Property Γ factors and the Hochschild cohomology problem

Erik Christensen[†], Florin Pop[‡], Allan M. Sinclair[§], and Roger R. Smith[¶‖]

[†]Institute for Mathematiske Fag, University of Copenhagen, Universitetsparken 5, DK 2100 Copenhagen, Denmark; [‡]Department of Mathematics, Wagner College, Staten Island, NY 10301; [§]Department of Mathematics, University of Edinburgh, Edinburgh EH9 3JZ, Scotland; and [¶]Department of Mathematics, Texas A&M University, College Station, TX 77843

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The main result of this article is that the *k*th continuous Hochschild cohomology groups $H^k(\mathcal{M}, \mathcal{M})$ and $H^k(\mathcal{M}, B(H))$ of a von Neumann factor $\mathcal{M} \subseteq B(H)$ of type II_1 with property Γ are 0 for all positive integers *k*. The method of proof involves the construction of hyperfinite subfactors with special properties and a new inequality of Grothendieck type for multilinear maps. We prove joint continuity in the $\|\cdot\|_2$ norm of separately ultraweakly continuous multilinear maps and combine these results to reduce to the case of completely bounded cohomology, which is already solved.

1. Introduction

The study of the continuous Hochschild cohomology groups $H^n(\mathcal{M}, \mathcal{M})$, $n \geq 1$, of a von Neumann algebra $\mathcal{M}$ with coefficients in itself was begun in a series of papers by Johnson, Kadison, and Ringrose (1–4). Their work was an outgrowth of the Kadison–Sakai theorem on derivations (5, 6), which proved, in an equivalent formulation, that $H^1(\mathcal{M}, \mathcal{M}) = 0$ for all von Neumann algebras. It was natural to conjecture that the higher cohomology groups $H^n(\mathcal{M}, \mathcal{M})$ also vanish, and this was settled affirmatively for hyperfinite von Neumann algebras in ref. 4. These authors established many general results on cohomology, some of which are reviewed below. One particular consequence is that it suffices to consider separately the cases when $\mathcal{M}$ is type I, II_1, II_∞, or III in the Murray–von Neumann classification scheme; the general von Neumann algebra is a sum of these four types. Because type I von Neumann algebras are hyperfinite (but by no means exhaust this class), attention has been focused on the remaining three types. Considerable progress on the problem has been made recently by the introduction of the completely bounded cohomology groups $H^n_{cb}(\mathcal{M}, \mathcal{M})$. E.C. and A.M.S. (7) used the structure theory of completely bounded multilinear maps (8) to show that $H^n_{cb}(\mathcal{M}, \mathcal{M}) = 0$ for all von Neumann algebras (see chapter 4 in ref. 9). These authors and Effros (10) also proved that the continuous and completely bounded cohomology groups coincide when $\mathcal{M}$ is type II_∞, III, or II_1 and stable under tensoring with the hyperfinite type II_1 factor, showing that $H^n(\mathcal{M}, \mathcal{M}) = 0$ in these cases. Thus the conjecture remains open only for type II_1 von Neumann algebras.

Within the class of type II_1 factors the main results are that $H^n(\mathcal{M}, \mathcal{M}) = 0$, for separable factors with a Cartan subalgebra and $n \geq 1$ (11, 12), and for factors with property Γ when $n = 2$. The latter result is due to E.C. and may be found in chapter 6 of ref. 9 (see ref. 13 for a later proof). The main result of this article, which builds on the techniques of ref. 13 is that $H^n(\mathcal{M}, \mathcal{M}) = 0$, $n \geq 1$, for type II_1 von Neumann factors that have property Γ. This was introduced by Murray and von Neumann (14) to distinguish certain nonisomorphic factors from one another. They defined this property by requiring that for each finite subset $\{x_i\}_{i=1}^n \in \mathcal{M}$ and each $\varepsilon > 0$, there should exist a unitary $u \in \mathcal{M}$ of zero trace satisfying

$$\|ux_i - x_iu\|_2 < \varepsilon, \quad 1 \leq i \leq n. \qquad \textbf{[1.1]}$$

We will work with a later characterization due to Dixmier (15), which we describe in Section 5. The class of Γ factors contains all that are stable under tensoring with the hyperfinite factor but also contains many factors that are not of this type.

