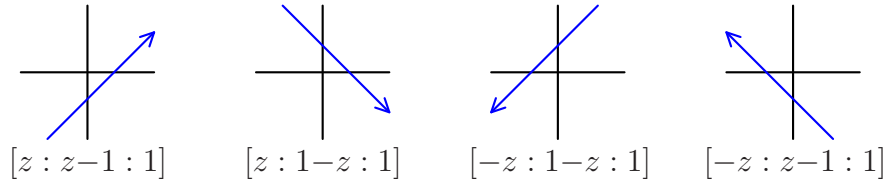
For these, the initial point p_1 is (π, π), $(\pi, 0)$, $(0, 0)$, and $(0, \pi)$, respectively. The four coamoeba chains are, in the fundamental domain $[-\pi, \pi]^2$,
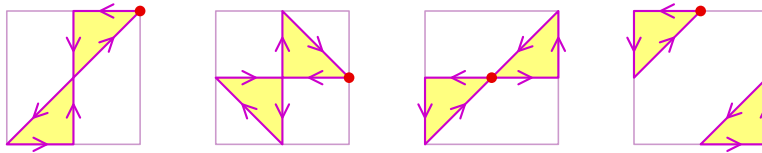
and the corresponding zonotope chains are as follows.

For each, the sum $\overline{\mathcal{A}(\ell)} + Z(\ell)$ of chains is a cycle. This cycle is homologous to zero for the first three, and it forms the fundamental cycle $\mathbf{e}_1 \wedge \mathbf{e}_2$ of $\mathbf{T}^2$ for the fourth.

Now suppose that $M = 1$. We may assume that $\xi_1 = 0$. Up to positive rescaling there are eight possibilities for the parametrization of ℓ,

$$[-z : -z : 1], \; [z : z : 1], \; [-z : 1 : 1], \; [z : 1 : 1],$$
$$[z : -z : 1], \; [z : -1 : 1], \; [-z : -1 : 1], \; [-z : z : 1].$$

For all of these, the coamoeba is one-dimensional. In the first four, the zonotope chain is one-dimensional. Table 1 gives the parametrization, the vertices of the coamoeba of the upper half plane, and the path $P(\ell) = \widetilde{p}_4, \widetilde{p}_3, \widetilde{p}_2, \widetilde{p}_1$ for these four.

TABLE 1. Coamoeba and zonotope chains.

ℓ	$\mathcal{A}(\ell)$	$P(\ell)$
$[-z : -z : 1]$	$(0,0)$, $(-\pi,-\pi)$	$(-\pi,-\pi)$, $(0,0)$, (π,π), $(0,0)$
$[z : z : 1]$	(π,π), $(0,0)$	$(0,0)$, $(-\pi,-\pi)$, $(0,0)$, (π,π)
$[-z : 1 : 1]$	$(0,0)$, $(-\pi,0)$	$(-\pi,0)$, $(0,0)$, $(\pi,0)$, $(0,0)$
$[z : 1 : 1]$	$(\pi,0)$, $(0,0)$	$(0,0)$, $(-\pi,0)$, $(0,0)$, $(\pi,0)$

The remaining parametrizations are more interesting. When ℓ is given by $z \mapsto [z : -z : 1]$, we have $p_1 = (\pi,0)$ and $p_2 = (0,-\pi)$, and $P(\ell)$ is

$$\widetilde{p}_4 = (0,-\pi), \; \widetilde{p}_3' = (\pi,0), \; \widetilde{p}_3 = (-\pi,0), \; \widetilde{p}_2 = (0,\pi), \; \widetilde{p}_1' = (-\pi,0), \quad \text{and} \quad \widetilde{p}_1 = (\pi,0),$$

and the zonotope chain is shown on the left in Figure 5. The path $\widetilde{p}_4 - \widetilde{p}_3' - \widetilde{p}_3 - \widetilde{p}_2 - \widetilde{p}_1' - \widetilde{p}_1 - \widetilde{p}_4$ zig-zags over itself, once in each direction, and consequently each triangle is covered twice, once with each orientation, and therefore $[Z(\ell)] = 0$ in homology.
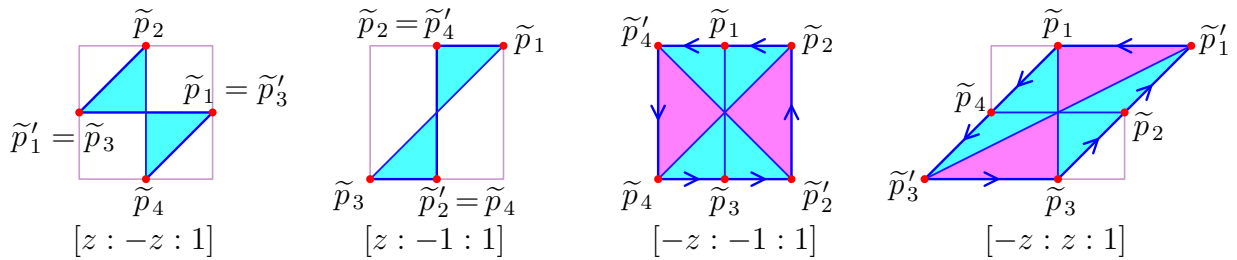

FIGURE 5. Four more zonotope chains.

When ℓ is given by $[z:-1:1]$, we have $p_1 = (\pi,\pi)$ and $p_2 = (0,\pi)$, and $P(\ell)$ is

$$\widetilde{p}'_4 = (0,\pi)\,, \ \widetilde{p}_4 = (0,-\pi)\,, \ \widetilde{p}_3 = (-\pi,-\pi)\,, \ \widetilde{p}'_2 = (0,-\pi)\,, \ \widetilde{p}_2 = (0,\pi)\,, \quad \text{and} \quad \widetilde{p}_1 = (\pi,\pi)\,,$$

and the zonotope chain is shown on the left center of Figure 5. As before, each triangle is covered twice, once with each orientation, and therefore $[Z(\ell)] = 0$ in homology.

