Type II_1 factors satisfying the spatial isomorphism conjecture

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This paper addresses a conjecture in the work by Kadison and Kastler [Kadison RV, Kastler D (1972) *Am J Math* 94:38–54] that a von Neumann algebra *M* on a Hilbert space $\mathcal{H}$ should be unitarily equivalent to each sufficiently close von Neumann algebra *N*, and, moreover, the implementing unitary can be chosen to be close to the identity operator. This conjecture is known to be true for amenable von Neumann algebras, and in this paper, we describe classes of nonamenable factors for which the conjecture is valid. These classes are based on tensor products of the hyperfinite II_1 factor with crossed products of abelian algebras by suitably chosen discrete groups.

Kadison-Kastler stability | perturbations | bounded group cohomology

In 1972, Kadison and Kastler (1) initiated the study of perturbation theory of operator algebras. The setting was a Hilbert space $\mathcal{H}$ and the collection of all von Neumann subalgebras of the bounded operators $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$, namely those *-closed subalgebras of $\mathcal{B}(\mathcal{H})$ that contain the identity operator and are closed in the strong operator topology. By applying the Hausdorff distance to the unit balls of two von Neumann algebras, Kadison and Kastler (1) equipped the collection of all von Neumann subalgebras with a metric $d(\cdot,\cdot)$. This metric can be described as the infimum of numbers $\lambda > 0$, for which each element of either unit ball is within a distance λ of an element of the other in the operator norm on $\mathcal{B}(\mathcal{H})$. Natural examples of close pairs of von Neumann algebras arise by fixing a von Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$ and considering a unitary $u \in \mathcal{B}(\mathcal{H})$. It is easy to see that

$$d(M, uMu^*) \le 2\|u - \mathrm{id}_{\mathcal{H}}\|,$$

and therefore, if u is chosen with $\|u - \mathrm{id}_{\mathcal{H}}\|$ being small, then uMu^* will be close to M. In this case, we refer to uMu^* as a *small unitary perturbation* of M. The work by Kadison and Kastler (1) proposed that such a small unitary perturbation should be essentially the only way of producing pairs of close von Neumann algebras, leading to the following conjecture.

Conjecture 1 (Kadison–Kastler). *For all $\varepsilon > 0$, there exists $\delta > 0$ with the property that, if $M, N \subseteq \mathcal{B}(\mathcal{H})$ are von Neumann algebras with $d(M,N) < \delta$, then there exists a unitary operator u on $\mathcal{H}$ with $uMu^* = N$ and $\|u - \mathrm{id}_{\mathcal{H}}\| < \varepsilon$.*

Initial progress on this conjecture focused on amenable von Neumann algebras, which, due to the work of Connes (2), may be characterized as inductive limits of finite dimensional von Neumann algebras. For these algebras, this conjecture was established in the late 1970s in the works by Christensen (3), Johnson (4), and Raeburn and Taylor (5) [see *Theorem 2* below]. In this paper, we will describe our examples of nonamenable von Neumann algebras that satisfy the conjecture. Full details and proofs will be available elsewhere in a longer account.

Background

As the Kadison–Kastler conjecture predicts that close operator algebras should be isomorphic, it is natural to ask whether they necessarily share the same invariants and structural properties. This was the primary focus of ref. 1, which examined the type decomposition of close von Neumann algebras. The foundational work of Murray and von Neumann (6) decomposes every von Neumann algebra M uniquely into a direct sum $M_{\mathrm{I}} \oplus M_{\mathrm{II}_1} \oplus M_{\mathrm{II}_\infty} \oplus M_{\mathrm{III}}$, where the summands have types I, II_1, II_∞, and III, respectively. In particular, every von Neumann factor (those von Neumann algebras that are maximally noncommutative in that the centers consist only of scalar multiples of the identity operator) is of one of these types. Our work is concerned with factors of type II_1, and a formulation equivalent to the original definition is that M should be infinite dimensional and possess a positive linear functional τ of norm 1 satisfying $\tau(ab) = \tau(ba)$ for $a, b \in M$. This functional is called a trace, and it is the counterpart of the standard trace on the algebra of $n \times n$ matrices that averages the diagonal entries. The main theorem of ref. 1 shows that, if M and N are close von Neumann algebras, then the projections onto the summands of each type are necessarily close. This work also shows that algebras close to factors are again factors, and therefore, any von Neumann algebra close to a II_1 factor is again a II_1 factor (1), a result that we will use subsequently.

It is also natural to consider perturbation theory for other classes of operator algebras. In ref. 7, the work by Phillips initiated the study of these questions in the context of norm closed self-adjoint algebras (C^*-algebras) and examined the ideal lattices of close algebras. A key difference in flavor between perturbation theory for C^*-algebras and the von Neumann algebra version was exposed in two critical examples: ref. 8 gives examples of arbitrarily close but nonisomorphic C^*-algebras, whereas ref. 9 gives examples of close unitarily conjugate separable C^*-algebras for which it is not possible to choose a unitary witnessing this conjugacy close to the identity. The counterexamples of ref. 8 are nonseparable, and therefore, the appropriate formulation of *Conjecture 1* for C^*-algebras is that sufficiently close separable C^*-algebras acting on a separable Hilbert space should be spatially isomorphic but without asking for control of the unitary implementing a spatial isomorphism. Special cases of this conjecture were established for separable approximately finite dimensional C^*-algebras (10, 11) and continuous trace algebras (12) in the early 1980s, and a complete analog of the perturbation results for amenable von Neumann algebras was recently given in refs. 13 and 14, which establish the conjecture for separable nuclear C^*-algebras. There has also been significant work on perturbation questions for nonself-adjoint algebras (see ref. 15, for example).

