

Kac-Moody and Virasoro Symmetries of Principal Chiral Sigma Models

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ABSTRACT

It is commonly asserted that there is a $\widehat{G} \times G$ centreless Kac-Moody extension of the manifest $G \times G$ global symmetry of the two-dimensional principal chiral model (PCM) for the group manifold G . Here, we show that the symmetry is in fact larger, namely $\widehat{G} \times \widehat{G}$, the full centreless Kac-Moody extension of the entire manifest $G \times G$ global symmetry. Extending previous results in the literature, we also obtain an explicit realisation of the Virasoro-like symmetry of the PCM, generated by $K_n = L_{n+1} - L_{n-1}$ for both positive and negative n . We show that these generators obey Sugarawara-type commutation relations with the two commuting copies of the Kac-Moody algebra \widehat{G} .

1 Introduction

String theory is probably the most successful approach to quantum gravity now available, yet there remains much that is not understood about it. In perturbative string theory, one starts by investigating how the string propagates in flat Minkowski spacetime. This amounts to an investigation of free fields propagating on the two-dimensional world-sheet of the physical string. One of the organising principles of such a theory is its classical invariance under two-dimensional Weyl transformations of the string world-sheet. Much of what is known about string theory comes from ensuring that this Weyl symmetry is preserved in the quantum domain. This leads to the possibility of examining strings propagating in backgrounds other than flat spacetime. For example, string theory on a product of d -dimensional Minkowski spacetime with a group manifold of suitable dimension is described by an exactly-solvable principal sigma model (PCM) with a critical Wess-Zumino term that makes the model conformally invariant at the quantum level [1]. The world-sheet supersymmetric version of the model, namely the spinning string on a group manifold, has also been studied [2]. Further extensions using R-R rather than NS-NS fields to support the background have been considered in recent years, which requires the use of the Green-Schwarz formalism.

An example of a background of this last type that has been much studied is $AdS_5 \times S^5$ in the type IIB string. This, however, goes beyond the remit of the present paper since its bosonic sector is not a group manifold. Another example, which is a group manifold, is $AdS_3 \times S^3 \times T^4$. Although this can be considered in either the type IIA or type IIB string theory, it is of more interest to consider it in type IIB, since then one can rotate between the use of RR and NS-NS flux for supporting the background. With only NS-NS flux non-vanishing, the superstring action contains an $SL(2, \mathbb{R}) \times SU(2)$ WZW model as a subsector [3]. Turning on the RR flux means that the coefficient of the WZ term changes, reducing to zero if the flux is rotated into purely the RR sector. The bosonic sector is then precisely described by a PCM. In order to study the intermediate cases, it becomes of interest to consider a PCM with an adjustable coefficient μ for the WZ term. In fact, away from criticality (which occurs at $\mu = 1$) a straightforward redefinition enables the model to be mapped into the pure $\mu = 0$ PCM case [4].

There exists a vast literature on the PCM and yet, it is remarkable that properties as basic as its symmetries have still not been fully settled. Before stating our new results, let us briefly summarize the known symmetries and related properties of the PCM. Although PCMs make an appearance in string theory, and are currently fashionable as a consequence, they have a much longer history (see [4] for a comprehensive historical summary of the

literature on hidden symmetries in two-dimensional PCM).

Firstly, it is known that the PCM can be derived from the integrability condition of first order equations [5, 6], known as a Lax pair, that amount to certain zero curvature conditions. This implies classical integrability and infinitely many nonlocal conserved charges that obey a Yangian algebra, which is essentially an enveloping algebra with a nontrivial coproduct rule (see [7] for a review). In addition, an infinite set of local charges with spins equal to the exponents of the associated Lie algebra modulo the Coxeter number exist, and they commute with each other as well as the Yangian charges [8].

The focus of the paper will be addressing the issue of Kac-Moody and Virasoro-like symmetries. The PCM has a manifest global $G_L \times G_R$ invariance in any dimensions. Half of this invariance is a gauge transformation which can be used to fix the value of g at some point in spacetime. The other half is a genuine symmetry of the theory, albeit a rather trivial one. What is remarkable however is that in two spacetime dimensions this symmetry becomes enhanced. This enhancement was first discovered by Luscher and Pohlmeyer [5] who identified an infinite set of conserved charges in the theory, and a collection of Bäcklund transformations that mapped one solution of the classical equations of motion into another. Subsequently, L. Dolan [10] showed that the modulus space of classical solutions of the PCM equations of motion admitted an action of half of the Kac-Moody algebra \widehat{G} . Her work was revisited, simplified and confirmed by Devchand and Fairlie [11]. Y.S. Wu showed that the symmetry discovered by Dolan could be extended to a direct product of a full Kac-Moody algebra \widehat{G} with a Lie algebra G , i.e. $\widehat{G} \times G$ [12].¹ Later, it was argued in [13] that the symmetry is instead described by a direct product of two commuting “half Kac-Moody” algebras.

In fact, as we shall demonstrate in this paper, the Kac-Moody symmetry of the PCM is actually larger than either the $\widehat{G} \times G$ algebra found in [12] or the product of two commuting “half Kac-Moody” algebras described in [13]. There is a simple argument, based on an observation by Schwarz [4], which shows that this must be so. It is known that a symmetric space sigma model (SSM), in which the scalar fields of a two-dimensional theory live on a symmetric coset space G/H , has a Kac-Moody symmetry \widehat{G} [17]. Now the PCM for a group manifold G can be equivalently viewed as an SSM for the symmetric coset space $(G \times G)/G$, and therefore from the known results for the SSM, it must be that the PCM has

¹Note that this Kac-Moody algebra has no central extension, and as such it is often referred to as a loop algebra. In common with much of the literature in this subject, we shall however use the terminology of Kac-Moody algebra in this paper.

the symmetry $\widehat{G} \times \widehat{G}$.² One of the main purposes of the present paper is to give an explicit construction of the full $\widehat{G} \times \widehat{G}$ symmetry transformations for the PCM. Whilst this obviously contains $\widehat{G} \times G$ as a subalgebra, this is, as we shall discuss in appendix A, different from the $\widehat{G} \times G$ symmetry that was claimed in [12, 4]. We shall in addition show in Appendix A how the product of two commuting “half Kac-Moody” algebras found in [13] is also a subalgebra of the full $\widehat{G} \times \widehat{G}$ symmetry.