When ℓ is given by $[-z:-1:1]$, we have $p_1 = (0,\pi)$ and $p_2 = (-\pi,\pi)$, and $P(\ell)$ is

$$\widetilde{p}'_4 = (-\pi,\pi)\,, \ \widetilde{p}_4 = (-\pi,-\pi)\,, \ \widetilde{p}_3 = (0,-\pi)\,, \widetilde{p}'_2 = (\pi,-\pi)\,, \ \widetilde{p}_2 = (\pi,\pi)\,, \quad \text{and} \quad \widetilde{p}_1 = (0,\pi)\,,$$

and the zonotope chain is shown on the right center of Figure 5. The triangles $\mathrm{conv}(\mathbf{0},\widetilde{p}_2,\widetilde{p}'_2)$ and $\mathrm{conv}(\mathbf{0},\widetilde{p}_4,\widetilde{p}'_4)$ are shaded differently. The zonotope chain is equal to the fundamental cycle of $\mathbf{T}^2$, with the standard positive orientation. Thus $[Z(\ell)] = \mathbf{e}_1 \wedge \mathbf{e}_2$ in homology.

Finally, when ℓ is given by $[-z:z:1]$, we have $p_1 = (0,\pi)$ and $p_2 = (-\pi,0)$, and $P(\ell)$ is

$$\widetilde{p}_4 = (-\pi,0)\,, \ \widetilde{p}'_3 = (-2\pi,-\pi)\,, \ \widetilde{p}_3 = (0,-\pi)\,, \widetilde{p}_2 = (\pi,0)\,, \ \widetilde{p}'_1 = (2\pi,\pi)\,, \quad \text{and} \quad \widetilde{p}_1 = (0,\pi)\,,$$

and the zonotope chain is shown on the right of Figure 5. Again, $[Z(\ell)] = [\mathbf{T}^2]$.

Observe that $\mathcal{A}(\ell) + Z(\ell)$ forms a cycle which is homologous to zero unless $\widetilde{p}_1 = (0,\pi)$, in which case it equals the fundamental cycle $\mathbf{e}_1 \wedge \mathbf{e}_2$ of $\mathbf{T}^2$. ◻

Proof of Theorem 2.6. We show that the two chains $\overline{\mathcal{A}(\ell)}$ and $Z(\ell)$ have the same boundary, but with opposite orientation, which implies that their sum is a cycle. We observed that the boundary of $\mathcal{A}(\ell)$ lies along the $M+1$ circles in which the jth contains p_j and p_{j+1} (with $p_{M+2} = p_1$) and has direction parallel to $\mathbf{f}_j$. On this jth circle the boundary of $\mathcal{A}(\ell)$ consists of the two semicircles oriented from p_j to p_{j+1}.

There are two types of edges forming the boundary of the zonotope cycle $Z(\ell)$. The first comes from the edges of $P(\ell)$ with direction $\pm\mathbf{f}_j$ connecting $\widetilde{p}_{j+1}$ to $\widetilde{p}'_j$ and $\widetilde{p}_{M+1+j+1}$ to $\widetilde{p}'_{M+1+j}$, and the second comes from edges connecting $\widetilde{p}'_j$ to $\widetilde{p}_j$, when $\widetilde{p}'_j \neq \widetilde{p}_j$.

The first type of edge gives a part of the boundary of $Z(\ell)$ which is equal to the boundary of $\mathcal{A}(\ell)$, but with opposite orientation. (The edges point from p_{j+1} to p_j.) The edges of the second type come in pairs which cancel each other. Indeed, when $\widetilde{p}_j \neq \widetilde{p}'_j$, then the edge from $\widetilde{p}'_j$ to $\widetilde{p}_j$ is the directed circle connecting p_j with itself and having direction $\pm\mathbf{g}_j$, which is equal to, but opposite from, the edge connecting $\widetilde{p}'_{M+1+j}$ to $\widetilde{p}_{M+1+j}$. Thus $\overline{\mathcal{A}(\ell)} + Z(\ell)$ forms a cycle in homology.

We determine the homology class $[\overline{\mathcal{A}(\ell)} + Z(\ell)]$ by computing its pushforward to each two-dimensional coordinate projection of $\mathbf{T}^N$. Let $1 \leq i < j \leq N$ be two coordinate directions and consider the projection onto the plane of the coordinates i and j, which is a map $pr\colon \mathbf{T}^N \to \mathbf{T}^2$. The image of ℓ under pr is parametrized by

$$(2.7) \qquad\qquad z \longmapsto [b_i(z) : b_j(z) : b_{N+1}(z)]\,.$$

If b_i, b_j, (and $b_{N+1} = 1$) all vanish at $\xi_{N+1} = \infty$, then the image of ℓ under pr is a point, and the image of $Z(\ell)$ is either a point or is one-dimensional, and so $pr_*[\mathcal{A}(\ell) + Z(\ell)] = 0$. In this case $(\widetilde{p}_{1,i},\widetilde{p}_{1,j})$ is either $(0,0)$, $(\pi,0)$, or (π,π), by (2.3) and (2.6).

Otherwise, the image of ℓ under the projection of $\mathbf{P}^N$ to the (i,j)-coordinate plane is the line ℓ' parameterized by (2.7). It is immediate from the definitions that

$$pr(\overline{\mathcal{A}(\ell)}) \;=\; \overline{\mathcal{A}(\ell')} \qquad \text{and} \qquad pr(Z(\ell)) \;=\; Z(\ell')\,.$$

When b_i and b_j have distinct (finite) zeroes, say ζ_a and ζ_b, then pr is injective on the interior of $\overline{\mathcal{A}(\ell)}$ and on the edges with directions $\pm\mathbf{f}_a$, $\pm\mathbf{f}_b$, and $\pm\mathbf{f}_{M+1}$ (sending them to edges with directions $\pm\mathbf{e}_1$, $\pm\mathbf{e}_2$, and $\pm(\mathbf{e}_1+\mathbf{e}_2)$) and collapsing the others to points. In the other cases, $\mathcal{A}(\ell')$ is a circle. However, in all cases pr is one-to-one over the interiors of each triangle in the image zonotope cycle $Z(\ell')$, collapsing the other triangles to line segments or to points. Thus

$$pr_*[\overline{\mathcal{A}(\ell)} + Z(\ell)] \;=\; [\overline{\mathcal{A}(\ell')} + Z(\ell')]\,.$$

Since the last vertex of the path $P(\ell')$ is $(\widetilde{p}_{1,i}, \widetilde{p}_{1,j})$, the theorem follows from the computation of Example 2.8. ▨