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A related notion of near-containments also plays a substantial role in our work. We say that $M \subseteq_\gamma N$ if each element of the unit ball of M is within a distance γ of an element of N (not required to be in the unit ball of N). Analogous to *Conjecture 1*, one might expect a sufficiently small near-inclusion of von Neumann algebras to arise from a small unitary perturbation of a genuine inclusion. That is, for each $\epsilon > 0$, does there exist $\delta > 0$ such that, if $M \subseteq_\delta N$ is a near inclusion of von Neumann algebras on $\mathcal{H}$, then there is a unitary u on $\mathcal{H}$ with $uMu^* \subseteq N$ and $\|u - \mathrm{id}_\mathcal{H}\| < \epsilon$? The work by Christensen (11) introduced this notion, with the twofold purpose of improving numerical estimates and extending perturbation results beyond the amenable von Neumann algebra setting. In particular, the work by Christensen (11) gave the following positive answer to the previous question when M is amenable, but N is arbitrary. It is easy to use *Theorem 2* to show that, if $d(M,N) < 1/101$ and M is amenable, then there is a unitary $u \in (M \cup N)''$ with $uMu^* = N$ and $\|u - \mathrm{id}_\mathcal{H}\| \le 150 d(M,N)$.

Theorem 2 (Spatial Embedding Theorem). *Let M and N be von Neumann algebras on a Hilbert space $\mathcal{H}$, and suppose that M is amenable. If $M \subset_\gamma N$ for a constant $\gamma < 1/100$, then there exists a unitary $u \in (M \cup N)''$ so that $\|u - \mathrm{id}_\mathcal{H}\| \le 150\gamma$, $d(M, uMu^*) \le 100\gamma$, and $uMu^* \subseteq N$.*

Embedding theorems are also possible in the setting of C^*-algebras; given a sufficiently close near-inclusion of a separable nuclear C^*-algebra A into a general C^*-algebra B, ref. 16 establishes the existence of an embedding $A \hookrightarrow B$.

The other general context in which perturbation results have been obtained is when we replace $\mathcal{B}(\mathcal{H})$ with a finite von Neumann algebra. Given unital von Neumann subalgebras B_1 and B_2 of a finite von Neumann algebra M with $d(B_1, B_2) < 1/8$, ref. 16 gives a unitary $u \in (B_1 \cup B_2)''$ with $uB_1u^* = B_2$ and $\|u - 1_M\| \le 7d(B_1, B_2)$.

In our longer account of the work surveyed in this paper, we keep track of the estimates involved at each step. Here, we simplify matters by describing our results qualitatively.

Kadison–Kastler Stability and the Similarity Problem

The spatial embedding theorem does not depend on the particular *-representation of M on a Hilbert space. Our search for positive answers to *Conjecture 1* is guided by this result, leading us to the following definition.

Definition 3. *Let M be a von Neumann algebra. Say that M is strongly Kadison-Kastler stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every faithful normal unital *-representation $\pi : M \to \mathcal{B}(\mathcal{H})$ and every von Neumann algebra N on $\mathcal{H}$ with $d(\pi(M), N) < \delta$, there is a unitary operator u on $\mathcal{H}$ with $u\pi(M)u^* = N$ and $\|u - \mathrm{id}_\mathcal{H}\| < \varepsilon$.*

We use this terminology, because it is the strongest of several versions of the conjecture that one could consider. For example, one could ask for spatial isomorphisms without requiring control of $\|u - \mathrm{id}_\mathcal{H}\|$ or isomorphisms between close algebras that are not necessarily spatial. Our methods also give examples of von Neumann algebras satisfying these weaker forms of the conjecture (*Theorems 7 and 8*). An ℓ^∞-direct sum argument can be used to show that *Conjecture 1* is equivalent to the statement that all von Neumann algebras are strongly Kadison-Kastler stable.

Conjecture 1 implies that the operation

$$M \mapsto M' = \{x \in \mathcal{B}(\mathcal{H}) : \forall y \in M, xy = yx\}$$

of taking commutants of von Neumann algebras in $\mathcal{B}(\mathcal{H})$ is continuous with respect to the Kadison–Kastler metric, and this implication would extend to C^*-algebras by an application of Kaplansky's density theorem. This question is equivalent to another long-standing question: the similarity problem. In 1955, motivated by work by Dixmier and Day on uniformly bounded group representations, in ref. 18 Kadison asked whether every bounded representation of a C^*-algebra on a Hilbert space is necessarily similar to a *-representation. Using the equivalence of the similarity and derivation problems in the work by Kirchberg (19), we recently observed (20) that the similarity problem is equivalent to the continuity of commutants. The arguments in ref. 21 also give a local version of this equivalence: a C^*-algebra A has the similarity property if the operation of taking commutants is continuous at A, uniformly over all representations of A (ref. 20 gives the precise statement). The following consequence is of particular relevance here (we restrict to II_1 factors, where it suffices to consider normal representations in the similarity property; see the proof of theorem 2.3 in ref. 22).

Proposition 4. *Every strongly Kadison–Kastler-stable II_1 factor satisfies the similarity property.*

The similarity problem is known to have positive answers for von Neumann algebras of types I_∞, II_∞, and III (23), but it remains open for finite algebras and particularly, factors of type II_1. Here, the only factors for which a positive answer is known are those factors with Murray and von Neumann's property gamma (the factors with property gamma are those containing non-trivial asymptotically centralizing sequences, and this property was introduced in ref. 6 to distinguish the hyperfinite II_1 factor from the free group factors) (22). In particular, McDuff factors (those factors M that absorb the hyperfinite II_1 factor R tensorially, meaning that $M \cong M \overline{\otimes} R$) have the similarity property. Thus, to produce new examples of strongly Kadison–Kastler-stable factors, we work with II_1 factors with property gamma.

