Bull. Math. Sci. (2013) 3:73–131
DOI 10.1007/s13373-013-0031-5

Amenable groups without finitely presented amenable covers

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Received: 10 June 2012 / Revised: 25 June 2012 / Accepted: 2 February 2013 /
Published online: 1 May 2013

Abstract The goal of this article is to study results and examples concerning finitely presented covers of finitely generated amenable groups. We collect examples of groups G with the following properties: (i) G is finitely generated, (ii) G is amenable, e.g. of intermediate growth, (iii) any finitely presented group with a quotient isomorphic to G contains non-abelian free subgroups, or the stronger (iii') any finitely presented group with a quotient isomorphic to G is large.

Keywords Finitely generated groups · Finitely presented covers · Self-similar groups · Metabelian groups · Soluble groups · Groups of intermediate growth · Amenable groups · Groups with non-abelian free subgroups · Large groups

Mathematics Subject Classification (2000) 20E08 · 20E22 · 20F05 · 43A07

Communicated by Dr. Efim Zelmanov.

We are grateful to the Swiss National Science Foundation for supporting a visit of the first author to Geneva. The first author is grateful to the department of mathematics, University of Geneva, for their hospitality during his stay. The first and the second authors are partially supported by NSF grant DMS – 1207699. The second author was supported by ERC starting grant GA 257110 "RaWG".

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1 Introduction

1.A. Motivation In the study of finiteness conditions on groups, the following type of question is natural:

Question 1.1 *Given a Property $(\mathcal{P})$ of groups, does any finitely generated group with $(\mathcal{P})$ have a finitely presented cover with $(\mathcal{P})$?*

A **cover** of a group G is a group E given together with an epimorphism $E \twoheadrightarrow G$. A *cover* fits into an *extension* $1 \to N \to E \twoheadrightarrow G \to 1$.

The answer can be positive for trivial reasons, for example when Property $(\mathcal{P})$ holds for free groups (such as exponential growth) or when Property $(\mathcal{P})$ implies finite presentation (such as nilpotency, or polynomial growth).The answer is also positive when $(\mathcal{P})$ is Kazhdan's Property $(\mathcal{T})$, by a non-trivial result of Shalom [137].

Here, we concentrate on Question 1.1 for amenability, a case in which the answer is negative. Indeed, the goal of this article is to study examples and results concerning *finitely generated amenable groups that do not have finitely presented amenable covers.*

Our motivation is better to understand the class of *finitely presented amenable groups*, and related groups. To describe the situation, it is convenient to introduce the class $\mathcal{AG}$ of amenable groups, and the subclass $\mathcal{EG}$ of elementary amenable groups, which is easier to understand (Appendix D is a reminder on these classes, and on growth).

On the one hand, many examples are known in $\mathcal{EG}$ of finitely presented soluble groups, including the standard examples described in the exposition [145], the Baumslag–Remeslennikov group, which is a finitely presented metabelian group with derived group free abelian of infinite rank (references in our Remark A.4.c), and the Kharlampovich group, which is a finitely presented soluble group with unsoluble word problem (references in the proof of Proposition A.5). There are also finitely presented groups in $\mathcal{EG}$ which are not virtually soluble (Example D.1).

In the complement $\mathcal{AG} \smallsetminus \mathcal{EG}$, a reasonable number of *finitely generated* examples are known, in particular the *groups of intermediate growth*. Also known are *finitely presented* examples, all closely related to self-similar groups. But not enough examples are known to allow one to guess at a general picture; this is illustrated by the following old and basic problem in the theory of growth of groups (see, e.g., [105], Problems 4.5.b, 9.8, and 10.11):

Basic problem 1.2 *Does there exist a finitely presented group of intermediate growth?*

The answer does not seem to be at hand. Here is a possibly easier problem:

Problem 1.3 *Does there exist a finitely generated group of intermediate growth that is the quotient of a finitely presented group without non-abelian free subgroups? or the quotient of a finitely presented amenable group?*

Finitely generated elementary amenable groups are never of intermediate growth [48], so that Problems 1.2 and 1.3 do not arise in $\mathcal{EG}$. Problem 1.3 involves *two* questions, because there exist finitely presented groups that are non-amenable and do not contain non-abelian free subgroups [126].

Strictly speaking, the solution of the analogue of Problem 1.2 is known for the class $\mathcal{AG} \smallsetminus \mathcal{EG}$. Indeed, one knows a few sporadic examples of finitely presented groups in this class, such as the HNN-extension $\widehat{\mathfrak{G}}$ of the Grigorchuk group [70], and the HNN-extension of the Brunner–Sidki–Vieira group that appears in [81, Proposition 6] (the latter group is amenable by [13] and not elementary amenable by [81]). The finitely presented HNN-extension of the basilica group that appears in [81, Theorem 1.7] is not even in the class $\mathcal{AG} \smallsetminus \mathcal{SG}$, defined in Appendix D; see also [10, Theorem 12]. But, despite these examples, very little is known about finitely presented groups in $\mathcal{AG} \smallsetminus \mathcal{EG}$, and it would be interesting to find methods providing new examples; a priori, one could hope and try finitely presented covers, but our paper shows that this does not look very promising.

Before stating the next problem, here is a definition: the **elementary amenable radical** $\mathrm{Rad}_{\mathrm{ea}}(G)$ of a group G is its unique maximal normal elementary amenable subgroup. Note that $\mathrm{Rad}_{\mathrm{ea}}(G)$ is contained in the amenable radical of G, that appears (but for its name) in [56, Lemma 1 of Section 4].

Problem 1.4 *Is there a finitely presented group G in $\mathcal{AG} \smallsetminus \mathcal{EG}$, with* $\mathrm{Rad}_{\mathrm{ea}}(G) = \{1\}$, *that has infinitely presented*[1] *quotients? or that has uncountably many pairwise non-isomorphic quotients?*

1.B. First examples When $(\mathcal{P})$ is amenability, we know two approaches to Question 1.1: One is based on the theory of Σ-invariants, developed in a series of papers by Bieri, Strebel, and Neumann (see Appendix C). The other involves self-similar groups and approximation methods (see Sections 2, 3).

The first result we quote is due to Bieri and Strebel [31, Theorem 5.5 and Corollary 5.6]. On the one hand, we reformulate it in a slightly weaker version, assuming that E is finitely presented (instead of assuming that E has Property FP_2, as in the original paper); on the other hand, we formulate it for *virtually* metabelian groups, because this follows immediately from the case of metabelian groups.

Theorem 1.5 (Bieri-Strebel) *Let G be a virtually metabelian group that is finitely generated and infinitely presented.*

Any finitely presented cover E of G has non-abelian free subgroups. In particular, E is non-soluble, indeed non-amenable.

The proof of Theorem 1.5 can now be understood in a much simpler way than in [31], as we indicate in Corollary C.4. The reason is that we can use better invariants, namely those defined in [32] and their later reformulations (see Strebel's exposition in progress [146]).

Examples of metabelian groups that are finitely generated and infinitely presented include matrix groups like $\begin{pmatrix} \left(\frac{\ell}{m}\right)^{\mathbf{Z}} & \mathbf{Z}\left[\frac{1}{\ell m}\right] \\ 0 & 1 \end{pmatrix}$ where $\ell, m \geq 2$ are coprime integers, wreath products $A \wr \mathbf{Z}$ where $A \neq \{1\}$ is a finitely generated abelian group, and free metabelian groups $\mathrm{FSol}(k, 2) := F_k/[[F_k, F_k], [F_k, F_k]]$, where F_k is the free group on $k \geq 2$ generators.

[1] A finitely generated group is "infinitely presented" if it is not finitely presented.

Appendix A includes a reminder on metabelian groups. Appendix B is a reminder on wreath products and lamplighter groups. Appendix C is a reminder on Bieri–Neumann–Strebel invariants. Proposition B.2, Corollary B.3, Proposition B.9, Corollary C.4, and Corollary C.6 give examples of finitely generated groups of which all finitely presented covers have non-abelian free subgroups.

We denote by $\mathfrak{G}$ the *Grigorchuk group*,[2] introduced in [66]; see Example 2.16. Recall here that $\mathfrak{G}$ is finitely generated, indeed generated by four involutions traditionally written a, b, c, d. This group has many remarkable properties, including that of being an infinite 2-group of intermediate growth; in particular it is amenable. The group $\mathfrak{G}$ has a presentation, due to Lysenok, with four generators and infinitely many relators. Appropriate finite subsets of these relators naturally define a sequence $(\mathfrak{G}_n)_{n\geq 0}$ of four-generated finitely presented groups converging to $\mathfrak{G}$ (Definition 5.2). There is a reminder on convergence of groups in Section 3.

Theorem 1.6 [78] *Any finitely presented cover of the group $\mathfrak{G}$ is large.*

Recall that a group is **large** if it contains a subgroup of finite index that has non-abelian free quotients. Note that large groups have non-abelian free subgroups.

In view of Proposition 3.3 below, Theorem 1.6 is a straightforward consequence of the main result of [78]; see also Theorem 2.14 and Example 2.16. More precisely, it was shown in [78] that each $\mathfrak{G}_n$ is virtually a direct product of finitely generated non-abelian free groups, this has been sharpened in [11], and can be further improved:

Theorem 1.7 (Section 5) *Let $\mathfrak{G}$ and $\mathfrak{G}_n$ be as above. For each $n \geq 0$, the group $\mathfrak{G}_n$ has a normal subgroup H_n of index $2^{2^{n+1}+2}$ that is isomorphic to the direct product of 2^n free groups of rank 3.*

Another way of sharpening Theorem 1.6 is given in Proposition 5.11, where the condition of finite presentability for the cover of $\mathfrak{G}$ is replaced by the weaker Condition FP_2.

We denote by $\mathfrak{B}$ the Basilica group. Recall here that $\mathfrak{B}$ can be generated by two elements, and is an amenable torsion-free group of exponential growth. See Example 2.17 and Appendix D for some other properties of $\mathfrak{B}$.

Theorem 1.8 (Erschler) *Any finitely presented cover of the Basilica group $\mathfrak{B}$ has non-abelian free subgroups.*

Given an invertible automaton (A, τ) over a finite alphabet X, Erschler introduces a notion of "automatically presented group $G^*(A, \tau)$ generated by the automaton" (see [60] for details). She shows that, if $G^*(A, \tau)$ is not virtually abelian, then any finitely presented cover of it has non-abelian free subgroups. For the Basilica automaton, the group $G^*(A, \tau)$ coincides with $\mathfrak{B}$, and Theorem 1.8 follows.

As noted in [60], these arguments do not apply to the group $\mathfrak{G}$ of Theorem 1.6; this is due to the fact that, for (A, τ) the automaton of $\mathfrak{G}$, the cover $G^*(A, \tau) \twoheadrightarrow \mathfrak{G}$ has a non-trivial kernel.

In our setting, for $G = \mathfrak{B}$, the universal contracting cover G_0 of Definition 2.5 is free of rank 2 (see Example 2.17), and Theorem 1.8 follows from our Theorem 2.14. Indeed, our argument shows that *any finitely presented cover of $\mathfrak{B}$ is large.*

[2] Two of the authors insist that we mention this terminology, often used today. The third author submits.

Notation: In this paper, the symbols $\mathfrak{G}$ and $\mathfrak{B}$ will be used *only* for the two groups introduced above. In Examples 2.18 to 2.21, we introduce four other groups, with their usual notation in this subject: the iterated monodromy group of $z^2 + i$ denoted by $\mathfrak{I}$, the Gupta–Sidki group $\mathfrak{G}\mathfrak{S}$, the Fabrykowski–Gupta group $\mathfrak{F}\mathfrak{G}$, and the Hanoi Towers group $\mathfrak{H}$.

We show in Section 2 that Theorem 2.14 implies Theorems 1.6 on $\mathfrak{G}$, 1.8 on $\mathfrak{B}$, and 2.22 on $\mathfrak{I}$, $\mathfrak{G}\mathfrak{S}$, $\mathfrak{F}\mathfrak{G}$, and $\mathfrak{H}$. We state now a shorthand version of 2.14. Notation and technical terms are defined in Section 2.

Theorem 1.9 *Let G be an infinite finitely generated self-similar group. Assume that G is contracting faithful self-replicating. Let G_0 denote a standard contracting cover of G, as in Definition* 2.8.

If G_0 has non-abelian free subgroups, then so does any finitely presented cover of G.

If G_0 is large, any finitely presented cover of G is large.

1.C. Infinitely more examples The group $\mathfrak{G}$ has uncountably many[3] relatives G_ω introduced in [68]. Each of these groups is generated by a set S_ω of four involutions. They are parameterized by the space $\Omega = \{0, 1, 2\}^{\mathbf{N}}$. We have $\mathfrak{G} = G_\omega$ when ω is the 3-periodic sequence $012012012\cdots$. Let Ω_+ denote the complement in Ω of the space of eventually constant sequences. Section 4 contains a description of the family $(G_\omega)_{\omega\in\Omega_+}$.

Theorem 1.10 *For $\omega \in \Omega_+$, the group G_ω is of intermediate growth, and any finitely presented cover of G_ω is large.*

Note also the following straightforward consequence of Theorems 1.6, 1.8, and 1.10:

Corollary 1.11 *Let H be a finitely generated cover of one of $\mathfrak{G}$ (as in Theorem* 1.6*), $\mathfrak{B}$ (as in Theorem* 1.8*), or G_ω, $\omega \in \Omega_+$ (as in Theorem* 1.10*). Any finitely presented cover of H is large.*

There are several interesting classes of groups that qualify to be the H of Corollary 1.11:

(i) The uncountably many groups of [59], which are finitely generated, of intermediate growth, and not residually finite, each one being a *central* cover of $\mathfrak{G}$.
(ii) The groups of [14], which are finitely generated groups of intermediate growth, with exactly known growth functions, each one being a cover of $\mathfrak{G}$.
(iii) Permutational wreath products of the form $A \wr_X G_\omega$, where $A \neq \{1\}$ is a finite group and G_ω is as in Theorem 1.10 [15].

There is an uncountable family of finitely generated amenable simple groups, which are topological full groups of minimal homeomorphisms of the Cantor space [100]. None of these groups is finitely presented (see [114, Theorem 5.7], as well as [86]). In our context, it is natural to formulate:

[3] Here and elsewhere, "uncountably many" groups means "uncountably many *pairwise non-isomorphic*" groups.

Problem 1.12 *Let G be one of the finitely generated amenable simple groups that appears in* [100]. *Does G have an amenable finitely presented cover?*

Remark 1.13 We state two kinds of results concerning appropriate covers: some establish that the covers are large (see Theorem 1.6), others, weaker, that they contain non-abelian free groups (see Theorem 1.5). Strong statements do not hold in all cases.

For example, let $G = \mathrm{Met}(k, \ell) \simeq \mathbf{Z}\left[\frac{1}{\ell m}\right] \rtimes_{\ell/m} \mathbf{Z}$, for two coprime integers k, $\ell \geq 2$ (Definition B.7). Then G satisfies the hypothesis of Theorem 1.5. The Baumslag–Solitar group $\mathrm{BS}(\ell, m)$ is a finitely presented cover of G; it is known that $\mathrm{BS}(\ell, m)$ has non-abelian free subgroups, but is not large (Proposition B.6.iv).

1.D. A dual to Question 1.1 Given a Property $(\mathcal{P})$ of groups, it is standard to ask whether any countable group with $(\mathcal{P})$ is a *subgroup* of some finitely generated group with $(\mathcal{P})$. For example, the answer is known to be positive if $(\mathcal{P})$ is solubility [122], or amenability, or elementary amenability [128, Corollary 1.3]. An earlier result of [96] also gives a positive answer to a similar question: for a given integer n, if G is a group that has a presentation with a countable number (possibly infinite) of generators and n relators, then G can be embedded in a finitely presented group with 2 generators and n relators. But the answer is negative if $(\mathcal{P})$ is nilpotency, because any subgroup of a finitely generated nilpotent group is finitely generated. It is also negative if $(\mathcal{P})$ is metabelianity: indeed, for any prime p, the abelian quasi-cyclic group $\mathbf{Z}(p^{\infty})$ cannot be embedded in a finitely generated metabelian group [122, Lemma 5.3].

As a digression from our main theme, and since recursively presented groups are mentioned in Appendix A, we formulate one more question, which is in some sense dual to Question 1.1.

Question 1.14 *Given a Property $(\mathcal{P})$ of groups, is any finitely generated recursively presented group with $(\mathcal{P})$ a subgroup of some finitely presented group with $(\mathcal{P})$?*

The answer to Question 1.14 is known to be positive if $(\mathcal{P})$ is metabelianity (Proposition A.2), or solubility of the word problem ([49], and also [117, Theorem 2.8]). The answer is not known if $(\mathcal{P})$ is amenability [128, Problem 1.7]. NP

1.E. Plan of the paper Section 2: Non-abelian free subgroups of finitely presented covers of contracting self-similar groups. Proofs of Theorems 1.9 and 2.14; proofs of Theorems 1.6, 1.8, and their analogues for $\mathfrak{J}$, $\mathfrak{G}\mathfrak{S}$, $\mathfrak{F}\mathfrak{G}$, and $\mathfrak{H}$.

Section 3: Marked groups and the Chabauty topology, a reminder.

Section 4: The analogue of Theorem 1.6 for the family $(G_{\omega})_{\omega \in \Omega}$ of [68], Theorem 1.10 and its proof.

Section 5: The group of intermediate growth $\mathfrak{G}$, and the proof of Theorem 1.7; this is a quantitative sharpening of Theorem 1.6. Proposition 5.11 on FP_2-covers of $\mathfrak{G}$.

- Appendix A: On soluble groups, metabelian groups, and finite presentations.
- Appendix B: On wreath products and lamplighter groups.
- Appendix C: On Bieri–Neumann–Strebel invariants.
- Appendix D: On growth and amenability.

2 Non-abelian free subgroups of finitely presented covers of contracting self-similar groups

Let G, H be two groups and X a H-set; here, this means that H acts on X *from the right*. The corresponding **permutational wreath product** is the semi-direct product

$$G \wr_X H := G^{(X)} \rtimes H.$$

We denote by $G^{(X)}$ the group of functions $(g_x)_{x\in X} : X \longrightarrow G, x \longmapsto g_x$ with finite support, and we consider the action *from the left* of H on $G^{(X)}$, for which the action of h on $(g_x)_{x\in X}$ is $(g_{xh})_{x\in X}$. Hence the product of two elements in $G \wr_X H$ is given by

$$((g_x)_{x\in X}, h)\left((g'_x)_{x\in X}, h'\right) = \left((g_x g'_{xh})_{x\in X}, hh'\right).$$

In case G acts *from the right* on some set W, the group $G \wr_X H$ acts naturally *from the right* on $W \times X$, by

$$(w, y)\,((g_x)_{x\in X}, h) = (wg_y, yh).$$

In expositions of wreath products and self-similar groups, choices of which groups actions are from the left and which from the right vary from one paper to the other, but a (sometimes hidden) mixture of left actions and right actions seems unavoidable.

If G, G' are two groups and $\alpha : G \longrightarrow G'$ a homomorphism, we have a natural homomorphism

$$\alpha \wr_X 1_H : G \wr_X H \longrightarrow G' \wr_X H, ((g_x)_{x\in X}, h) \longmapsto ((\alpha(g_x))_{x\in X}, h), \qquad (1)$$

where 1_H stands for the identity automorphism of H.

If X is clear from the context, we write "$\wr$" for "$\wr_X$". In particular, with $X = \{0, 1, \ldots, d-1\}$ and S_d the full symmetric group of X, we write $G \wr S_d$ for $G \wr_X S_d$. Also, we write 1_d for 1_{S_d}, and

$$((g_x)_{x\in X}, \tau) = (g_0, \ldots, g_{d-1}, \tau) \quad \text{with} \quad g_0, \ldots, g_{d-1} \in G \quad \text{and} \quad \tau \in S_d$$

for a typical element of $G \wr S_d = G^X \rtimes S_d$.

The **iterated wreath products** with S_d are defined inductively, for each integer $n \geq 0$, by

$$G \wr^n S_d = \begin{cases} G & \text{for } n = 0 \\ \left(G \wr^{n-1} S_d\right) \wr S_d & \text{for } n \geq 1. \end{cases}$$

We have the following **associativity of permutational wreath products**: for a H-set X and a K-set Y, the canonical mapping

$$\begin{cases} (G \wr_X H) \wr_Y K & \longrightarrow G \wr_{X\times Y} (H \wr_Y K) \\ \left(\left((g_{x,y})_{x\in X}, h_y\right)_{y\in Y}, k\right) & \longmapsto \left(\left(g_{x,y}\right)_{(x,y)\in X\times Y}, \left((h_y)_{y\in Y}, k\right)\right) \end{cases}$$

is an isomorphism of groups (this is standard, see e.g. [115, Chapter 1, Theorem 3.2]). In particular, for $n \geq 1$, we have

$$G \wr^n S_d = \left(G \wr^{n-1} S_d\right) \wr S_d = (G \wr S_d) \wr^{n-1} S_d = G^{X^n} \rtimes S_d^{(n)},$$

where $S_d^{(n)} = (\cdots(S_d \wr S_d) \wr \cdots) \wr S_d = S_d^{(n-1)} \wr S_d$ is the appropriate permutation group of X^n, acting here *from the right*. We write

$$((g_v)_{v \in X^n}, \tau) \quad \text{with} \quad g_v \in G \quad \text{for all} \quad v \in X^n \quad \text{and} \quad \tau \in S_d^{(n)}$$

for a typical element of $G \wr^n S_d$.