Recent approaches to cohomology (9–13) have focused on proving that the relevant cocycles are completely bounded as multilinear maps, and that is the method that we use here. It has often been the case that complete boundedness of maps of a factor into itself could be proved but only by techniques that were ineffective for larger ranges. For Γ factors, the situation is different, and our results also apply to the cohomology groups with coefficients in any containing $B(H)$. For $n = 1$ and $n = 2$, see refs. 16 and 17, respectively.

In Section 2 of the article we review the basic definitions of cohomology theory and we include a brief discussion of completely bounded maps. Section 3 presents a new version (*Theorem 3.1*) of the Pisier–Haagerup–Grothendieck inequality, which has been important in earlier work on cohomology, and also discusses some results on the joint continuity in the $\|\cdot\|_2$ norm for multilinear maps (*Theorem 3.3*). Section 4 expands on a theorem of Popa (17), which states that hyperfinite subfactors of trivial relative commutant can be found in any type II_1 factor with separable predual. When property Γ is assumed, we show that the hyperfinite subfactor can be chosen to contain certain auxiliary projections (*Theorem 4.1*). In Section 5 we describe how the results of the previous two sections can be combined to prove the complete boundedness of certain multilinear maps (*Theorem 5.1*), and this leads to our main result on the cohomology groups of Γ factors (*Theorem 5.2*). Complete details will appear elsewhere.

We refer the reader to ref. 18 for an early survey of cohomology theory and to a later account in ref. 9, which contains all the necessary background material for this article, as well as a discussion of applications.

2. Preliminaries

The matrix algebras $\mathbb{M}_n(\mathcal{A})$, $n \geq 1$, over a C^*-algebra $\mathcal{A} \subseteq B(H)$ carry natural norms, defined by viewing $\mathbb{M}_n(\mathcal{A})$ as a subalgebra of $\mathbb{M}_n(B(H))$ and identifying the latter algebra with $B(H \oplus \cdots \oplus H)$ (n-fold direct sum). Thus a bounded linear map $\phi: \mathcal{A} \to B(H)$ induces a family $\phi^{(n)}: \mathbb{M}_n(\mathcal{A}) \to \mathbb{M}_n(B(H))$, $n \geq 1$, of bounded maps on the matrix algebras by applying ϕ in each entry, and ϕ is said to be completely bounded if

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[‖]To whom correspondence should be addressed. E-mail: rsmith@math.tamu.edu.

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$$\sup\{\|\phi^{(n)}\|: n \geq 1\} < \infty. \quad \textbf{[2.1]}$$

This supremum then defines the completely bounded norm $\|\phi\|_{cb}$. There is substantial literature on completely bounded maps (see refs. 9 and 19 and references therein). Now let $\mathcal{A}^k$ denote the k-fold Cartesian product of copies of $\mathcal{A}$. A k-linear map ϕ: $\mathcal{A}^k \to B(H)$ may be lifted to a k-linear map $\phi^{(n)}$: $\mathbb{M}_n(\mathcal{A})^k \to \mathbb{M}_n(B(H))$, $n \geq 1$. For clarity we take $k = 2$, because this case contains the essential ideas. For matrices $X = (x_{ij})$, $Y = (y_{ij}) \in \mathbb{M}_n(\mathcal{A})$, the (i, j) entry of $\phi^{(n)}(X, Y) \in \mathbb{M}_n(B(H))$ is defined to be $\Sigma_{r=1}^{n}\ \phi(x_{ir}, y_{rj})$. Following the linear case, the completely bounded norm is also defined by Eq. **2.1**. Such maps have important applications in cohomology theory (9).

We will require the notion of multimodular maps below. If $\mathcal{R} \subseteq \mathcal{M} \subseteq B(H)$ is an inclusion of algebras then $\mathcal{R}$-multimodularity of ϕ: $\mathcal{M}^k \to B(H)$ is defined by the equations

$$r\phi(x_1, x_2, \ldots, x_k) = \phi(rx_1, x_2, \ldots, x_k), \quad \textbf{[2.2]}$$

$$\phi(x_1, \ldots, x_{k-1}, x_k)r = \phi(x_1, \ldots, x_{k-1}, x_k r), \quad \textbf{[2.3]}$$

$$\phi(x_1, \ldots, x_i r, x_{i+1}, \ldots, x_k) = \phi(x_1, \ldots, x_i, rx_{i+1}, \ldots, x_k), \quad \textbf{[2.4]}$$

where $r \in \mathcal{R}$ and $x_i \in \mathcal{M}$ for $1 \leq i \leq k$. A simple but important consequence of the definitions is that $\phi^{(n)}$ is $\mathbb{M}_n(\mathcal{R})$-multimodular, for all $n \geq 1$, when ϕ is $\mathcal{R}$-multimodular.