The role played by the similarity property in obtaining examples of strongly Kadison-Kastler stable factors is encapsulated in the following result, which dates back to ref. 17.

Proposition 5. *Let A be a C^*-algebra satisfying the similarity property, and suppose that $\theta_1, \theta_2 : A \to \mathcal{B}(\mathcal{H})$ are two *-representations with $\|\theta_1 - \theta_2\|$ sufficiently small. Then, there exists a unitary u on $\mathcal{H}$ such that $\theta_2 = \mathrm{Ad}(u) \circ \theta_1$. Furthermore, one can control $\|u - \mathrm{id}_\mathcal{H}\|$ in terms of $\|\theta_1 - \theta_2\|$ and quantitative estimates on how well A satisfies the similarity property.*

In the presence of the similarity property, if we can show that two close von Neumann algebras M and N on $\mathcal{H}$ are *-isomorphic through an isomorphism θ close to the inclusion map $M \hookrightarrow \mathcal{B}(\mathcal{H})$, then it will follow that θ is spatially implemented by a unitary close to $\mathrm{id}_\mathcal{H}$. Consequently, M will be strongly Kadison-Kastler stable.

Twisted Crossed Products

Our examples of strongly Kadison-Kastler stable factors arise from the crossed product construction that goes back to Murray and von Neumann. Consider a countable infinite discrete group Γ acting by measure-preserving transformations on a probability space (X, μ), and write α for the induced action of Γ on the abelian von Neumann algebra $L^\infty(X)$. A unitary-valued normalized 2-cocycle is a function $\omega : \Gamma \times \Gamma \to \mathcal{U}(L^\infty(X))$ with $\omega(g,e) = \omega(e,g) = 1_{L^\infty(X)}$ for all $g \in \Gamma$, which satisfies the cocycle identity

$$\alpha_g(\omega(h,k))\omega(gh,k)^*\omega(g,hk)\omega(g,h)^* = 1_{L^\infty(X)}, \quad g,h,k \in \Gamma.$$

Two such 2-cocycles ω_1, ω_2 are cohomologous if there exists $\nu : \Gamma \to \mathcal{U}(L^\infty(X))$ with $\nu(e) = 1_{L^\infty(X)}$ and (Eq. **1**) holds:

$$\omega_2(g,h) = \left(\alpha_g(\nu(h))\nu(gh)^*\nu(g)\right)\omega_1(g,h), \quad g,h \in \Gamma. \qquad [1]$$

Given a unitary-valued normalized 2-cocycle ω, the twisted crossed product

$$L^\infty(X) \rtimes_{\alpha,\omega} \Gamma$$

is a von Neumann algebra generated by a copy of $L^\infty(X,\mu)$ and unitaries $(u_g)_{g\in\Gamma}$, satisfying (Eq. **2**):

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$$u_g f u_g^* = \alpha_g(f), \quad u_g u_h = \omega(g,h) u_{gh}, \quad f \in L^\infty(X), g,h \in \Gamma. \quad [2]$$

Because the action is measure-preserving, we obtain a trace τ on the twisted crossed product by extending (Eq. **3**)

$$\tau\left(\sum_{g\in\Gamma} f_g u_g\right) = \int f_e \mathrm{d}\mu \quad [3]$$

from the dense *-subalgebra of finite linear combinations $\sum_{g\in\Gamma} f_g u_g$ with $f_g \in L^\infty(X,\mu)$; therefore, the twisted crossed product is of type II_1. The two conditions (Eqs. **2** and **3**) characterize twisted crossed products, and we will use these conditions to recognize factors close to a twisted crossed product as again of this form, albeit via a possibly different 2-cocycle.

We will impose two additional conditions on the action $\Gamma \curvearrowright X$ in addition to preserving a standard probability measure.

i) Essential freeness: For $g \neq e$, the stabilizer $\{x \in X : g \cdot x = x\}$ is required to be null. This requirement ensures that the copy of $L^\infty(X)$ is a maximal abelian subalgebra of the twisted crossed product $L^\infty(X) \rtimes_{\alpha,\omega} \Gamma$.

ii) Ergodicity: This condition requires any Γ-invariant subset to be either null or conull. In the presence of freeness, the twisted crossed product $L^\infty(X) \rtimes_{\alpha,\omega} \Gamma$ is a factor if and only if the action is ergodic.

Combining these assumptions, the twisted crossed products $L^\infty(X) \rtimes_{\alpha,\omega} \Gamma$ are always II_1 factors.

We are now in position to state our main result. Recall that $SL_n(\mathbb{Z})$ denotes the group of $n \times n$ matrices with integer entries and determinant equal to one.

Theorem 6. *Let (X,μ) be a standard probability space, and suppose that $SL_n(\mathbb{Z})$ acts freely and ergodically by measure-preserving transformations on (X,μ) for $n \geq 3$. Then, the II_1 factor (Eq.* **4***)*

$$M = \left(L^\infty(X,\mu) \rtimes_\alpha SL_n(\mathbb{Z})\right) \overline{\otimes} R \quad [4]$$

is strongly Kadison-Kaster stable.