Cheng [14], Hou and Li [15], Li, and Hao, Hou and Li [16] found evidence for the existence of some kind of Virasoro like symmetry acting on the classical moduli space. The subject was re-invigorated by Schwarz in 1995 when, stimulated by the string theoretic applications, he re-examined the whole problem, and presented arguments in support of the $\widehat{G} \times G$ symmetry proposed by Wu [4]. In addition, he found that a particular subalgebra of the Virasoro algebra also acts on the classical moduli space. We shall also discuss a further Virasoro-like symmetry of the model.

2 Kac-Moody Symmetries

The principal chiral model starts with a field $g(x)$ which map the spacetime with coordinates x^μ into some representation R of a Lie group G . Suppose that the Lie algebra of G is denoted by \mathcal{G} and T_i are the generators of G in this representation, then these generators obey the commutation relation

$$[T_i, T_j] = f_{ij}^k T_k \quad (2.1)$$

where f_{ij}^k are the structure constants of the group G . If it is required, any object in the Lie algebra can be decomposed into its components by using T_i are a basis, thus for example if X is Lie algebra valued, then

$$X = X^i T_i \quad (2.2)$$

defines its components X^i . From $g(x)$ we can construct a gauge field A , a connection, that is a one-form that takes its values in the Lie algebra \mathcal{G} . Explicitly,

$$A = g^{-1} dg \quad (2.3)$$

The curvature of any gauge field is given by

$$F = dA + A \wedge A \quad (2.4)$$

²Schwarz actually used this argument in reverse, observing that if one accepts the result $\widehat{G} \times G$ for a PCM, then the SSM with coset G/H should have a symmetry smaller than the full \widehat{G} . It was subsequently shown in [17] that this smaller symmetry for the SSM, exhibited explicitly in [4], is actually augmented by additional symmetries not found in [4], to give the full \widehat{G} Kac-Moody symmetry for the SSM.

Thus, by the Maurer-Cartan equation, the connection is flat, $F = 0$, *i.e.*

$$dA + A \wedge A = 0. \quad (2.5)$$

The action for the PCM is

$$I = -\frac{1}{2} \int d^n x \operatorname{Tr}(*A \wedge A) \quad (2.6)$$

Variation of this action with respect to $g(x)$, for which it is useful to record the lemma

$$\delta A = d\Delta g + [A, \Delta g], \quad \Delta g \equiv g^{-1} \delta g, \quad (2.7)$$

gives the equation of motion

$$d*A = 0. \quad (2.8)$$

A is a left-invariant one-form since if one considers the transformation $g \rightarrow hg$ where h is a constant element of G , A is invariant. The theory can be reformulated in terms of a right-invariant one-form $\bar{A} = -dgg^{-1}$, so that A can be rewritten in terms of \bar{A} by the substitution

$$A = -g^{-1} \bar{A} g. \quad (2.9)$$

Under the transformation $g \rightarrow gk$ where k is a constant element of G , \bar{A} is invariant. The curvature \bar{F} of \bar{A} is again zero as can be seen from its definition

$$\bar{F} = d\bar{A} + \bar{A} \wedge \bar{A} \quad (2.10)$$

The action for the PCM can be written in terms of \bar{A} instead of A and is

$$I = -\frac{1}{2} \int d^n x \operatorname{Tr}(*\bar{A} \wedge \bar{A}). \quad (2.11)$$

Under variations of g one finds the equation of motion

$$d*\bar{A} = 0 \quad (2.12)$$

which is equivalent to the original equation of motion (2.8).

So far, this discussion of the PCM has been applicable to an arbitrary spacetime dimension n . We now specialise to the two-dimensional case, for which the global symmetry is much larger than the $G \times G$ of a generic dimension. In order to establish the symmetry of the two-dimensional model, it is convenient to note that both $F = dA + A \wedge A = 0$ and the

field equation $d*A = 0$ can be derived from the integrability condition from either of the first-order Lax equations

$$dXX^{-1} = \frac{t}{1-t^2} *A + \frac{t^2}{1-t^2} A, \quad (2.13)$$

$$d\bar{X}\bar{X}^{-1} = \frac{t}{1-t^2} *\bar{A} + \frac{t^2}{1-t^2} \bar{A}, \quad (2.14)$$

where t is a constant spectral parameter, and $X = X(x; t)$ and $\bar{X} = \bar{X}(x; t)$. Writing $d = dx^+ \partial_+ + dx^- \partial_-$, each of the above equations yields two differential equations known as a Lax pair.

Taking the exterior derivative of equation (2.13) gives

$$d*A + t(dA + A \wedge A) = 0. \quad (2.15)$$

Since this must hold for all t , we see that this implies the equation of motion $d*A = 0$ and the flat curvature condition $F \equiv dA + A \wedge A = 0$.

The Lax equations (2.13) and (2.14) can be integrated to give solutions to X and \bar{X} . Alternatively, we can expand the X and \bar{X} around $t = 0$. Since X and \bar{X} are constants at $t = 0$, they can be expanded in non-negative powers of t :

$$X(x; t) = \sum_{n \geq 0} \Phi_n(x) t^n, \quad \bar{X}(x; t) = \sum_{n \geq 0} \bar{\Phi}_n(x) t^n. \quad (2.16)$$

From now on we shall suppress the explicit indication of the x dependence of X and \bar{X} , but it will often be important to indicate their t dependence explicitly. Thus we shall sometimes write X as $X(t)$.

Substituting (2.16) into the Lax equation (2.13) and performing a Taylor expansion in t , we obtain the infinite hierarchy of equations

$$\begin{aligned} d\Phi_0 &= 0, \\ d\Phi_1 &= *A \Phi_0, \\ d\Phi_2 &= *A \Phi_1 + A \Phi_0, \\ &\vdots \end{aligned} \quad (2.17)$$

Thus the functions Φ_0, Φ_1, \dots can be viewed as an infinite number of auxiliary fields, satisfying first-order coupled equations.

The PCM action (2.6) has the manifest global $G_L \times G_R$ symmetry given by

$$\delta g = g \epsilon - \bar{\epsilon} g, \quad (2.18)$$

where ϵ and $\bar{\epsilon}$ are Lie-algebra valued infinitesimal constant matrices.