3. Structure of discriminant coamoebas in dimension two

Suppose now that $B \subset \mathbf{Z}^2$ is a multiset of $N{+}1$ vectors which span $\mathbf{R}^2$ and have sum $\mathbf{0} = (0,0)$. We use $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_{N+1}\}$ to define a rational map $\mathbf{C}^2 -\!\rightarrow \mathbf{C}^2$

$$(3.1) \qquad\qquad z \;\longmapsto\; \Big(\prod_{i=1}^{N+1} \langle \mathbf{b}_i, z \rangle^{\mathbf{b}_{i,1}}, \; \prod_{i=1}^{N+1} \langle \mathbf{b}_i, z \rangle^{\mathbf{b}_{i,2}} \Big)\,.$$

Since $\sum_i \mathbf{b}_i = \mathbf{0}$, each coordinate is homogenous of degree 0, and so (3.1) induces a rational map $\Psi_B \colon \mathbf{P}^1 \to \mathbf{P}^2$ (where the image has distinguished coordinates). Define D_B to be the image of this map (3.1). When B consists of distinct vectors that span $\mathbf{Z}^2$, then it is Gale dual to a set of vectors of the form $(1, \mathbf{a})$ for $\mathbf{a} \in A \subset \mathbf{Z}^{n+2}$. In this case, (3.1) is the Horn-Kapranov parametrization [7] of the reduced A-discriminant. We use Theorem 2.6 to study the coamoeba $\mathcal{A}_B$ of D_B and its complement, for any multiset B.

The results of Section 2 are applicable because the map (3.1) factors,

$$\mathbf{C}^2 \ni z \;\longmapsto\; (\langle \mathbf{b}_1, z \rangle, \langle \mathbf{b}_2, z \rangle, \ldots, \langle \mathbf{b}_{N+1}, z \rangle) \in \mathbf{C}^{N+1}$$

$$\mathbf{C}^{N+1} \ni (x_1, x_2, \ldots, x_{N+1}) \;\longmapsto\; \Big(\prod_{i=1}^{N+1} x_i^{\mathbf{b}_{i,1}}, \; \prod_{i=1}^{N+1} x_i^{\mathbf{b}_{i,2}} \Big) \in \mathbf{C}^2$$

The first map, Φ_B, is linear and the second, β, is a monomial map. They induce maps $\mathbf{P}^1 \to \mathbf{P}^N -\!\rightarrow \mathbf{P}^2$, with the second a rational map. Let ℓ_B be the image of Φ_B in $\mathbf{P}^N$, which is a real line as in Section 2. The map $\mathrm{Arg}(\beta)$ is the homomorphism $\mathbf{T}^N \to \mathbf{T}^2$ induced by the linear map on the universal covers, (also written $\mathrm{Arg}(\beta)$),

$$\mathrm{Arg}(\beta) \colon \; \mathbf{R}^N \ni \mathbf{e}_i \;\longmapsto\; \mathbf{b}_i \in \mathbf{R}^2\,,$$

and the following is immediate.

Lemma 3.1. *The coamoeba $\mathcal{A}_B$ is the image of the coamoeba $\mathcal{A}(\ell_B)$ under the map $\mathrm{Arg}(\beta)$.*

Example 3.2. Let B be the vector configuration $\{(1,0), (-2,1), (1,-2), (0,1)\}$. Observe that $\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 + \mathbf{b}_4 = 0$ and $3\mathbf{b}_1 + 2\mathbf{b}_2 + \mathbf{b}_3 = 0$, thus B is Gale dual to the vector configuration $\{(1,3), (1,2), (1,1), (1,0)\} \subset \{1\} \times \mathbf{Z}$. So A is simply $\{0, 1, 2, 3\}$ if we identify $\mathbf{Z}$ with $\{1\} \times \mathbf{Z}$. We show these two configurations.
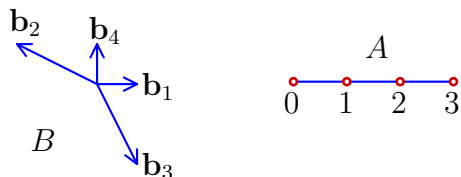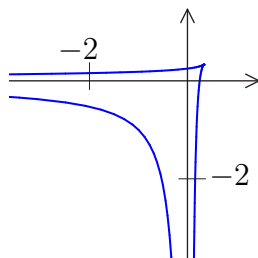

Observe that the convex hull of A has volume $d_B = 3$.

The map (3.1) becomes

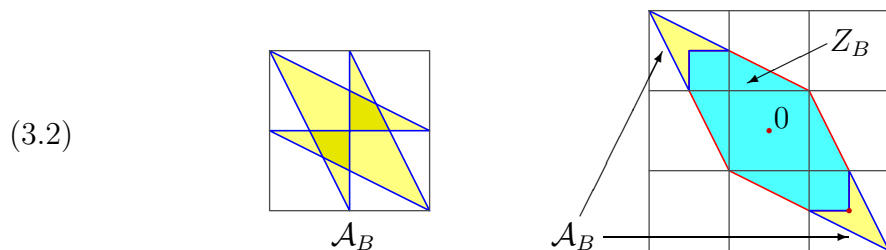$$(x,y) \longmapsto \left(\frac{x(x-2y)}{(y-2x)^2}, \frac{y(y-2x)}{(x-2y)^2} \right),$$

whose image is the curve below.

The line ℓ_B is the line of Example 2.2 and so $\mathcal{A}_B$ is the image of the coamoeba of Figure 2 under the map

$$\mathrm{Arg}(\beta) : \ (\theta_1, \theta_2, \theta_3) \longmapsto (\theta_1 - 2\theta_2 + \theta_3, \theta_2 - 2\theta_3).$$

We display this image below, first in the fundamental domain $[\pi, \pi]^2$ of $\mathbf{T}^2$, and then in universal cover $\mathbf{R}^2$ of $\mathbf{T}^2$ (each square is one fundamental domain).

(3.2)

In the picture on the left, the darker shaded regions are where the argument map is two-to-one. The octagon on the right is the zonotope Z_B generated by B and it is the image of the zonotope chain of Figure 3 under the map $\mathrm{Arg}(\beta)$. Observe that the union of the coamoeba and the zonotope covers the fundamental domain $d_B = 3$ times.

What we observe in this example is in fact quite general. We first use Lemma 3.1 to describe the coamoeba $\mathcal{A}_B$ more explicitly, then study the zonotope Z_B generated by B, before making an important definition and giving our proof of Theorem 1.