We recall from refs. 1 and 9 the basic definitions of Hochschild cohomology theory. Let $\mathcal{M}$ be a von Neumann algebra and denote by $\mathcal{L}^k(\mathcal{M}, \mathcal{X})$ the space of k-linear bounded maps ϕ: $\mathcal{M}^k \to \mathcal{X}$ into a Banach $\mathcal{M}$-bimodule $\mathcal{X}$. The coboundary operator ∂: $\mathcal{L}^k(\mathcal{M}, \mathcal{X}) \to \mathcal{L}^{k+1}(\mathcal{M}, \mathcal{X})$ is defined by

$$\partial\phi(x_1, \ldots, x_{k+1}) = x_1\phi(x_2, \ldots, x_{k+1}) + \sum_{i=1}^{k}(-1)^i\phi(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_k)$$
$$+ (-1)^{k+1}\phi(x_1, \ldots, x_k)x_{k+1} \quad \textbf{[2.5]}$$

for $x_1, \ldots, x_{k+1} \in \mathcal{M}$. Then ϕ is a k cocycle if $\partial\phi = 0$, whereas ϕ is a k coboundary if $\phi = \partial\psi$ for some $\psi \in \mathcal{L}^{k-1}(\mathcal{M}, \mathcal{X})$. A short algebraic calculation shows that $\partial\partial = 0$, and thus coboundaries are cocycles. The cohomology group $H^k(\mathcal{M}, \mathcal{X})$ then is defined to be the space of k cocycles modulo the space of k coboundaries (for $k \geq 2$). For $k = 1$, $H^1(\mathcal{M}, \mathcal{X})$ is defined to be the space of bounded derivations modulo the space of inner derivations. The definition gives rise to a related family of cohomology groups by imposing further restrictions on the bounded maps. We might require ultraweak continuity ($H^k_w(\mathcal{M}, \mathcal{X})$), $\mathcal{R}$-multimodularity ($H^k(\mathcal{M}, \mathcal{X}$:/$\mathcal{R}$)), complete boundedness ($H^k_{cb}(\mathcal{M}, \mathcal{X})$), or any combination of these. The interplay between these various cohomology theories gives an important tool for the determination of $H^k(\mathcal{M}, \mathcal{X})$. The following theorem summarizes the work of several authors (2–4, 7, 10), and the various parts are described in ref. 9.

Theorem 2.1. *Let $\mathcal{M} \subseteq B(H)$ be a von Neumann algebra with a hyperfinite subalgebra $\mathcal{R}$, and let $\mathcal{X}$ be a dual normal $\mathcal{M}$-bimodule.*

(*i*) *For $k \geq 1$,*

$$H^k(\mathcal{M}, \mathcal{X}) = H^k_w(\mathcal{M}, \mathcal{X}) = H^k_w(\mathcal{M}, \mathcal{X}: /\mathcal{R}). \quad \textbf{[2.6]}$$

(*ii*) *For $k \geq 1$ and either $\mathcal{X} = \mathcal{M}$ or $\mathcal{X} = B(H)$,*

$$H^k_{cb}(\mathcal{M}, \mathcal{X}) = H^k_{wcb}(\mathcal{M}, \mathcal{X}) = H^k_{wcb}(\mathcal{M}, \mathcal{X}: /\mathcal{R}) = 0. \quad \textbf{[2.7]}$$

The second part of this theorem shows that vanishing of cohomology for these two bimodules can be established by proving that each cocycle is cohomologous to one that is completely bounded, whereas the first part shows that attention can be restricted to those cocycles that are separately normal and $\mathcal{R}$-multimodular for a suitably chosen $\mathcal{R}$. This is now the standard approach to such problems, and we adopt it below.