It has been established that in the special case of a two-dimensional spacetime, there exists an infinite dimensional extension of this symmetry. Let us first consider the right action on g . The infinite-dimensional hierarchy of symmetries is obtained by replacing the parameter ϵ by a quantity of the general form $X\epsilon X^{-1}$. To be precise, we now take ϵ to be t -dependent (but still, of course, constant in spacetime), with an expansion in non-positive powers of t :

$$\epsilon(t) = \sum_{n \geq 0} \epsilon_{(n)} t^{-n}. \quad (2.19)$$

The right-acting transformations are then given by

$$\delta g = g \oint X(t)\epsilon(t)X(t)^{-1} \frac{dt}{2\pi i t}, \quad (2.20)$$

where the contour of integration is a small loop that encloses the origin. The transformation parameter $\epsilon_{(0)}$ describes the original Lie-algebra symmetry, while the higher parameters $\epsilon_{(n)}$ with $n \geq 1$ describe the hierarchy of additional symmetries.

The contour integration in (2.20) is simply serving the purpose of extracting the t^0 terms in the Laurent expansion of $X(t)\epsilon(t)X(t)^{-1}$. This gives us the infinite hierarchy of symmetry transformations, with an independent Lie-algebra valued parameter $\epsilon_{(n)}$ at each level n . In practice, it is generally more convenient to re-express the transformations, as was done in [4], in the form

$$\delta(\epsilon, t)g = g\eta, \quad \eta(t) = X(t)\epsilon X(t)^{-1}, \quad (2.21)$$

where ϵ is taken to be independent of t , and

$$\delta(\epsilon, t) = \sum_{n \geq 0} t^n \delta_{(n)}(\epsilon). \quad (2.22)$$

By equating the coefficients of a given power of t on the two sides of (2.21) (bearing in mind that g itself is independent of t), we read off the n 'th level symmetry transformation of g , with Lie-algebra valued parameter ϵ .³

It is straightforward to verify that

$$\delta A = \frac{1}{t} *d\eta, \quad (2.23)$$

³Although this way of writing the n 'th level transformation is quite convenient, one should be careful not to be misled by the slightly "informal" notation. In particular, it should be emphasised that the Lie-algebra valued parameter for the transformation at level n can be chosen independently of the parameter at level m , for all $n \neq m$.

it follows that the equation of motion $d*A = 0$ is invariant under the above transformation.

To show that the Lax equation (2.13) is also invariant, and to obtain the transformation for X , let us consider the following general transformation rule

$$\delta_1 X_2 = U X_2. \quad (2.24)$$

Here, we need to distinguish between the spectral parameter used in the expansion of the transformations $\delta(\epsilon, t)$ and the spectral parameter in $X(t)$. We call these t_1 and t_2 respectively, and use the notation

$$\delta_1 \equiv \delta(\epsilon_1, t_1), \quad \text{and} \quad X_2 \equiv X(t_2). \quad (2.25)$$

Under the transformation (2.24), the Lax equation (2.13) becomes

$$dU + [U, dX_2 X_2^{-1}] = \frac{t_2}{1-t_2^2} * \delta A + \frac{t_2^2}{1-t_2^2} \delta A. \quad (2.26)$$

This is a first-order differential equation for U , which has a one-parameter family of solutions given by

$$U = U_0 + \frac{t_2}{t_1 - t_2} \eta_1, \quad (2.27)$$

where $\eta_1 = X_1 \epsilon_1 X_1^{-1}$ and U_0 satisfies the homogeneous equation $dU_0 + [U_0, dX_2 X_2^{-1}] = 0$, which gives

$$U_0 = X_2 \epsilon_0 X_2^{-1}, \quad (2.28)$$

where ϵ_0 is a Lie-algebra valued parameter, arising as a constant of integration. Thus the transformation associated with U_0 acts only on the (auxiliary) field X , but not the original PCM field g . By requiring that the Lax equation (2.14) be invariant under the transformation, we can also obtain the transformation rule for \bar{X} . To summarise, the complete transformation of the right action is given by

$$\begin{aligned} \delta g &= g\eta, & \eta &= X\epsilon X^{-1}, & \tilde{\delta} g &= 0, \\ \delta_1 X_2 &= \frac{t_2}{t_1 - t_2} (\eta_1 X_2 - X_2 \epsilon_1), & \tilde{\delta}_1 X_2 &= \frac{t_1 t_2}{1 - t_1 t_2} X_2 \epsilon_1, \\ \delta_1 \bar{X}_2 &= \frac{t_1 t_2}{t_1 t_2 - 1} g \eta_1 g^{-1} \bar{X}_2, & \tilde{\delta} \bar{X}_2 &= 0, \end{aligned} \quad (2.29)$$

Here the $\tilde{\delta}$ transformation is the homogeneous one associated with U_0 , and it acts only on X . We also have used U_0 in $\delta_1 X_2$ transformation so as to remove the pole at $t_1 = t_2$, by choosing $\epsilon_0 = -t_2/(t_1 - t_2) \epsilon_1$. (Note that since all expansions in the spectral parameters are performed around the origin, the only pole of concern to us is that associated with the $t_1 - t_2$ denominator.)

Analogously, we can obtain the transformation rules for the left action, given by

$$\begin{aligned}
\bar{\delta}g &= -\bar{\eta}g, & \bar{\eta} &= \bar{X}\bar{\epsilon}\bar{X}^{-1}, & \tilde{\tilde{\delta}}g &= 0, \\
\bar{\delta}_1\bar{X}_2 &= \frac{t_2}{t_1-t_2}(\bar{\eta}_1\bar{X}_2 - \bar{X}_2\bar{\epsilon}_1), & \tilde{\tilde{\delta}}_1\bar{X}_2 &= \frac{t_1t_2}{1-t_1t_2}\bar{X}_2\bar{\epsilon}_1, \\
\bar{\delta}_1X_2 &= \frac{t_1t_2}{t_1t_2-1}g^{-1}\bar{\eta}_1gX_2, & \tilde{\tilde{\delta}}_1X_2 &= 0.
\end{aligned} \tag{2.30}$$

Having obtained the complete set of transformation rules, it is straightforward to calculate their commutators. The commutators for δ and $\tilde{\delta}$ are given by

$$\begin{aligned}
[\delta_1, \delta_2] &= \frac{t_1}{t_1-t_2}\delta(\epsilon_{12}, t_1) - \frac{t_2}{t_1-t_2}\delta(\epsilon_{12}, t_2), \\
[\delta_1, \tilde{\delta}_2] &= \frac{t_1t_2}{1-t_1t_2}\delta(\epsilon_{12}, t_1) + \frac{1}{1-t_1t_2}\tilde{\delta}(\epsilon_{12}, t_2), \\
[\tilde{\delta}_1, \tilde{\delta}_2] &= \frac{t_2}{t_1-t_2}\tilde{\delta}(\epsilon_{12}, t_1) - \frac{t_1}{t_1-t_2}\tilde{\delta}(\epsilon_{12}, t_2),
\end{aligned} \tag{2.31}$$

where $\epsilon_{12} \equiv [\epsilon_1, \epsilon_2]$. The commutators for the barred transformations $\bar{\delta}$ and $\tilde{\tilde{\delta}}$ take the identical form. Furthermore, the barred and unbarred transformations commute.