The line ℓ_B is parametrized by the forms $z \mapsto \langle \mathbf{b}_i, z \rangle$, for $i = 1, \ldots, N+1$. Let $\xi_i \in \mathbf{RP}^1$ be the zero of the ith form, and suppose these are in a weakly increasing cyclic order on $\mathbf{RP}^1$,

$$\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{N+1}.$$

Next, identify $\mathbf{P}^1 \smallsetminus \{\xi_{N+1}\}$ with $\mathbf{C}$, so that ξ_{N+1} is the point ∞ at infinity, and suppose that the distinct zeroes are

$$\zeta_1 < \zeta_2 < \cdots < \zeta_M < \zeta_{M+1} = \infty.$$

By the description of the coamoeba $\mathcal{A}(\ell_B)$ of Section 2 and Lemma 3.1, we see that the coamoeba $\mathcal{A}_B$ is composed of two components, each bounded by polygonal paths that are the images of the boundary of $\mathcal{A}(\ell_B)$ under the map $\mathrm{Arg}(\beta)$. For each $j = 1, \ldots, M+1$, set

$$\mathbf{c}_j := \mathrm{Arg}(\beta)(\mathbf{f}_j) = \sum_{i\,:\,\langle \mathbf{b}_i, \zeta_j \rangle = 0} \mathbf{b}_i.$$

The components of $\mathcal{A}_B$ correspond to the half planes of $\mathbf{P}^1$, and the boundary along each is the polygonal path with edges $\pm\pi\mathbf{c}_1, \ldots, \pm\pi\mathbf{c}_{M+1}$ with the $+$ signs for the upper half plane and $-$ signs for the lower half plane. The complete description requires the following proposition, which is explained in [10, § 2].

Proposition 3.3. *Suppose that $M > 1$. Then the composition*

$$\mathbf{P}^1 \smallsetminus \{\zeta_1, \ldots, \zeta_{M+1}\} \xrightarrow{\Psi_B} D_B \xrightarrow{\mathrm{Arg}} \mathcal{A}_B \subset \mathbf{T}^2$$

is an immersion when restricted to $\mathbf{P}^1 \smallsetminus \mathbf{RP}^1$ (in fact it is locally a covering map).

The edges $\pm\pi\mathbf{c}_1, \ldots, \pm\pi\mathbf{c}_{M+1}$ decompose $\mathbf{T}^2$ into polygonal regions. Over each polygonal region the map of Proposition 3.3 has a constant number of preimages. This number of preimages equals the winding number of the polygonal path around that region. Then the pushforward $\mathrm{Arg}(\beta)_*(\overline{\mathcal{A}(\ell_B)})$ of the coamoeba chain of the line ℓ_B is the chain in $\mathbf{T}^2$ where the multiplicity of a region is this number of preimages/winding number. This equals the coamoeba chain of D_B. We will write $\overline{\mathcal{A}_B}$ for this chain $\mathrm{Arg}(\beta)_*(\overline{\mathcal{A}(\ell_B)})$, as our arguments use the pushforward.

There is another natural chain we may define from the vector configuration B. Let $\overline{\mathbf{0}, \pi\mathbf{b}_i}$ be the directed line segment in $\mathbf{R}^2$ connecting the origin to the endpoint of the vector $\pi\mathbf{b}_i$. Let $Z_B \subset \mathbf{R}^2$ be the Minkowski sum of the line segments $\overline{\mathbf{0}, \pi\mathbf{b}_i}$ for $\mathbf{b}_i \in B$. This is a centrally symmetric zonotope as $\sum_i \mathbf{b}_i = \mathbf{0}$. We will also write Z_B for its image in $\mathbf{T}^2$, considered now as a chain. For any $\mathbf{v} \in \mathbf{R}^2$, the points

$$q := \sum_{\langle \mathbf{b}_i, \mathbf{v} \rangle > 0} \mathbf{b}_i \qquad \text{and} \qquad q' := \sum_{\langle \mathbf{b}_i, \mathbf{v} \rangle \geq 0} \mathbf{b}_i$$

are vertices of Z_B which are extreme in the direction of $\mathbf{v}$. These differ only if the line $\mathbf{R}\mathbf{v}$ represents a zero ζ_j of one of the forms, and then the edge between them is $\pi\mathbf{d}_j$, where

$$(3.3) \qquad\qquad \mathbf{d}_j \ := \sum_{i:\ \langle\mathbf{b}_i,\mathbf{v}\rangle=0} \mathrm{sign}(\langle\mathbf{b}_i,\mathbf{w}\rangle)\,\mathbf{b}_{i,},$$

where $\mathbf{w}$ is a vector such that $\langle-\mathbf{w},q\rangle > \langle-\mathbf{w},q'\rangle$ and $\mathrm{sign}(x) \in \{\pm1\}$ is the sign of the real number x. Thus $\mathbf{d}_j$ is the vector parallel to any $\mathbf{b}_i$ with $\langle\mathbf{b}_i,\zeta_j\rangle = 0$ whose length is the sum of the lengths of these vectors and its direction is such that $\langle\mathbf{d}_j,\mathbf{w}\rangle > 0$.

Starting at a vertex of Z_B and moving, say clockwise, the successive edge vectors will be the vectors $\{\pm\pi\mathbf{d}_1,\ldots,\pm\pi\mathbf{d}_M,\pm\pi\mathbf{d}_{M+1}\}$ occuring in a cyclic clockwise order. This may be seen on the right in (3.2), where Z_B is the octagon. Its southeastern-most vertex is $\pi\mathbf{b}_1+\pi\mathbf{b}_3$ (corresponding to the vector $\mathbf{v}_1 = -\mathbf{b}_2$, and the edges encountered from there in clockwise order are $-\pi\mathbf{b}_1,\pi\mathbf{b}_2,-\pi\mathbf{b}_3,\pi\mathbf{b}_4,\pi\mathbf{b}_1,-\pi\mathbf{b}_2,\pi\mathbf{b}_3,-\pi\mathbf{b}_4$. (Here, $\mathbf{d}_j = \mathbf{b}_j$)

Before giving our proof of Theorem 1, we make an important definition. Let $B = \{\mathbf{b}_1,\ldots,\mathbf{b}_{N+1}\}$ be a multiset of vectors in $\mathbf{Z}^2$ that span $\mathbf{R}^2$ and whose sum is $\mathbf{0}$. Write $\mathrm{cone}(\mathbf{b}_i,\mathbf{b}_j)$ for the cone generated by the vectors $\mathbf{b}_i,\mathbf{b}_j$. Suppose that $\mathbf{v}$ is any vector in $\mathbf{R}^2$ not pointing in the direction of a vector in B, and set

$$(3.4) \qquad\qquad d_{B,\mathbf{v}} \ := \sum_{\mathbf{v}\in\mathrm{cone}(\mathbf{b}_i,\mathbf{b}_j)} |\mathbf{b}_i \wedge \mathbf{b}_j|\ .$$

Here $|\mathbf{b}_i \wedge \mathbf{b}_j|$ is the absolute value of the determinant of the matrix whose columns are the two vectors, which is the area of the parallelogram generated by $\mathbf{b}_i$ and $\mathbf{b}_j$.