Definition 2.1 Let G be a group and $d \geq 2$ an integer. A **self-similar structure of degree** d on G is a homomorphism

$$\Phi : G \longrightarrow G \wr S_d. \tag{2}$$

A **self-similar group** is such a pair (G, Φ); when Φ is clear from the context, we write also "G is a self-similar group".

If (G, Φ) is a self-similar group, the construction (1) gives rise to a sequence of homomorphisms

$$\Phi_n : G \overset{\Phi_{n-1}}{\longrightarrow} G \wr^{n-1} S_d \overset{\Phi \wr 1_{d^{n-1}}}{\longrightarrow} (G \wr S_d) \wr^{n-1} S_d = G \wr^n S_d \tag{3}$$

for $n \geq 2$; we write $\Phi_0 = \mathrm{id}_G$ and $\Phi_1 = \Phi$. Note that, if Φ is injective, so is Φ_n for all $n \geq 0$. It is routine to check that Φ_{m+n} is the composition

$$\Phi_{m+n} : G \overset{\Phi_n}{\longrightarrow} G \wr^n S_d \overset{\Phi_m \wr 1_{d^n}}{\longrightarrow} \left(G \wr^m S_d\right) \wr^n S_d = G \wr^{m+n} S_d \tag{4}$$

for all $m, n \geq 0$.

The composition of Φ_n and the quotient map $G \wr^n S_d \longrightarrow S_d^{(n)}$ is a homomorphism

$$G \longrightarrow S_d^{(n)}, g \longmapsto \tau_g^{(n)}. \tag{5}$$

Thus, introducing the v-coordinates of $\Phi_n(\cdot)$, we have

$$\Phi_n(g) = \left((g_v)_{v \in X^n}, \tau_g^{(n)}\right) \in G \wr_{X^n} S_d^{(n)} = G \wr^n S_d$$

for all $g \in G$. Note that

$$\tau_g^{(n)} = \left(\left(\tau_{g_x}^{(n-1)}\right)_{x \in X}, \tau_g^{(1)}\right) \in S_d^{(n)} \tag{6}$$

for all $g \in G$ and $n \geq 1$.

Let $X^* = \bigsqcup_{n \geq 0} X^n$ be the free monoid over $X = \{0, 1, \dots, d-1\}$. The **length** of $v \in X^*$ is the integer $n = |v|$ such that $v \in X^n$. The d-**regular rooted tree** is the tree with vertex set X^*, and with edges connecting pairs of vertices of the form $(x_1 \cdots x_n, x_1 \cdots x_n x_{n+1})$, with $n \geq 0$ and $x_1, \dots, x_{n+1} \in X$; abusively, we denote this tree also by X^*. The homomorphisms of (5) define an action *from the right* of G on the tree X^*.

For $n \geq 1$ and $v \in X^n$, we define the **stabilizer of** v to be the subgroup

$$\mathrm{Stab}_G(v) = \left\{ g \in G \mid v\tau_g^{(n)} = v \right\}, \tag{7}$$

and we have a homomorphism

$$\Phi_v : \mathrm{Stab}_G(v) \longrightarrow G, g \longmapsto g_v \tag{8}$$

where $g_v = \Phi_v(g)$ is the v-coordinate of $\Phi_n(g)$.

Lemma 2.2 *With the notation above,*

$$g_{vw} = (g_v)_w, (gh)_v = g_v h_{v\tau_g^{(n)}}, \quad \textit{and} \quad (h^{-1})_v = \left(h_{v\tau_{h^{-1}}^{(n)}} \right)^{-1},$$

for all $g, h \in G$, $n \geq 1$, $v \in X^n$, and $w \in X^*$.

Proof To illustrate the fact that G acts on X^* *from the right*, we spell out the proof of the second equality, writing vg for $v\tau_g^{(n)}$. We have on the one hand

$$((vw)g)\,h = ((vg)(wg_v))\,h = (vgh)(wg_v h_{vg})$$

and on the other hand

$$(vw)(gh) = (vgh)\,(w(gh)_v))\,.$$

Hence $(gh)_v = g_v h_{vg}$. □

Definition 2.3 A self-similar group (G, Φ) is **faithful** if its action on the tree X^* described above is faithful; this implies that the homomorphism Φ is injective (but the converse does not hold[4]).

A self-similar group (G, Φ) is **contracting** if there is a finite subset $\mathcal{M} \subset G$ with the following property:

for all $g \in G$, there exists an integer $k \geq 0$
such that $g_v \in \mathcal{M}$ for all $v \in X^*$ with $|v| \geq k$.

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4 Set $H = \Phi^{-1}(G^X)$. The kernel of the action of G on X^* is the largest normal subgroup N of G that is contained in H and such that $\Phi(N) \subset N^X$.

The smallest such $\mathcal{M}$, namely

$$\mathcal{N} := \bigcup_{g \in G} \bigcap_{k \geq 0} \{g \in G \mid \exists h \in G, \ell \geq k, v \in X^\ell \quad \text{with} \quad h_v = g\}$$

is called the **nucleus** of (G, Φ).

A self-similar group (G, Φ) is **self-replicating**[5] if, for all $g \in G$ and $x \in X$, there exists $h \in \mathrm{Stab}_G(x)$ such that $h_x = g$, namely if, for all $x \in X$, the homomorphism Φ_x of (8) is onto. When this is so, it is easy to check by induction on the level that, for all $g \in G$, $n \geq 1$, and $v \in X^n$, there exists $h \in \mathrm{Stab}_G(v)$ such that $\Phi_v(h) = g$, namely Φ_v is onto.

Observe that, by definition, we have $1 \in \mathcal{N}$. Moreover for $g \in G$, we have $g \in \mathcal{N}$ if and only if $g^{-1} \in \mathcal{N}$, by Lemma 2.2.

The following proposition records basic facts about the nucleus of a contracting group.

Proposition 2.4 *Let (G, Φ) be a contracting self-similar group with nucleus $\mathcal{N}$, as above.*

(i) *For $g \in \mathcal{N}$ and $x \in X$, we have $g_x \in \mathcal{N}$.*
(ii) *If (G, Φ) is self-replicating and G is finitely generated, then $\mathcal{N}$ generates G.*

Proof For $g \in \mathcal{N}$, there exist $h \in G$, $k \geq 0$, and $v \in X^k$ such that $h_v = g$ and $h_w \in \mathcal{N}$ for all $w \in X^*$ with $|w| \geq k$ (otherwise, $\mathcal{N}$ would not be minimal). Hence $g_x = (h_v)_x = h_{vx} \in \mathcal{N}$ for all $x \in X$. This proves (i).

For (ii), we paraphrase [118, Lemma 2.11.3]. Denote by $\langle \mathcal{N} \rangle$ the subgroup of G generated by $\mathcal{N}$. Let S be a symmetric finite generating set of G. For all $s \in S$, there exists $k_s \geq 0$ such that $s_v \in \mathcal{N}$ for all $v \in X^*$ with $|v| \geq k_s$. Set $k = \max\{k_s \mid s \in S\}$.

Let $g \in G$ and $v \in X^*$ with $|v| \geq k$. There exist $s_1 \dots, s_m \in S$ with $g = s_1 \cdots s_m$, so that

$$\begin{aligned} g_v &= (s_1)_v (s_2 \cdots s_m)_{vs_1} = \cdots \\ &= (s_1)_v (s_2)_{vs_1} (s_3)_{vs_1 s_2} \cdots (s_m)_{vs_1 \cdots s_{m-1}} \in \langle N \rangle, \end{aligned}$$

where the last inclusion follows from $|v| = |vs_1| = \cdots |vs_1 \cdots s_{m-1}| \geq k$. In particular, the image of Φ_v, as defined in (8), is contained in $\langle \mathcal{N} \rangle$.

If (G, Φ) is self replicating, then Φ_v is onto for all $v \in X^*$ with $|v| \geq 1$. The conclusion follows. □

The next proposition was inspired to us by [118, Lemma 2.13.4]. We need some notation and a definition; our exposition borrows from [7].

Definition 2.5 Let (G, Φ) be a self-replicating contracting self-similar group, with nucleus $\mathcal{N} = \{n_1, \dots, n_\ell\}$. Let $S = \{s_1, \dots, s_\ell\}$ be a finite set given with a bijection

[5] Or **recurrent**, or **fractal**, as in [118, Definition 2.8.1]. See [9] for relations of such groups with fractal sets.

$s_j \leftrightarrow n_j$ with $\mathcal{N}$. Let R be the set of relators in the letters of S of one the forms

$$\begin{array}{ll} s_i = 1 & \text{if } n_i = 1 \in G, \\ s_i s_j = 1 & \text{if } n_i n_j = 1 \in G, \\ s_i s_j s_k = 1 & \text{if } n_i n_j n_k = 1 \in G. \end{array}$$

Note that these relators are of length at most 3; they are indexed by a subset of $\mathcal{N} \sqcup \mathcal{N}^2 \sqcup \mathcal{N}^3$.

Assume furthermore that G is finitely generated. The **universal contracting cover** of G is the finitely presented group G_0^{un} defined by the presentation with S as set of generators and R as set of relators. The assignment $\pi^{\text{un}}(s_i) = n_i$ extends to a group homomorphism

$$\pi^{\text{un}} : G_0^{\text{un}} = \langle S \mid R \rangle \longrightarrow G, \tag{9}$$

because $\pi^{\text{un}}(r) = 1$ for any $r \in R$.

Note that π^{un} is onto, by Proposition 2.4. We define finally

$$\widehat{\pi}^{\text{un}} = \pi^{\text{un}} \wr 1_d : G_0^{\text{un}} \wr S_d \longrightarrow G \wr S_d. \tag{10}$$

Remark 2.6 In particular examples, and for simplicity, it is often convenient to delete from S the generator corresponding to $1 \in \mathcal{N}$, to delete s_k if there exist $i, j \in \{1, \ldots, \ell\}$ with $n_k = n_i n_j$, and to delete one generator of every pair corresponding to $\{n, n^{-1}\} \subset \mathcal{N}$. For example, in Example 2.17, we have $\mathcal{N} = \{1, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$ with 7 elements, and $c = a^{-1}b$, but $S = \{a, b\}$ with 2 elements.

Note however that, in Example 2.16, we keep d in the generating set $\{a, b, c, d\}$ of G_0, even though $d = bc$.

Proposition 2.7 *Let (G, Φ) be a self-replicating contracting self-similar group of degree d, with nucleus $\mathcal{N}$. Assume that G is finitely generated. Let $G_0^{\text{un}} = \langle S \mid R \rangle$ and $\pi^{\text{un}} : G_0^{\text{un}} \twoheadrightarrow G$ be the universal contracting cover and the projection of Definition 2.5.*

Then there exists a homomorphism

$$\varphi_1^{\text{un}} : G_0^{\text{un}} \longrightarrow G_0^{\text{un}} \wr S_d$$

such that the self-similar group $(G_0^{\text{un}}, \varphi_1^{\text{un}})$ is contracting, with nucleus S. Moreover the diagram

$$\begin{array}{ccc} G_0^{\text{un}} & \xrightarrow{\varphi_1^{\text{un}}} & G_0^{\text{un}} \wr S_d \\ \pi^{\text{un}} \downarrow & & \downarrow \widehat{\pi}^{\text{un}} \\ G & \xrightarrow{\Phi} & G \wr S_d \end{array} \tag{11}$$

commutes.

Proof Step 1, definition of φ_1^{un}. Denote by ℓ the order of $\mathcal{N}$, and write $\mathcal{N} = \{n_1, \dots, n_\ell\}$, as above. Let $i \in \{1, \dots, \ell\}$. By Proposition 2.4, there exist $i_0, \dots, i_{d-1} \in \{1, \dots, \ell\}$ and $\tau_i \in S_d$ such that

$$\Phi(n_i) = (n_{i_0}, \dots, n_{i_{d-1}}, \tau_i).$$

We set

$$\varphi_1^{\mathrm{un}}(s_i) = (s_{i_0}, \dots, s_{i_{d-1}}, \tau_i) \in G_0^{\mathrm{un}} \wr S_d,$$

and we claim that this extends to a group homomorphism φ_1^{un} as in (11).

Consider a relator as in Definition 2.5, say $s_i s_j s_k = 1$ (shorter relators are dealt with similarly); hence $n_i n_j n_k = 1 \in G$. Choose $x \in X$; recall that X stands for $\{0, \dots, d-1\}$. There exist $p, q, r \in \{1, \dots, \ell\}$ and $\tau_p, \tau_q, \tau_r \in S_d$ such that the x-coordinate and the last coordinate of $\Phi(n_i n_j n_k)$ can be written as

$$\left(n_i n_j n_k\right)_x = n_p n_q n_r \quad \text{and} \quad \tau_{n_i n_j n_k} = \tau_p \tau_q \tau_r.$$

Since $n_i n_j n_k = 1 \in G$, we have

$$n_p n_q n_r = 1 \in G \; \forall x \in X \quad \text{and} \quad \tau_p \tau_q \tau_r = 1 \in S_d.$$

Hence $\varphi_1^{\mathrm{un}}(s_i)\varphi_1^{\mathrm{un}}(s_j)\varphi_1^{\mathrm{un}}(s_k) = 1 \in G_0^{\mathrm{un}}$. The claim is proven.

Step 2: $(G_0^{\mathrm{un}}, \varphi_1^{\mathrm{un}})$ *is a contracting group with nucleus* S. For any word w in the letters of S, we have to show that there exists a vertex $v \in X^*$ such that $(w)_v \in S$. By induction on the word length, and by Lemma 2.2, it is enough to show this for a word of length 2.

Let $s_i, s_j \in S$ and $v \in X^*$ be such that $(n_i n_j)_v \in \mathcal{N}$, say $(n_i n_j)_v = n_k$. We have

$$(n_i)_v (n_j)_{v\tau_{n_i}^{(|v|)}} = n_k \text{ in } G,$$

which is a relator of length at most 3. Hence the corresponding relator $(s_i s_j)_v = s_k$ holds in S.

It follows that S is the nucleus of the group $(G, \varphi_1^{\mathrm{un}})$.

Step 3, commutativity of the diagram. This can be checked on the set S of generators of G_0^{un}. □

The universal contracting cover $(G_0^{\mathrm{un}}, \varphi_1^{\mathrm{un}})$ of (G, Φ) is uniquely defined by (G, Φ), and contracting. But we believe it need not be self-replicating (even though we do not know of any specific example). In all cases, $(G_0^{\mathrm{un}}, \varphi_1^{\mathrm{un}})$ has quotients by finite sets of relations that are self-replicating contracting covers of (G, Φ), as described in the Definition 2.8 and Proposition 2.9. Note however that these quotients are no more uniquely defined by (G, Φ), since choices are involved.

In each of Examples 2.16 to 2.21 below, the universal contracting cover is self-replicating.

Definition 2.8 Let (G, Φ) be a self-replicating contracting self-similar group, with nucleus $\mathcal{N} = \{n_1, \ldots, n_\ell\}$; assume that G is finitely generated. Let $S = \{s_1, \ldots, s_\ell\}$ be in bijection with $\mathcal{N}$, and $(G_0^{\mathrm{un}}, \varphi_1^{\mathrm{un}})$ the universal contracting cover of (G, Φ), as in Definition 2.5. Let $\pi^{\mathrm{un}} : G_0^{\mathrm{un}} \longrightarrow G$ be as in (9).

Let $x \in X$ and $n_i \in \mathcal{N}$. Since the pair (G, Φ) is self-replicating, there exists[6] $g(x, n_i) \in \mathrm{Stab}_G(x)$ such that $(g(x, n_i))_x = n_i$. Since π^{un} is onto, there exists $h(x, n_i) \in G_0^{\mathrm{un}}$ such that $\pi^{\mathrm{un}}(h(x, n_i)) = g(x, n_i)$; moreover, since $\widehat{\pi}^{\mathrm{un}}$ is the identity on the permutations of the wreath product, we have $h(x, n_i) \in \mathrm{Stab}_{G_0^{\mathrm{un}}}(x)$. By commutativity of Diagram (11), we have $\pi^{\mathrm{un}}((h(x, n_i))_x) = n_i$. Set $w(x, n_i) = (h(x, n_i))_x s_i^{-1}$; then $w(x, n_i)$ belongs to the kernel of π^{un}.

Again, by commutativity of (11), we have $(w(x, n_i))_v \in \ker(\pi^{\mathrm{un}})$ for all $v \in X^*$. Since $(G_0^{\mathrm{un}}, \varphi_1^{\mathrm{un}})$ is contracting, the subset

$$E(x, n_i) = \{g \in G_0^{\mathrm{un}} \mid g = (w(x, n_i))_v \quad \text{for some } v \in X^*\}$$

of G_0^{un} is finite. Define

$$E = \bigcup_{x \in X, n \in \mathcal{N}} E(x, n) \quad \text{and} \quad H = \langle\langle E \rangle\rangle \subset G_0^{\mathrm{un}},$$

where $\langle\langle E \rangle\rangle$ denote the *normal* subgroup of G_0^{un} generated by E.

A **standard contracting cover** of G is a quotient group of the form $G_0 = G_0^{\mathrm{un}}/H$, with H as above; the image of S in G is a generating set, that we denote again (abusively) by S. Note that E is a finite subset of G_0^{un}, and consequently that G_0 is a finitely presented group.

The epimorphism π^{un} factors through a homomorphism $\pi : G_0 \longrightarrow G$, because E is a subset of $\ker \pi^{\mathrm{un}}$. It follows from the definition that the restriction of π to the generating set S of G_0 is injective.

The following proposition is the analogue of proposition 2.7 for G_0.

Proposition 2.9 *Let (G, Φ) be a self-replicating contracting self-similar group of degree d, with nucleus $\mathcal{N}$. Assume that G is finitely generated. Let G_0 and $\pi : G_0 \twoheadrightarrow G$ be a standard contracting cover of (G, Φ) and its projection to G, as in Definition 2.8.*

Then there exists a homomorphism

$$\varphi_1 : G_0 \longrightarrow G_0 \wr S_d$$

such that the self-similar group (G_0, φ_1) is contracting and self-replicating with nucleus S. Moreover the diagram

[6] Note that a choice is involved here.

$$\begin{array}{ccc} G_0 & \xrightarrow{\varphi_1} & G_0 \wr S_d \\ \pi \downarrow & & \downarrow \hat{\pi} \\ G & \xrightarrow{\Phi} & G \wr S_d \end{array} \tag{12}$$

commutes.

Proof By construction of the set E, for any element $g \in E$ and any $x \in X$, we have $g_x \in E$. Hence the homomorphism $\varphi_1^{\text{un}} : G_0^{\text{un}} \longrightarrow G_0^{\text{un}} \wr S_d$ induces a homomorphism $\varphi_1 : G_0 \longrightarrow G_0 \wr S_d$. Since $(G_0^{\text{un}}, \varphi_1^{\text{un}})$ is contracting with nucleus S, the self-similar group (G_0, φ_1) is contracting with nucleus S.

Let $x \in X$, and $n_i \in \mathcal{N}$. We continue with the notation of Definition 2.8. By construction of G_0, the relation $h(x, n_i) = s_i$ holds in G_0; moreover $h(x, n_i)$ is an element of $\text{Stab}_{G_0}(x)$. This shows that the pair (G_0, φ_1) is self-replicating.

The commutativity of diagram (12) can be checked on the generators of G_0. □

From here to Corollary 2.15, we keep the same notation as in Definition 2.8 and Proposition 2.9 for G_0, π, and φ_1, in relation with a given contracting self-replicating self-similar group (G, Φ), with G finitely generated.

Definition 2.10 For an integer $n \geq 0$, define

(i) the homomorphism $\varphi_n : G_0 \longrightarrow G_0 \wr^n S_d$ as in (3),
(ii) its kernel $N_n = \ker(\varphi_n)$ and the quotient $G_n = G_0/N_n$,
(iii) the homomorphism

$$\widehat{\pi}_n = \pi \wr 1_{d^n} : G_0 \wr^n S_d \longrightarrow G \wr^n S_d \tag{13}$$

as in (1); note that $\widehat{\pi}_1$ is the $\widehat{\pi}$ of (10).

We have $\Phi_n \pi = \widehat{\pi}_n \varphi_n$, i.e. the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{\varphi_n} & G_0 \wr^n S_d \\ \pi \downarrow & & \downarrow \hat{\pi}_n \\ G & \xrightarrow{\Phi_n} & G \wr^n S_d \end{array} \tag{14}$$

commutes. Observe that $N_0 \subset \cdots \subset N_n \subset N_{n+1} \subset \cdots$ and define

$$N = \bigcup_{n=0}^{\infty} N_n.$$

Remark 2.11 As noted in Definition 2.8, the restriction of π to S is injective. More generally, in Definition 2.10, the restriction of $\widehat{\pi}_n$ to the subset $(S^{X^n}, 1)$ of $G_0 \wr^n S_d = G_0^{X^n} \rtimes S_d^{(n)}$ is injective.