We may now consider the mode expansions of these transformations. Writing the Lie-algebra valued parameter ϵ as $\epsilon = \epsilon^i T_i$, where T^i are the generators of the Lie algebra, satisfying $[T_i, T_j] = f_{ij}{}^k T_k$, we write⁴

$$\begin{aligned}
\delta(\epsilon, t) &= \sum_{n=0}^{\infty} \epsilon_i J_n^i t^n, & \tilde{\delta}(\epsilon, t) &= \sum_{n=1}^{\infty} \epsilon_i J_{-n}^i t^n, \\
\bar{\delta}(\epsilon, t) &= \sum_{n=0}^{\infty} \bar{\epsilon}_i \bar{J}_{-n}^i t^n, & \tilde{\tilde{\delta}}(\epsilon, t) &= \sum_{n=1}^{\infty} \bar{\epsilon}_i \bar{J}_n^i t^n.
\end{aligned} \tag{2.32}$$

Substituting these expansions into the commutators of the transformations, we obtain the algebra

$$[J_m^i, J_n^j] = f^{ij}{}_k J_{m+n}^k, \quad [\bar{J}_m^i, \bar{J}_n^j] = f^{ij}{}_k \bar{J}_{m+n}^k, \quad [J_m^i, \bar{J}_n^j] = 0. \tag{2.33}$$

This is the full Kac-Moody algebra $\widehat{G}_L \times \widehat{G}_R$ (*i.e.* two commuting copies of the Kac-Moody extension \widehat{G} of the Lie algebra G).

The Kac-Moody extension of the original $G_L \times G_R$ Lie algebra symmetries that we have obtained here is larger than the commonly claimed result $\widehat{G} \times G$.⁵ It is also larger than the

⁴We remind the reader at this point about the cautionary remark about the labeling of parameters in the previous footnote.

⁵It is not clear precisely what is meant in the literature by $\widehat{G} \times G$; in particular, whether the Kac-Moody extended factor \widehat{G} is being associated specifically with left-acting transformations or with right-acting transformations, or neither. Clearly the correct answer must be completely symmetric between left and right. Since in any case the $\widehat{G} \times G$ claim is incorrect (as demonstrated by the fact that a PCM is a special case of an SSM), establishing a precise interpretation of the wrong claims is inessential.

two commuting “half Kac-Moody” algebras found in [13]. It is useful, therefore, to make a comparison between our result and the earlier results in the literature. We shall do this in appendix A.

3 Virasoro Symmetry

Virasoro transformation

The Kac-Moody symmetries that we obtained in the previous section were an infinite-dimensional extension of the manifest $G_L \times G_R$ Lie algebra symmetries of the PCM. There are also additional infinite-dimensional symmetries with parameters that are singlets under the original $G_L \times G_R$, and it is to these that we now turn. These algebra of these symmetries is closely related to the centreless Virasoro algebra.

We begin by discussing the realisation of the symmetries on g . First, consider the transformation

$$\delta' g = g \xi', \quad \text{where} \quad \xi'(t) \equiv -(1 - t^2)\dot{X}(t) X(t)^{-1}. \quad (3.1)$$

A straightforward calculation shows that this implies

$$\delta'(d*A) = -A \wedge A. \quad (3.2)$$

Although the non-zero right-hand side means that δ' is not a symmetry transformation, we see that the right-hand side is independent of t . We can therefore obtain a symmetry transformation by subtracting out a $\delta'(t)$ transformation at any fixed value of t . We shall take this subtraction to be at $t = 0$, and define $\delta^V(t) \equiv \delta'(t) - \delta'(0)$. Thus we define

$$\xi(t) \equiv -(1 - t^2)\dot{X}(t) X(t)^{-1} + I, \quad \text{where} \quad I \equiv \dot{X}(0)X(0)^{-1}, \quad (3.3)$$

and the symmetry transformation is

$$\delta^V g = g \xi. \quad (3.4)$$

Note that the subtraction using I removes the t^0 term in the Taylor expansion of $\delta'(t)$.

Further, independent, symmetry transformations can also be found, given by

$$\bar{\delta}^V g = -\bar{\xi} g, \quad \text{where} \quad \bar{\xi} \equiv -(1 - t^2)\dot{\bar{X}}(t)\bar{X}(t)^{-1} + \bar{I}, \quad (3.5)$$

with

$$\bar{I} \equiv \dot{\bar{X}}(0)\bar{X}(0)^{-1}. \quad (3.6)$$

(We have already performed the analogous zero-mode subtraction.) It is straightforward to see that $\bar{\delta}^V (d*\bar{A}) = 0$, and that therefore the $\bar{\delta}^V$ transformations are also symmetries of the equations of motion.

There is also a zero-mode symmetry transformation, given by

$$\delta_{(0)}^V g = -(gI - \bar{I}g) \equiv g\zeta \equiv -\bar{\zeta}g. \quad (3.7)$$

Proceeding in a similar fashion to the Kac-Moody calculations, seeking transformations that leave the Lax equations invariant, we may now determine the transformation rules for the fields X and \bar{X} , finding

$$\begin{aligned} \delta_1^V X_2 &= \frac{t_2}{t_1 - t_2} \xi_1 - \frac{t_1}{t_1 - t_2} \xi_2, & \delta_1^V \bar{X}_2 &= \frac{t_1 t_2}{t_1 t_2 - 1} (g \xi_1 g^{-1} - \bar{\xi}_2 - \bar{\zeta}) \bar{X}_2, \\ \bar{\delta}_1^V \bar{X}_2 &= \frac{t_2}{t_1 - t_2} \bar{\xi}_1 - \frac{t_1}{t_1 - t_2} \bar{\xi}_2, & \bar{\delta}_1^V X_2 &= \frac{t_1 t_2}{t_1 t_2 - 1} (g^{-1} \bar{\xi}_1 g - \xi_2 - \zeta) X_2, \\ \delta_{(0)}^V X &= \xi X, & \delta_{(0)}^V \bar{X} &= \bar{\xi} \bar{X}. \end{aligned} \quad (3.8)$$