Lemma 3.4. *The sum* (3.4) *is independent of choice of* $\mathbf{v}$.

Proof. The rays generated by elements of B divide $\mathbf{R}^2$ into regions. The sum (3.4) depends only upon the region containing $\mathbf{v}$—it is a sum over all cones containing the given region. To show its independence of region, let $\mathbf{v},\mathbf{v}'$ lie in adjacent regions with $\mathbf{u}$ a vector generating the ray separating the regions. Suppose that the vectors in B are indexed so that $\mathbf{b}_\kappa,\mathbf{b}_{\kappa+1},\ldots,\mathbf{b}_{\mu-1}$ are the vectors with direction $-\mathbf{u}$ and $\mathbf{b}_\mu,\mathbf{b}_{\mu+1},\ldots,\mathbf{b}_\lambda$ are the vectors with direction $\mathbf{u}$. Then the sums for $d_{B,\mathbf{v}}$ and $d_{B,\mathbf{v}'}$ both include the sum over all cones whose relative interior contains $\mathbf{u}$, but have different terms involving cones with one generator among $\mathbf{b}_\mu,\ldots,\mathbf{b}_\lambda$. All such cones appear, and up to a sign, the difference $d_{B,\mathbf{v}} - d_{B,\mathbf{v}'}$ is equal to

$$\big(\mathbf{b}_\mu + \cdots + \mathbf{b}_\lambda\big) \wedge \big(\mathbf{b}_1 + \cdots + \mathbf{b}_{\kappa-1} \ + \ \mathbf{b}_{\lambda+1} + \cdots + \mathbf{b}_{N+1}\big)$$
$$= \big(\mathbf{b}_\mu + \cdots + \mathbf{b}_\lambda\big) \wedge \big(\mathbf{b}_1 + \cdots + \mathbf{b}_{N+1}\big) \ = \ 0\,,$$

which proves the lemma. ⧄

Remark 3.5. The sum (3.4) is known to coincide with the normalized volume of the convex hull of the vector configuration A that is Gale dual to B (see [3]), so Lemma 3.4 also follows from this fact. We will henceforth write d_B for this volume/sum. ⧄

Example 3.6. Consider the sum (3.4) for the vector configuration B of Example 3.2. There are four choices for the vector $\mathbf{v}$ as indicated below

(3.5)

The vector $\mathbf{v}_1$ lies only in $\mathrm{cone}(\mathbf{b}_2, \mathbf{b}_3)$, and we have $\mathbf{b}_2 \wedge \mathbf{b}_3 = \left|\begin{smallmatrix} -2 & 1 \\ 1 & -2 \end{smallmatrix}\right| = 3$. The vector $\mathbf{v}_2$ lies in $\mathrm{cone}(\mathbf{b}_3, \mathbf{b}_1)$ and $\mathrm{cone}(\mathbf{b}_3, \mathbf{b}_4)$, and we have $\mathbf{b}_3 \wedge \mathbf{b}_1 + \mathbf{b}_3 \wedge \mathbf{b}_4 = \left|\begin{smallmatrix} 1 & -2 \\ 1 & 0 \end{smallmatrix}\right| + \left|\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right| = 2+1 = 3$. Similarly, $\mathbf{v}_3$ lies in $\mathrm{cone}(\mathbf{b}_3, \mathbf{b}_4)$, $\mathrm{cone}(\mathbf{b}_1, \mathbf{b}_4)$, and $\mathrm{cone}(\mathbf{b}_1, \mathbf{b}_2)$, and $\mathbf{b}_3 \wedge \mathbf{b}_4 + \mathbf{b}_1 \wedge \mathbf{b}_4 + \mathbf{b}_1 \wedge \mathbf{b}_2 = 1+1+1 = 3$, and the calculation for $\mathbf{v}_4$ is the mirror-image of that for $\mathbf{v}_2$. In every case, $d_{B, \mathbf{v}_i} = 3$, and so $d_B = 3$.

Theorem 3.7. *The sum, $\overline{\mathcal{A}_B} + Z_B$, of the coamoeba chain of D_B and the B-zonotope chain is a cycle in $\mathbf{T}^2$ which equals $d_B[\mathbf{T}^2]$.*

Proof. We will show that $\mathrm{Arg}(\beta)_*[Z(\ell_B)] = [Z_B]$, which implies that

$$[\overline{\mathcal{A}_B} + Z_B] \;=\; \mathrm{Arg}(\beta)_*[\overline{\mathcal{A}(\ell_B)} + Z(\ell_B)]$$

is a cycle, as $\mathrm{Arg}(\beta)_*[\overline{\mathcal{A}(\ell_B)}] = [\overline{\mathcal{A}_B}]$. Since $\mathrm{Arg}(\beta)_*(\mathbf{e}_i \wedge \mathbf{e}_j) = \mathbf{b}_i \wedge \mathbf{b}_j \cdot [\mathbf{T}^2]$, the formula of Theorem 2.6 will give us the homology class of $[\overline{\mathcal{A}_B} + Z_B]$. We will use (3.4) and Lemma 3.4 to show that it equals $d_B[\mathbf{T}^2]$. This will imply the theorem as we will show that there is an ordering of the vectors B such that the map $\mathrm{Arg}(\beta) \colon Z(\ell_B) \to Z_B$ in the universal covers $\mathbf{R}^N \to \mathbf{R}^2$ is injective.

Recall that $\xi_1, \ldots, \xi_{N+1}$ are points of $\mathbf{RP}^1$ with $\langle \mathbf{b}_i, \xi_i \rangle = 0$ and $\zeta_1, \ldots, \zeta_{M+1}$ are the distinct points among them. Let $\mathbf{0} \neq \mathbf{v} \in \mathbf{R}^2$ represent $\xi_{N+1} = \zeta_{M+1}$ (so that $\langle \mathbf{b}_{N+1}, \mathbf{v} \rangle = 0$) and choose $x \in \mathbf{R}^2$ to be a point with $\langle \mathbf{b}_{N+1}, x \rangle = 1$. Then $t \mapsto x + t\mathbf{v}$ gives a parametrization of $\mathbf{RP}^1$ with $\infty = \zeta_{M+1}$, and identifies $\mathbf{R}$ with $\mathbf{RP}^1 \setminus \{\infty\}$.