The key property of the group $SL_n(\mathbb{Z})$ used in the proof of *Theorem 6* is cohomological. By combining the results in the works of Burger and Monod (24, 25) and Monod and Shalom (26) with later results in the work by Monod (27), it follows that the bounded cohomology groups

$$H_b^2\left(SL_n(\mathbb{Z}), L_{\mathbb{R}}^\infty(X,\mu)\right)$$

vanish for $n \geq 3$ [a key difficulty, which is overcome in ref. 27, is that the module $L_{\mathbb{R}}^\infty(X,\mu)$ is a nonseparable Banach space]. In *Theorem 6*, the groups $SL_n(\mathbb{Z})$ can be replaced by any discrete group Γ, for which $H_b^2(\Gamma, L_{\mathbb{R}}^\infty(X,\mu)) = 0$; the works (24–27) also establish a vanishing result for the bounded cohomology in degree 2 of certain other irreducible higher-rank lattices. The effect of the vanishing of this bounded cohomology group is that the open mapping theorem gives a constant $K > 0$ with the property that, for any two unitary 2-cocycles $\omega_1, \omega_2 : \Gamma \times \Gamma \to \mathcal{U}(L^\infty(X))$ with

$$\sup_{g,h\in\Gamma} \left\|\omega_1(g,h) - \omega_2(g,h)\right\| < \sqrt{2},$$

we can find $\nu : \Gamma \to \mathcal{U}(L^\infty(X))$ such that Eq. **1** holds, and

$$\sup_{g\in\Gamma} \left\|\nu(g) - 1_{L^\infty(X)}\right\| \leq K \sup_{g,h\in\Gamma} \left\|\omega_1(g,h) - \omega_2(g,h)\right\|.$$

For the purpose of finding examples to which *Theorem 6* applies, it is useful to note that, for measure-preserving actions of $SL_n(\mathbb{Z})$ with $n \geq 3$ on nonatomic standard probability spaces (X,μ), ergodicity implies freeness by ref. 28.

Examples of suitable actions of $\Gamma = SL_n(\mathbb{Z})$ are given by Bernoulli shifts. Given a base probability space (Y,ν) (which could be atomic but is not a singleton), form the infinite product space $X = \prod_{g\in\Gamma} Y$ indexed by the group, and let μ be the product measure on X. Then, Γ acts on X by shifting the indices: $h \cdot (x_g)_{g\in\Gamma} = (x_{hg})_{g\in\Gamma}$. When Γ is infinite, it induces a free ergodic probability measure-preserving action. By suitably varying the base space (Y,ν) and using the results in the works by Bowen (29) and Popa (30, 31), one obtains an uncountable family of pairwise nonisomorphic factors of the form (Eq. **4**) to which *Theorem 6* applies.

The role of the hyperfinite II_1 factor R in *Theorem 6* is to ensure that the tensor product $(L^\infty(X,\mu) \rtimes_\alpha SL_n(\mathbb{Z})) \overline{\otimes} R$ has the similarity property. Indeed, if one could construct a free ergodic probability measure-preserving action $\alpha : SL_n(\mathbb{Z}) \curvearrowright (X,\mu)$ for $n \geq 3$ so that the resulting crossed product factor $L^\infty(X,\mu) \rtimes_\alpha SL_n(\mathbb{Z})$ has the similarity property, then this crossed product will be strongly Kadison-Kastler stable. However, the only known method for establishing the similarity property for a II_1 factor is to establish property gamma. By combining results from refs. 32 and 33, the presence of Kazhdan's property (T) (34) for $SL_n(\mathbb{Z})$ $(n \geq 3)$ provides an obstruction to property gamma for the crossed product factors $L^\infty(X,\mu) \rtimes_\alpha SL_n(\mathbb{Z})$.

Outline of the Proof of Theorem 6

In the light of *Proposition 5*, to prove *Theorem 6*, it suffices to show that, if N is close to a II_1 factor M of the form (Eq. **4**), then there is a *-isomorphism of M onto N which is close to the inclusion of M into the containing $\mathcal{B}(\mathcal{H})$. Our strategy involves three main steps.

i) Because M takes the form $M_0 \overline{\otimes} R$ (where $M_0 = L^\infty(X) \rtimes_\alpha \Gamma$ and R is the hyperfinite II_1 factor), we show that N is also a McDuff factor and after a small unitary perturbation, that the factorizations of M and N are compatible. To do this work, we use the spatial embedding theorem to produce a small unitary perturbation N_1 of N, which contains R, and then, we define $N_0 = (R' \cap N_1)$. One can check that $d(M_0, N_0)$ is small. To identify N_1 as $N_0 \overline{\otimes} R$, we need to show that N_1 is generated by N_0 and R.

ii) To obtain an isomorphism between M_0 and N_0, we transfer the crossed product structure of M_0 to N_0. Given a II_1 factor N_0, which is sufficiently close to a crossed product factor $M_0 = L^\infty(X) \rtimes_\alpha \Gamma$, it is possible to use *Theorem 2* repeatedly to find a copy of $L^\infty(X)$ inside N_0 close to the copy in M_0 and unitaries $v_g \in N_0$ normalizing $L^\infty(X)$ and inducing the same action as the u_g. We must then show that N_0 is generated by $L^\infty(X)$ and the unitaries v_g. Once this is achieved, it follows that N_0 is a twisted crossed product

$$L^\infty(X) \rtimes_{\alpha,\omega} \Gamma,$$

where ω is a 2-cocycle measuring the failure of multiplicitivity of the map $g \mapsto v_g$.

iii) In the previous step, each v_g can be chosen close to the corresponding u_g, and therefore,

$$\omega(g,h) = v_g v_h v_{gh}^* \approx u_g u_h u_{gh}^* = 1_{L^\infty(X)}, \quad g,h \in \Gamma.$$

Our cohomological assumption then ensures that ω is cohomologous to a trivial cocycle, which induces a *-isomorphism between M_0 and N_0. Moreover, the fact that we ask for the bounded cohomology group $H_b^2(\Gamma, L_{\mathbb{R}}^\infty(X))$ to vanish [and not just for $H^2(\Gamma, L_{\mathbb{R}}^\infty(X))$ to vanish] gives additional information: one can find a surjective *-isomorphism $\theta : M_0 \to N_0$ such that $\|\theta(fu_g)-fu_g\|$ is small for all $f\in L^\infty(X)$ with $\|f\|\leq 1$ and all $g\in\Gamma$. In general, there is no reason to expect $\|\theta(y)-y\|$ to be uniformly small for all y in the unit ball of M_0, but we are able to use extra ingredients to achieve this result.