After some tedious algebra, we find that the commutators of the Virasoro-type transformations are given by

$$\begin{aligned} [\delta_1^V, \delta_2^V] &= \frac{t_1 t_2}{t_1 - t_2} \left(\partial_1 \left(\frac{1 - t_1^2}{t_1} \delta_1^V \right) + \partial_2 \left(\frac{1 - t_2^2}{t_2} \delta_2^V \right) \right) - \frac{2t_1 t_2}{(t_1 - t_2)^2} \left(\frac{1 - t_1^2}{t_1} \delta_1^V - \frac{1 - t_2^2}{t_2} \delta_2^V \right), \\ [\delta_1^V, \bar{\delta}_2^V] &= \frac{t_1 t_2}{t_1 t_2 - 1} \left(\partial_{t_2} (1 - t_2^2) \bar{\delta}_2^V - \partial_{t_1} ((1 - t_1^2) \delta_1^V) \right) \\ &\quad + \frac{t_1 t_2 + 1}{(t_1 t_2 - 1)^2} \left(t_2 (1 - t_1^2) \delta_1^V - t_1 (1 - t_2^2) \bar{\delta}_2^V \right) \\ &\quad + \frac{t_1 t_2}{(t_1 t_2 - 1)^2} \left((t_1 - t_2) (t_1 t_2 - 3) \right) \delta_0^V \\ [\bar{\delta}_1^V, \bar{\delta}_2^V] &= \frac{t_1 t_2}{t_1 - t_2} \left(\partial_1 \left(\frac{1 - t_1^2}{t_1} \bar{\delta}_1^V \right) + \partial_2 \left(\frac{1 - t_2^2}{t_2} \bar{\delta}_2^V \right) \right) - \frac{2t_1 t_2}{(t_1 - t_2)^2} \left(\frac{1 - t_1^2}{t_1} \bar{\delta}_1^V - \frac{1 - t_2^2}{t_2} \bar{\delta}_2^V \right), \\ [\delta_0^V, \delta^V] &= -(1 - t^2) \dot{\delta}^V + \left(t + \frac{1}{t} \right) \delta^V + t \delta_0^V \\ [\delta_0^V, \bar{\delta}^V] &= -(1 - t^2) \dot{\bar{\delta}}^V + \left(t + \frac{1}{t} \right) \bar{\delta}^V + t \delta_0^V \end{aligned} \quad (3.9)$$

We can also calculate the commutators of the Virasoro-type transformations with the Kac-Moody transformations. For these, we find

$$\begin{aligned} [\delta_0^V, \delta] &= -(1 - t^2) \dot{\delta} + \frac{1}{t} (\delta - \delta(0)) + t \delta, \\ [\delta_0^V, \tilde{\delta}] &= t \delta(0) + \partial_t \left((t^2 - 1) \tilde{\delta} \right) - \frac{1}{t} (t^2 - 1) \tilde{\delta}, \\ [\delta_1^V, \delta_2] &= \frac{t_1 (1 - t_2^2)}{t_1 - t_2} \dot{\delta}_2 + \frac{t_2 (1 - t_1^2)}{(t_1 - t_2)^2} (\delta_2 - \delta_1) - \frac{1}{t_2} (\delta_2 - \delta(0)), \\ [\delta_1^V, \tilde{\delta}_2] &= \frac{t_2}{(1 - t_1 t_2)^2} (1 - t_1^2) \delta_1 - t_2 \delta(0) \end{aligned} \quad (3.10)$$

$$\begin{aligned}
& + \frac{t_1 t_2}{1 - t_1 t_2} \partial_2 \left((t_2^2 - 1) \tilde{\delta}_2 \right) + \frac{t_1^2 t_2}{(1 - t_1 t_2)^2} (t_2^2 - 1) \tilde{\delta}_2, \\
[\delta_1^V, \bar{\delta}_2] &= \frac{t_1 t_2}{t_1 t_2 - 1} \partial_2 \left((1 - t_2^2) \bar{\delta}_2 \right) - \frac{t_1^2 t_2}{(t_1 t_2 - 1)^2} (1 - t_2^2) \bar{\delta}_2 + \frac{(1 - t_1^2) t_2}{(1 - t_1 t_2)^2} \tilde{\delta}_1, \\
[\delta_1^V, \tilde{\delta}_2] &= \frac{t_1}{t_1 - t_2} \partial_2 \left((1 - t_2^2) \tilde{\delta}_2 \right) - \frac{t_1(t_1 - 2t_2)}{t_2(t_1 - t_2)^2} (1 - t_2^2) \tilde{\delta}_2 - \frac{t_2}{(t_1 - t_2)^2} (1 - t_1^2) \tilde{\delta}_2,
\end{aligned}$$

together with the ‘‘conjugate’’ commutators where the rôles of barred and unbarred transformations are exchanged:

$$\begin{aligned}
[\delta_0^V, \bar{\delta}] &= -(1 - t^2) \dot{\bar{\delta}} + \frac{1}{t} (\bar{\delta} - \bar{\delta}(0)) + t \bar{\delta}, \\
[\delta_0^V, \tilde{\delta}] &= t \bar{\delta}(0) + \partial_t \left((t^2 - 1) \tilde{\delta} \right) - \frac{1}{t} (t^2 - 1) \tilde{\delta}, \\
[\bar{\delta}_1^V, \bar{\delta}_2] &= \frac{t_1(1 - t_2^2)}{t_1 - t_2} \dot{\bar{\delta}}_2 + \frac{t_2(1 - t_1^2)}{(t_1 - t_2)^2} (\bar{\delta}_2 - \bar{\delta}_1) - \frac{1}{t_2} (\bar{\delta}_2 - \bar{\delta}(0)), \\
[\bar{\delta}_2^V, \tilde{\delta}_2] &= \frac{t_2}{(1 - t_1 t_2)^2} (1 - t_1^2) \bar{\delta}_1 - t_2 \bar{\delta}(0) \\
&\quad + \frac{t_1 t_2}{1 - t_1 t_2} \partial_2 \left((t_2^2 - 1) \tilde{\delta}_2 \right) + \frac{t_1^2 t_2}{(1 - t_1 t_2)^2} (t_2^2 - 1) \tilde{\delta}_2, \\
[\bar{\delta}_1^V, \delta_2] &= \frac{t_1 t_2}{t_1 t_2 - 1} \partial_2 \left((1 - t_2^2) \delta_2 \right) - \frac{t_1^2 t_2}{(1 - t_1 t_2)^2} (1 - t_2^2) \delta_2 + \frac{t_2}{(1 - t_1 t_2)^2} (1 - t_1^2) \tilde{\delta}_1, \\
[\bar{\delta}_1^V, \tilde{\delta}_2] &= \frac{t_1}{t_1 - t_2} \partial_2 \left((1 - t_2^2) \tilde{\delta}_2 \right) - \frac{t_1(t_1 - 2t_2)}{t_2(t_1 - t_2)^2} (1 - t_2^2) \tilde{\delta}_2 - \frac{t_2}{(t_1 - t_2)^2} (1 - t_1^2) \tilde{\delta}_2.
\end{aligned} \tag{3.11}$$