To agree with Definition 2.1, we suppose that the points of B are ordered so that (2.1) and (2.2) hold. Thus there are integers $1 = m_1 < \cdots < m_{M+1} < m_{M+2} = N+2$ such that

$$\langle \mathbf{b}_i, \zeta_j \rangle = 0 \iff m_j \leq i < m_{j+1}.$$

We further suppose that B is ordered so that one of (2.3) or (2.4) holds for every $j = 1, \ldots, M+1$. Specifically, let $\mathbf{w} := x + \tau\mathbf{v}$ for some fixed $\tau < \zeta_1$. Then there exist integers $n_1, \ldots, n_{M+1}$ such that for each $j = 1, \ldots, M+1$ we have $m_j < n_j \leq m_{j+1}$ and either

$$\langle \mathbf{b}_{m_j}, \mathbf{w} \rangle, \;\; \ldots, \;\; \langle \mathbf{b}_{n_j-1}, \mathbf{w} \rangle \;\; < \; 0 \; < \;\; \langle \mathbf{b}_{n_j}, \mathbf{w} \rangle, \;\; \ldots, \;\; \langle \mathbf{b}_{m_{j+1}-1}, \mathbf{w} \rangle, \qquad \text{or}$$
$$\langle \mathbf{b}_{m_j}, \mathbf{w} \rangle, \;\; \ldots, \;\; \langle \mathbf{b}_{n_j-1}, \mathbf{w} \rangle \;\; > \; 0 \; > \;\; \langle \mathbf{b}_{n_j}, \mathbf{w} \rangle, \;\; \ldots, \;\; \langle \mathbf{b}_{m_{j+1}-1}, \mathbf{w} \rangle.$$

For $i = 1, \ldots, N+1$, let $\mathrm{sgn}_i \in \{\pm 1\}$ be the sign of $\langle \mathbf{b}_i, \mathbf{w} \rangle$. Note that $\mathrm{sgn}_{N+1} = 1$.

Define $\mathbf{f}_j, \mathbf{g}_j, \mathbf{h}_j$ as in Definition 2.1,

$$\mathbf{f}_j \ := \ \sum_{i=m_j}^{m_{j+1}-1} \mathbf{e}_i\,, \qquad \mathbf{g}_j \ := \ \sum_{i=m_j}^{n_j-1} \mathbf{e}_i\,, \qquad \text{and} \qquad \mathbf{h}_j \ := \ \sum_{i=m_j}^{m_{j+1}-1} \mathrm{sgn}_i\,\mathbf{e}_i\,.$$

Consider now the following affine parametrization of $\ell_B \subset \mathbf{P}^N$,

$$\Phi_B \ : \ t \ \longmapsto \ [\langle\mathbf{b}_1, x+t\mathbf{v}\rangle \ : \ \cdots \ : \ \langle\mathbf{b}_N, x+t\mathbf{v}\rangle \ : \ \langle\mathbf{b}_{N+1}, x+t\mathbf{v}\rangle = 1]\,.$$

Let $\widetilde{p}_1 \in \{0,\pi\}^N \in \mathbf{R}^N$ be the point whose ith coordinate is $\arg(\mathrm{sgn}_i)$. Its image $p_1 \in \mathbf{T}^N$ is the point on the coamoeba of ℓ_B coming from the real points $\Phi_B(-\infty, \zeta_1)$.

We describe $\mathrm{Arg}(\beta)(Z(\ell_B))$ in the universal cover $\mathbf{R}^2$ of $\mathbf{T}^2$. For each $j = 1, \ldots, 2M+2$, set $\widetilde{q}_j := \mathrm{Arg}(\beta)(\widetilde{p}_j)$ and $\widetilde{q}_j' := \mathrm{Arg}(\beta)(\widetilde{p}_j')$. Since

$$(3.6) \hspace{4cm} \widetilde{p}_{1,i} \ = \ \left\{ \begin{array}{ll} \pi & \text{if } \langle\mathbf{b}_i, \mathbf{w}\rangle < 0 \\ 0 & \text{if } \langle\mathbf{b}_i, \mathbf{w}\rangle > 0 \end{array} \right.,$$

we have

$$\widetilde{q}_1 \ = \ \pi \cdot \sum_{\langle\mathbf{b}_i, \mathbf{w}\rangle < 0} \mathbf{b}_i\,,$$

and so $\widetilde{q}_1$ is a vertex of Z_B which is extreme in the direction of $-\mathbf{w}$.

The zonotope chain $Z(\ell_B)$ is a union of the triangles

$$(3.7) \hspace{2cm} \mathrm{conv}(\mathbf{0}, \widetilde{p}_{j+1}, \widetilde{p}_j') \qquad \text{and} \qquad \mathrm{conv}(\mathbf{0}, \widetilde{p}_j', \widetilde{p}_j) \qquad \text{for } j = M+2, \ldots, 1\,,$$

where the second is degenerate if $\widetilde{p}_j = \widetilde{p}_j'$. Thus $\mathrm{Arg}(\beta)(Z(\ell_B))$ will be the union of the (possibly degenerate) triangles

$$(3.8) \hspace{2cm} \mathrm{conv}(\mathbf{0}, \widetilde{q}_{j+1}, \widetilde{q}_j') \qquad \text{and} \qquad \mathrm{conv}(\mathbf{0}, \widetilde{q}_j', \widetilde{q}_j) \qquad \text{for } j = M+2, \ldots, 1\,,$$

For $j \leq M+1$, $\widetilde{p}_{j+1} = \widetilde{p}_j + \pi\mathbf{h}_j$, so

$$\widetilde{q}_{j+1} \ = \ \widetilde{q}_j + \pi\,\mathrm{Arg}(\beta)(\mathbf{h}_j) \ = \ \widetilde{q} + \pi\mathbf{d}_j\,,$$

which we see by (3.3) (with the vector $\mathbf{w} = x + \tau\mathbf{v}$) and our definition of sgn_i. If we fix the orientation so that $\mathbf{v}$ is clockwise of $\mathbf{b}_{N+1}$, then by our choice of ordering of the zeroes ζ_j, the lines $\mathbf{R}\mathbf{d}_1, \ldots, \mathbf{R}\mathbf{d}_{M+1}$ occur in clockwise order. Since $\langle\mathbf{d}_j, \mathbf{w}\rangle > 0$ and $\widetilde{q}_1$ is extreme in the direction of $-\mathbf{w}$, the vectors $\pi\mathbf{d}_1, \ldots, \pi\mathbf{d}_{M+1}$ will form the edges of the zonotope starting at $\widetilde{q}_1$ and moving clockwise. It follows from the discussion following (3.3) that $\widetilde{q}_1, \ldots, \widetilde{q}_{2M+2}$ form the vertices of the zonotope Z_B. This implies that no $\widetilde{q}_j$ coincides with the origin $\mathbf{0}$.