3. Grothendieck's Inequality and Joint Continuity

One of the most useful tools for cohomology theory has been the noncommutative generalization of Grothendieck's inequality for bilinear forms. This was proved first by Pisier (20) under mild restrictions and then by Haagerup (21) in full generality. For a bilinear form θ: $\mathcal{M} \times \mathcal{M} \to \mathbb{C}$ on a von Neumann algebra $\mathcal{M}$, separately normal in each variable, the appropriate formulation is as follows. There exist four normal states $f_1, f_2, g_1, g_2 \in \mathcal{M}_*$ such that

$$|\theta(x, y)| \leq \frac{1}{2}\|\theta\|(f_1(x^*x) + f_2(xx^*) + g_1(y^*y) + g_2(yy^*)) \quad \textbf{[3.1]}$$

for $x, y \in \mathcal{M}$. This inequality is valid for all von Neumann algebras; we assume for the rest of the section that $\mathcal{M}$ is a type II_1 factor, and we describe some more specialized inequalities that then are available. If θ above has the additional property of being inner $\mathcal{R}$-modular, in the sense that

$$\theta(xr, y) = \theta(x, ry), \quad x, y \in \mathcal{M}, \quad \textbf{[3.2]}$$

for r in a hyperfinite subfactor $\mathcal{R}$ with trivial relative commutant $\mathcal{R}' \cap \mathcal{M} = \mathbb{C}1$, then Eq. **3.1** takes the form

$$|\theta(x, y)| \leq \|\theta\|(F(xx^*) + G(y^*y)), \quad x, y \in \mathcal{M}, \quad \textbf{[3.3]}$$

where F and G are normal states. This is accomplished by replacing x and y in Eq. **3.1** by xu and u^*y, respectively, where u lies in an amenable group $\mathcal{G}$ of unitaries, which generates $\mathcal{R}$, and then averaging the right-hand side of the inequality over $\mathcal{G}$. The x^*x and yy^* terms in Eq. **3.1** become $tr(x^*x)1$ and $tr(yy^*)1$ after averaging and are then replaced by $tr(xx^*)1$ and $tr(y^*y)1$, explaining why only two types of terms appear on the right-hand side of Eq. **3.3**. Further replacement of x by tx and y by $t^{-1}y$ for $t \in \mathbb{R}^+$ in Eq. **3.3**, followed by minimization over t, leads to

$$|\theta(x, y)| \leq 2\|\theta\|F(xx^*)^{1/2}G(y^*y)^{1/2}, \quad x, y \in \mathcal{M}. \qquad [3.4]$$

There is no direct generalization of Eq. **3.4** to multilinear forms, but nevertheless certain inequalities can be obtained, based on the bilinear equalities above, by fixing all but two of the variables. We illustrate this for three variables. We observe that a bilinear map ϕ: $\mathcal{M} \times \mathcal{M} \to B(H)$ may be studied by applying a vector functional $\langle \cdot \xi, \eta \rangle$ and examining the bilinear form

$$\theta(x, y) = \langle \phi(x, y)\xi, \eta \rangle, \quad x, y \in \mathcal{M}. \qquad [3.5]$$

For a trilinear map ϕ: $\mathcal{M}^3 \to B(H)$, various auxiliary bilinear maps are used to focus on each variable separately. For the outer variables we use maps of the type

$$\psi_1(x, y) = \phi(x^*, x_2, x_3)^*\phi(y, x_2, x_3), \quad x, y \in \mathcal{M}, \qquad [3.6]$$

and

$$\psi_3(x, y) = \phi(x_1, x_2, x)\phi(x_1, x_2, y^*)^*, \quad x, y \in \mathcal{M}, \qquad [3.7]$$

whereas the inner variable is handled by

$$\psi_2(x, y) = \phi(x, y, x_3), \quad x, y \in \mathcal{M}. \qquad [3.8]$$

The x_i values in these equations are taken to be arbitrary but fixed, and an assumption of $\mathcal{R}$-multimodularity on ϕ ensures that each ψ_i, $1 \leq i \leq 3$, is inner $\mathcal{R}$-modular. This approach applies also to the nth amplification $\phi^{(n)}$ of ϕ to $\mathbb{M}_n(\mathcal{M})$, and we now state an inequality for k-linear maps that results from applying the bilinear Grothendieck inequality repeatedly to bilinear maps of the above types. We have indicated that the trace is implicit in Eqs. **3.3** and **3.4**; the new inequality will contain it explicitly. We let tr_n denote the unique normalized normal trace on $\mathbb{M}_n(\mathcal{M})$, and we introduce a new norm on $\mathbb{M}_n(\mathcal{M})$ by

$$\rho_n(X) = (\|X\|^2 + n\ tr_n(X^*X))^{1/2}, \quad X, Y \in \mathbb{M}_n(\mathcal{M}). \qquad [3.9]$$