3.1 Mode expansions

We now perform a mode expansion for the Virasoro-like transformations, which is given by

$$\delta^V(t) = \sum_{n=1}^{\infty} K_n t^n, \quad \bar{\delta}^V(t) = \sum_{n=1}^{\infty} K_{-n} t^n, \quad \delta_{(0)}^V = K_0. \tag{3.12}$$

Substituting this, and the previous Kac-Moody mode expansion (2.32), into the various commutators of transformations, we obtain the algebra

$$\begin{aligned}
[J_m^i, J_n^i] &= f^{ij}{}_k J_{m+n}^k, & [\bar{J}_m^i, \bar{J}_n^i] &= f^{ij}{}_k \bar{J}_{m+n}^k, & [J_m^i, \bar{J}_n^j] &= 0, \\
[K_m, K_n] &= (m - n) (K_{m+n+1} - K_{m+n-1}), \\
[K_m, J_n^i] &= -n (J_{m+n+1}^i - J_{m+n-1}^i), \\
[K_m, \bar{J}_n^i] &= -n (\bar{J}_{m+n+1}^i - \bar{J}_{m+n-1}^i).
\end{aligned} \tag{3.13}$$

The K_m generate a subalgebra of the centreless Virasoro algebra, as may be seen by noting that if we were to define

$$K_m = L_{m+1} - L_{m-1}, \tag{3.14}$$

where the L_m satisfy $[L_m, L_n] = (m - n)L_{m+n}$, then we would obtain precisely the algebra given in the second line of (3.13). It should be emphasised, however, that the relation (3.14) cannot be inverted to express the Virasoro generators in terms of the generators K_m of the symmetry transformations we have exhibited. In fact, only two extra generators would be needed in order to be able to invert (3.14). If, for example, we had symmetry transformations corresponding directly to L_0 and L_1 , then the inversion would be possible. (Alternatively, it would suffice to have an additional symmetry associated with an L_n for any even n , and an additional symmetry associated with an L_n for any odd n .)

3.2 Towards a full Virasoro symmetry

It has been asserted by Devchand and Schiff that the Virasoro-like symmetry studied above can actually be extended to a full centreless Virasoro algebra [18], at least in the case that $G = U(N)$. The method they employ is based on aspects of the symmetries as transformations that act on the solution space. For the reader's convenience, we outline very briefly the basic set up. To begin with, one considers special solutions of the PCM equations that take the form

$$A^{(0)} = A(x^+)dx^+ + B(x^-)dx^- , \quad (3.15)$$

where $A(x^+)$ and $B(x^-)$ are arbitrary matrices lying in the Cartan subalgebra of $U(N)$. The Lax equation is then readily solved, to yield

$$X^{(0)}(x^+, x^-, \lambda) = e^{M(x^+, x^-, \lambda)} X_0(\lambda) , \quad (3.16)$$

where $\lambda = 1/t$ and

$$M(x^+, x^-, \lambda) = \frac{1}{\lambda - 1} \int_{x_0^+}^{x^+} A(y^+) dy^+ - \frac{1}{\lambda + 1} \int_{x_0^-}^{x^-} B(y^-) dy^- , \quad (3.17)$$

and $X_0(\lambda)$ is an unconstrained element of G serving as the initial condition. Next, it is assumed that any group element can be factorized as $G = G_- G_+$ where (a) G_- denotes the group of analytic maps inside the contour \mathcal{C} in the λ -plane that is a union of two small contours centered around $\lambda = \pm 1$ such that $\lambda = 0$ remains outside both of them, and approaching the identity at $\lambda = \infty$; and (b) G_+ denotes the the group of maps analytic in the region complementary to the contour \mathcal{C} [18].⁶ Using this factorization, the solution

⁶This rather non-trivial decomposition goes by the name of Birkhoff factorisation. For the case presented, a proof that this is always in principle possible, even if not entirely practical, is contained in some results of Pressley and Segal [19].

(3.16) is written as

$$X^{(0)} = S^{-1}Y , \quad (3.18)$$

where $S^{-1} : M \rightarrow G_-$ and $Y : M \rightarrow G_+$. Then the element S is expanded as

$$S = \sum_{n=0}^{\infty} s_n(x^+, x^-)(1 + \lambda)^n = \sum_{n=0}^{\infty} \tilde{s}_n(x^+, x^-)(1 - \lambda)^n . \quad (3.19)$$

Finally, a general solution to the PCM is constructed as [18]

$$A_+ = s_0 A(x^+) s_0^{-1} , \quad A_- = \tilde{s}_0 B(x^-) \tilde{s}_0^{-1} . \quad (3.20)$$

The symmetries of the PCM are then viewed as various transformations of the free fields $A(x^+), B(x^-), U_0(\lambda)$, i.e. as symmetries that act on the solution space. In particular, Devchand and Schiff argue that the Virasoro-like symmetries we exhibited explicitly in section 3 correspond to the reparameterisations

$$\delta_m^V X_0(\lambda) = \epsilon_m \lambda^{m+1} X'_0(\lambda) , \quad (3.21)$$

with the condition that the points $\lambda = \pm 1$ are held fixed [18]. It is then argued that at the level of infinitesimal symmetries the need to fix ± 1 is superfluous and that the *full* Virasoro algebra arises upon relaxing this condition.

According to Devchand and Schiff, the complete Virasoro algebra acts as a solution generating symmetry of the PCM. They do not, however, exhibit the action of the complete Virasoro algebra directly on the original fields g of the PCM, nor on the auxiliary fields X , and as far as we are aware, there is no local way of doing so.