All that remains is to understand the two triangles (3.8) for those j when $\widetilde{q}_j' \neq \widetilde{q}_j$. In this case, $\widetilde{p}_j' = \widetilde{p}_j + 2\pi\,\mathrm{sgn}_{m_j}\mathbf{g}_j$, and so

$$\widetilde{q}_j' \ = \ \widetilde{p}_j \ + \ 2\pi\,\mathrm{sgn}_{m_j} \sum_{i=m_j}^{n_j-1} \mathbf{b}_i \ = \ \widetilde{p}_j \ + \ 2\pi \sum_{i=m_j}^{n_j-1} \mathrm{sgn}_i\,\mathbf{b}_i\,.$$

Since $\mathbf{b}_{m_j}, \ldots, \mathbf{b}_{m_{j+1}-1}$ are parallel, $\widetilde{q}_j, \widetilde{q}_j'$, and $\widetilde{q}_{j+1}$ are collinear. This implies that
$$\mathrm{Arg}(\beta)_*[\mathrm{conv}(\mathbf{0}, \widetilde{p}_j', \widetilde{p}_j) + \mathrm{conv}(\mathbf{0}, \widetilde{p}_{j+1}, \widetilde{p}_j')] \;=\; [\mathrm{conv}(\mathbf{0}, \widetilde{q}_{j+1}, \widetilde{q}_j)]\,,$$
which shows that $\mathrm{Arg}(\beta)_*[Z(\ell_B)] = [Z_B]$.

Indeed, if $\widetilde{q}_j'$ lies between $\widetilde{q}_j$ and $\widetilde{q}_{j+1}$ then $\mathrm{Arg}(\beta)$ preserves the orientation of the triangles (3.7) and is therefore injective over their images, whose union is $\mathrm{conv}(\mathbf{0}, \widetilde{q}_{j+1}, \widetilde{q}_j)$. Otherwise, the two triangles (3.8) have opposite orientations and
$$\mathrm{conv}(\mathbf{0}, \widetilde{q}_j', \widetilde{q}_j) \;\supset\; \mathrm{conv}(\mathbf{0}, \widetilde{q}_{j+1}, \widetilde{q}_j')\,,$$
so that $\mathrm{Arg}(\beta)_*[\mathrm{conv}(\mathbf{0}, \widetilde{p}_j', \widetilde{p}_j) + \mathrm{conv}(\mathbf{0}, \widetilde{p}_{j+1}, \widetilde{p}_j')]$ equals
$$[\mathrm{conv}(\mathbf{0}, \widetilde{q}_j', \widetilde{q}_j)] - [\mathrm{conv}(\mathbf{0}, \widetilde{q}_{j+1}, \widetilde{q}_j')] \;=\; [\mathrm{conv}(\mathbf{0}, \widetilde{q}_{j+1}, \widetilde{q}_j)]\,.$$

Theorem 2.6, Equation (3.6), and $\mathrm{Arg}(\beta)_*(\mathbf{e}_i \wedge \mathbf{e}_j) = \mathbf{b}_i \wedge \mathbf{b}_j \cdot [\mathbf{T}^2]$, show that
$$\mathrm{Arg}(\beta)_*[\overline{\mathcal{A}(\ell_B)} + Z(\ell_B)] \;=\; [\mathbf{T}^2] \cdot \sum_{\substack{1 \le i < j \le N \\ \langle \mathbf{b}_i, \mathbf{w} \rangle > 0 > \langle \mathbf{b}_j, \mathbf{w} \rangle}} \mathbf{b}_i \wedge \mathbf{b}_j \,.$$

We will show that this equals $d_B[\mathbf{T}^2]$. Observe that if $\mathbf{b}_i$ and $\mathbf{b}_j$ are parallel, then $\mathbf{b}_i \wedge \mathbf{b}_j = 0$ and they do not contribute to the sum. We will consider the sum with the restriction that the vectors $\mathbf{b}_i$ and $\mathbf{b}_j$ are not parallel.

Set $\mathbf{w}^\perp := -\mathbf{b}_{N+1} + \mathbf{w}/\langle \mathbf{w}, \mathbf{w} \rangle$, which is orthogonal to $\mathbf{w}$. Suppose that $\mathbf{v}$ is clockwise of $\mathbf{b}_{N+1}$, as below.
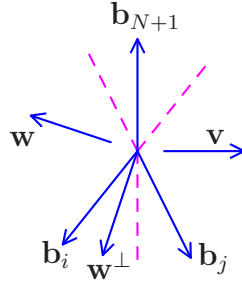
By our choice of $\mathbf{w}$, the lines $\mathbf{Rw}^\perp, \mathbf{Rb}_1, \ldots, \mathbf{Rb}_{N+1}$ occur in weak clockwise order with $\mathbf{Rw}^\perp$ distinct from the rest. Suppose now that $1 \le i < j \le N$ where

(3.9) $$\langle \mathbf{b}_i, \mathbf{w} \rangle \;>\; 0 \;>\; \langle \mathbf{b}_j, \mathbf{w} \rangle\,,$$

and $\mathbf{b}_i$ and $\mathbf{b}_j$ are not parallel. The cone spanned by $\mathbf{b}_i$ and $\mathbf{b}_j$ meets a half ray of $\mathbf{Rw}^\perp$, with $\mathbf{b}_i$ to the left of $\mathbf{Rw}^\perp$ and $\mathbf{b}_j$ to the right of $\mathbf{Rw}^\perp$, by (3.9). Since $\mathbf{Rw}^\perp, \mathbf{Rb}_i$, and $\mathbf{Rb}_j$ occur in clockwise order, we must have that $\mathbf{w}^\perp \in \mathrm{cone}(\mathbf{b}_i, \mathbf{b}_j)$, which shows that
$$\sum_{\substack{1 \le i < j \le N \\ \langle \mathbf{b}_i, \mathbf{w} \rangle < 0 < \langle \mathbf{b}_j, \mathbf{w} \rangle}} \mathbf{b}_i \wedge \mathbf{b}_j \;=\; \sum_{\substack{1 \le i < j \le N \\ \mathbf{w}^\perp \in \mathrm{cone}(\mathbf{b}_i, \mathbf{b}_j)}} \mathbf{b}_i \wedge \mathbf{b}_j \;=\; d_{B, \mathbf{w}^\perp} \;=\; d_B \,.$$