Lemma 2.12 *Let (G, Φ) be a self-similar group; assume that G is finitely generated and that (G, Φ) is faithful contracting self-replicating. With the notation above, we have*

$$N = \ker \pi, \quad \textit{namely} \quad G_0/N = G,$$

so that

$$\lim_{n\to\infty} G_n = G$$

in the space of marked groups on $|S|$ generators (in the sense of Section 3).

Proof Let $g \in N$. Let $n \geq 1$ be such that $g \in \ker(\varphi_n)$. Then $\Phi_n\pi(g) = \widehat{\pi}_n\varphi_n(g) = 1$, hence $g \in \ker(\pi)$ by the faithfulness assumption.

Conversely, let $k \in \ker(\pi)$. On the one hand, since (G_0, φ) is contracting, there exists $n \geq 0$ such that $k_v \in S$ for all $v \in X^n$. On the other hand, $\pi(k) = 1$ implies $\pi(k_v) = 1$ for all $v \in X^n$; moreover, the $S_d^{(n)}$-coordinate of $\varphi_n(k)$ is 1, by commutativity of Diagram (14). Hence, by Remark (2.11), we have $k_v = 1$ for all $v \in X^n$, and therefore $k \in N_n = \ker(\varphi_n)$, a fortiori $k \in N$. □

Lemma 2.13 *In the situation of the previous lemma, for all $n \geq 1$, we have*

$$N_n = \varphi_1^{-1}\left(N_{n-1}^d\right)$$

so that $\varphi_1 : G_0 \longrightarrow G_0 \wr S_d$ induces a homomorphism

$$\psi_n : \begin{cases} G_n & \longrightarrow G_{n-1} \wr S_d \\ gN_n & \longmapsto \left((\varphi_1(g)_x N_{n-1})_{x\in X}, \tau_g^{(1)}\right). \end{cases}$$

Moreover ψ_n is injective.

Proof For $g \in G$, write

$$\varphi(g) = \left((g_x)_{x\in X}, \tau_g^{(1)}\right) \quad \text{and} \quad \varphi_n(g) = \left((g_v)_{v\in X^n}, \tau_g^{(n)}\right). \tag{15}$$

Assume first that $g \in N_n$. Thus $(g_x)_{v'} = 1$ and $\tau_{g_x}^{(n-1)} = 1$ for all $x \in X$ and $v' \in X^{n-1}$. This can be written

$$\varphi_{n-1}(g_x) = \left(((g_x)_{v'})_{v'\in X^{n-1}}, \tau_{g_x}^{(n-1)}\right) = 1 \ \forall x \in X,$$

namely $g_x \in N_{n-1}\ \forall x \in X$. We have checked that $\varphi_1(N_n) \subset N_{n-1}^d$, and $N_n \subset \varphi_1^{-1}(N_{n-1}^d)$ follows.

Assume now that $g \in \varphi_1^{-1}(N_{n-1}^d)$, namely that $(g_x)_{v'} = 1$ and $\tau_{g_x}^{(n-1)} = 1$ for all $x \in X$ and $v' \in X^{n-1}$. This can be written $g_v = 1$ for all $v \in X^n$ and $\tau_g^{(n)} = 1$, namely $g \in N_n$. Hence $\varphi_1^{-1}(N_{n-1}^d) \subset N_n$. □

The next theorem is a detailed version of Theorem 1.9.

Theorem 2.14 *Let (G, Φ) be a self-similar group; assume that G is finitely generated and that (G, Φ) is faithful contracting self-replicating. Let G_0 be a standard contracting cover, as in Definition 2.8.*

Assume that G_0 contains non-abelian free subgroups. Then, for each $n \geq 0$, the group G_n of Definition 2.10 contains non-abelian free subgroups. More generally, every finitely presented cover of G contains non-abelian free subgroups.

Assume moreover that G_0 is large. Then every finitely presented cover of G is large.

Proof Let $S_d^{(0)}$ be the subgroup of S_d of permutations fixing the letter $x = 0$. For $n \geq 1$, let H_n be the finite index subgroup of G_n defined by

$$H_n = \psi_n^{-1}(G_{n-1}^{\{0,1,\ldots,d-1\}} \rtimes S_d^{(0)}),$$

where ψ_n is as in Lemma 2.13. Projection onto the first coordinate (i.e. the coordinate $x = 0$)

$$p_n^{(0)} : G_n^{\{0,1,\ldots,d-1\}} \rtimes S_d^{(0)} \longrightarrow G_n$$

defined by

$$p_n^{(0)}((g_x N_n)_{x \in X}, \tau) = g_0 N_n$$

is a group homomorphism. It turns out that the composition

$$q_n^{(0)} : H_n \xrightarrow{\psi_n} \psi_n(H_n) \xrightarrow{p_{n-1}^{(0)}} G_{n-1}$$

defines a group homomorphism from H_n to G_{n-1}.

Given a generator sN_{n-1} of G_{n-1} (where s is a generator of G_0), using the self-replicating property of (G_0, φ_1), let $h \in St_{G_0}(0)$ be such that $\varphi_1(h)_0 = s$. It turns out that $q^{(0)}(hN_n) = sN_{n-1}$ which shows that $q_n^{(0)}$ is onto G_{n-1}. The conclusion is that for each $n \geq 1$, G_n contains a finite index subgroup H_n which maps onto G_{n-1}.

Therefore, if G_0 contains non-abelian free subgroups (respectively is large), by induction on n, each G_n will contain non-abelian free subgroups (respectively will be large). Then by Lemma 2.12 and Corollary 3.4 below, every finitely presented cover of G will contain non-abelian free subgroups (respectively will be large). □

Corollary 2.15 *Let G be as in Theorem 2.14. If G_0 contains non-abelian free subgroups, then G is infinitely presented.*

Proof Since G_0 does contain non-abelian free subgroups, by assumption, and G does not, by [119, Theorem 4.2], G cannot be finitely presented, by the previous theorem. □

Consider integers $d, \ell \geq 1$, and a system of relations

$$\begin{cases} s_1 &= ((s_1)_0, \ldots, (s_1)_{d-1})\, \tau_1 \\ \ldots\ldots \\ s_\ell &= ((s_\ell)_0, \ldots, (s_\ell)_{d-1})\, \tau_\ell \end{cases} \tag{16}$$

with

$$(s_j)_x \in \{s_1, \ldots, s_\ell\} \quad \text{and } \tau_j \in S_d$$
$$\text{for } j \in \{1, \ldots, \ell\} \quad \text{and } x \in X = \{0, \ldots, d-1\}.$$

By induction on $n \geq 0$, the system (16) defines a set S of automorphisms of the d-regular rooted tree X^*, again denoted by $s_1, \ldots, s_\ell$. These generate a group $G = \langle S \rangle$ of automorphisms of the tree X^*, and (16) define a self-similarity structure Φ on G; thus G is a self-similar group as in Definition 2.1, and moreover G is faithful. The system (16) is often denoted by Φ again.

We review below some classical examples of self-similar groups defined this way.

Example 2.16 The 4-generated group $\mathfrak{G} = \langle a, b, c, d \rangle$ of Theorem 1.6 and [66,67] is a self-similar group of degree 2, with $\Phi : \mathfrak{G} \longrightarrow \mathfrak{G} \wr C_2$ defined by

$$\Phi(a) = (1, 1)\tau,\ \Phi(b) = (a, c),\ \Phi(c) = (a, d),\ \Phi(d) = (1, b).$$

Here, $C_2 = \{1, \tau\}$ denotes the cyclic group of order 2 (written S_2 in Definition 2.1). The self-similar group $(\mathfrak{G}, \Phi)$ is faithful, contracting, and self-replicating. The group $\mathfrak{G}$ is of intermediate growth.

The nucleus is

$$\mathcal{N} = \{1, a, b, c, d\}.$$

The universal contracting cover of Definition 2.5 has the presentation

$$G_0 = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd \rangle \simeq C_2 * V,$$

where C_2 is now the group $\{1, a\}$ and V the Klein Vierergruppe $\{1, b, c, d\}$, isomorphic to $C_2 \times C_2$. The sign $\simeq$ indicates an isomorphism of groups. It can easily be checked that this cover is self-replicating, so that a cover as in Definition 2.8 is not needed here.

Proofs of these facts, and of other properties of $\mathfrak{G}$, can be found in [66], [93, Chapter VIII], or [118, Section 1.6], to quote *some* of the existing expositions only; see also our Section 5. The group $\mathfrak{G}$ can be viewed as the IMG of an orbifold version of the *tent map* $T : [0, 1] \longrightarrow [0, 1]$, defined by $T(x) = 2x$ for $x \leq 1/2$ and $T(x) = 2 - 2x$ for $x \geq 1/2$ [120, Section 5.3]. Here and below, "IMG" stands for "**Iterated Monodromy Group**" (see [118]).

Observe that $C_2 * V$ is virtually a non-abelian free group, because it is a free product of finite groups, distinct from $C_2 * C_2$ (see for example [136, Proposition 4 in Number

I.1.3, p. 14]). It contains a free subgroup F_3 of index 8; one easy way to check this involves computing virtual Euler-Poincaré characteristics, as in [136, Section 1.8]: if

$$F_x \;=\; \ker(p : C_2 * V \overset{\text{canonical}}{\longrightarrow} C_2 \times V),$$

where F_x stands for the free group of rank x, then

$$\chi(C_2 * V) \;=\; \frac{1}{2} + \frac{1}{4} - 1 \;=\; \frac{1}{[C_2 * V : \ker p]}\chi(F_x) \;=\; \frac{1}{8}(1 - x),$$

and therefore $x = 3$.

Hence Theorem 1.6 is a particular case of Theorem 2.14.

Example 2.17 The 2-generated **Basilica group** $\mathfrak{B} = \langle a, b \rangle$ is a self-similar group of degree 2, with homomorphism $\Phi : \mathfrak{B} \longrightarrow \mathfrak{B} \wr C_2$ defined by

$$\Phi(a) = (b, 1)\tau, \, \Phi(b) = (a, 1).$$

The self-similar group $(\mathfrak{B}, \Phi)$ is faithful, contracting, and self-replicating. The group $\mathfrak{B}$ is of exponential growth.

The nucleus is

$$\mathcal{N} \;=\; \{1, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}, \text{ where } c = a^{-1}b.$$

The universal contracting cover has the presentation

$$G_0 \;=\; \langle a, b \mid \emptyset \rangle \;\simeq\; F_2.$$

It is self-replicating.

The group $\mathfrak{B}$ has been introduced in [80,81]. The name "Basilica" was given by Mandelbrot to the Julia set of the quadratic transformation $z \longmapsto z^2 - 1$ of the complex plane, in honour of the *Basilica Cattedrale Patriarcale di San Marco,* and its reflection in Venetian waters [112, p. 254]. The group $\mathfrak{B}$ was identified as $\mathrm{IMG}(z^2 - 1)$ in [9, Theorem 5.8],[7] and the group was named "Basilica" in [10,101,118].

Our notation for $\Phi(a)$ and $\Phi(b)$ is essentially that of [118, p. 208]; the roles of a and b are exchanged in [80].

Again, Theorem 1.8 is a particular case of Theorem 2.14.

Incidentally, since $\mathfrak{B}$ is amenable (references in Appendix D), Theorem 1.8 shows that $\mathfrak{B}$ is *not* finitely presented. Since we could not find references for a direct proof the latter statement in the literature, let us allude to two other simple ways to show that $\mathfrak{B}$ is not finitely presented. One, suggested by Julia Bartsch (private communication), uses the infinite presentation of $\mathfrak{B}$ given in [81] and obtained together with Laurent Bartholdi; then a nice argument concludes that this presentation is minimal (erasing any of its relators would change the group). The other uses the contracting property of the Basilica group established in [80] and follows the idea indicated in [66] for $\mathfrak{G}$.

[7] As acknowledged in [9], part of the credit for this is due to Richard Pink.

Example 2.18 The **IMG of** $z^2 + i$

$$\mathfrak{I} = \mathrm{IMG}(z^2 + i) = \langle a, b, c \rangle$$

is defined by

$$\Phi(a) = (1, 1)\tau, \Phi(b) = (a, c), \Phi(c) = (b, 1).$$

It was studied in detail in [84], and was shown to be of intermediate growth in [40].

Its nucleus is

$$\mathcal{N} = \{1, a, b, c\},$$

and the only non-trivial relators of length ≤ 3 among elements of $\mathcal{N}$ are $a^2 = b^2 = c^2 = 1$ (this can best be checked with the GAP package (http://www.gap-system.org/Packages/automgrp.html). Thus the universal contracting cover

$$G_0 = \langle a, b, c \mid a^2, b^2, c^2 \rangle \simeq C_2 * C_2 * C_2$$

has a free subgroup of finite index (indeed a subgroup F_5 of index 8). It is self-replicating.

Example 2.19 The **Gupta–Sidki group** $\mathfrak{GS} = \langle a, b \rangle$ is the 2-generated group of automorphisms of the ternary rooted tree defined by

$$\Phi(a) = (1, 1, 1)\tau, \Phi(b) = (a, a^{-1}, b),$$

where $\tau \in S_3$ is the cyclic permutation $(0, 1, 2)$. It is the infinite 3-group introduced in [88]; it is just infinite [8, Proposition 8.3]; it can be viewed as an IMG [120, Section 4.5].

Its nucleus is

$$\mathcal{N} = \{1, a, a^{-1}, b, b^{-1}\}.$$

The universal contracting cover is

$$G_0 = \langle a, b \mid a^3, b^3 \rangle \simeq C_3 * C_3,$$

and contains a free subgroup F_4 of index 9. It is self-replicating.

The growth type of $\mathfrak{GS}$ is not known.

Example 2.20 The **Fabrykowski–Gupta group** $\mathfrak{FG} = \langle a, b \rangle$ is the 2-generated group of automorphisms of the ternary rooted tree defined by

$$\Phi(a) = (1, 1, 1)\tau, \Phi(b) = (a, 1, b),$$

with τ as in Example 2.19 [61,62]. It is of intermediate growth (see the original papers, and an exposition with improved estimates of growth in [12]), it is just infinite

[8, Proposition 6.2], and it is the IMG of the cubic polynomial $z^3(-\frac{3}{2} + i\frac{\sqrt{3}}{2}) + 1$ [120, Section 5.4].

As in the previous example, the nucleus is

$$\mathcal{N} = \{1, a, a^{-1}, b, b^{-1}\}.$$

The universal contracting cover is

$$G_0 = \langle a, b \mid a^3, b^3 \rangle \simeq C_3 * C_3.$$

It is self-replicating.

Example 2.21 The ternary **Hanoi Towers group** $\mathfrak{H} = \langle a, b, c \rangle$ is the 3-generated group of automorphisms of the ternary rooted tree defined by

$$\Phi(a) = (a, 1, 1)\tau_{1,2}, \Phi(b) = (1, b, 1)\tau_{0,2}, \Phi(c) = (1, 1, c)\tau_{0,1}$$

where $\tau_{1,2}$ is the transposition of S_3 exchanging 1 and 2, and similarly for $\tau_{0,2}, \tau_{0,1}$. It was introduced in [82] as a model for the well-known Hanoi Towers problem; it is known to be of exponential growth ([83, Subsection 6.1] and [74]), and isomorphic to $\mathrm{IMG}(z^2 - \frac{16}{27z})$ [83, Example 8].

The nucleus is

$$\mathcal{N} = \{1, a, b, c\}.$$

The universal contracting cover is

$$G_0 = \langle a, b, c \mid a^2, b^2, c^2 \rangle \simeq C_2 * C_2 * C_2.$$

It is self-replicating.

Theorem 2.22 *Any finitely presented cover of one of the groups $\mathfrak{J}, \mathfrak{G}\mathfrak{S}, \mathfrak{F}\mathfrak{G}, \mathfrak{H}$, of the four previous examples is large.*

This is a straightforward consequence of Theorem 2.14. In Section 4, we will show how to modify 2.14 to cover uncountably many examples.

Remark 2.23 Groups of interest here are often known to have rather few quotients, of special kinds. Let us illustrate this as follows.

(i) A group is **just infinite** if all its proper quotients are finite. The group $\mathfrak{G}$ is just infinite. More generally, with the notation of section 4, the group G_ω is just infinite for all $\omega \in \Omega_0$ (as we repeat below in Proposition 4.2.ii).

(ii) Without recalling here the technical definitions, let us mention the following property of a finitely generated group G assumed to be branch, or even weakly branch: for any normal subgroup $N \neq \{1\}$ of G, there exists an integer $n \geq 1$ such that N contains the derived group of the rigid stabilizer $\mathrm{Rist}_G(n)$; this follows from the proof of [72, Theorem 4].

As a consequence, if G is branch, then any proper quotient of G is virtually abelian. In particular, any proper quotient of one of the groups $\mathfrak{J}$, $\mathfrak{GS}$, $\mathfrak{FG}$, and $\mathfrak{H}$, is virtually abelian.

(This does *not* apply to $\mathfrak{B}$, which is weakly branch but *not* branch group. This applies to $\mathfrak{G}$, but it is of little interest in this case since the property of (i) is strictly stronger.)

(iii) It is shown in [70] that $\mathfrak{G}$ has a finitely presented HNN-extension $\widehat{\mathfrak{G}}$ which is in $\mathcal{SG} \setminus \mathcal{EG}$. Any proper quotient of $\widehat{\mathfrak{G}}$ is metabelian and virtually abelian [133, Theorem 2.3].

(iv) The Basilica group $\mathfrak{B}$ is **just non-soluble**, which means that all its proper quotients are soluble [80, Proposition 6].

(v) Recall however that there exist groups of intermediate growth with uncountably many quotients: see [69] and Definition 4.13.

3 Marked groups and the Chabauty topology

For k a positive integer, let F_k denote the free group of rank k, given together with an ordered free basis $(s_1, \dots, s_k)$ of generators. A **marked group of rank** k is a pair (G, S) where G is a group and S an ordered set of k generators (for distinct $s, t \in S$, equalities $s = 1$ and $s = t \in G$ are allowed). To such a pair corresponds a free cover $\pi_G : F_k \twoheadrightarrow G$, with $\pi_G(s_j)$ being the jth generator of $S (1 \leq j \leq k)$. We denote by $\mathcal{M}_k$ the set of marked groups on k generators, identified here with the set of normal subgroups of F_k via the bijection $(G, S) \longleftrightarrow \ker \pi_G$.

The idea to furnish a space of (sub)groups with a topology goes back at least to Chabauty [45], and has been revisited on many occasions, among others by Bourbaki [36, chapitre VIII, § 5], Gromov [87, final remarks], one of us [68], Stepin [144], Champetier [46], Champetier and Guirardel [47], and Ceccherini-Silberstein and Coornaert [43].

The **Chabauty topology** on $\mathcal{M}_k$, also called the **Cayley topology**, is that defined by the basis[8]

$$\mathcal{O}_{K,K'} = \left\{N \lhd F_k : N \cap K = \emptyset \text{ and } K' \subset N\right\}, \tag{17}$$

with K, K' finite subsets in F_k. This topology makes $\mathcal{M}_k$ a totally disconnected compact space. It is also completely metrisable, as we now recall. For two subsets A, A' in F_k, let $v(A, A')$ denote the largest integer n such that $A \cap B_S^{F_k}(n) = A' \cap B_S^{F_k}(n)$, where $S = (s_1, \dots, s_k)$ in F_k is as above, and where balls $B_S^{F_k}(n)$ are as in Appendix D. Set $d(A, A') = \exp(-v(A, A'))$. Then d is a metric (indeed an ultrametric) and makes the set 2^{F_k} of subsets of F_k a totally discontinuous compact metric space, in

[8] There is an equivalent definition in terms of the subbasis

$$\mathcal{O}_{K,V} = \{N \lhd F_k : N \cap K = \emptyset \text{ and } N \cap V \neq \emptyset\},$$

indexed by pairs (K, V) where K is a finite subset of F_k and V a subset of F_k. With K compact and V open, it has the advantage to carry over to the space of closed subgroups of a locally compact group G.

which the space $\mathcal{M}_k$ of marked groups on k generators (namely the space of normal subgroups of F_k) is closed. The topology induced by d on $\mathcal{M}_k$ coincides with that defined by (17).

Here is an elementary and basic fact about this topology. The earliest written reference we know for it is [53, Lemma 1.3 and Lemma 1].

Proposition 3.1 *Let k, ℓ be two positive integers. Let $(G, S) \in \mathcal{M}_k$ and $(G, T) \in \mathcal{M}_\ell$ be two marked groups with the same underlying group. Then there exist neighbourhoods $U \subset \mathcal{M}_k$ of (G, S) and $V \subset \mathcal{M}_\ell$ of (G, T) that are homeomorphic.*

In loose words, local properties of (G, S) are properties of G itself.