4 PCM with WZ Term

As we mentioned in the introduction, if one considers an $\text{AdS}_3 \times S^3 \times T^4$ in the type IIB string, then by allowing the 3-form flux to be sourced partly by the RR field and partly by the NS-NS field, then the bosonic sector of the theory will be described by a PCM in which the WZ term is non-vanishing, but with an adjustable coefficient μ that can lie in the range $-1 \leq \mu \leq 1$. The case $\mu = 0$ corresponds to pure RR flux, while $\mu = \pm 1$, the ‘‘critical cases,’’ correspond to the chiral or antichiral WZW model, with pure NS-NS flux.

To be more precise, the 3-form flux in the $\text{AdS}_3 \times S^3 \times T^4$ background can be written as

$$F = i(\mu + i\nu)(\Omega_{\text{AdS}_3} + \Omega_{S^3}) , \quad (4.1)$$

where Ω_{AdS_3} and Ω_{S^3} are the volume forms on AdS_3 and S^3 , the constants μ and ν satisfy

$$\mu^2 + \nu^2 = 1, \quad (4.2)$$

and the complex 3-form F is constructed from the RR and NS-NS 3-forms according to

$$F = F^{RR} + iF^{NS}. \quad (4.3)$$

The bosonic part of the superstring action contains a WZ term constructed from the NS-NS 2-form potential B^{NS} . The RR field F^{RR} couples only to fermionic terms in the action, and thus it is absent from the bosonic sector.

As was discussed in [4], the PCM with a WZ term where μ takes a non-critical value can in fact be mapped by means of an invertible field redefinition into the pure PCM case with $\mu = 0$. To see this, we shall demonstrate the converse, namely that a pure PCM can be mapped into a PCM with non-critical WZ term by means of an invertible field redefinition.

We begin with a standard PCM whose group manifold G is parameterised by $g(x)$, and so $A = g^{-1}dg$ satisfies the standard Maurer-Cartan and field equations

$$dA + A \wedge A = 0, \quad d*A = 0. \quad (4.4)$$

We then introduce $Y(\mu)$, which is defined to obey the equation

$$dY Y^{-1} = \frac{\mu^2}{1 - \mu^2} *A + \frac{\mu}{1 - \mu^2} A. \quad (4.5)$$

From this, we define a new group element g' , and gauge connection A' , by writing

$$g' = gY, \quad A' = g'^{-1}dg'. \quad (4.6)$$

Clearly, from its definition, A' satisfies the same Maurer-Cartan equation as does A , namely

$$dA' + A' \wedge A' = 0. \quad (4.7)$$

A straightforward calculation shows that in consequence of (4.4), the new connection A' satisfies the equation of motion

$$d*A' + \mu A' \wedge A' = 0. \quad (4.8)$$

This equation can be derived by varying the action

$$I_{WZ} = -\frac{1}{2} \int_{\partial M} \text{Tr}(*A' \wedge A') - \frac{\mu}{3} \int_M \text{Tr}(A' \wedge A' \wedge A') \quad (4.9)$$

with respect to the fundamental field g' . This action is precisely the one that describes a PCM with a WZ term.

Note from (4.5) that we can generate such a model for any value of μ except for the critical values $\mu = \pm 1$ that arise in the WZW model.

Of course the PCM with non-critical WZ term has the same $\widehat{G} \times \widehat{G}$ Kac-Moody symmetry as does the pure $\mu = 0$ PCM.

5 Conclusions

In this paper we have studied the Kac-Moody extension of the manifest $G \times G$ global symmetry of a two-dimensional principal chiral model based on the group manifold G . We demonstrated that the symmetry algebra is the full $\widehat{G} \times \widehat{G}$ centreless Kac-Moody extension of $G \times G$, and we obtained explicit transformation rules for the complete algebra. These results go beyond those presented previously in the literature, where the smaller symmetries $\widehat{G} \times G$ [12] or two commuting “half Kac-Moody” algebras [13] were proposed. The $\widehat{G} \times \widehat{G}$ symmetry that we find is consistent with the fact that a two-dimensional symmetric space sigma model G/H has a \widehat{G} Kac-Moody symmetry, since a group manifold G can be viewed as the symmetric space $(G \times G)/G$, where the denominator lies in the diagonal subgroup of $G \times G$.

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A Comparison with Previous Literature

Earlier literature on the infinite-dimensional symmetries of the PCM has in common with our construction the subalgebra of transformations $\delta g = g\eta$ given in (2.29). We may denote this right-acting “half” Kac-Moody algebra, which includes the ordinary Lie algebra G_R as the “zero modes,” by \widehat{G}_R^+ . The earlier approaches and our approach diverge in the attempt to find further infinite-dimensional symmetries over and above \widehat{G}_R^+ .

In our case, we find the additional “homogeneous” transformations $\tilde{\delta}$, which leave g inert but not X , and which complete the algebra \widehat{G}_R^+ to give the full Kac-Moody symmetry

\widehat{G}_R . Furthermore, we find that this right-acting Kac-Moody symmetry has a completely independent left-acting counterpart, giving in total a $\widehat{G}_L \times \widehat{G}_R$ Kac-Moody symmetry. An important feature of our results is that all transformations, namely $\delta(t)$, $\tilde{\delta}(t)$ acting on the right, and $\bar{\delta}(t)$ and $\tilde{\tilde{\delta}}(t)$ acting on the left, involve the use of spectral parameters that are expanded in Taylor series around $t = 0$. Thus in our approach it is never necessary to make use of the Riemann-Hilbert transformation that invokes a relation between $X(t)$ and $X(1/t)$.

In previous works [12, 4], by contrast, it was claimed that the full Kac-Moody extension of the manifest $G_L \times G_R$ global symmetry was just $\widehat{G} \times G$, where the \widehat{G} was associated with the $\delta(t)$ transformations together with further transformations that we shall call $\hat{\delta}(t)$, where $\hat{\delta}(t) = \delta(1/t)$. The expansion of $\hat{\delta}(t)$ around $t = 0$ is therefore equivalent to an expansion of $\delta(t)$ around $t = \infty$. It is not clear to us whether the claim of a $\widehat{G} \times G$ symmetry carries with it any implication that the \widehat{G} is to be associated with the left action, and the G with the right action (or *vice versa*). In any case, clearly neither of these could be correct as the *full* symmetry algebra, since there is a complete left-right symmetry in the original PCM.