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Theorem 3.1. *Let $\mathcal{M} \subseteq B(H)$ be a type II_1 factor containing a hyperfinite subfactor $\mathcal{R}$ with trivial relative commutant. Let ϕ: $\mathcal{M}^k \to B(H)$ be a bounded $\mathcal{R}$-multimodular k-linear map that is separately normal in each variable. Then*

$$\|\phi^{(n)}(X_1, \ldots, X_k)\| \leq 2^{k/2}\|\phi\|\rho_n(X_1)\cdots\rho_n(X_k) \qquad [3.10]$$

for all $X_1, \ldots, X_k \in \mathbb{M}_n(\mathcal{M})$ and $n \in \mathbb{N}$.

If the tracial term in Eq. **3.9**, and thus in Eq. **3.10**, could be removed, then *Theorem 3.1* would give complete boundedness of maps that satisfy the hypotheses. The following corollary is a step in this direction.

Corollary 3.2. *In addition to the hypotheses of* Theorem 3.1, *let $P \in \mathbb{M}_n(\mathcal{M})$ be a projection of trace n^{-1}. Then*

$$\|\phi^{(n)}(X_1P, \ldots, X_kP)\| \leq 2^k\|\phi\|\|X_1\|\cdots\|X_k\|, \qquad [3.11]$$

for $X_1, \ldots, X_k \in \mathbb{M}_n(\mathcal{M})$.

This inequality is a simple consequence of Eq. **3.10** by estimating that

$$\rho_n(XP)^2 = \|XP\|^2 + n\ tr_n(PX^*XP) \leq \|X\|^2(1 + n\ tr_n(P)) = 2\|X\|^2 \qquad [3.12]$$

for $X \in \mathbb{M}_n(\mathcal{M})$.

We now turn to joint continuity of multilinear separately normal maps. We will need the topology of $\|\cdot\|_2$-norm convergence, where $\|\cdot\|_2$ is defined by

$$\|x\|_2 = (tr(x^*x))^{1/2}, \quad x \in \mathcal{M}. \qquad [3.13]$$

To see the necessity for this, consider the free group $\mathbb{F}_2$ with generators a and b and its associated type II_1 factor $VN(\mathbb{F}_2)$. The group element a becomes a unitary in this factor and has the property that $\lim_{n\to\infty} a^{\pm n} = 0$ ultraweakly. For the separately normal bilinear multiplication map $m(x, y) = xy$, defined on any factor, the equation $\lim_{n\to\infty} m(a^n, a^{-n}) = 1$ precludes joint ultraweak continuity. The simple estimates

$$\|m(x, y)\|_2 \leq \|x\|\|y\|_2, \|x\|_2\|y\|, \quad x, y \in \mathcal{M}, \qquad [3.14]$$

when restricted to the unit ball of $\mathcal{M}$, lead to

$$\|m(x, y)\|_2 \leq \min\{\|x\|_2, \|y\|_2\} \leq \|x\|_2^{1/2}\|y\|_2^{1/2}, \quad \|x\|, \|y\| \leq 1, \qquad [3.15]$$

and this inequality generalizes to

$$\|x_1\cdots x_k\|_2 \leq \min_{1\leq i\leq k}\{\|x_i\|_2\}\leq\|x_1\|_2^{1/k}\cdots\|x_k\|_2^{1/k}, \quad \text{for } \|x_i\|\leq 1. \quad [3.16]$$

This suggests joint $\|\cdot\|_2$-norm continuity on bounded balls in any type II_1 factor. The example of a sequence of projections $\{p_n\}_{n=1}^{\infty}$ with $tr(p_n) = n^{-4}$ and thus satisfying

$$\lim_{n\to\infty}\|np_n\|_2 = 0, \quad \|m(np_n, np_n)\|_2 = 1 \quad [3.17]$$

shows that joint continuity on the whole factor cannot be expected. These remarks indicate that the following result is optimal.