This proposition justifies the following definitions: a property $(\mathcal{P})$ of finitely generated groups is **open** [respectively **closed]** if, for any positive integer k, the subset of $\mathcal{M}_k$ of marked groups (G, S) such that G has Property $(\mathcal{P})$ is open [respectively closed]. A finitely generated group G is **isolated** if, for any ordered generating set $S = (s_1, \ldots, s_k)$ of G, the point (G, S) is isolated in $\mathcal{M}_k$. We collect a few examples as follows:

Proposition 3.2 *For $k \geq 2$, in the space $\mathcal{M}_k$ of marked groups of rank k:*

(i) *"Being abelian" is both an open and a closed property; more generally, for $d \geq 1$, "being nilpotent of nilpotent class at most d" is both open and closed. "Being nilpotent" is open and non-closed.*
(ii) *"Being soluble of solubility class at most k" is closed and non-open. "Being soluble" is neither open nor closed .*
(iii) *"Being finite" and "having torsion" are open and non-closed.*
(iv) *If $(G, S) \in \mathcal{M}_k$ is a marking of a finitely presented group G, there exists a neighbourhood of (G, S) in $\mathcal{M}_k$ containing only marked quotients of (G, S).*
(v) *A necessary condition for (G, S) to be an isolated point in $\mathcal{M}_k$ is that G is finitely presented. Finite groups and finitely presented simple groups are isolated.*
(vi) *There exists an isolated group that is 3-soluble and non-Hopfian; the group $\widehat{\mathfrak{G}}$ mentioned in Remark 2.23.iii is isolated.*
(vii) *Amenability is neither open nor closed.*
(viii) *Kazhdan Property (T) is open in $\mathcal{M}_k$.*
(ix) *Serre Property (FA) is not open in $\mathcal{M}_k$.*

On the proof Claims (i) to (v) are elementary; most of them appear explicitly in [47, Section 2.6 and Lemma 2.3]. For (i), note moreover that "being nilpotent" is open by (iv), because nilpotent groups are finitely presented. For (ii), note that "being soluble" is non-open, because metabelian groups like $\mathbf{Z} \wr \mathbf{Z}$ are limits of non-soluble groups (see Example B.2, say).

"Being nilpotent", "being soluble", "being finite", "being amenable" and "having torsion" are non-closed properties, because non-abelian free groups are residually finite p-groups, for any prime p (due to [89], see also [149]).

For (v), observe that a finitely generated infinitely presented group G is always a limit of finitely presented groups G_n; more precisely

$$G = \langle s_1, \ldots, s_k \mid (r_i)_{i \geq 1} \rangle = \lim_{n \to \infty} G_n$$
$$\text{with } G_n = \langle s_1, \ldots, s_k \mid r_1, \ldots, r_n \rangle.$$

Necessary *and sufficient* conditions for isolated points are known in terms of the existence of "finite discriminating subsets"; we refer to [53, Proposition 2]; see also [73, Theorem 2.1]. The class of isolated groups contains considerably more groups than the finite groups and the finitely presented simple groups [53].

The first part of Claim (vi) is [53, Proposition 10]; the second part is implicit in [133], and explicit in [53, Proposition 5.18]). "Being amenable" is non-open, again because $\mathbf{Z} \wr \mathbf{Z}$ is a limit of groups with non-abelian free subgroups (Example B.2). Claim (viii) is a result of [137], and (ix) of [57]. □

The Chabauty topology on $\mathcal{M}_k$ plays an important role in connection with many group properties including "Property LEF" and "soficity". The two latter properties define subspaces in $\mathcal{M}_k$ that are closed [43, Propositions 7.3.7 and 7.5.13].

Note the contraposition of (v): for $(G, S) \in \mathcal{M}_k$ with G infinitely presented, there exists a sequence $((G_n, S_n))_{n\geq 1}$ of pairwise distinct points in $\mathcal{M}_k$ such that $\lim_{n\to\infty}(G_n, S_n) = (G, S)$.

The simplest examples of converging sequences in $\mathcal{M}_k$ are of the following kind. Let

$$N_1 \subset \cdots \subset N_n \subset N_{n+1} \subset \cdots \subset N := \bigcup_{n\geq 1} N_n$$

be a nested sequence of normal subgroups in F_k. Let S_0 be a free basis of F_k. Denote by $p_n : F_k \longrightarrow G_n := F_k/N_n$ $(n \geq 1)$ and $p : F_k \longrightarrow G := F_k/N$ the canonical projections. Set $S_n = p_n(S_0)$ and $S = p(S_0)$. Then $((G_n, S_n))_{n\geq 1}$ is a sequence in $\mathcal{M}_k$ converging to (G, S). In this case we often suppress the emphasis on the generating sets and write simply that the sequence $(G_n)_{n\geq 1}$ converges to G in $\mathcal{M}_k$.

Converging sequences in $\mathcal{M}_k$ need not be of this special kind, with G a quotient of G_n for all n large enough. See below, Proposition 4.2.vi.

The following observation about $\mathcal{M}_k$ and covers, basic for us, is well-known; see e.g. [54, Proposition 3.3]. We provide a proof for the convenience of the reader.

Proposition 3.3 *Let $((G_n, S_n))_{n\geq 1}$ be a converging sequence in $\mathcal{M}_k$; set $(G, S) = \lim_{n\to\infty}(G_n, S_n)$. Let Γ be a finitely presented group; assume there exists a cover $\pi : \Gamma \twoheadrightarrow G$.*

Then Γ is a cover of G_n for n large enough.

Note. In case G itself is finitely presented, this lemma is an immediate consequence of Proposition 3.2.v.

Proof Denote as above by $(s_1, \ldots, s_k)$ an ordered free basis of F_k. Let $p_n : F_k \twoheadrightarrow G_n$ and $p : F_k \twoheadrightarrow G$ be the free covers corresponding to (G_n, S_n) and (G, S) respectively. Set $N_n = \ker(p_n)$ and $N = \ker(p)$. Let $(t_1, \ldots, t_\ell)$ an ordered generating set of Γ. Consider the free group F_ℓ on an ordered basis $U = (u_1, \ldots, u_\ell)$ and the free cover $q : F_\ell \twoheadrightarrow \Gamma$ defined by $q(u_j) = t_j$ for $j = 1, \ldots, \ell$.

Since Γ is finitely presented, there exists a finite subset $R \subset F_\ell$ of words $v_1, \ldots, v_m$ in the letters of $U \cup U^{-1}$ such that $\ker(q)$ is the normal subgroup of F_ℓ generated by R, namely such that $\langle U \mid R\rangle$ is a presentation of Γ. For $j \in \{1, \ldots, \ell\}$, choose a word

w_j in the letters $p(s_1), \ldots, p(s_k)$ and their inverses such that $\pi(t_j) = w_j$. Let $\tilde{w}_j$ be the word in $\{s_1, s_1^{-1}, \ldots, s_k, s_k^{-1}\}$ obtained by substitution of $s_i^{\pm 1}$ in place of $p(s_i)^{\pm 1}$; observe that $p(\tilde{w}_j) = w_j = \pi(t_j)$. Consider the homomorphism

$$h \,:\, F_\ell \longrightarrow F_k \quad \text{defined by} \quad h(u_j) \,=\, \tilde{w}_j \quad (1 \leq j \leq \ell).$$

Then $ph(u_j) = p(\tilde{w}_j) = w_j = \pi(t_j) = \pi q(u_j)$ for all j, so that $ph = \pi q$, and therefore $h(R) \subset N$.

The last inclusion means that the open subset

$$\mathcal{O}' \,:=\, \{M \triangleleft F_k \,:\, h(R) \subset M\} \,=\, \bigcap_{i=1}^{m} \mathcal{O}_{\emptyset, \{h(r_i)\}}$$

is a neighbourhood of N in $\mathcal{M}_k$. Hence, for n large enough, we have $N_n \in \mathcal{O}'$ and therefore $h(R) \subset N_n$.

Denote by $\langle\langle T \rangle\rangle$ the *normal* subgroup of a group H generated by a subset $T \subset H$. Let

$$h_1 \,:\, \Gamma = F_\ell / \langle\langle R \rangle\rangle \longrightarrow F_k / \langle\langle h(R) \rangle\rangle$$

be the cover induced by h, and

$$h_2 \,:\, F_k / \langle\langle h(R) \rangle\rangle \longrightarrow F_k / N_n \,=\, G_n$$

that defined by the inclusion $\langle\langle h(R) \rangle\rangle \subset N_n$ (for $n \gg 1$). The composition $h_2 h_1$ is a cover $\Gamma \twoheadrightarrow G_n$, and this concludes the proof. □

An immediate consequence of the previous proposition is the following corollary, of very frequent use in our work.

Corollary 3.4 *Consider the three following group properties:*

(NA) *non-amenability,*
(Fr) *containing non-abelian free groups,*
(La) *being large.*

Let $k \geq 2$ and $((G_n, S_n))_{n \geq 1}$ be a converging sequence in $\mathcal{M}_k$, with limit (G, S).

If, for all n large enough, G_n has one of the three properties above, then any finitely presented cover of G has the same property.

Proof The point is that a group that has a quotient with one of the properties (NA), (Fr), (La) has itself the same property. □

4 The analogue of Theorem 1.6 for the family $(G_\omega)_{\omega \in \Omega}$ of [68]

Let Ω be the Cantor space $\{0, 1, 2\}^{\mathbf{N}}$ of all sequences of 0's, 1's and 2's, with the product topology. Denote by Ω_- the countable subspace of eventually constant sequences, by Ω_+ its complement, and by Ω_0 the subspace of sequences with infinitely many occurrences of each of 0, 1, 2; thus

$$\Omega_0 \subset \Omega_+ \subset \Omega = \Omega_+ \sqcup \Omega_-.$$

We denote by σ the shift on Ω, defined by $(\sigma(\omega))_n = \omega_{n+1}$ for all $n \geq 1$.

We will recall the construction of [68], which is a generalisation of that of Section 2. It associates with each point $\omega \in \Omega$ a marked group (G_ω, S_ω) with S_ω consisting of 4 generators of order 2; for example, $\mathfrak{G} = G_{\overline{012}}$, where $\overline{012}$ stands for the 3-periodic sequence $012012012\cdots$. In this section, set

$$X = \{0, 1\}$$

and identify X^* with the 2-regular rooted tree. We proceed to define for all $\omega \in \Omega$ a marked group $(G_\omega.S_\omega) \in \mathcal{M}_4$ of automorphisms of X^*.

Definition 4.1 The flip $a \in \mathrm{Aut}(X^*)$ is defined by

$$a(0v) = 1v \text{ and } a(1v) = 0v \text{ for all } v \in X^*.$$

Set

$$\begin{array}{lll} a_{\beta(0)} = a & a_{\beta(1)} = a & a_{\beta(2)} = 1 \\ a_{\gamma(0)} = a & a_{\gamma(1)} = 1 & a_{\gamma(2)} = a \\ a_{\delta(0)} = 1 & a_{\delta(1)} = a & a_{\delta(2)} = a. \end{array}$$

Define for each $\omega = (\omega_n)_{n\geq 1} \in \Omega$ a set $S_\omega = \{a, b_\omega, c_\omega, d_\omega\}$ of four automorphisms of X^* by

$$\begin{aligned} b_\omega &= \left(a_{\beta(\omega_1)}, b_{\sigma(\omega)}\right) \\ c_\omega &= \left(a_{\gamma(\omega_1)}, c_{\sigma(\omega)}\right) \\ d_\omega &= \left(a_{\delta(\omega_1)}, d_{\sigma(\omega)}\right). \end{aligned}$$

It is easy to check that

$$\begin{aligned} ac_\omega a &= \left(b_{\sigma(\omega)}, a_{\beta(\omega_1)}\right) \\ ad_\omega a &= \left(c_{\sigma(\omega)}, a_{\gamma(\omega_1)}\right) \\ ab_\omega a &= \left(d_{\sigma(\omega)}, a_{\delta(\omega_1)}\right) \\ a^2 = b_\omega^2 &= c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = 1. \end{aligned} \tag{18}$$

We define the group

$$G_\omega = \langle a, b_\omega, c_\omega, d_\omega \rangle$$

generated by S_ω; it is a subgroup of $\mathrm{Aut}(X^*)$. It follows from the last line of (18) that any element of G_ω can be written as

$$(*)a * a * \cdots a(*) \tag{19}$$

with $* \in \{b_\omega, c_\omega, d_\omega\}$, $(*) \in \{1, b_\omega, c_\omega, d_\omega\}$, and $n \geq 0$ occurrences of a.

Observe that any permutation τ of $\{0, 1, 2\}$ induces a permutation of Ω, again denoted by τ; the groups $G_{\tau(\omega)}$ and G_ω are isomorphic.

In [68, Section 6], there is moreover a modified construction providing a marked group $(\widetilde{G}_\omega, \widetilde{S}_\omega)$; we refer to the original paper. Note that (v) below holds for the modified groups, but not for the groups G_ω.

Proposition 4.2 *Let $\Omega = \{0, 1, 2\}^{\mathbf{N}}$. For $\omega \in \Omega$, let G_ω and $\widetilde{G}_\omega$ be as above.*

(i) *For $\omega \in \Omega$, the groups G_ω and $\widetilde{G}_\omega$ are both infinite, and $\widetilde{G}_\omega$ is infinitely presented.*
(ii) *For $\omega \in \Omega_+$, the marked groups (G_ω, S_ω) and $(\widetilde{G}_\omega, \widetilde{S}_\omega)$ are isomorphic; the group G_ω is of intermediate growth.*
For $\omega \in \Omega_0$, the group G_ω is an infinite 2-group, and is just infinite.[9]
(iii) *For $\omega \in \Omega_-$, the group G_ω is virtually free abelian, and consequently finitely presented of polynomial growth, while the group $\widetilde{G}_\omega$ is virtually metabelian and of exponential growth.*
(iv) *For $\omega, \omega' \in \Omega_+$, the groups G_ω and $G_{\omega'}$ are isomorphic if and only if $\omega' = \tau(\omega)$ for some permutation τ of $\{0, 1, 2\}$.*
(v) *The mapping $\Omega \longrightarrow \mathcal{M}_4$, $\omega \longmapsto (\widetilde{G}_\omega, \widetilde{S}_\omega)$ is a homeomorphism onto its image.*
(vi) *For a converging sequence $(\omega_{(n)})_{n \geq 1}$ of points in Ω_+ with a limit ω in Ω_+, we have $\lim_{n\to\infty}(G_{\omega_{(n)}}, S_{\omega_{(n)}}) = (G_\omega, S_\omega)$ in $\mathcal{M}_4$. If, moreover, $\omega_{(n)} \in \Omega_0$ for all $n \geq 1$ and $\omega \notin \Omega_0$, then, for all $n \geq 1$, the group G_ω is not a quotient of $G_{\omega_{(n)}}$.*

On the proof Most of this is proved in [68]; more precisely:

(i) G_ω is infinite [68, Theorem 2.1] and $\widetilde{G}_\omega$ is infinitely presented [68, Theorem 6.2].
(ii) For $\omega \in \Omega_+$, we have $(G_\omega, S_\omega) = (\widetilde{G}_\omega, \widetilde{S}_\omega)$ [68, observation just before Theorem 6.1], and G_ω is of intermediate growth [68, Corollary 3.2]. For $\omega \in \Omega_0$, G_ω is an infinite 2-group that is just infinite [68, Theorems 2.1 and 8.1].
(iii) G_ω is virtually free abelian [68, Theorem 2.1.(3)], while $\widetilde{G}_\omega$ is virtually metabelian and of exponential growth [68, Theorem 6.1].

About (iv), see [118, Theorem 2.10.13]. A weaker statement is proved in [68, Section 5].

For (v), see [68, Proposition 6.2].

For (vi), given any $n \geq 1$, note that G_ω is neither isomorphic to $G_{\omega_{(n)}}$, by (iv), nor a non-trivial quotient of $G_{\omega_{(n)}}$, by (ii). □

For the main result of this section (Theorem 4.5), we will need an analogue in the present context of the homomorphisms (2) and (3) of Section 2. Recall that we have

[9] Let Ω_1 be the subset of Ω_+ of sequences containing infinitely many occurrences of two of 0, 1, 2, and finitely many occurrences of the third, so that $\Omega_+ = \Omega_0 \sqcup \Omega_1$. For $\omega \in \Omega_1$, the group G_ω is not a 2-group, indeed it has elements of infinite order.

a natural *isomorphism*

$$\Phi_X \,:\, \mathrm{Aut}(X^*) \xrightarrow{\simeq} \mathrm{Aut}(X^*) \wr S_2.$$

We keep the notation of Definition 4.1.

Definition 4.3 Let $\omega \in \Omega$. The restriction to G_ω of the isomorphism Φ_X provides an injective homomorphism

$$\Phi_\omega^{(1)} = \Phi_\omega \,:\, G_\omega \longrightarrow G_{\sigma(\omega)} \wr S_2.$$

On the generators, we have

$$\begin{aligned}
\Phi_\omega(a) &= (1,1)\tau \\
\Phi_\omega(b_\omega) &= (a_{\beta(\omega_2)}, b_{\sigma(\omega)}) \\
\Phi_\omega(c_\omega) &= (a_{\gamma(\omega_2)}, c_{\sigma(\omega)}) \\
\Phi_\omega(d_\omega) &= (a_{\delta(\omega_2)}, d_{\sigma(\omega)})
\end{aligned}$$

(recall that $S_2 = \{1, \tau\}$). The sequence of homomorphisms $\left(\Phi_\omega^{(n)}\right)_{n\geq 1}$ is defined inductively by

$$\Phi_\omega^{(n)} \,:\, G_\omega \xrightarrow{\Phi_\omega^{(n-1)}} G_{\sigma^{n-1}(\omega)} \wr^{n-1} S_2 \xrightarrow{\Phi^{(1)}_{\sigma^{n-1}(\omega)} \wr 1_{d^{n-1}}} G_{\sigma^n(\omega)} \wr^n S_2.$$

Lemma 4.4 (contraction in G_ω) *Let $\omega \in \Omega$. We keep the notation above.*

(i) *For each $n \geq 1$, the homomorphism $\Phi_\omega^{(n)}$ is injective.*
(ii) *For all $g \in G_\omega$, there exists an integer $n \geq 1$ such that*

$$\Phi_\omega^{(n)}(g) = \left((g_v)_{v\in X^n}, \tau_g^{(n)}\right)$$

with $g_v \in \{1, a, b_{\sigma^n(\omega)}, c_{\sigma^n(\omega)}, d_{\sigma^n(\omega)}\}$ $\forall v \in X^n$ and $\tau_g^{(n)} \in S_2$.

Proof By induction on the length of g, in the sense of (19). □

In Theorem 1.10 of the Introduction, the claim on intermediate growth is a repetition of part of Proposition 4.2, and the claim on covers is the theorem below.

Theorem 4.5 *For $\omega \in \Omega_+$, any finitely presented cover of G_ω is large.*

Remark 4.6 (1) Let $\omega \in \Omega_-$. Any finitely presented cover of the infinitely presented group $\widetilde{G}_\omega$ contains non-abelian free groups, by Theorem 1.5. As recorded in Proposition 4.2.iii, the group G_ω is virtually free abelian, and finitely presented. For example, if ω is the constant sequence $000\cdots$, then G_ω is the infinite dihedral group.

(2) If we replace "is large" by "contains non-abelian free subgroups" in Theorem 4.5, the resulting statement has a short proof. Indeed:

For any $\omega \in \Omega$, any finitely presented cover of $\widetilde{G}_\omega$ has non-abelian free subgroups. Indeed, let $(\omega_n)_{n\geq 1}$ be a sequence of eventually constant sequences converging to ω in Ω. Then $\widetilde{G}_{\omega_n}$ is virtually metabelian and infinitely presented for all $n \geq 1$ (Claims (i) and (iii) in Proposition 4.2), and $(\widetilde{G}_{\omega_n})_{n\geq 1}$ converges to $\widetilde{G}_\omega$ (Proposition 4.2.v). Let E be a finitely presented cover of $\widetilde{G}_\omega$. Then E is a cover of G_{ω_n} for n large enough (Proposition 3.3). Hence Bieri–Strebel Theorem 1.5 shows that E contains non-abelian free subgroups.

From now on, we assume that

$$\omega \in \Omega_+.$$

Our strategy for the proof of Theorem 4.5 is to adapt to the present context the steps of Section 2.

The following definition should be compared with Definition 2.10. Note however that G_0 has not quite the same meaning here and there.

Definition 4.7 Set again

$$G_0 = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd\rangle \simeq C_2 * V,$$

as in Example 2.16. Observe that any element of G_0 can be written as

$$(*)a * a * \cdots a(*) \tag{20}$$

with $* \in \{b, c, d\}$, $(*) \in \{1, b, c, d\}$, and $n \geq 0$ occurrences of a (compare with Equation (19)).

For $i \in \{0, 1, 2\}$, set

$$\varphi_i(a) = (1, 1)\tau \quad \text{for all } i \in \{0, 1, 2\}$$

and

$$\begin{array}{lll} \varphi_0(b) = (a, b) & \varphi_1(b) = (a, b) & \varphi_2(b) = (1, b) \\ \varphi_0(c) = (a, c) & \varphi_1(c) = (1, c) & \varphi_2(c) = (a, c) \\ \varphi_0(d) = (1, d) & \varphi_1(d) = (a, d) & \varphi_2(d) = (a, d). \end{array}$$

It is easy to check that these formulas define homomorphisms

$$\varphi_i : G_0 \longrightarrow G_0 \wr S_2 \quad (i = 0, 1, 2).$$

Set $\varphi_\omega^{(1)} = \varphi_{\omega_1}$ and define, inductively for $n \geq 2$, homomorphisms

$$\varphi_\omega^{(n)} : G_0 \stackrel{\varphi_\omega^{(n-1)}}{\longrightarrow} G_0 \wr^{n-1} S_2 \stackrel{\varphi_{\omega_n} \wr 1_{2^n}}{\longrightarrow} G_0 \wr^n S_2.$$

For $n \geq 1$, set

$$N_{\omega,n} = \ker(\varphi_\omega^{(n)}) \quad \text{and} \quad G_{\omega,n} = G_0/N_{\omega,n}.$$

We have natural homomorphisms

$$\begin{aligned} \pi_\omega \ : \ G_0 &\longrightarrow G_\omega, \\ \widehat{\pi}_\omega = \widehat{\pi}_{\omega,1} \ : \ G_0 \wr S_2 &\longrightarrow G_{\sigma(\omega)} \wr S_2, \\ \widehat{\pi}_{\omega,n} \ : \ G_0 \wr^n S_2 &\longrightarrow G_{\sigma^n(\omega)} \wr^n S_2. \end{aligned}$$

(Compare with (9), (10), and (13), but note that $\widehat{\pi}_{\omega,1} = \pi_\omega \wr 1_2$ *does not hold* here.)