Obviously there is, in a rather trivial sense, a genuine $\widehat{G} \times G$ symmetry of the PCM, since $\widehat{G} \times G$ is a subalgebra of the $\widehat{G} \times \widehat{G}$ symmetry that we have exhibited in this paper. This, however, is not the symmetry that is discussed in [12, 4]. One way of seeing this is that if one looks just at the zero-mode sector, the resulting $G \times G$ symmetry described in [12, 4] is given by

$$\delta^{(0)}g = g\epsilon, \quad \hat{\delta}^{(0)}g = -g_0 \hat{\epsilon} g_0^{-1} g, \quad (\text{A.1})$$

where g_0 is the value of $g(x)$ at some arbitrarily-selected point x_0 in the two-dimensional spacetime.⁷ The $\delta^{(0)}$ transformation is a standard right action of G on the group manifold. However, the $\hat{\delta}^{(0)}$ transformation of $g(x)$ is highly non-local, since it depends not only on the value of g at the point x but also on the value of g at the point x_0 . Although one might be tempted to view $\hat{\delta}^{(0)}$ simply as a left action of G of the form $\hat{\delta}^{(0)} = -\hat{\epsilon}' g$, where the constant parameter $\hat{\epsilon}' \equiv g_0 \hat{\epsilon} g_0^{-1}$ is just a conjugation of $\hat{\epsilon}$ by the constant group element g_0 , the fact that g_0 itself transforms under $\delta^{(0)}$ and $\hat{\delta}^{(0)}$ means that $\hat{\epsilon}'$ itself transforms also. In fact

$$\delta^{(0)}g_0 = g_0\epsilon, \quad \hat{\delta}^{(0)}g_0 = -g_0 \hat{\epsilon}, \quad (\text{A.2})$$

so at the spacetime point x_0 the $\hat{\delta}^{(0)}$ is actually a *right* translation, and not a left translation.

⁷Note that the following discussion of the zero-mode sector of the algebra found in [12, 4] need not be restricted to the case of a two-dimensional spacetime; it is equally applicable in *any* spacetime dimension.

The $\hat{\delta}^{(0)}$ transformation defined in (A.1) is just one of an infinity of non-local transformation laws that one could introduce,⁸ but there does not appear to be a strong motivation for doing so since they would all lie outside the class of transformations we normally wish to consider. Thus, the $\hat{\delta}^{(0)}$ transformations in (A.1) amount to a somewhat artificial introduction of a non-standard non-local G symmetry that lies outside the usual $G_L \times G_R$ locally-defined symmetry of a group-manifold sigma model. One striking way of seeing this is to note that if we do also consider standard left translations, defined by

$$\delta_L g = \epsilon g, \quad (\text{A.3})$$

then a simple calculation shows that $\delta^{(0)}$, $\hat{\delta}^{(0)}$ and δ_L all commute, and thus there is apparently a $G \times G \times G$ symmetry in any group-manifold sigma model (in any spacetime dimension)!

The moral that we draw from the above discussion is that the introduction of a preferred point x_0 in spacetime, with transformation rules for fields at the point x depending also on the value of fields at the point x_0 , leads to an unnecessary profusion of extra non-local symmetry transformations, even at the zero-mode level, that are not particularly germane to the original problem under investigation. By contrast, all the transformations of the $\hat{G}_L \times \hat{G}_R$ symmetries that we have obtained in this paper act locally on the fields g and X .

Finally, we consider the results presented in [13], in which a symmetry algebra was obtained that is described as corresponding to two commuting copies of a ‘‘half Kac-Moody’’ algebra. Specifically, the algebra, given in equation (21) of [13], has the commutation relations

$$\begin{aligned} [Q_m^i, Q_n^j] &= f^{ij}{}_k Q_{m+n}^k, & m, n \geq 0, \\ [Q_m^i, Q_n^j] &= f^{ij}{}_k Q_m^k, & m \geq 0 \geq n, \\ [Q_m^i, Q_n^j] &= f^{ij}{}_k (-Q_{m+n}^k + Q_m^k + Q_n^k), & m, n < 0. \end{aligned} \quad (\text{A.4})$$

(We have adjusted the notation to match that which we are using in this paper.) If we now define new generators P_m^i by

$$\begin{aligned} P_m^i &= Q_m^i, & m \geq 0, \\ P_m^i &= Q_0^i - Q_m^i, & m < 0, \end{aligned} \quad (\text{A.5})$$

⁸For example, one could introduce transformations involving conjugations of the parameter $\hat{\epsilon}$ by an arbitrarily large number of group elements $g(x_i)$ at points x_i in spacetime.

then we see that these satisfy

$$\begin{aligned}
[P_m^i, P_n^j] &= f^{ij}_k P_{m+n}^k, & m, n \geq 0, \\
[P_m^i, P_n^j] &= 0, & m \geq 0 > n, \\
[P_m^i, P_n^j] &= f^{ij}_k (P_{m+n}^k - P_m^k - P_n^k), & m, n < 0.
\end{aligned}
\tag{A.6}$$

Thus the non-negative modes of P_m^i generate a half Kac-Moody algebra that commutes with the negative modes. The negative modes themselves generate an algebra that is “nearly” another half Kac-Moody algebra.

This can be made more precise by expressing the generators Q_m^i of the algebra obtained in [13] in terms of the $\hat{G} \times \hat{G}$ generators J_m^i and \bar{J}_m^i that we introduced in (2.32) and (2.33). Specifically, the algebra (A.4) found in [13] is obtained if we define

$$\begin{aligned}
Q_m^i &= J_m^i, & m \geq 0, \\
Q_m^i &= -\bar{J}_m^i + J_0^i + \bar{J}_0^i, & m \leq 0.
\end{aligned}
\tag{A.7}$$

In terms of the redefined generators P_m^i of (A.6), we then have

$$\begin{aligned}
P_m^i &= J_m^i, & m \geq 0, \\
P_m^i &= \bar{J}_m^i - \bar{J}_0^i, & m < 0.
\end{aligned}
\tag{A.8}$$

As well as making manifest that the generators P_m^i for $m \geq 0$ commute with those for $m < 0$, it also shows that the algebra (A.4) of [13] is a subalgebra of $\hat{G}_L \times \hat{G}_R$. Furthermore, it can be seen that if the generators \bar{J}_0^i were included also, the algebra would be precisely two commuting copies of a half Kac-Moody algebra.⁹

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⁹In fact the generators \bar{J}_0^i must certainly give symmetries of the PCM, since they just correspond to the action of the right-handed factor in the manifest $G_L \times G_R$ Lie algebra symmetry of the PCM.

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