Theorem 3.3. *Let $\mathcal{M} \subseteq B(H)$ be a type II_1 factor, let $k \in \mathbb{N}$, and let ϕ: $\mathcal{M}^k \to B(H)$ be a bounded separately normal k-linear map.*

(*i*) *If the range of ϕ is contained in $\mathcal{M}$, then the restriction of ϕ to the product of closed balls is jointly $\|\cdot\|_2$-norm continuous.*
(*ii*) *If ξ, $\eta \in H$, then the same conclusion holds for the k-linear form*

$$\psi(x_1, \ldots, x_k) = \langle\phi(x_1, \ldots, x_k)\xi, \eta\rangle, \quad x_i \in \mathcal{M}. \quad [3.18]$$

For $k = 1$, a normal linear map ϕ: $\mathcal{M} \to \mathcal{M}$ induces a separately normal bilinear form

$$\psi(x, y) = tr(\phi(x)\phi(y^*)^*), \quad x, y \in \mathcal{M}, \quad [3.19]$$

to which Grothendieck's inequality can be applied. The $\|\cdot\|_2$-norm continuity of ϕ on balls is a consequence of this (using corollary 5.4.3 of ref. 9). The proof of the theorem for $k \geq 2$ is by induction. The first step is to obtain joint continuity at a single point, and this is a special case of a theorem in ref. 22, which relies on the Baire category theorem. A separate argument is required to deduce joint continuity at the origin, from which joint continuity at all points in a product of balls follows. *ii* is a simple consequence of *i* by treating the range $\mathbb{C}$ as $\mathbb{C}1 \subseteq \mathcal{M}$.

4. Hyperfinite Subfactors

The purpose of this section is to combine two results, due to Popa (17) and Dixmier (15), respectively. If we assume that a type II_1 factor $\mathcal{M}$ has a separable predual, then the first states that we may find a hyperfinite subfactor $\mathcal{R}$ with trivial relative commutant. Any hyperfinite subfactor $\mathcal{R}$ may be expressed as the ultraweak closure of a union of an increasing sequence of matrix subfactors $\mathcal{A}_n$, $n \geq 1$. Denoting their unitary groups by $\mathcal{U}_n$ and then integrating with respect to normalized Haar measure leads to

$$\mathbb{E}_{\mathcal{R}'\cap\mathcal{M}}(x) = \lim_{n\to\infty}\int_{\mathcal{U}_n} uxu^*du, \quad x \in \mathcal{M}, \quad [4.1]$$

where $\mathbb{E}_{\mathcal{R}'\cap\mathcal{M}}$ is the trace preserving conditional expectation of $\mathcal{M}$ onto $\mathcal{R}' \cap \mathcal{M}$, and the limit is taken ultraweakly (see lemma 5.4.4 of ref. 9). Consequently, the relative commutant will be trivial precisely when the limit in Eq. **4.1** is $tr(x)1$ for all $x \in \mathcal{M}$, giving a criterion for $\mathcal{R}' \cap \mathcal{M}$ to be $\mathbb{C}1$.

Dixmier's result is an equivalent formulation of property Γ: given $\varepsilon > 0$, $n \in \mathbb{N}$, and $\{x_1, \ldots, x_m\} \in \mathcal{M}$, there exist n orthogonal projections $\{p_i\}_{i=1}^n$, each of trace n^{-1}, so that

$$\|[p_i, x_j]\|_2 < \varepsilon, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m, \quad [4.2]$$

where $[\cdot, \cdot]$ denotes the commutator. For Γ factors these results can be combined.

Theorem 4.1. *Let $\mathcal{M}$ be a type II_1 factor with property Γ and a separable predual. Then $\mathcal{M}$ has a hyperfinite subfactor $\mathcal{R}$ with trivial relative commutant, and such that Eq.* **4.2** *can always be satisfied by projections from $\mathcal{R}$.*

This subfactor is built inductively from matrix subfactors $\mathcal{A}_k$, $k \geq 1$. A $\|\cdot\|_2$-norm dense sequence $\{m_i\}_{i=1}^{\infty}$ is fixed in the unit ball of $\mathcal{M}$. At the kth stage of the induction, $\mathcal{A}_{k-1}$ is tensored with a large matrix subfactor to form $\mathcal{B}_k$ in such a way that this algebra contains projections satisfying Eq. **4.2** for $\varepsilon = k^{-1}$, for any subset of $\{m_1, \ldots, m_k\}$, and for any $n \in \{1, \ldots, k\}$. Then $\mathcal{A}_k$ is formed by tensoring $\mathcal{B}_k$ with another matrix subfactor, chosen to ensure that the limit in Eq. **4.1** is $tr(x)1$ for all $x \in \mathcal{M}$. This last step requires Popa's theorem on the existence of hyperfinite subfactors with trivial relative commutant.

5. Main Results

We are now able to state the main results of the article. In the first theorem we note that there is no restriction on the dimension of the Hilbert space.