A common feature of the first two steps is the need to show that, if we are given close von Neumann algebras, one of which is generated by a certain collection of elements, then the second can be generated by suitably chosen elements close to the original generators. Because the set of generators of a von Neumann algebra is not open in the norm topology, we approach this problem indirectly by changing representations to standard position and working at the Hilbert space level. This problem is the subject of the next two sections, and the techniques developed are also used to ensure that $\theta(y)$ is uniformly close to y across the unit ball of M_0 in step *iii*.

The steps above can be used to prove additional stability results; we give two examples. In *Theorem 7*, we use the fact that free groups have cohomological dimension one, and therefore, $H^2(\mathbb{F}_r,L^\infty_{\mathbb{R}}(X,\mu))=0$. This result enables us to untwist the cocycle ω in step *iii*; however, because $H^2_b(\mathbb{F}_r,\mathbb{R})\neq 0$, we cannot obtain any information about how the resulting isomorphism behaves on the canonical unitaries. In *Theorem 8*, cohomological methods do not apply, and instead, we use the recent work by Popa and Vaes (35) on the uniqueness (up to unitary conjugacy) of the Cartan masa in a crossed product by a hyperbolic group. The results of ref. 35 are valid for a more general class of groups, and *Theorem 8* holds for this class.

Theorem 7. *Suppose that $\mathbb{F}_r\curvearrowright(X,\mu)$ is a free ergodic measure-preserving action of a free group on a standard probability space. Write $M=L^\infty(X)\rtimes_\alpha\mathbb{F}_r$. Then, there exists $\delta>0$ such that if $M\subseteq\mathcal{B}(\mathcal{H})$ is a normal unital representation of M and $N\subseteq\mathcal{B}(\mathcal{H})$ is a von Neumann algebra with $d(M,N)<\delta$, then $N\cong M$. Additionally, if we assume that the action is not strongly ergodic (i.e., every sequence of asymptotically invariant subsets of X is approximately null or conull), then such an isomorphism $N\cong M$ is necessarily spatial.*

Theorem 8. *There exists $\delta>0$ with the following property. Suppose that $\Gamma_i\curvearrowright(X_i,\mu_i)$ for $i=1,2$ are two free ergodic probability measure-preserving actions of hyperbolic groups on standard probability spaces, and write $M_i=L^\infty(X_i)\rtimes\Gamma_i$. If $d(M_1,M_2)<\delta$, then $M_1\cong M_2$.*

Changing Representations, Standard Position, and the Basic Construction

The theory of normal representations of von Neumann algebras is easy to describe; any two faithful normal representations of a von Neumann algebra are unitarily equivalent after an amplification. Thus, given faithful unital normal representations $\pi_1:M\to\mathcal{B}(\mathcal{H}_1)$ and $\pi_2:M\to\mathcal{B}(\mathcal{H}_2)$, we can find a Hilbert space $\mathcal{K}$ and a unitary isomorphism $U:\mathcal{H}_1\otimes\mathcal{K}\to\mathcal{H}_2\otimes\mathcal{K}$ such that $U(\pi_1(x)\otimes\mathrm{id}_\mathcal{K})=(\pi_2(x)\otimes\mathrm{id}_\mathcal{K})U$ for all $x\in M$. In this way, representations of a II_1 factor M with separable predual on a separable Hilbert space are classified up to unitary equivalence by the coupling constant or M-dimension of the space. Suppose that $M\subseteq\mathcal{B}(\mathcal{H})$ is a unital normal representation on a separable Hilbert space. The commutant M' is a type II factor, so is either type II_∞, where we define $\dim_M(\mathcal{H})=\infty$, or type II_1, in which case we define $\dim_M(\mathcal{H})=\tau_{M'}(e^M_\xi)/\tau_M(e^{M'}_\xi)$, where τ_M and $\tau_{M'}$ are the normalized traces on M and M', ξ is a unit vector in $\mathcal{H}$, e^M_ξ is the projection in M' onto $\overline{M\xi}$ and $e^{M'}_\xi$ is the projection in M onto $\overline{M'\xi}$. This quantity is independent of the choice of ξ. In *Lemma 9*, when M and N have separable preduals, we can always reduce to the situation where they act on a separable Hilbert space by cutting by a projection with range $\overline{(M\cup N)''\xi}$ for some $\xi\in\mathcal{H}$, which lies in $M'\cap N'$.