The sum equals $d_{B, \mathbf{w}^\perp}$ because if $\mathbf{b}_j$ is counter clockwise from $\mathbf{b}_i$ by (3.9) and the condition that $\mathbf{w}^\perp \in \mathrm{cone}(\mathbf{b}_i, \mathbf{b}_j)$ with $i < j$. Thus $\mathbf{b}_i \wedge \mathbf{b}_j > 0$.

We complete the proof by noting that $\widetilde{q}'_j$ will lie between $\widetilde{q}_j$ and $\widetilde{q}_{j+1}$ if either $n_j = m_{j+1}$, so that $\mathbf{g}_j = \mathbf{f}_j$, or if

$$\|\mathbf{g}_j\| \;=\; \|\sum_{i=m_j}^{n_j-1} \mathbf{b}_j\| \;=\; \sum_{i=m_j}^{n_j-1} \|\mathbf{b}_j\| \;\leq\; \sum_{i=n_j}^{m_{j+1}-1} \|\mathbf{b}_j\| \;=\; \|\mathbf{f}_j - \mathbf{g}_j\|\,,$$

as $\mathbf{b}_{m_j}, \ldots, \mathbf{b}_{n_j-1}$ have the same direction which is opposite to the (common) direction of $\mathbf{b}_{n_j}, \ldots, \mathbf{b}_{m_{j+1}-1}$. If this does not occur for our given order, then we simply reverse the vectors $\mathbf{b}_{m_j}, \ldots, \mathbf{b}_{m_{j+1}-1}$, replacing $\mathbf{g}_j$ with $\mathbf{f}_j - \mathbf{g}_j$.

Example 3.8. The last point in the proof about the injectivity of

$$\mathrm{Arg}(\beta) : \; Z(\ell_B) \;\longrightarrow\; Z_B$$

(and more generally the arguments when B has parallel vectors) is geometrically subtle. We expose this subtlety in the following two examples. Suppose that B consists of the vectors $(1,0), (0,1), (-2,-2)$, and $(1,1)$,
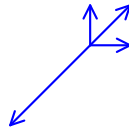
When $\mathbf{v} = (1,-1)$ and $x = (\frac{1}{2}, \frac{1}{2})$, then ℓ_B has the parametrization

(3.10) $$z \;\longmapsto\; [\tfrac{1}{2} + z \; : \; \tfrac{1}{2} - z \; : \; -2 \; : \; 1]\,,$$

which is the second line in our running Examples 2.3, 2.5, and 2.7. In this case the image $\mathrm{Arg}(\beta)(Z(\ell_B))$ is shown on the left of Figure 6. It is superimposed over a fundamental domain and dashed lines $\theta_1, \theta_2 = n\pi$ for $n \in \mathbf{Z}$. The segments $\widetilde{q}_3, \widetilde{q}'_2$ and $\widetilde{q}_6, \widetilde{q}'_5$ are covered in both directions as $\mathrm{Arg}(\beta)(P(\ell_B))$ backtracks over these segments. In fact, the triangles

$$\mathrm{conv}(\mathbf{0}, \widetilde{q}_3, \widetilde{q}'_2) \qquad \text{and} \qquad \mathrm{conv}(\mathbf{0}, \widetilde{q}_6, \widetilde{q}'_5)$$

have orientation opposite of the other triangles. The medium shaded parts (near $\widetilde{q}'_2$ and $\widetilde{q}'_5$) are covered twice and the darker shaded parts near $\mathbf{0}$ are covered thrice.
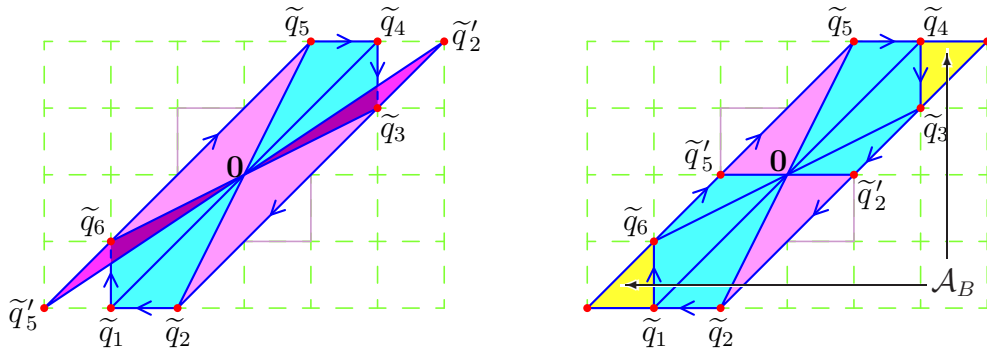

FIGURE 6. Images of $\mathrm{Arg}(\beta)(Z(\ell_B))$

Now suppose that the vectors in B are in the order $(1,0)$, $(1,1)$, $(-2,-2)$, and $(0,1)$, and $v = (-1,0)$ and $x = (0,1)$. Then ℓ_B is parametrized by

$$z \longmapsto \left[-z \,:\, 1-z \,:\, 2z-2 \,:\, 1\right],$$

In this case the image $\mathrm{Arg}(\beta)(Z(\ell_B))$ is equal to the zonotope Z_B, and is shown on the right of Figure 6, together with the coamoeba $\mathcal{A}_B$. As explained in the proof of Theorem 3.7, the image equals the zonotope because in the pair of parallel vectors $(1,1)$ and $(-2,-2)$, the shorter comes first in this case, while in the previous case, the shorter one came second.

In both cases (which are just different parametrizations of the same line) $\mathrm{Arg}(\beta)_*[Z(\ell_B)] = [Z_B]$ as shown in the proof of Theorem 3.7, and the coamoebas coincide. Furthermore, $[\overline{\mathcal{A}_B + Z_B}] = 2[\mathbf{T}^2]$ for both, as $d_B = 2$.

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