The next lemma is about diagrams analogous to (11) and (14). Its proof uses an argument similar to one in the proof of Proposition 2.7, and will be omitted.

Lemma 4.8 *The diagram*

$$\begin{array}{ccc} G_0 & \xrightarrow{\varphi_\omega^{(n)}} & G_0 \wr^n S_2 \\ \pi_\omega \downarrow & & \downarrow \ \hat{\pi}_{\omega,n} \\ G & \xrightarrow{\Phi_\omega^{(n)}} & G_{\sigma^n(\omega)} \wr^n S_2 \end{array} \tag{21}$$

commutes for each $n \geq 1$.

The next lemma is analogous to Step 2 in the proof of Proposition 2.7.

Lemma 4.9 (contraction in G_0) *For all* $k \in G_0$*, there exists an integer* $n \geq 1$ *such that*

$$\varphi_\omega^{(n)}(k) = \left((k_v)_{v \in X^n}, \tau_k^{(n)}\right)$$

with $k_v \in \{1, a, b, c, d\}$ $\forall v \in X^n$ *and* $\tau_k^{(n)} \in S_2$.

Proof by induction on the length of k, in the sense of (20). □

Define now

$$N_\omega = \bigcup_{n \geq 1} N_{\omega,n}$$

(compare with Definition 2.10). The two following lemmas are appropriate modifications of Lemmas 2.12 and 2.13; we repeat the proof for the first one, and not for the second one.

Lemma 4.10 *We have*

$$N_\omega = \ker\left(\pi_\omega : G_0 \longrightarrow G_\omega\right), \quad \textit{namely } G_\omega \simeq G_0/N_\omega,$$

so that

$$\lim_{n\to\infty} G_{\omega,n} = G_\omega$$

in the space of marked groups on 4 *generators.*

Proof Let $g \in N$. Let $n \geq 1$ be such that $g \in \ker(\varphi_\omega^{(n)})$. Since $\Phi_\omega^{(n)}\pi_\omega(g) = \widehat{\pi}_{\omega,n}\varphi_\omega^{(n)}(g)$, we have $\pi_\omega(g) = 1$ by Lemma 4.4.i.

Conversely, let $k \in G_0$. There exists $n \geq 0$ such that $\left(\varphi_\omega^{(n)}(k)\right)_v \in \{1, a, b, c, d\}$ for all $v \in X^n$, by Lemma 4.9. Assume that $k \in \ker(\pi_\omega)$. Then $\widehat{\pi}_{\omega,n}(\varphi_\omega^{(n)}(k)) = 1$. As $\widehat{\pi}_{\omega,n}$ is injective "on generators" (in a sense similar to that of Remark 2.11), we have $\varphi_\omega^{(n)}(k) = 1$, and therefore $k \in N_{\omega,n} \subset N_\omega$.

(Note that the hypothesis "$\omega \in \Omega_+$" is necessary for the previous argument. If ω_n were eventually constant, one of $b_{\sigma^n(\omega)}, c_{\sigma^n(\omega)}, d_{\sigma^n(\omega)}$ would be the identity of $G_{\sigma^n(\omega)}$ for n large enough). □

Lemma 4.11 *In the situation of the previous lemma, we have for all* $n \geq 1$

$$\varphi_\omega^{(1)}(N_{\omega,n}) \subset N_{\omega,n-1}^2 \subset G_0 \wr S_2 \quad \textit{and} \quad \left(\varphi_\omega^{(1)}\right)^{-1}(N_{\omega,n-1}^2) \subset N_{\omega,n}.$$

It follows that $\varphi_\omega^{(1)} : G_0 \longrightarrow G_0 \wr S_2$ *induces a homomorphism*

$$\psi_\omega^{(n)} : \begin{cases} G_{\omega,n} & \longrightarrow G_{\omega,n-1} \wr S_2 \\ gN_{\omega,n} & \longmapsto \left(\left((\varphi_{\omega_n}(g))_x N_{\omega,n-1}\right)_{x\in X}, \tau_g^{(1)}\right) \end{cases}$$

which is injective.

Proposition 4.12 *For each* $\omega \in \Omega_+$ *and* $n \geq 0$*, the group* $G_{\omega,n}$ *is large.*

Proof The group $G_0 = C_2 * V$ has a free subgroup of finite index, indeed a subgroup isomorphic to F_3 of index 8. For $n \geq 1$, because of the previous lemma and as in the proof of Theorem 2.14, there exists a subgroup of index 2 in $G_{\omega,n}$ and a homomorphism from this subgroup onto $G_{\omega,n-1}$. It follows by induction on n that $G_{\omega,n}$ is large. □

End of proof of Theorem 4.5 Since $G_{\omega,n}$ is large for n large enough, it follows from Lemma 4.10 and Corollary 3.4 that any cover of G_ω is large. □

Definition 4.13 For $\omega \in \Omega$, let M_ω denote the kernel of the defining cover $F_4 \twoheadrightarrow G_\omega$; in other terms, M_ω is the inverse image of N_ω by the epimorphism $F_4 \twoheadrightarrow G_0$ mapping the four generators of F_4 onto $a, b, c, d \in G_0$. For a subset Ψ of Ω, the **Ψ-universal group** is the group

$$\mathcal{U}_\Psi = F_4 / \bigcap_{\omega\in\Psi} M_\omega.$$

For example, $\mathcal{U}_\emptyset = \{1\}$, and $\mathcal{U}_{\{\omega\}} = G_\omega$ for all $\omega \in \Omega$.

The terminology is justified by cases that have appeared in the literature, with Ψ large. For example, let Λ denote the subset of Ω_0 of sequences that are concatenations of blocks 012, 120, 201. Then $\mathcal{U}_\Lambda$ has uncountably many quotients (a consequence of Proposition 4.2.iv); it has intermediate growth, and therefore is amenable (established in [75, Theorem 9.7]).

Suppose that Ψ contains some $\omega \in \Omega_+$. Then any cover of $\mathcal{U}_\Psi$ is a cover of G_ω. Theorem 4.5 implies:

Corollary 4.14 *For any $\Psi \subset \Omega$ such that $\Psi \cap \Omega_+ \neq \emptyset$, the Ψ-universal group $\mathcal{U}_\Psi$ is infinitely presented, and any finitely presented cover of it is large.*

In particular, this corollary solves the first part of Problem 9.5 in [73], by showing that $\mathcal{U}_\Omega$ is infinitely presented.

5 The group of intermediate growth $\mathfrak{G}$

Let $\mathfrak{G}$ be the self-similar group of degree 2 of Example 2.16. On the one hand, $\mathfrak{G}$ is a group of the family studied in the previous section: $\mathfrak{G} = G_{\overline{012}}$; thus Theorem 4.5 "contains" Theorem 1.6. On the other hand, in this particular case, we can describe much more precisely a sequence of finitely presented covers converging to $\mathfrak{G}$, and this is the subject of the present section. Note however that, even if the cover $\mathfrak{G}_{-1}$ below is the same as G_0 in Example 2.16, the sequence $(\mathfrak{G}_n)_{n\geq 0}$ *is not* the particular case for $\mathfrak{G}$ of the sequence $(G_n)_{n\geq 1}$ (even shifted) of Section 2.

Immediately after its discovery it was observed that $\mathfrak{G}$ is infinitely presented. Then, Lysenok found a presentation that we recall below.

Set

$$\mathfrak{G}_{-1} = \left\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1 \right\rangle \simeq C_2 * V,$$

and denote by S the system of four involutions $\{a, b, c, d\}$ generating $\mathfrak{G}_{-1}$. Elements in $\mathfrak{G}_{-1}$ are in natural bijection with "reduced words" of the form

$$t_0 a t_1 a \cdots a t_{k-1} a t_k$$

with $k \geq 0$, $t_1, \ldots, t_{k-1} \in \{b, c, d\}$, and $t_0, t_k \in \{\emptyset, b, c, d\}$. Throughout the remainder of this section, we use the same symbol to denote an element of $\mathfrak{G}_{-1}$ and its image in any quotient of $\mathfrak{G}_{-1}$, in particular in $\mathfrak{G}$; thus, $S = \{a, b, c, d\}$ denotes a set of generators in $\mathfrak{G}_{-1}$ *and* in any quotient of $\mathfrak{G}_{-1}$.

The substitution σ defined by

$$\sigma(a) = aca, \quad \sigma(b) = d, \quad \sigma(c) = b, \quad \sigma(d) = c$$

extends to reduced words, for example $\sigma(abac) = acadacab$, and the resulting map

$$\sigma : \mathfrak{G}_{-1} \longrightarrow \mathfrak{G}_{-1}$$

is a group endomorphism. Define

$$u_0 = (ad)^4 \quad u_n = \sigma^n(u_0) \quad \forall n \geq 0$$
$$v_0 = (adacac)^4 \quad v_n = \sigma^n(v_0) \quad \forall n \geq 0.$$

Theorem 5.1 [108,109] *The group $\mathfrak{G}$ has a presentation*

$$\left\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1, u_n = v_n = 1 \quad \forall n \geq 0 \right\rangle.$$

Note. It is moreover known that this presentation is minimal [71]. Lysenok's presentation if the prototype of what is now called an *L-presentation* [6].

Definition 5.2 For $n \geq 0$, define a pair $(\mathfrak{G}_n, S) \in \mathcal{M}_4$ by

$$\mathfrak{G}_n = \left\langle a, b, c, d \;\middle|\; \begin{matrix} a^2 = b^2 = c^2 = d^2 = bcd = 1 \\ u_0 = \cdots = u_n = v_0 = \cdots = v_{n-1} = 1 \end{matrix} \right\rangle$$
$$S = \{a, b, c, d\} \subset \mathfrak{G}_n.$$

Observe that $\lim_{n\to\infty}(\mathfrak{G}_n, S) = (\mathfrak{G}, S)$ in $\mathcal{M}_4$, and that there are natural surjections $\mathfrak{G}_{-1} \twoheadrightarrow \mathfrak{G}_n \twoheadrightarrow \mathfrak{G}$ for all $n \geq 0$.

Theorem 5.3 *For each $n \geq 0$, the group $\mathfrak{G}_n$ has a normal subgroup H_n of index $2^{2^{n+1}+2}$ which is isomorphic to the direct product of 2^n free groups of rank 3.*

Remark 5.4 (i) A weaker result was first established in [78]: For each $n \geq 0$, $\mathfrak{G}_n$ contains a subgroup of finite index isomorphic to the direct product of 2^n copies of finitely generated non-abelian free groups. This by itself implies that any finitely presented cover of $\mathfrak{G}$ contains non-abelian free subgroups.

(ii) The result of [78] was improved in [11]: For each $n \geq 0$, the group $\mathfrak{G}_n$ has a normal subgroup H_n of index 2^{α_n}, where $\alpha_n \leq (11 \cdot 4^n + 1)/3$, and H_n is a subgroup of index 2^{β_n} in a finite direct product of 2^n non-abelian free groups of rank 3, where $\beta_n \leq (11 \cdot 4^n - 8)/3 - 2^n$.

(iii) Our proof of Theorem 5.3 is split in several lemmas, until 5.10.

If $x, \ldots, y$ are elements of a group H, we denote by $\langle x, \ldots, y\rangle_H$ the subgroup of H they generate, and by $\langle\langle x, \ldots, y\rangle\rangle_H$ the *normal* subgroup of H they generate. Define first

$$B_0 = \langle\langle b\rangle\rangle_{\mathfrak{G}_0},$$
$$\Xi_0 = \langle b, c, d, aba, aca, ada\rangle_{\mathfrak{G}_0},$$
$$D_0 = \langle a, d\rangle_{\mathfrak{G}_0},$$
$$D_0^{\text{diag}} = \langle (a, d), (d, a)\rangle_{\mathfrak{G}_0}.$$

It is easy to check that $D_0^{\text{diag}} \cap (B_0 \times B_0) = \{1\}$, and that D_0^{diag} normalizes $B_0 \times B_0$. The assignment

$$\begin{array}{ll} b \mapsto (a, c) & aba \mapsto (c, a) \\ c \mapsto (a, d) & aba \mapsto (d, a) \\ d \mapsto (1, b) & aba \mapsto (b, 1) \end{array}$$

extends to a group homomorphism $\psi_0 : \Xi_0 \longrightarrow \mathfrak{G}_0 \times \mathfrak{G}_0$ [78, Proposition 1]. For each $n \geq 0$, define now

$$\begin{aligned} N_n &= \langle\langle u_0, \ldots, u_n, v_0, \ldots, v_{n-1} \rangle\rangle_{\mathfrak{G}_0}; \text{ observe that} N_n \subset \Xi_0; \\ \mathfrak{G}_n &= \mathfrak{G}_0 / N_n \quad \text{and} \quad \pi_n : \mathfrak{G}_0 \twoheadrightarrow \mathfrak{G}_n \text{ the canonical projection}; \\ B_n &= \langle\langle b \rangle\rangle_{\mathfrak{G}_n} = \pi_n(B_0); \\ \Xi_n &= \langle b, c, d, aba, aca, ada \rangle_{\mathfrak{G}_n} = \pi_n(\Xi_0); \\ D_n^{\text{diag}} &= \langle (a, d), (d, a) \rangle_{\mathfrak{G}_n \times \mathfrak{G}_n}; \\ \sigma_n &: \mathfrak{G}_{n-1} \longrightarrow \mathfrak{G}_n, gN_{n-1} \longmapsto \sigma(g)N_n \qquad (\text{for } n \geq 1 \text{ only}). \end{aligned}$$

For the definition of the homomorphism σ_n, note that $\sigma(N_{n-1}) \subset N_n$.

Lemma 5.5 ([78], Lemma 3) *Let B_0 denote the* normal *subgroup of $\mathfrak{G}_0$ generated by b. Then:*

(i) *B_0 is of index* 8 *in $\mathfrak{G}_0$;*
(ii) *B_0 is generated by the four elements* $\xi_1 := b, \xi_2 := aba, \xi_3 := dabad, \xi_4 := adabada;$
(iii) *B_0 has the presentation $\langle \xi_1, \xi_2, \xi_3, \xi_4 \mid \xi_1^2 = \xi_2^2 = \xi_3^2 = \xi_4^2 = 1 \rangle$;*
(iv) *B_0 contains N_n for all $n \geq 1$.*

Lemma 5.6 ([78], mostly Proposition 10)

(i) *The kernel and the image of the homomorphism ψ_0 are given by*

$$\begin{aligned} \ker(\psi_0) &= \langle\langle u_1, v_0 \rangle\rangle_{\Xi_0}, \\ \operatorname{Im}(\psi_0) &= (B_0 \times B_0) \rtimes D_n^{\text{diag}} \textit{ of index } 8 \textit{ in } \mathfrak{G}_0 \times \mathfrak{G}_0. \end{aligned}$$

(ii) *For $n \geq 1$, the homomorphism ψ_0 induces an isomorphism*

$$\psi_n : \Xi_n \xrightarrow{\simeq} (B_{n-1} \times B_{n-1}) \rtimes D_{n-1}^{\text{diag}} <_8 \mathfrak{G}_{n-1} \times \mathfrak{G}_{n-1}$$

where $<_8$ indicates that the left-hand side is a subgroup of index 8 in the right-hand side.

Set $K_0 = \langle\langle (ab)^2 \rangle\rangle_{\mathfrak{G}_0}$; observe that $K_0 \subset B_0$.

Lemma 5.7 (i) *The subgroup K_0 is of index* 2 *in B_0. It is generated by*

$$t = (ab)^2 \quad v = (bada)^2 \quad w = (abad)^2$$

Moreover K_0 contains N_n for $n \geq 1$.
(ii) *The group K_0 is a free group of rank 3.*

Proof (i) This follows from [93, p. 230]. Since B_0 contains N_n and each u_n, v_n is a fourth power, necessarily N_n is contained in K_0.

For (ii), see [11, Proposition 4], where the proof uses Kurosh's theorem. Alternatively one can use the Reidemeister–Schreier method to find a presentation for K_0 and see that it is indeed free of rank 3. □

Lemma 5.8 *If g is an element of B_{n-1} then*

$$\psi_n(\sigma_n(g)) = (1, g) \quad \text{and} \quad \psi_n(a\sigma_n(g)a) = (g, 1).$$

Proof For the generators of B_{n-1} that are images of those of Lemma 5.5 for B_0, we have

$$\begin{aligned} \psi_n(\sigma_n(b)) &= \psi_n(d) = (1, b), \\ \psi_n(\sigma_n(aba)) &= \psi_n(acadaca) = (d^2, aba) = (1, aba), \\ \psi_n(\sigma_n(dabad)) &= \psi_n(cacadacac) = (ad^2a, dabad) = (1, dabad), \\ \psi_n(\sigma_n(adabada)) &= \psi_n(acacacadacacaca) = (1, adabada), \end{aligned}$$

and this shows the first equality. The second follows because, if $\psi_n(h) = (h_0, h_1)$, then $\psi_n(aha) = (h_1, h_0)$. □

Let $K_n = K_0/N_n$. It is a normal subgroup of $\mathfrak{G}_n$ contained in B_n.

Lemma 5.9 *Let $n \geq 1$.*

(i) *We have $\sigma_n(K_{n-1}) \subset K_n \subset B_n$.*
(ii) *If H_{n-1} is a subgroup of K_{n-1}, then $\psi_n^{-1}(H_{n-1} \times H_{n-1}) \subset K_n$.*

Proof (i) Let t, v, w be now the canonical images in K_n of the elements of K_0 denoted by the same symbols in Lemma 5.7. On the one hand, we have $\psi_n(\sigma_n(t)) = (1, t)$ by Lemma 5.8. On the other hand, we have

$$\psi_n(w) = \psi_n(aba)\psi_n(d)\psi_n(aba)\psi_n(d) = (cc, abab) = (1, t)$$

by the definitions of ψ_n and w. Hence $\sigma_n(t) = w \in K_n$ by Lemma 5.6.ii.

Let $g_1 \in \mathfrak{G}_{n-1}$. From the definition of ψ_n, we see that the composition $\Xi_n \longrightarrow \mathfrak{G}_{n-1}$ of ψ_n with a projection onto one of the factors is onto. Hence there exists $g \in \Xi_n$ and $g_0 \in \mathfrak{G}_{n-1}$ such that $\psi_n(g) = (g_0, g_1)$. We have as above[10] $\psi_n(\sigma_n(t^{g_1})) = (1, t^{g_1})$ and

[10] Remember that $t^h = h^{-1}th$.

$$\psi_n(w^g) = \psi_n(w)^{\psi_n(g)} = (1,t)^{\psi_n(g)} = (1, t^{g_1}),$$

and therefore $\sigma_n(t^{g_1}) = w^g$. Since K_n is a normal subgroup of $\mathfrak{G}_n$ containing w, we have $\sigma_n(t^{g_1}) \in K_n$ for all $g_1 \in \mathfrak{G}_{n-1}$. The inclusion $\sigma_n(K_{n-1}) \subset K_n$ follows, because K_{n-1} is generated by t as a normal subgroup of $\mathfrak{G}_{n-1}$.

(ii) Let $(h_0, h_1) \in H_{n-1} \times H_{n-1}$. We have

$$\psi_n^{-1}(h_0, h_1) \;=\; a\sigma_n(h_0)a\sigma_n(h_1)$$

by Lemma 5.8, and the right-hand side is in K_n by (i). □

Set $H_0 = K_0$. For $n \geq 1$, define inductively

$$H_n \;=\; \psi_n^{-1}(H_{n-1} \times H_{n-1}).$$

The definition makes sense by Lemma 5.9.ii. The following lemma finishes the proof of Theorem 5.3.

Lemma 5.10 *Let $n \geq 0$, and the notation be as above.*

(i) *H_n is a normal subgroup of $\mathfrak{G}_n$ contained in K_n.*
(ii) *The group H_n is a direct product of 2^n free groups of rank 3.*
(iii) *Its index is given by $[\mathfrak{G}_n : H_n] = 2^{(2^{n+1}+2)}$.*

Proof For $n = 0$, the three claims follow from Lemmas 5.5 and 5.7. We suppose now that $n \geq 1$ and that the lemma holds for $n-1$.

(i) The group H_n is clearly normal in Ξ_n, by Lemma 5.6.ii. To show that H_n is normal in $\mathfrak{G}_n$, it suffices to check that $aH_na \subset H_n$, because $\mathfrak{G}_n$ is generated by Ξ_n (of index 2 in $\mathfrak{G}_n$) and a. Let $h \in H_n$. Let $h_0, h_1 \in H_{n-1}$ be defined by $\psi_n(h) = (h_0, h_1)$. Then $\psi_n(aha) = (h_1, h_0) \in H_{n-1} \times H_{n-1}$, and therefore $aha \in H_n$.