Lemma 9. *Suppose that M and N are II_1 factors acting on a separable Hilbert space $\mathcal{H}$ with $d(M,N)$ small. Let $\pi_M:M\to\mathcal{B}(\mathcal{K})$ be a unital normal representation on another separable Hilbert space. Then, there exists a unital normal representation $\pi_N:N\to\mathcal{B}(\mathcal{K})$ with $d(\pi_M(M),\pi_N(N))\leq O(d(M,N)^{1/2})$. When M has the similarity property, this estimate can be improved to $d(\pi_M(M),\pi_N(N))\leq O(d(M,N))$.*

Sketch Proof of Lemma 9

We can assume that $\dim_M(\mathcal{H})=\infty$, because if not the case, we can simultaneously amplify both M and N [that is, replace $\mathcal{H}$ by $\mathcal{H}\otimes\ell^2(\mathbb{N})$, M by $M\otimes\mathrm{id}_{\ell^2(\mathbb{N})}$, and N by $N\otimes\mathrm{id}_{\ell^2(\mathbb{N})}$] to reach this situation without changing the distance between M and N. If $\dim_{\pi_M(M)}(\mathcal{K})=\infty$, then π_M is unitarily equivalent to the initial representation of M on $\mathcal{H}$, and we can use a unitary implementing this equivalence to define π_N. Otherwise, we can find a projection $e\in M'$ such that $x\mapsto xe$ is a unital normal representation of N on $e(\mathcal{H})$, which is unitarily equivalent to π_M. When M has the similarity property, M' and N' are close, and therefore, e is close to a projection f in N'. We can then find a unitary u close to $\mathrm{id}_\mathcal{H}$ with $ueu^*=f$. This gives us a normal unital representation of N on $e(\mathcal{H})$ by $y\mapsto u^*yue$ for $y\in N$, and uNu^*e is close to Me on $e(\mathcal{H})$. We define π_N by conjugating the representation $y\mapsto u^*yuw$ by the same unitary used to show that $x\mapsto xe$ is equivalent to π_M.

In the case that M does not have the similarity property, after the initial amplification, it will not always be possible to approximate an arbitrary projection in M' by a projection in N'. However, using work on the derivation problem in the presence of a cyclic vector, which dates back to the work in ref. 36, we can show that, given $e\in M'$ such that M has a cyclic vector for $e(\mathcal{H})$, then it is possible to find a nonzero subprojection $p\leq e$ in M', which is close to N'. By choosing a projection in N' close to p, we obtain close representations of M and N on $p(\mathcal{H})$ as above. At this point in the argument, we are only able to obtain estimates of the form $O\left(d(M,N)^{1/2}\right)$ in contrast with the $O(d(M,N))$ estimate that one obtains in the presence of the similarity property. Our methods do not enable us to get a lower bound on $\dim_M(p(\mathcal{H}))$, which could be very small, but we can take a further subprojection of p to ensure that $\dim_{Mp}(p(\mathcal{H}))=\dim_{\pi_M(M)}(\mathcal{K})/n$ for some $n\in\mathbb{N}$. In this way, we can make a suitable amplification of our representations on $p(\mathcal{H})$ such that the resulting representation of M is unitarily equivalent to π_M. This completes the proof of *Lemma 9*.

A II_1 factor M is said to be in standard position on a Hilbert space $\mathcal{K}$ if $\dim_M(\mathcal{K})=1$. In this case, there exists a unit vector $\xi\in\mathcal{K}$ such that the vector state $\langle\cdot\,\xi,\xi\rangle$ restricts to the traces on M and M'. This vector has the properties that $x\xi=0$ for $x\in M$ implies that $x=0$ (ξ is separating for M), and $M\xi$ is dense in $\mathcal{K}$ (ξ is cyclic for M). These properties also hold for M'. One defines the modular conjugation operator J_M with respect to ξ by extending the map $x\xi\mapsto x^*\xi$ for $x\in M$ to a conjugate linear isometry on $\mathcal{K}$. The commutant M' takes the form J_MMJ_M, and therefore, we have an anti-isomorphism $x\mapsto J_MxJ_M$ between M and M'.

By applying *Lemma 9* to a pair of close II_1 factors M and N on $\mathcal{H}$, we can find new close representations on a Hilbert space $\mathcal{K}$, where M is now in standard position. Our objective is to show that N is also in standard position on $\mathcal{K}$. To establish this result, we first extend the work in ref. 21 (section 3) to show that N is almost in standard position in the sense that $\dim_N(\mathcal{K})\approx 1$; it follows that M' and N' are close on $\mathcal{K}$ (this result is automatic when M has the similarity property). Now, given an amenable subalgebra $P\subseteq M$, we have $P\subset_\gamma N$ and $J_MPJ_M\subset_\gamma N'$ for some small γ, and we can use the spatial embedding theorem (*Theorem 2*) two times to replace N by a small unitary perturbation such that $P\subseteq N$ and $J_MPJ_M\subseteq N'$. In this way, we can apply *Lemma 10* to see that N is in standard position.

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Lemma 10. *Suppose that M is a II_1 factor in standard position on $\mathcal{K}$ with respect to $\xi \in \mathcal{K}$, and suppose that $A \subseteq M$ is a maximal abelian subalgebra (masa) in M. Suppose that N is another II_1 factor on $\mathcal{K}$ such that $A \subseteq N$, $J_M A J_M \subseteq N'$, and $d(M,N)$ is sufficiently small. Then, ξ is a tracial vector for N and N', and therefore, N is also in standard position on $\mathcal{K}$.*

Sketch Proof of Lemma 10

The lemma is proved by using the unique trace preserving expectation E_A^N from N onto A. It is easy to check that, because A is maximal abelian in M, it is also maximal abelian in N, and then, the form of E_A^N is known: $E_A^N(x)$ lies in the strong*-closed convex hull of the set $\{uxu^* : u \in \mathcal{U}(A)\}$ of unitary conjugates of x by A for $x \in N$. The assumption $J_M A J_M \subseteq N'$ gives

$$\langle uxu^* \xi, \xi \rangle = \langle x J_M u J_M \xi, J_M u J_M \xi \rangle = \langle x\xi, \xi \rangle, \quad u \in \mathcal{U}(A),$$

and therefore, $\langle E_A^N(x)\xi, \xi\rangle = \langle x\xi, \xi\rangle$ for all $x \in N$. Because $E_A^N(x) \in A \subseteq M$ and ξ is tracial for M, we have $\tau_M\left(E_A^N(x)\right) = \langle x\xi, \xi\rangle$. However, it is not hard to check that, because M and N are close, τ_M and τ_N agree on A, and therefore, $\tau_N\left(E_A^N(x)\right) = \langle x\xi, \xi\rangle$ for $x \in N$. Because E_A^N is τ_N-preserving, this shows that ξ is tracial for N.