Theorem 5.1. *Let $\mathcal{M} \subseteq B(H)$ be a type II_1 factor with property Γ and a separable predual. Let $\mathcal{R}$ be a hyperfinite subfactor satisfying the conclusions of* Theorem 4.1, *and let ϕ: $\mathcal{M}^k \to B(H)$ be a separately normal $\mathcal{R}$-multimodular k-linear map. Then ϕ is completely bounded and $\|\phi\|_{cb} \leq 2^k\|\phi\|$.*

The argument is best illustrated by $k = 2$. If $p \in \mathcal{R}$ is a projection, then $\mathcal{R}$-multimodularity is used to establish the following algebraic identity, which has more complicated counterparts for $k \geq 3$:

$$p\phi(x, y) = \phi(p[p, x], y) + \phi([p, x], p[p, y]) + \phi(x, p[p, y]) + p\phi(xp, yp)p, \quad x, y \in \mathcal{M}. \quad [5.1]$$

Now fix $n \in \mathbb{N}$ and two unit vectors ξ, $\eta \in H^n$. Choose sets of projections $\{p_{i,j} \in \mathcal{R}: 1 \leq j \leq n, i \geq 1\}$ of trace n^{-1} such that

$$\sum_{j=1}^{n} p_{i,j} = 1, \quad \lim_{i\to\infty} \|[p_{i,j}, x]\|_2 = 0, \quad x \in \mathcal{M}, \qquad [5.2]$$

and let $P_{i,j} = I_n \otimes p_{i,j} \in \mathbb{M}_n(\mathcal{R})$. Note that for each j, the sequence $\{P_{i,j}\}_{i=1}^{\infty}$ asymptotically commutes in the $\|\cdot\|_2$ norm with the elements of $\mathbb{M}_n(\mathcal{M})$. Because $\phi^{(n)}$ is $\mathbb{M}_n(\mathcal{R})$-multimodular, we may replace ϕ by $\phi^{(n)}$ in Eq. **5.1** while also substituting $X, Y \in \mathbb{M}_n(\mathcal{M})$ for $x, y \in \mathcal{M}$ and $P_{i,j} \in \mathbb{M}_n(\mathcal{R})$ for $p \in \mathcal{R}$. Then apply the vector functional $\langle \cdot\xi, \eta\rangle$ to the resulting equation, sum over j, and let $i \to \infty$. On the left-hand side we obtain $\langle \phi^{(n)}(X, Y)\xi, \eta\rangle$. On the right-hand side, the term $\sum_{j=1}^{n} \langle P_{i,j}\phi^{(n)}(XP_{i,j}, YP_{i,j})P_{i,j}\xi, \eta\rangle$ is estimated by

$$\max_{1\le j\le n} \|\phi^{(n)}(XP_{i,j}, YP_{i,j})\| \le 2^2\|\phi\|\|X\|\|Y\|, \qquad [5.3]$$

using *Corollary 3.2* and the orthogonality of $\{P_{i,j}\}_{j=1}^{n}$ for $i \ge 1$. The remaining terms all contain commutators and thus vanish in the limit, by *Theorem 3.3*, leaving

$$|\langle \phi^{(n)}(X, Y)\xi, \eta\rangle| \le 2^2\|\phi\|\|X\|\|Y\|, \quad X, Y \in \mathbb{M}_n(\mathcal{M}). \qquad [5.4]$$

The inequality $\|\phi\|_{cb} \le 2^2\|\phi\|$ follows immediately, because the right-hand side of Eq. **5.4** is independent of n.

We have already remarked at the end of Section 2 that, in determining cohomology groups of von Neumann algebras, it suffices to restrict to normal $\mathcal{R}$-multimodular cocycles and prove that they are completely bounded. When $\mathcal{M}$ has a separable predual, the following theorem is a consequence of *Theorem 5.1.* The general case is then deduced by showing that any type II_1 factor with property Γ is a union of factors with this property, each of which has a separable predual.

Theorem 5.2. *Let $\mathcal{M} \subseteq B(H)$ be a type II_1 factor with property Γ. Then, for $n \ge 1$,*

$$H^n(\mathcal{M}, \mathcal{M}) = 0, \quad H^n(\mathcal{M}, B(H)) = 0. \qquad [5.5]$$

As we have noted previously, the cases $n = 1, 2$ of this theorem are already known (5, 6, 13, 16).

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