(ii) This is a straightforward consequence of the isomorphism $H_n \simeq H_{n-1} \times H_{n-1}$, see again Lemma 5.6.

(iii) By the induction hypothesis, we have

$$\begin{aligned}
&[(B_{n-1} \times B_{n-1}) \rtimes D_{n-1}^{\mathrm{diag}} : H_{n-1} \times H_{n-1}] \\
&\quad = \frac{[\mathfrak{G}_{n-1} \times \mathfrak{G}_{n-1} : H_{n-1} \times H_{n-1}]}{[\mathfrak{G}_{n-1} \times \mathfrak{G}_{n-1} : (B_{n-1} \times B_{n-1}) \rtimes D_{n-1}^{\mathrm{diag}}]} \\
&\quad = \frac{2^{2^n+2} \times 2^{2^n+1}}{2^3} \;=\; 2^{2^{n+1}+1}.
\end{aligned}$$

Thus the commutative diagram

$$\begin{array}{lcl} & \mathfrak{G}_n & & \mathfrak{G}_{n-1} \times \mathfrak{G}_{n-1} \\ & | & & | \\ & 2 & & 2^3 \\ & | & & | \\ \psi_n : & \Xi_n & \overset{\cong}{\to} & (B_{n-1} \times B_{n-1}) \rtimes D_{n-1}^{diag} \\ & | & & | \\ & 2^{(2^{n+1}+1)} & & 2^{(2^{n+1}+1)} \\ & | & & | \\ & H_n & \longrightarrow & H_{n-1} \times H_{n-1} \end{array}$$

shows that H_n has index $2^{(2^{n+1}+2)}$ in $\mathfrak{G}_n$. □

The proof of Theorem 5.3 is now complete.

Recall that a group G is called of type FP$_n$ if the trivial $\mathbf{Z}[G]$-module $\mathbf{Z}$ has a projective resolution, namely if there exists an exact sequence

$$\cdots \longrightarrow P_{j+1} \longrightarrow P_j \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbf{Z}$$

where the P_j 's are projective $\mathbf{Z}[G]$-modules for all $j \geq 0$, and finitely generated projective $\mathbf{Z}[G]$-modules for all $j \leq n$. It is known that

(i) a group is of type FP$_1$ if and only if it is finitely generated;
(ii) finitely presented groups are of type FP$_2$;
(iii) Condition FP$_2$ is *strictly* weaker than finite presentability;
(iv) a group is of type FP$_2$ if and only if it is the quotient of some finitely presented group by a *perfect* normal subgroup.

For (i) and (ii), see for example the notes in which "type FP$_n$" was first defined [28]; see [27] for (iii) and [38, Section VII.5, Exercise 3] for (iv). The following question is natural:

Does $\mathfrak{G}$ have an amenable cover of type FP$_2$?

The answer is due to Yves de Cornulier (unpublished). We reproduce it here, with our thanks to him.

Proposition 5.11 (de Cornulier) *Any cover of type FP$_2$ of the group $\mathfrak{G}$ is large.*

Proof Let $E \twoheadrightarrow \mathfrak{G}$ be a cover, with E of type FP$_2$. By (iv) above, there exists a finitely presented group F and a perfect normal subgroup P of F such that F/P is isomorphic to E. For n large enough and $\mathfrak{G}_n$ as in Theorem 5.3, there exists by Proposition 3.3 a normal subgroup K_n of F such that F/K_n is isomorphic to $\mathfrak{G}_n$. Observe that the group

$$F/PK_n \simeq (F/K_n)/(P/P \cap K_n) \simeq \mathfrak{G}_n/(P/P \cap K_n)$$

is a quotient of $\mathfrak{G}_n$ by a perfect normal subgroup.

Since $\mathfrak{G}_n$ has a subgroup of index a power of 2 which is a direct product of free groups (Theorem 5.3, or [11]), $\mathfrak{G}_n$ is residually soluble. It follows that the only perfect subgroup of $\mathfrak{G}_n$ is $\{e\}$, so that $F/PK_n \simeq \mathfrak{G}_n$. Hence F/PK_n is large, and so is its cover $F/P \simeq E$. □

Appendix A: On soluble groups, metabelian groups, and finite presentations

The existence of groups that are finitely generated and infinitely presented was established by B. H. Neumann in 1937. More precisely, he constructed *uncountably many 2-generator groups* [120, Theorem 14]; at most countably many of them are finitely presented. Later it was checked that none of them is finitely presented (see the last proof of Appendix C, as well as [26, Theorem C]).

Infinitely generated *soluble* groups are equally abundant, as we recall below after having fixed some notation.

The groups of the derived series of a group G are defined inductively by $D^0G = G$ and $D^{\ell+1}G = [D^\ell G, D^\ell G]$. The **free soluble group of rank** k **and solubility class** ℓ is the quotient $\mathrm{FSol}(k, \ell) = F_k/D^\ell F_k$, where F_k stands for the free group of rank k. Any k-generated soluble group of solubility class at most ℓ is a quotient of $\mathrm{FSol}(k, \ell)$. A group G is **metabelian** if $D^2G = 1$, namely if it is a cover of an abelian group with abelian kernel. The group $\mathrm{FSol}(k, 2)$ is the **free metabelian group of rank** k.

Philip Hall established the existence of *uncountably many finitely generated soluble groups*. His result is much more precise [90, Theorem 6]: given any countable abelian group $A \neq 1$, there exist uncountably many groups G such that

$$d(G) = 2, \qquad Z(G) \simeq A, \qquad [G, D^2G] = 1.$$

Here $d(G)$ stands for the minimal number of generators of G, and $Z(G)$ for its centre. The condition $[G, D^2G] = 1$ can be translated in words: G is a *centre-by-metabelian group*. It is moreover known that there are uncountably many finitely generated soluble groups which are not quasi-isometric to each other [55, Corollary 1.8].

On the contrary, there are only countably many finitely generated metabelian groups (this is repeated as Proposition A.3 below), and more generally[11] abelian-by-polycyclic groups ([90, Corollary 2 to Theorem 3], see also [107, Corollary 4.2.5]). Before comparing soluble groups in general with metabelian groups in particular, we collect some well-known facts in the following lemma.

Recall that a group G satisfies **Max-n**, **the maximal condition for normal subgroups**, if any increasing sequence of normal subgroups of G is ultimately stationary, or equivalently if any normal subgroup of G is finitely generated *as normal subgroup*.

Lemma A.1 *Let G be a finitely generated group, N a normal subgroup, and Z a central subgroup.*

(i) *If G/N is finitely presented, there exists a finite subset $S \subset N$ such that N is the smallest normal subgroup of G containing S.*

[11] Let $\mathcal{P}$ and $\mathcal{Q}$ be group properties. A group G is $\mathcal{P}$-by-$\mathcal{Q}$ if G has a normal subgroup N with Property $\mathcal{P}$ such that G/N has Property $\mathcal{Q}$.

(ii) *If G/Z is finitely presented, then Z is finitely generated.*
(iii) *If G has uncountably many normal subgroups, then G has uncountably many pairwise non-isomorphic quotients.*
(iv) *Suppose that G is finitely presented and satisfies Max-n. Then G/N is finitely presented.*

Proof Claim (i) is [132, Lemma 14.1.3]. It is a simple consequence of the following fundamental observation of B.H. Neumann: let S, S' be two finite generating sets of a group G; assume that G has a finite presentation $\langle S \mid R \rangle$ involving S and a finite set R of relators; then there exists a finite set R' of relators in the letters of S' such that $\langle S' \mid R' \rangle$ is also a finite presentation of G [121, Lemma 8].

Claim (ii) is the special case of (i) for a central subgroup.

For Claim (iii), consider an uncountable family $(N_\alpha)_{\alpha \in A}$ of distinct normal subgroups of G. Fix $\alpha \in A$. Let B be a subset of A such that, for each $\beta \in B$, there exists an isomorphism $\phi_\beta : G/N_\beta \longrightarrow G/N_\alpha$. It suffices to show that B is countable.

For $\beta \in B$, let π_β denote the composition of the canonical projection $G \longrightarrow G/N_\beta$ with ϕ_β. Since G is finitely generated and G/N_α countable, there are only countably many homomorphisms from G to G/N_α. As $N_\beta = \ker(\pi_\beta)$, the set B is countable.

For Claim (iv), consider a finite presentation of G, namely a free group F on a finite set S and a normal subgroup M of F generated as normal subgroup by a finite subset R of F, such that $G = F/M$. Since G satisfies Max-n, there exists a finite subset R' of F of which the image in G generates N as a normal subgroup. Then $\langle S \mid R \cup R' \rangle$ is a finite presentation of G/N.

Note that Claim (ii) is a special case of [90, Lemma 2]. Our argument for Claim (iii) can be found in [90, p. 433], and that for Claim (iv) is "a well-known principle" cited in [90, p. 420]. □

Finitely generated metabelian groups are "well-behaved" in many ways:

Proposition A.2 (Hall, Baumslag, Remeslennikov) *Let G be a finitely generated metabelian group.*

(i) *G satisfies Max-n. In particular, the centre of G is finitely generated.*
(ii) *If G is finitely presented, so is any quotient of G.*
(iii) *G is residually finite.*
(iv) *G has a soluble word problem.*
(v) *G can be embedded into a finitely presented metabelian group.*
(vi) *G is recursively presented.*

References Claim (i) is [90, Theorem 3], Claim (ii) follows by Lemma A.1.iv, and Claim (iii) is [91, Theorem 1].

For the particular case of free metabelian groups, Claim (iii) follows from linearity: it is known that $\mathrm{FSol}(k, 2)$ is a subgroup of $\mathrm{GL}_2(\mathbf{C})$.

This is a form of the "Magnus embedding theorem" ; see [110], and also [151, Theorem 2.11]. (On the contrary, $\mathrm{FSol}(k, \ell)$ is not linear when $\ell \geq 3$; see Remark A.6.)

Claim (iv) can be found in [20]. It is also a consequence of (a particular case of) a result of Wehrfritz: any finitely generated metabelian group is quasi-linear, namely is a subgroup of a group of the form $\prod_{i=1}^{r} \mathrm{GL}_n(F_i)$, where $F_1, \ldots, F_r$ are fields [152]. More generally, several algorithmic problems are known to be soluble in finitely generated metabelian groups [25].

Claim (v) was proved by Baumslag [19] and Remeslennikov [131], independently. See also [107, Proposition 11.3.2].

Claim (vi) is [20, Corollary A1]; it also follows from Claim (v). Note that Claim (vi) is contained in Claim (iv), but we add it for comparison with Proposition A.5. □

Proposition A.3 (P. Hall) *There are countably many finitely generated metabelian groups.*

Remark A.4 (a) Let G be a finitely generated metabelian group G; let $k \geq 0$ be such that G can be generated by k elements, so that G is a quotient of the free metabelian group $\mathrm{FSol}(k, 2)$. Though it need not be finitely presented (examples are shown below), the group G is finitely presented *as a metabelian group*, because $\mathrm{FSol}(k, 2)$ satisfies Max-n; in other terms, we can write

$$G = \mathrm{FSol}(k, 2)/\langle\langle r_1, \ldots, r_n \rangle\rangle,$$

where the notation $\langle\langle \cdots \rangle\rangle$ indicates a normal subgroup generated *as such* by elements $r_1, \ldots, r_n$ in $\mathrm{FSol}(k, 2)$. Proposition A.3 follows.

Note that Proposition A.3 is also a straightforward consequence of Claim (vi) in the previous proposition.

(b) Some of the claims in Proposition A.2 can be improved. For example, (i) holds for finitely generated abelian-by-polycyclic groups, and (iii) holds for finitely generated abelian-by-nilpotent groups (Hall). Moreover (iii) holds for abelian-by-polycyclic groups, as shown by Roseblade and Jategaonkar in 1973 and 1974 (see [134], or Chapter 7 and in particular Theorem 7.2.1 in [107]).

(c) Until the early 70's, there were rather few known examples of finitely presented metabelian groups. The 3-generator 3-relator group

$$H = \langle a, s, t \mid a^t = aa^s, [s, t] = 1 = [a, a^s] \rangle$$

appeared independently in papers by Baumslag [18] and Remeslennikov [131]; see also [145, Theorem A]. It is metabelian, its derived group is free abelian of infinite rank, and it contains the wreath product $\mathbf{Z} \wr \mathbf{Z}$ as a subgroup [20, Pages 72–73]. It was quite a surprise at this time [18, first lines] to find a finitely presented group containing a normal abelian subgroup of infinite rank. More recently, the quotient group $H/(a^2 = 1)$ was the main character in [77].

For groups of higher solubility degrees, the picture is substantially different, even under the stronger hypothesis of finite presentability. Each of the claims of the next proposition is meant to be compared with the corresponding claim of Proposition A.2.

Proposition A.5 *Let G be a finitely presented soluble group.*

(i) *G need not satisfy Max-n. Indeed, the centre of G need not be finitely generated.*

(ii) *G may have uncountably many quotients, and in particular infinitely presented quotients.*
(ii') *Any metabelian quotient of G is finitely presented.*
(iii) *G need not be residually finite.*
(iv) *G need not have a soluble word problem.*

Let G be now a finitely generated soluble group.

(v) *G need not be recursively presented.*
(vi) *G need not embed into any finitely presented group.*

On the proof Let p be a prime. For $n \geq 2$, consider the group A_n of n-by-n triangular matrices of the form

$$\begin{pmatrix} 1 & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

with upper-triangular entries $a_{i,j}$, $1 \leq i < j \leq n$, in $\mathbf{Z}[\frac{1}{p}]$, and diagonal entries $a_{2,2}, \ldots, a_{n-1,n-1}$ in $p^{\mathbf{Z}}$. This group is soluble. Its center $Z(A_n)$ is isomorphic to $\mathbf{Z}[\frac{1}{p}]$, that is to A_2, and therefore is not finitely generated; it follows that A_n does not satisfy Max-n. It is easy to check that A_n is finitely generated when $n \geq 3$.

For $n \geq 4$, the group A_n is finitely presented ([1] for $n = 4$ and [2] for $n \geq 4$). This justifies Claim (i). The existence of a finitely presented soluble group without Max-n solves a problem of P. Hall; Remeslennikov had an earlier claim for this [130] which was apparently unjustified [148].

Note that A_3 is infinitely presented. This was most likely known to P. Hall, and can be found in [2]. But it is also a consequence of Bieri–Strebel Theorem 1.5; indeed, since $Z(A_3)$ is not finitely generated, $A_3/Z(A_3)$ is infinitely presented (Lemma A.1), and the soluble group A_3 cannot be a finitely presented cover of the metabelian group $A_3/Z(A_3)$.

Since $Z(A_n) \simeq \mathbf{Z}[\frac{1}{p}]$ is not finitely generated, the quotient $A_n/Z(A_n)$, with $n \geq 3$, is finitely generated non-finitely presented, by Lemma A.1.ii. When $n \geq 4$, this justifies the second part of Claim (ii).

Claim (ii') follows from Theorem 1.5.

For $n \geq 3$, the quotient of A_n by the central subgroup

$$\left\{ (z_{i,j})_{1 \leq i,j \leq n} \in A_n \,\middle|\, \begin{array}{l} z_{i,j} = \delta_{i,j} \quad \text{for} \quad (i,j) \neq (1,n) \\ z_{1,n} \in \mathbf{Z} \end{array} \right\} \simeq \mathbf{Z}$$

(where $(\delta_{i,j})_{1 \leq i,j \leq n}$ denotes the unit matrix) is finitely generated non-Hopfian[12] (the argument of [92] for $n = 3$ carries over to all $n \geq 3$), and therefore non-residually finite. When $n \geq 4$, this justify Claim (iii).

[12] A group G is **non-Hopfian** if there exists a surjective endomorphism of G onto itself that is not injective. A finitely generated residually finite group is Hopfian [111].

Still for $n \geq 4$, it is known that the quotient $A_n/Z(A_n)$ does satisfy Max-n and does not have any minimal presentation [35, Lemma 3.2 and Corollary 3.6]. The last statement means that any presentation of $A_n/Z(A_n)$ contains redundant relators; in particular, the finitely related group A_n has a quotient that is not finitely related. The group A_n itself has only countably many quotients; see [109, Theorem 1] and [35, Corollary 3.4].

Concerning Claim (iv), finitely presented soluble groups with unsoluble word problems have been constructed by Kharlampovich in [102] and by Baumslag, Gildenhuys and Strebel in [24]. Groups in [24] have centers that are not finitely generated, and therefore have infinitely presented quotients (see again Claim (ii)).

Earlier, Meskin had constructed a finitely generated recursively presented residually finite soluble group with unsoluble word problem [116].

For any prime p, there exists a finitely presented soluble group G with centre $(C_p)^{(\infty)}$, an infinite direct sum of cyclic groups of order p (see [103], as well as [104, Lemma 4.14]). Hence G has uncountably many quotients, by Lemma A.1. iii. This justifies the first part of Claim (ii).

Claim (v) follows from the existence of uncountably many finitely generated soluble groups, because there are only countably many recursively presented groups.

(Digression: Let G be a finitely generated soluble group. Assume that G is of finite Prüfer rank, namely that there exists an integer $d \geq 1$ such that any finitely generated subgroup of G can be generated by d elements. Then G is recursively presented if and only if G has a soluble word problem. For this, and for examples of G with and without soluble word problem, see [41].)

Claim (vi) follows from Claim (v) because a finitely generated subgroup of a finitely presented group is recursively presented. (This is straightforward; see the first page of [95], where Higman establishes the famous non-trivial converse; alternatively, see [117, Lemma 2.1].) □

Remark A.6 As noted parenthetically just after Proposition A.2, free solvable groups $\mathrm{FSol}(k, \ell)$ are not linear when $k \geq 2$ and $\ell \geq 3$ [141,142]. Here is a proof.

For $k \geq 3$, here is first a short reduction to a more standard result. Let $\mathfrak{C}$ denote the class of (nilpotent-by-abelian)-by-finite groups; observe that quotients of groups in $\mathfrak{C}$ are in $\mathfrak{C}$. The Lie-Kolchin-Mal'cev theorem (see for example [132, Section 15.1]) establishes that soluble linear groups are in $\mathfrak{C}$. If $\mathrm{FSol}(k, \ell)$ were linear, hence in $\mathfrak{C}$, so would be any quotient, in particular the iterated wreath product $(\mathbf{Z} \wr \mathbf{Z}) \wr \mathbf{Z}$; but this is not [132, 15.1.5].

The following argument, shown to us by Ralph Strebel, holds for any $k \geq 2$. Denote by $x_1, \ldots, x_k$ a set of free generators of $\mathrm{FSol}(k, \ell)$. Let H be a subgroup of finite index in $\mathrm{FSol}(k, \ell)$; there exists an integer $m \geq 1$ such that $x_1^m, x_2^m \in H$; let U denote the subgroup of $\mathrm{FSol}(k, \ell)$ generated by x_1^m and x_2^m, so that $U \subset H \subset_{\text{finite index}} \mathrm{FSol}(k, \ell)$. The group U is isomorphic to $\mathrm{FSol}(k, \ell)$, by a result due independently to Gilbert Baumslag [17, Theorem 2] and Shmel'kin [138, Theorem 5.2].

If $\mathrm{FSol}(k, \ell)$ was linear, it would have a nilpotent-by-abelian subgroup of finite index, say H. By the lines above, $\mathrm{FSol}(2, \ell)$ would be nilpotent-by-abelian. Hence any 2-generated soluble group of solubility class at least 3 would be nilpotent-by-abelian. But this is not true.

Indeed, consider the symmetric group S_4 on four letters. It is generated by two elements, a transposition and a 4-cycle. It is soluble of class 3, with $D^1 S_4 = A_4$ (the alternating group on four letters), $D^2 S_4 = V$ (the Klein Vierergruppe), and $D^3 S_4 = \{1\}$. It is not nilpotent-by-abelian, namely A_4 is not nilpotent: $[A_4, V] = V$.

The argument of Smirnov is different. It relies on the bi-orderability of the groups $\mathrm{FSol}(k, \ell)$.

Appendix B: On wreath products and lamplighter groups

Permutational wreath products have been defined in the beginning of Section 2. The **standard wreath product** $G \wr_H H$ refers to the action of H on itself by left multiplications.

Proposition B.1 *Consider two groups G, H, a non-empty H-set X, and the permutational wreath product $G \wr_X H$. We assume that $G \neq \{1\}$.*

(i) *$G \wr_X H$ is finitely generated if and only if G, H are finitely generated and H has finitely many orbits on X.*
(ii) *$G \wr_X H$ is finitely presented if and only if G, H are finitely presented, the H_x's $(x \in X)$ are finitely generated, and there are finitely many orbits in $X \times X$ for the diagonal action of H (where $H_x = \{h \in H \mid h(x) = x\}$).*
(iii) *In particular, as soon as H is infinite, the standard wreath product $G \wr_H H$ is not finitely presented.*

References Claim (i) is standard, and easy; if necessary, see [52, Proposition 2.1]. For Claim (ii), see [52, Theorem 1.1]. Claim (iii) is the main result of [16]. □

As a particular case of Proposition B.1, if G is finitely-generated abelian and $G \neq 1$, the group $G \wr \mathbf{Z}$ is metabelian, finitely generated, and infinitely presented. When G is finite abelian and $H \simeq \mathbf{Z}$ infinite cyclic, we will call $G \wr \mathbf{Z}$ the **lamplighter group** for G. (For this terminology, precise assumptions on G and H vary from one author to the other; some ask that $G = \mathbf{Z}/2\mathbf{Z}$.)