To see that ξ is also tracial for N', interchange the roles of the algebras M and N and their commutants. Here, we use the standard position of M to ensure that $d(M', N')$ is small. This completes the proof.

In fact, we immediately get additional information: in the situation of *Lemma 10*, the inclusions $A \subseteq M$ and $A \subseteq N$ induce the same basic construction. This construction, developed extensively in ref. 37, is the starting point for Jones's theory of subfactors, and it plays a key role in the perturbation results for subalgebras of finite von Neumann algebras (30, 38). Given a subalgebra A of M, write e_A for the projection on $\mathcal{K}$ with range $\overline{A\xi}$. The basic construction of $A \subseteq M$ is the von Neumann algebra $(M \cup \{e_A\})''$ obtained by adjoining e_A to M, and it is denoted $\langle M, e_A\rangle$. This algebra satisfies $\langle M, e_A\rangle = (J_M A J_M)'$.

Corollary 11. *With the same hypotheses as in Lemma 10, we have*

$$\langle M, e_A\rangle = \langle N, e_A\rangle.$$

Proof of Corollary 11

We have $J_M A J_M \subseteq N' = J_N N J_N$ by hypothesis. Standard properties of the basic construction from ref. 37 show that e_A commutes with A and J_M, and therefore, $J_M A J_M \subseteq J_N N J_N \cap \{e_A\}' = J_N(N \cap \{e_A\}')J_N = J_N A J_N$ (using the fact that $N \cap \{e_A\}' = A$). As a result,

$$J_M A J_M \subseteq J_N A J_N \subseteq J_N N J_N = N'.$$

Now, $J_M A J_M$ is a masa in M'; moreover, M' and N' are close, and it follows that $J_M A J_M$ is also maximal abelian in $J_N N J_N = N'$. Hence, $J_M A J_M = J_N A J_N$, and the result follows by taking commutants. This completes the proof.

After we have reached this point of our argument, we can replace A in *Corollary 11* by an amenable subalgebra $P \subseteq M$ with $P' \cap M \subseteq P$ using a technical theorem in the work by Popa (39). This replacement enables us to formulate versions of our main results for suitable actions of discrete groups on the hyperfinite II_1 factor: any factor of the form $(R \rtimes_\alpha SL_n(\mathbb{Z})) \overline{\otimes} R$ for a properly outer action α and $n \geq 3$ is strongly Kadison-Kaster stable.

Using the Basic Construction to Prove Theorem 6

A considerable amount of information regarding an inclusion $A \subseteq M$ of finite von Neumann algebras is encoded in the basic construction algebra $\langle M, e_A\rangle$. Of particular relevance here is the result in the work by Popa (ref. 40, proposition 1.4.3), which shows that a masa A in a II_1 factor M is Cartan in the sense of ref. 41 [i.e., the group of normalizers $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ generates M as a von Neumann algebra] if and only if $A' \cap \langle M, e_A\rangle$ is generated by projections that are finite in $\langle M, e_A\rangle$. As the spatial embedding theorem, *Lemma 9*, *Lemma 10*, and *Corollary 11* combine to show that close inclusions of masas into II_1 factors can be adjusted by a small unitary perturbation to give the same basic construction algebras (albeit possibly on a different Hilbert space), we obtain the next result.

Proposition 12. *Let $A \subseteq M$ be a Cartan masa in a II_1 factor acting on a Hilbert space $\mathcal{H}$. Any inclusion $B \subseteq N$ with $d(M,N)$ and $d(A,B)$ sufficiently small is also an inclusion of a Cartan masa in a II_1 factor.*

Given a crossed product II_1 factor $M_0 = L^\infty(X) \rtimes \Gamma$ arising from a free ergodic probability measure-preserving action $\alpha : \Gamma \curvearrowright (X, \mu)$ and another factor N_0 close to M_0, the assumption of freeness ensures that $L^\infty(X)$ is a maximal abelian subalgebra of M_0. In step *ii* of *Theorem 6*, we use the spatial embedding theorem to assume that $A = L^\infty(X) \subseteq N_0$ and find unitary normalizers $\{v_g\}_{g\in\Gamma}$ in N_0 close to the canonical unitary normalizers $\{u_g\}_{g\in\Gamma}$ in M_0. The previous proposition shows that N_0 is generated by all normalizers of A, but in fact, N_0 is generated by $A \cup \{v_g : g \in \Gamma\}$ as required for step *ii* of the proof of *Theorem 6*. Once we convert to standard position so that $A \subseteq M_0$ and $A \subseteq N_0$ induce the same basic construction, one first notes that $\{u_g e_A u_g^*\}_{g\in\Gamma}$ are pairwise orthogonal and sum to $1_{M_0} = 1_{N_0}$. Because u_g and v_g are close, we must have $v_g e_A v_g^* \approx u_g e_A u_g^*$, but in fact, these projections are equal [because they both are in $Z(A' \cap \langle M, e_A\rangle)$]. The equation $1_N = \sum_{g\in\Gamma} v_g e_A v_g^*$ can then be used to see that finite linear combinations $\sum_{g\in\Gamma} v_g f_g$ (for $f_g \in A$) are dense in N_0.

A similar argument, working at the Hilbert space level, is used in step *i* to show that, if M is a McDuff factor of the form $M_0 \overline{\otimes} R$ and N is close to M and contains R, then N is generated by the commuting subalgebras $R' \cap N$ and R.

The fact that $\sum_{g\in\Gamma} u_g e_A u_g^* = 1_{M_0} = 1_{N_0}$ in $Z(A' \cap \langle M_0, e_A\rangle)$ is also vital in step *iii* of the proof of *Theorem 6*. At this point, using our earlier results, we have a crossed product $M_0 = L^\infty(X) \rtimes_\alpha \Gamma$ acting in standard position on $\mathcal{H}$ with respect to ξ and an isomorphic copy N_0 of M_0 on $\mathcal{H}$ with $A = L^\infty(X) \subseteq N_0$ and $J_{M_0} A J_{M_0} = J_{N_0} A J_{N_0} \subseteq N_0'$. The isomorphism $\theta : M_0 \to N_0$ is obtained from step *ii* using the vanishing of the bounded cohomology group $H_b^2(\Gamma, L_{\mathbb{R}}^\infty(X))$, and so it satisfies $\theta(f) = f$ for $f \in A$ and that $\|\theta(u_g) - u_g\|$ is small for all $g \in \Gamma$. Because *Lemma 10* shows that M_0 and N_0 are both in standard position on $\mathcal{H}$, the isomorphism θ is spatially implemented on $\mathcal{H}$ by W, where W is given by extending the map $W(x\xi) = \theta(x)\xi$ for $x \in M_0$. Since $\theta(f) = f$ for $f \in A$, it follows that $W \in A'$, and similarly, the assumption that $\theta(M_0) = N_0 \subseteq (J_{M_0} A J_{M_0})' = (J_{N_0} A J_{N_0})'$ ensures that $W \in (J_{M_0} A J_{M_0})' = \langle M_0, e_A\rangle$. That is, $W \in A' \cap \langle M_0, e_A\rangle$.

Write $P_g = u_g e_A u_g^*$. It is a standard fact that $(A' \cap \langle M_0, e_A\rangle)P_g = AP_g$ for each $g \in \Gamma$, and so W decomposes as $W = \sum_{g\in\Gamma} w_g P_g$ for some unitary operators $w_g \in A$. For each $g \in \Gamma$, the condition that $\theta(u_g) \approx u_g$ translates to $\alpha_g(w_e) \approx w_g$, and therefore (using the centrality of P_g), $W \approx W_1 = \sum_{g\in\Gamma} \alpha_g(w_e) P_g$. However, $J_{M_0} w_e J_{M_0} P_g = \alpha_g(w_e) P_g$, and therefore, $W_1 \in J_{M_0} A J_{M_0} \subseteq (M_0 \cup N_0)'$. Thus, $\theta = \mathrm{Ad}(W) = \mathrm{Ad}(W_1^* W)$ has $\|\theta - \mathrm{id}_{M_0}\|_{cb} \leq 2\|W - W_1\|$, giving us uniform control on $\|\theta(x) - x\|$ across the unit ball of M_0.

Concluding Remarks and Open Questions

We end with some questions and possible future directions.

It is not hard to use *Lemma 10* to show that, for each $K > 0$, there exists $\delta > 0$ with the property that, if $M, N \subset \mathcal{B}(\mathcal{H})$ are II_1 factors with $d(M,N) < \delta$ and $\dim_M(\mathcal{H}) \leq K$, then $\dim_N(\mathcal{H}) = \dim_M(\mathcal{H})$. However, we have not been able to show that sufficiently close II_1 factors necessarily have the same coupling constant in general. One consequence of a positive answer to this question would be that, in *Theorem 7*, the isomorphism would automatically be spatial without the assumption of a non-strongly ergodic action.

Question 1. *Does there exist $\delta > 0$ such that whenever $M, N \subset \mathcal{B}(\mathcal{H})$ are II_1 factors with $d(M,N) < \delta$, then $\dim_M(\mathcal{H}) = \dim_N(\mathcal{H})$?*

In *Theorem 8*, we use uniqueness results for Cartan masas from the work in ref. 35 to obtain an isomorphism. In contrast with *Theorems 6* and *7*, this method relies on imposing structural hypotheses on both M and N. Furthermore, there are hyperbolic groups Γ for which *Theorem 8* applies, but our cohomological methods do not. Such factors provide a suitable test case for future developments.

Question 2. *Let $M = L^\infty(X) \rtimes_\alpha \Gamma$ be a crossed product factor such that $L^\infty(X)$ is the unique Cartan masa up to unitary conjugacy but the comparison map*

$$H_b^2(\Gamma, L_{\mathbb{R}}^\infty(X)) \to H^2(\Gamma, L_{\mathbb{R}}^\infty(X))$$

is not zero [*i.e.*, *there are nontrivial bounded 2-cocycles*, *which are not trivial in $H^2(\Gamma, L_{\mathbb{R}}^\infty(X))$*]. *Does there exist $\delta > 0$ such that any II_1 factor N with $d(M,N) < \delta$ is isomorphic to M?*

Is it possible to use the methods in the work in ref. 42 to find stability results for factors that are completely close [i.e., $d_{cb}(M,N) = \sup d(\mathbb{M}_n(M), \mathbb{M}_n(N))$ is small]?

Finally, what is the analogous statement to *Theorem 6* in the category of C^*-algebras? A major difficulty here is that the known embedding theorem for separable nuclear C^*-algebras from ref. 16 is not as strong as *Theorem 2*, because it does not guarantee that the resulting embedding is spatial. Because of the counterexamples in ref. 8, it cannot give uniform control on the embedding.

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