Proposition B.2 *Consider two finitely presented groups G, H, an H-set X such that H has finitely many orbits on X and infinitely many orbits on $X \times X$, and the permutational wreath product $G \wr_X H$. We assume that $G \neq \{1\}$.*

(i) *$G \wr_X H$ is finitely generated and is infinitely presented.*
(ii) *For any finitely presented cover $\pi : E \twoheadrightarrow G \wr_X H$, the group E has non-abelian free subgroups.*

Proof Claim (i) is a particular case of Proposition B.1. In Proposition 2.10 of [52], it is shown that the kernel of π contains non-abelian free subgroups. □

Note that, in the particular case of two abelian groups G and H, Claim (ii) is also a consequence of Theorem 1.5.

On other proofs of Proposition B.2 in the case of $W := \mathbf{Z} \wr_{\mathbf{Z}} \mathbf{Z}$. (We have $G = H = \mathbf{Z}$ and $X = \mathbf{Z}$.) We have a presentation

$$W = \langle s, t \mid [s^{t^i}, s^{t^j}] \; \forall i, j \in \mathbf{Z} \rangle;$$

indeed, any element in the right-hand side can be written as

$$\begin{aligned} & s^{m_1} t^{n_1} s^{m_2} t^{n_2} s^{m_3} t^{n_3} \cdots s^{m_\ell} t^{n_\ell} \\ & \quad = s^{m_1} \left(s^{t^{-n_1}} \right)^{m_2} \left(s^{t^{-n_1-n_2}} \right)^{m_3} \cdots \left(s^{t^{-n_1-\cdots-n_{\ell-1}}} \right)^{m_\ell} t^{n_1+\cdots+n_\ell} \end{aligned}$$

for some $m_1, n_1, \ldots, m_\ell, n_\ell \in \mathbf{Z}$, and therefore as

$$\left(s^{t^{j_1}} \right)^{i_1} \left(s^{t^{j_2}} \right)^{i_2} \cdots \left(s^{t^{j_k}} \right)^{i_k} t^N$$

for appropriate $i_1, j_1, \ldots, i_k, j_k, N \in \mathbf{Z}$ with $t_1 < t_2 < \cdots < t_k$. It follows that the natural homomorphism

$$\langle s, t \mid [s^{t^i}, s^{t^j}] \; \forall i, j \in \mathbf{Z} \rangle \longrightarrow W$$

is an isomorphism.

Since $t^i [s, s^{t^k}] t^{-i} = [s^{t^i}, s^{t^{i+k}}]$, we have a second presentation

$$W = \langle s, t \mid [s, s^{t^i}] \; \forall i \in \mathbf{N} \rangle.$$

For a positive integer n, define

$$W_n = \langle s, t \mid [s, s^{t^i}], i = 0, \ldots, n \rangle.$$

Note that $\lim_{n \to \infty} W_n = W$ in $\mathcal{M}_2$. We have a third presentation

$$W_n = \left\langle s_0, \ldots, s_n, t \;\middle|\; \begin{array}{ll} [s_i, s_j], & 0 \le i, j \le n, \\ s_k^t = s_{k+1}, & 0 \le k \le n-1 \end{array} \right\rangle.$$

Indeed, it can be checked that the assignments

$$\begin{aligned} \varphi_1 &: s \longmapsto s_0, t \longmapsto t \\ \varphi_2 &: s_i \longmapsto s^{t^i}, t \longmapsto t \quad (0 \le i \le n) \end{aligned}$$

define, between the groups of the two previous presentations, isomorphisms that are inverse to each other.

Let H_n be the free abelian subgroup of W_n generated by $s_0, \ldots, s_n$. Denote by K_n the subgroup of H_n generated by $s_0, \ldots, s_{n-1}$, and by L_n that generated by $s_1, \ldots, s_n$; observe that $K_n \simeq L_n \simeq \mathbf{Z}^n$. Let $\psi_n : K_n \longrightarrow L_n$ be the isomorphism defined by $\psi(s_{i-1}) = s_i$ for $i = 1, \ldots, n$. Then W_n is clearly the HNN-extension corresponding to the data $(H_n, \psi_n : K_n \overset{\simeq}{\to} L_n)$. By Britton's lemma, W_n contains non-abelian free subgroups.

It follows from Corollary 3.4 that any finitely presented cover of W contains non-abelian free groups.

Let us finally allude to another argument showing that W is infinitely presented. Let H denote the subgroup of $\mathrm{GL}_3(\mathbf{C})$ generated by the three matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

where t is some transcendental number. The centre $Z(H)$ of H is free abelian of infinite rank, and the quotient $H/Z(H)$ is isomorphic to $M := \mathbf{Z}[t, t^{-1}]^2 \rtimes Z$, where $\rtimes$ refers to the action of the generator $1 \in \mathbf{Z}$ by $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$. There is an isomorphism of M onto a subgroup of index 2 in $Z[t, t^{-1}] \rtimes_t \mathbf{Z} \simeq \mathbf{Z} \wr \mathbf{Z}$, given by $(P(t), Q(t), n) \mapsto (P(t^2) + tQ(t^{-2}), 2n)$. It follows from Lemma A.1.ii that $\mathbf{Z} \wr \mathbf{Z}$ is not finitely presented. We are grateful to Adrien Le Boudec for correcting an earlier version of our argument at this point.

The isomorphism of $H/Z(H)$ with a subgroup of index 2 in $\mathbf{Z} \wr \mathbf{Z}$ is essentially due to P. Hall. See [90, Theorem 7] and [42, Lemma 3.1]. □

Concerning Proposition B.2, let us add one more remark about the particular case $W = (\mathbf{Z}/h\mathbf{Z}) \wr \mathbf{Z}$, with $h \geq 2$: it is known that any finitely presented cover of W is large [21, Section IV.3, Theorem 7].

Recall that the free soluble group $\mathrm{FSol}(k, \ell) = F_k/D^\ell F_k$ of rank k and solubility class ℓ has been defined in Appendix A.

Corollary B.3 *For $k, \ell \geq 2$, the group $\mathrm{FSol}(k, \ell)$ is infinitely presented, and any finitely presented cover of it contains non-abelian free subgroups.*

Note. (i) That $\mathrm{FSol}(k, \ell)$ is infinitely presented is a result due to [139]. See also [52, Proposition 2.10 and Corollary 2.14].

(ii) Corollary B.3 and our proof carry over to free polynilpotent groups

$$\mathrm{FPolynilp}(k, \ell_1, \ldots, \ell_n) := F_k/C^{\ell_k}(C^{\ell_{k-1}}(\cdots C^{\ell_1}(F_k)\cdots))$$

for any $k \geq 2$, $n \geq 2$, and $\ell_1, \ldots, \ell_n \geq 2$ (where $C^\ell G$ denotes the ℓth group of the lower central series of a group G, defined by $C^1 G = G$, and $C^{\ell+1}G = [G, C^\ell G]$). These groups are the subject of [138].

Proof Since $\mathbf{Z} \wr \mathbf{Z}$ is a two-generator metabelian group, we have an epimorphism $\mathrm{FSol}(2, 2) \twoheadrightarrow \mathbf{Z} \wr \mathbf{Z}$. Indeed, we have a sequence of natural epimorphisms

$$\mathrm{FSol}(k, \ell) \twoheadrightarrow \mathrm{FSol}(k, 2) \twoheadrightarrow \mathrm{FSol}(2, 2) \twoheadrightarrow \mathbf{Z} \wr \mathbf{Z}.$$

Hence any finitely presented cover of $\mathrm{FSol}(k, \ell)$ is also one of $\mathbf{Z} \wr \mathbf{Z}$. If $\mathrm{FSol}(k, \ell)$ was finitely presented, it would contain non-abelian free subgroup by Proposition B.2, but this cannot be in a soluble group. □

In the situation of Proposition B.2, suppose moreover that H is infinite residually finite, G has at least one non-trivial finite quotient, and consider the standard wreath product ($X = H$). The following strengthening of Claim (ii) is shown in [54, Theorem 1.5]: any finitely presented cover of $G \wr H$ is *large*. In particular:

any finitely presented cover of $\mathbf{Z} \wr \mathbf{Z}$ *is large.*

By the proof of Corollary B.3, it follows that,

for $k, \ell \geq 2$, *any finitely presented cover of* $\mathrm{FSol}(k, \ell)$ *is large*.

The following notion provides interesting examples of metabelian groups, as we will illustrate with Baumslag-Solitar groups.

Definition B.4 The **metabelianization** of a group G is the metabelian quotient group G/D^2G.

Definition B.5 ([22]) For $\ell, m \in \mathbf{Z} \smallsetminus \{0\}$, the **Baumslag-Solitar group** is defined by the two-generators one-relator presentation

$$\mathrm{BS}(\ell, m) \;=\; \langle s, t \mid t^{-1}s^{\ell}t = s^m \rangle.$$

We collect three well-known properties of these groups as follows.

Proposition B.6 *Let* $\ell, m \in \mathbf{Z} \smallsetminus \{0\}$ *and* $\mathrm{BS}(\ell, m)$ *be as above.*

(i) BS (ℓ, m) *is abelian if and only if BS* (ℓ, m) *is nilpotent, if and only if* $\ell = m = \pm 1$.
(ii) BS (ℓ, m) *is metabelian if and only if BS* (ℓ, m) *is soluble, if and only if BS* (ℓ, m) *does not contain non-abelian free subgroups, if and only if* $|\ell| = 1$ *or* $|m| = 1$.
(iii) If ℓ, m *satisfy* $\ell, m \geq 2$ *and are coprime, then BS* (ℓ, m) *is non-Hopfian.*
(iv) For ℓ, m *as in (iii), the group BS* (ℓ, m) *contains non-abelian free groups, but is not large.*

On the proof It is easy to check that the four groups $\mathrm{BS}(\ell, m), \mathrm{BS}(m, \ell), \mathrm{BS}(-\ell, -m)$, $\mathrm{BS}(-m, -\ell)$ are isomorphic. For simplicity, let us assume from now on that ℓ and m are positive. (For the general case, with all details, we refer to [143]).

It is an exercise to check that $\mathrm{BS}(1, m) \simeq \mathbf{Z}\left[\frac{1}{m}\right] \rtimes_m \mathbf{Z}$ for any $m \geq 1$. It follows that $\mathrm{BS}(\ell, m)$ is metabelian if $\ell = 1$ or $m = 1$, and abelian if and only if $\ell = m = 1$. If $m \geq 2$, note that $\mathrm{BS}(1, m)$ is not nilpotent, because its subgroup $\mathbf{Z}\left[\frac{1}{m}\right]$ is not finitely generated. If $\ell \geq 2$ and $m \geq 2$, the subgroup generated by $s^{-1}ts$ and t is free of rank 2, by Britton's Lemma.

Claim (iii) is the main reason for the celebrity of these groups. It is straightforward to check that the assignments $\varphi(s) = s^{\ell}$ and $\varphi(t) = t$ define an endomorphism φ of $\mathrm{BS}(\ell, m)$. The image of φ contains t and s^{ℓ}, hence $t^{-1}s^{\ell}t^{-1} = s^m$, and therefore s; hence φ is onto. On the one hand, $\varphi([t^{-1}st, s]) = [t^{-1}s^{\ell}t, s^{\ell}] = [s^m, s^{\ell}] = 1$; on the other hand, $[t^{-1}st, s] = t^{-1}stst^{-1}s^{-1}ts^{-1} \neq 1$, where the last inequality holds by Britton's Lemma; hence φ is not one-to-one.

More generally, we know necessary and sufficient conditions on ℓ, m for $\mathrm{BS}(\ell, m)$ to be non-Hopfian; see [22,50,51].

The first part of Claim (iv) is standard (it also follows from Theorem 1.5, see B.8.ii below). For the second part, see Example 3.2 and Theorem 6 in [58]. □

Definition B.7 For two coprime positive integers ℓ, m, not both 1, let

$$\mathrm{Met}(\ell, m) = \begin{pmatrix} \left(\frac{\ell}{m}\right)^{\mathbf{Z}} & \mathbf{Z}\left[\frac{1}{\ell m}\right] \\ 0 & 1 \end{pmatrix} \simeq \mathbf{Z}\left[\frac{1}{\ell m}\right] \rtimes_{\ell/m} \mathbf{Z}$$

be the group of triangular matrices generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{\ell}{m} & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$\mu_{\ell,m} : \mathrm{BS}(\ell, m) \twoheadrightarrow \mathrm{Met}(\ell, m)$$

denote the epimorphism defined by

$$\mu_{\ell,m}(s) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \mu_{\ell,m}(t) = \begin{pmatrix} \frac{\ell}{m} & 0 \\ 0 & 1 \end{pmatrix}.$$

The following proposition collects facts on $\mathrm{BS}(\ell, m)$ and $\mathrm{Met}(\ell, m)$. Claims (iii) to (v) constitute a digression from our theme.

Proposition B.8 *Let the notation be as just above, and ℓ, m be two coprime positive integers, not both* 1. *For (ii) to (v), assume furthermore that* $\min\{l, m\} \geq 2$.

(i) *$\mu_{\ell,m}$ is an isomorphism if and only if $\ell = 1$ or $m = 1$.*
(ii) $\mathrm{Met}(\ell, m)$ *is infinitely presented.*
(iii) *The multiplicator group* $H_2(\mathrm{Met}(\ell, m), \mathbf{Z})$ *is trivial.*
(iv) $\mathrm{Met}(\ell, m)$ *is of cohomological dimension* 3.
(v) *For $x \in \mathbf{C}$ transcendental, the matrices* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ *generate a group isomorphic to* $\mathbf{Z} \wr \mathbf{Z}$.

On proofs Claim (i) has already been given as an exercise, in the proof of Proposition B.6. Claim (ii) is a consequence of a particular case of the main result of [23], or a consequence of [30, Theorem C]; see also [107, Proposition 11.4.3]. Claim (iv) is [65, Theorem 4]. Claim (v) is [107, Proposition 3.1.4].

For (iii), see [23, No 1.8]. Recall that, if a group G is finitely presented, then its multiplicator group $H_2(G, \mathbf{Z})$ is finitely generated; this is a simple consequence of the so-called *Schur-Hopf Formula*, for a group $G = F/R$ presented as a quotient of a free group F, which reads "$H_2(G, \mathbf{Z}) = (R \cap [F, F])/[R, F]$". Claim (iii) is one of the standard examples showing that the converse *does not* hold. □

Let ℓ, m be coprime positive integers, with $\ell, m \geq 2$. We denote by $p_{\ell,m} : F_2 \twoheadrightarrow \mathrm{BS}(\ell, m)$ the defining cover of the corresponding Baumslag-Solitar group, namely the cover mapping a basis of the free group of rank 2 onto $\{s, t\}$. Let $\varphi : \mathrm{BS}(\ell, m) \twoheadrightarrow$

$\mathrm{BS}(\ell, m)$ be the usual non-injective surjective endomorphism, as in Proposition B.6. For $n \geq 1$, set

$$M_n = \ker(\varphi^n), N_n = \ker(\varphi^n p_{m,l}), G_n = \mathrm{BS}(\ell, m)/M_n = F_2/N_n.$$

Observe that the sequence $(M_n)_{n\geq 1}$ is strictly increasing, yet G_n is isomorphic to $\mathrm{BS}(\ell, m)$ for each $n \geq 1$.

Proposition B.9 *Let ℓ, m be coprime integers, with $\ell, m \geq 2$, and* $\mathrm{Met}(\ell, m)$ *as in Definition* B.7. *Let $(G_n)_{n\geq 1}$ be as above.*

(i) $\mathrm{Met}(\ell, m)$ *is isomorphic to* $\mathrm{BS}(\ell, m)_{\mathrm{metab}}$.
(ii) *With the notation above, we have an isomorphism*

$$\mathrm{Met}(\ell, m) \simeq F_2/\Big(\bigcup_{n\geq 1} N_n\Big),$$

so that $\lim_{n\to\infty} G_n = \mathrm{Met}(\ell, m)$ *in* $\mathcal{M}_2$.
(iii) *Any finitely presented cover of* $\mathrm{Met}(\ell, m)$ *contains non-abelian free subgroups.*

Proof Claim (i) is part of [20, Theorem G].

Claim (ii) is [23, see 1.8] or [76, Theorem 3], there for $(\ell, m) = (2, 3)$, but the argument carries over to the case stated here. (When working on [76], the authors were not aware of [23].)

Since $\mathrm{BS}(\ell, m)$ contains non-abelian free subgroups by Proposition B.6.ii, and since $\lim_{n\to\infty} G_n = \mathrm{Met}(\ell, m)$, Claim (iii) follows by Corollary 3.4. □

Appendix C: On Bieri–Neumann–Strebel invariants

Example C.1 Consider the two metabelian groups

$$\mathrm{Met}(1, 6) = \mathrm{BS}(1, 6) = \mathbf{Z}[1/6] \rtimes_{1/6} \mathbf{Z}$$
$$\mathrm{Met}(2, 3) = \mathrm{BS}(2, 3)_{\mathrm{metab}} = \mathbf{Z}[1/6] \rtimes_{2/3} \mathbf{Z}.$$

The first group is finitely presented. By contrast, the second group is infinitely presented (Propositions B.8). This spectacularly different behavior of two superficially similar-looking groups was the initial motivation of Bieri and Strebel (later joined by Neumann and others) for the work that lead to the Σ-invariants; see [29], as well as [20, Problem 1]. We come back to these two examples in Example C.3.

Let G be a finitely generated group, S a finite generating set, and $\mathrm{Cay}(G, S)$ the corresponding **Cayley graph**, with vertex set G and edge set $\{\{g, h\} \mid g^{-1}h \in S \cup S^{-1}\}$. The set of **characters**, namely of group homomorphisms from G to $\mathbf{R}$, is a real vector space $\mathrm{Hom}(G, \mathbf{R})$ isomorphic to $\mathbf{R}^n$, where n is the torsion-free rank of the abelian group G/D^1G. Let $S(G)$ denote the **character sphere** of G, quotient of the set of non-zero characters by the natural action of the group of positive reals; we have

$S(G) \approx \mathbf{S}^{n-1}$, and we write $[\chi]$ the class in $S(G)$ of a character $\chi \neq 0$ in $\mathrm{Hom}(G, \mathbf{R})$. (The sign $\approx$ indicates a homeomorphism; $\mathbf{S}^k$ denotes the k-sphere, and $\mathbf{S}^{-1} = \emptyset$.) For such a $[\chi]$, set

$$G_{[\chi]} = \{g \in G \mid \chi(g) \geq 0\},$$

and let $\mathrm{Cay}_{[\chi]}(G, S)$ be the subgraph of $\mathrm{Cay}(G, S)$ spanned by $G_{[\chi]}$. The **Bieri–Neumann–Strebel invariant**, or shortly **BNS-invariant**, of G is defined to be

$$\Sigma^1(G) = \big\{[\chi] \in S(G) \mid \mathrm{Cay}_{[\chi]}(G, S) \text{ is connected}\big\} \subset S(G).$$

The superscript indicates that Σ^1 is just one out of many "geometric invariants" (see [33, 146]).

Observe that there is an antipodal map $[\chi] \longmapsto -[\chi] := [-\chi]$ defined on $S(G)$. Let G_0 be a finitely generated group and $p : G_0 \twoheadrightarrow G$ an epimorphism; then p induces a map $p^* : S(G) \longrightarrow S(G_0)$ which is a homeomorphism onto its image and which intertwines the antipodal maps.

On the one hand, the invariant of [32] involves one more group A on which D^1G acts in an appropriate way; here, we particularize to $A = D^1G$. On the other hand, the definition above is a reformulation of the original definition, see for example [146].

Theorem C.2 *Let G be a finitely generated group, and $\Sigma^1(G) \subset S(G)$ as above.*

(i) *$\Sigma^1(G)$ is independent of the choice of S.*
(ii) *$\Sigma^1(G)$ is open in $S(G)$.*
(iii) *$\Sigma^1(G) = S(G)$ if and only if the derived group D^1G is finitely generated.*
(iv) *Suppose that G is metabelian. Then G is finitely presented if and only if $\Sigma^1(G) \cup (-\Sigma^1(G)) = S(G)$, and G is polycyclic if and only if $\Sigma^1(G) = S(G)$.*
(v) *If G is finitely presented and has no non-abelian free subgroup, then $\Sigma^1(G) \cup (-\Sigma^1(G)) = S(G)$.*
(vi) *If G_0 is a finitely generated group and $p : G_0 \twoheadrightarrow G$ an epimorphism, then $\Sigma^1(G_0) \cap p^*(S(G)) \subset p^*(\Sigma^1(G))$. In particular:*

$$\Sigma^1(G) \cup (-\Sigma^1(G)) \subsetneqq S(G) \Longrightarrow \Sigma^1(G_0) \cup (-\Sigma^1(G_0)) \subsetneqq S(G_0).$$

On the proof For (i) see [146, Theorem A2.3].

For (ii), see [32, Theorem A] and [146, Theorem A3.3].

(iii) If D^1G is finitely generated, then $\Sigma^1(G) = S(G)$, [146, Proposition A2.6]. For the "iff", see [32, Theorem B1].

For (iv), see [31, Theorem A] and [146, Subsection B3.2c].

For (v), see [32, Theorem C] and [146, Theorem A5.1].

Claim (vi) is rather straightforward from the definitions [146, Proposition A4.5]. □

Example C.3 (i) $\Sigma^1(G) = S(G) \approx \mathbf{S}^{n-1}$ for G a finitely generated abelian group of torsion-free rank n.
(ii) $\Sigma^1(F_n) = \emptyset \subset S(F_n) \approx \mathbf{S}^{n-1}$ for the non-abelian free group F_n of rank $n \geq 2$; see [146, Item A2.1a].

(iii) For the soluble Baumslag-Solitar group $\mathrm{BS}(1,n)$, the invariant $\Sigma^1(\mathrm{BS}(1,n))$ is one of the two points of the sphere $S(\mathrm{BS}(1,n)) \approx \mathbf{S}^0$; the argument of [146, Item A2.1a] for $n = 2$ carries over to $n \geq 2$.
(iv) For two coprime integers $\ell, m \geq 2$, we have $\Sigma^1(\mathrm{Met}(\ell, m)) = \emptyset \subset \mathbf{S}^0$ [146, Item A3.6].
(v) If G is a semi-direct product $H \rtimes \mathbf{Z}$ of an infinite locally finite group H by an infinite cyclic group, then $\Sigma^1(G) = \emptyset \subset \mathbf{S}^0$ [146, Lemma B3.1].
(vi) For any $n \geq 1$ and any rational polyhedral subset P of $\mathbf{S}^{n-1}$, there exists a finitely presented group G with $S(G) \approx \mathbf{S}^{n-1}$ and a homeomorphism $p^* : \mathbf{S}^{n-1} \longrightarrow S(G)$ such that $\Sigma^1(G) = p^*(S(G) \smallsetminus P)$.

Some comment is in order for (vi), cited here to show that the invariant $\Sigma^1(G)$ can be more complicated than those of Examples (i) to (v). For a finitely generated group G, a non-zero character $\chi \in \mathrm{Hom}(G, \mathbf{R})$ is *rational* if $\chi(G)$ is an infinite cyclic subgroup of $\mathbf{R}$. Denote by $\mathrm{Hom}^*_{\mathbf{Q}}(G, \mathbf{R})$ the set of non-zero rational characters on G, and by $S_{\mathbf{Q}}(G)$ its image in $S(G)$; then $S_{\mathbf{Q}}(G)$ is a dense subset of $S(G)$ [146, Lemma B3.3].

Consider a positive integer n and the sphere $\mathbf{S}^{n-1} = S(\mathbf{Z}^n)$. A *rational hemisphere* of $\mathbf{S}^{n-1}$ is the closure of the image of the half-space $\{\chi \in \mathrm{Hom}^*_{\mathbf{Q}}(G, \mathbf{R}) \mid \chi(z) \geq 0\}$, for some $z \in \mathbf{Z}^n \smallsetminus \{0\}$. A *rational convex polyhedral subset* of $\mathbf{S}^{n-1}$ is a finite intersection of rational hemispheres. A *rational polyhedral subset* of $\mathbf{S}^{n-1}$ is a finite union of rational convex polyhedral subsets.

Given an integer $n \geq 1$ and a rational polyhedral subset $P \subset \mathbf{S}^{n-1}$, there is a finitely presented group G and an epimorphism $p : G \twoheadrightarrow \mathbf{Z}^n$ such that $p^* : S(\mathbf{Z}^n) \longrightarrow S(G)$ is a homeomorphism, and $\Sigma^1(G) = p^*(S(\mathbf{Z}^n) \smallsetminus P)$. See [32, Corollary 7.6] and [34, Chapter IV, Section 1.1].

Corollary C.4 *Let G be a finitely generated group and E a finitely presented cover of G. If $\Sigma^1(G) \cup (-\Sigma^1(G)) \subsetneqq S(G)$, then E contains non-abelian free subgroups. This holds in particular:*

when G is metabelian and infinitely presented,

when $G = H \rtimes \mathbf{Z}$ with H infinite locally finite, as in Example C.3.v.

Proof This is straightforward from Claims (iii) to (v) of Theorem C.2 and from Example C.3.v. □

In Corollary C.4, the claim concerning metabelian groups is precisely Theorem 1.5.

Example C.5 (B.H. Neumann) Let $\mathcal{V} = \{2 \leq v_1 < v_2 < v_3 < \cdots\}$ be an infinite increasing sequence of integers. Set

$$X_{\mathcal{V}} = \{(i, j) \in \mathbf{Z}^2 \mid i \geq 1, -v_i \leq j \leq v_i\}.$$

For each $i \geq 1$, denote by $X_{\mathcal{V},\,i}$ the subset $\{(i, j) \in X_{\mathcal{V}} \mid -v_i \leq j \leq v_i\}$, of cardinal $2v_i + 1$. Define two permutations $\alpha_{\mathcal{V}}, \beta_{\mathcal{V}}$ of the set $X_{\mathcal{V}}$ as follows: for each $i \geq 1$, they preserve $X_{\mathcal{V},\,i}$, and

($\alpha_{\mathcal{V}}$) the restriction of $\alpha_{\mathcal{V}}$ to $X_{\mathcal{V},i}$ is the $(2v_i + 1)$-cycle $((i, -v_i), (i, -v_i + 1), \ldots, (i, v_i))$,

($\beta_{\mathcal{V}}$) the restriction of $\beta_{\mathcal{V}}$ to $X_{\mathcal{V},\,i}$ is the 3-cycle $((i,-1),(i,0),(i,1))$.

The **Neumann group** corresponding to $\mathcal{V}$ is the group $G_{\mathcal{V}}$ of permutations of $X_{\mathcal{V}}$ generated by $\alpha_{\mathcal{V}}$ and $\beta_{\mathcal{V}}$.

Let $\mathrm{Alt}_f(\mathbf{Z})$ denote the group of permutations of finite supports of $\mathbf{Z}$ that are even on their support. Let $\mathrm{Alt}_f(\mathbf{Z}) \rtimes_{\mathrm{shift}} \mathbf{Z}$ denote its semi-direct product with $\mathbf{Z}$, where $\mathbf{Z}$ acts on itself by shifts, the generator 1 acting by $\alpha : j \longmapsto j+1$. Observe that $\mathrm{Alt}_f(\mathbf{Z}) \rtimes_{\mathrm{shift}} \mathbf{Z}$ is generated by α and the 3-cycle $\beta = (-1, 0, 1)$. It is easy to check that the assignment $\alpha_{\mathcal{V}} \longmapsto \alpha, \beta_{\mathcal{V}} \longmapsto \beta$ extends to an epimorphism $\pi_{\mathcal{V}} : G_{\mathcal{V}} \twoheadrightarrow \mathrm{Alt}_f(\mathbf{Z}) \rtimes_{\mathrm{shift}} \mathbf{Z}$. Neumann has shown that the kernel of $\pi_{\mathcal{V}}$ is the restricted product $\prod_{i=1}^{\prime\infty} \mathrm{Alt}(2v_i + 1)$; see [121], as well as [93, Complement III.35].

This has two straightforward consequences. On the one hand, any minimal finite normal subgroup of $G_{\mathcal{V}}$ is one of the $\mathrm{Alt}(2v_i + 1)$. Thus, for two distinct sequences $\mathcal{V}$ and $\mathcal{V}'$, the groups $G_{\mathcal{V}}$ and $G_{\mathcal{V}'}$ are not isomorphic. In particular, there are uncountably many pairwise non-isomorphic 2-generator groups; hence these are infinitely presented, except possibly for a countable number of them (but see below). On the other hand, the groups $G_{\mathcal{V}}$ are elementary amenable.

Corollary C.6 *Let $\mathcal{V}$ be a sequence of integers as in Example C.5. Any finitely presented cover of $G_{\mathcal{V}}$ contains non-abelian free subgroups.*

In particular $G_{\mathcal{V}}$ is not finitely presented.

Proof The claim of Corollary C.4 concerning $H \rtimes \mathbf{Z}$ applies to the group $\mathrm{Alt}_f(\mathbf{Z}) \rtimes_{\mathrm{shift}} \mathbf{Z}$. Since any cover of $G_{\mathcal{V}}$ is a cover of $\mathrm{Alt}_f(\mathbf{Z}) \rtimes_{\mathrm{shift}} \mathbf{Z}$, Corollary C.6 follows from Corollary C.4. □

Let us also indicate how the last statement of Corollary C.6 is a straightforward consequence of the first paper [30] on BNS-invariants.

Proof that $G_{\mathcal{V}}$ is not finitely presented. It follows from the definition of $G_{\mathcal{V}}$ that the kernel $N_{\mathcal{V}}$ of the composition $G_{\mathcal{V}} \twoheadrightarrow \mathrm{Alt}_f(\mathbf{Z}) \rtimes_{\mathrm{shift}} \mathbf{Z} \twoheadrightarrow \mathbf{Z}$ is locally finite. Suppose (ab absurdo) that $G_{\mathcal{V}}$ is finitely presented. By [30, Theorem A], we have $G_{\mathcal{V}} = \mathrm{HNN}(H, \varphi : K \overset{\simeq}{\to} L)$ for a finitely generated subgroup H of $N_{\mathcal{V}}$ and an isomorphism φ between two subgroups K, L of H. Since non-ascending extensions contain non-abelian free subgroups (by Britton's lemma), $G_{\mathcal{V}}$ is an ascending HNN-extension; we may assume that $K = H$. Since $N_{\mathcal{V}}$ is locally finite, the finitely generated subgroup H of $N_{\mathcal{V}}$ is finite. It follows that $K = H = L$. Hence $G_{\mathcal{V}} = H \rtimes_{\varphi} \mathbf{Z}$ and the kernel of $G_{\mathcal{V}} \twoheadrightarrow \mathbf{Z}$ is finite. This is preposterous, and the proof is complete. □

Appendix D: On growth and amenability

Let G be a group generated by a finite set S. For an integer $n \geq 0$, let $B_S^G(n)$ denote the "ball of radius n around the origin", namely the set of those elements $g \in G$ that can be written as words $g = s_1 \cdots s_n$, with $s_1, \ldots, s_n \in S \cup S^{-1} \cup \{1\}$. Let $\gamma_S^G(n)$ denote the cardinality of $B_S^G(n)$. Then G is said to be

(pol) of **polynomial growth** if there exist constants $a, d > 0$ such that $\gamma_S^G(n) \leq an^d$ for all $n > 0$, (exp) of **exponential growth** if there exist a constant $c > 1$ such that $\gamma_S^G(n) \geq c^n$ for all $n \geq 0$,

(int) of **intermediate growth** in other cases.

It is easy to check that this trichotomy depends only on G, not on the finite generating set S. For information on the growth of groups, we refer to the books [93] and [113].

A group G is **amenable** if there exists a left-invariant finitely additive probability measure defined on all subsets of G (there are many other equivalent definitions). Two basic results are important here: (i) amenability of groups is preserved by the four operations of taking subgroups, quotients, directs limits, and extensions with amenable kernels (already in [123]), and (ii) groups of intermediate growth are amenable (this goes back to [3], and is also a straightforward consequence of Følner's Criterion [63]). These results make it natural to define three classes of groups:

$\mathcal{AG}$ is the **class of amenable groups**, defined in [123].
$\mathcal{EG}$ is the **class of elementary amenable groups**, defined in [56]; it is the smallest class of groups containing the easiest examples, that are finite groups and abelian groups, and stable by the four operations listed above.
$\mathcal{SG}$ is the **class of subexponentially amenable groups** (see below for an historical comment on this definition); it is the smallest class of groups containing $\mathcal{EG}$ and the next easiest examples, that are the groups of intermediate growth.

We have a partition

$$\mathcal{AG} = (\mathcal{AG} \smallsetminus \mathcal{SG}) \sqcup (\mathcal{SG} \smallsetminus \mathcal{EG}) \sqcup \mathcal{EG}.$$

Let us mention a few groups in each of these three parts.

The class $\mathcal{EG}$ contains all virtually soluble groups and all locally finite groups; other examples are cited in Section 1.A.

There are finitely generated groups in $\mathcal{EG}$ which are not virtually soluble: for example all Neumann groups $G_{\mathcal{V}}$ discussed in Example C.5, or an example in [97]. As already mentioned in Section 1.D, any countable elementary amenable group embeds in a finitely generated elementary amenable group; the same hold for "amenable" instead of "elementary amenable" [128, Corollary 1.3].

Let us describe a family of finitely presented elementary amenable groups that are not virtually soluble.

Example D.1 Consider an integer $n \geq 1$, the set

$$S_n = \{(j, k) \in \mathbf{Z}^2 \mid j \geq 0, 1 \leq k \leq n\}$$

of n parallel half-intervals in the square lattice, and the **Houghton group** H_n of all permutations h of S_n such that, for each $k \in \{1, \ldots, n\}$, there exists a translation $t_k \in \mathbf{Z}$ such that $h(j, k) = (j + t_k, k)$ for all j large enough [98]. Denote by $\mathrm{Sym}_f(S_n)$ the group of permutations of S_n with finite supports, clearly a normal subgroup of H_n, and set

$$A = \left\{ (t_1, \ldots, t_n) \in \mathbf{Z}^n \,\middle|\, \sum_{k=1}^{n} t_k = 0 \right\}.$$

We have a short exact sequence

$$1 \longrightarrow \mathrm{Sym}_f(S_n) \longrightarrow H_n \stackrel{\pi}{\longrightarrow} A \longrightarrow 1$$

where, with the notation above, $\pi(h) = (t_1, \ldots, t_n)$. Since $\mathrm{Sym}_f(S_n)$ is locally finite and A abelian, H_n is elementary amenable.

It is known that the group H_n is of type FP_{n-1} but not FP_n [39, Theorem 5.1]. In particular, for $n \geq 3$, the group H_n is finitely presented.

Finitely generated groups in the class $\mathcal{EG}$ are either of polynomial growth or of exponential growth [48]; this has been sharpened: a finitely generated group in the class $\mathcal{EG}$ has either polynomial growth or *uniform* exponential growth [129] (see also [37]). By a famous theorem of Gromov [87], a finitely generated group of polynomial growth is virtually nilpotent, and in particular finitely presented.

The class $\mathcal{SG} \smallsetminus \mathcal{EG}$ contains the class of finitely generated groups of intermediate growth. Historically, the group $\mathfrak{G}$ of Theorem 1.6 and Example 2.16 was the first group shown to be of intermediate growth [67]. This class also contains finitely presented groups, such as the group with 5 generators and 11 relators of [70], later shown to have another presentation with 2 generators and 4 relators (due to Bartholdi, see [44, Number 12]).

The history of early papers on the class $\mathcal{SG}$ is worth a few lines. This class was first implicitly introduced in a paper on 4-manifold topology, more precisely on 4-manifold surgery and 5-dimensional s-cobordism theorems, [64] (see also [106]), and then explicitly in [70]. Freedman and Teichner introduce a class of groups that they call "good", defined as the groups for which the "π_1-Null Disk Lemma" holds; this lemma establishes the existence of 2-discs bounding some closed curves in 4-manifolds of a certain kind. Good groups include finitely generated groups in the class $\mathcal{SG}$ [64, Theorem 0.1 and Lemma 1.2].

The class $\mathcal{AG} \smallsetminus \mathcal{SG}$ contains the **Basilica group** $\mathfrak{B}$ of Example 2.17, which was first shown to be not in $\mathcal{SG}$ [80], and later shown to be amenable [10]. The method of Bartholdi and Virag was streamlined and generalized by Kaimanovich in [101], in terms of entropy and the legendary "Münchhausen's trick". This and later papers show the amenability of $\mathfrak{B}$ and of many other non elementary amenable groups (see [5,13], building among other things on [140]). The class $\mathcal{AG} \smallsetminus \mathcal{SG}$ contains also the finitely generated amenable simple groups that appear in [100], and in Problem 1.12.

Non-amenable groups include non-abelian free groups, more generally groups containing non-abelian free subgroups [123].

We conclude this report by Question D.2, due to Tullio Ceccherini-Silberstein. Before this, we recall some background.

A group G has a **paradoxical decomposition** if there exist integers $p, q \geq 2$, subsets $X_1, \ldots, X_p, Y_1, \ldots, Y_q$ of G, and elements $g_1, \ldots, g_p, h_1, \ldots, h_q$ in G such that

$$\begin{aligned} G &= X_1 \sqcup \cdots \sqcup X_p \sqcup Y_1 \sqcup \cdots Y_q \\ &= g_1 X_1 \sqcup \cdots \sqcup g_p X_p = h_1 Y_1 \sqcup \cdots \sqcup h_q Y_q, \end{aligned}$$

where $\sqcup$ denotes disjoint union. Tarski has shown that G is non-amenable if and only if G has a paradoxical decomposition [147]; see also [150] and [94]. For G non-amenable, the **Tarski number** $\mathcal{T}(G)$ of G is the minimum of the sum $p + q$, over all paradoxical decompositions of G. When G has non-abelian free subgroups, it is easy to show that $\mathcal{T}(G) = 4$; in particular, if G is a non-elementary Gromov-hyperbolic group, then $\mathcal{T}(G) = 4$. As a student of Tarski in the 1940's, Jonsson has shown that, conversely, $\mathcal{T}(G) = 4$ implies that G has non-abelian free subgroups. It is also easy to show that $\mathcal{T}(G) \geq 6$ for a non-amenable torsion group G. See [150] or [44, Propositions 20 and 21]. We do not know any example of a non-amenable group G without non-abelian free subgroups for which the exact value of $\mathcal{T}(G)$ has been computed.

Let $\mathcal{M}_m^{\mathrm{na}}$ be the subspace of $\mathcal{M}_m$ of those pairs (G, S) with G non-amenable. For $m \geq 2$, the Tarski number function

$$\mathcal{M}_m^{\mathrm{na}} \longrightarrow \mathbf{N}, (G, S) \longmapsto \mathcal{T}(G)$$

is not continuous. Indeed, there are sequences of non-elementary Gromov-hyperbolic groups (with Tarski number 4) converging in $\mathcal{M}_m$ to non-amenable torsion groups (with Tarski number at least 6). See [46, Théorème 1.3], or one of the following classes of examples.

(1) For integers $m, n \geq 2$, let $\mathrm{B}(m, n)$ denote the free Burnside group of rank m and exponent n, that is the quotient of the free group F_m by the set of relators $(w^n = 1)_{w \in F_m}$. For n odd and $n \geq 665$, Adyan has shown that $\mathrm{B}(m, n)$ is non-amenable [4]; moreover, it is known that $6 \leq \mathcal{T}(\mathrm{B}(m, n)) \leq 14$ [44, Theorem 61].

Such groups are limits in $\mathcal{M}_m$ of non-elementary Gromov-hyperbolic groups. More precisely, for n odd and large enough, the group $\mathrm{B}(m, n)$ is a limit in $\mathcal{M}_m$ of a sequence $(\mathrm{B}(m, n, i))_{i \geq 1}$ of non-elementary Gromov-hyperbolic groups. A similar fact is shown by Ivanov in [99, see Lemma 21.1], in the much more difficult case of n even, with $n \geq 2^{48}$ and n divisible by 2^9; his proof adapts to the case needed here, with important simplifications (compare with [85], in particular Theorems 1.10 and 1.7, and recall that a finitely presented group with a subquadratic Dehn function is hyperbolic). In particular, for $m \geq 2$ and n odd large enough, the free Burnside group $\mathrm{B}(m, n)$, of Tarski number between 6 and 14, is a limit in $\mathcal{M}_m$ (indeed in $\mathcal{M}_m^{\mathrm{na}}$) of groups of Tarski number 4.

(2) Ol'shanskii has worked out several "Tarski monster groups". In particular, for any prime p large enough, he has constructed a 2-generated non-amenable torsion group $\mathrm{TM}(p)$ in which any proper non-trivial subgroup is of order p. Moreover $\mathrm{TM}(p)$, of Tarski number ≥ 6, is a limit in $\mathcal{M}_2$ of non-elementary Gromov hyperbolic groups, of Tarski number 4 [124, Lemma 10.7a].

In several other papers by Ol'shanskii and co-authors, there are several other classes of examples of such Burnside type limits of hyperbolic groups. Let us only quote one paper [127], and the book [125].

Question D.2 *For an integer $k \geq 4$ and a finitely generated non-amenable group G with $\mathcal{T}(G) = k$, does there exist a finitely presented cover E of G with $\mathcal{T}(E) = k$?*

The answer is clearly positive when $\mathcal{T}(G) = 4$ (with E free). If we define $\mathcal{T}(G) = \infty$ when G is amenable, the previous question for $k = \infty$ coincides with Question 1.1, and the answer is negative.

Acknowledgments We are most grateful to Yves de Cornulier and Ralph Strebel, for generous comments on preliminary versions of this paper, as well as to Laurent Bartholdi, Gilbert Baumslag, Marc Burger, Tullio Ceccherini-Silberstein, Anna Erschler, Sergei Ivanov, Olga Kharlampovich, Volodymyr Nekrashevych, Alexander Ol'shanskii, Vitaliĭ Roman'kov, and Mark Sapir, for useful discussions and mail exchanges.

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