DAMTP-2001-25 CTP TAMU-06/02 UPR-986-T MCTP-02-20
March 2002 hep-th/0204064

General Metrics of G_2 Holonomy and Contraction Limits

Z.W. Chong [‡], M. Cvetič [†], G.W. Gibbons [♯], H. Lü [⋆], C.N. Pope [‡] and P. Wagner [‡]

[†] *Department of Physics and Astronomy, University of Pennsylvania*
Philadelphia, PA 19104, USA

[♯] *DAMTP, Centre for Mathematical Sciences, Cambridge University*
Wilberforce Road, Cambridge CB3 OWA, UK

[⋆] *Michigan Center for Theoretical Physics, University of Michigan*
Ann Arbor, MI 48109, USA

[‡] *Center for Theoretical Physics, Texas A&M University, College Station, TX 77843, USA*

ABSTRACT

We obtain first-order equations for G_2 holonomy of a wide class of metrics with $S^3 \times S^3$ principal orbits and $SU(2) \times SU(2)$ isometry, using a method recently introduced by Hitchin. The new construction extends previous results, and encompasses all previously-obtained first-order systems for such metrics. We also study various group contractions of the principal orbits, focusing on cases where one of the S^3 factors is subjected to an Abelian, Heisenberg or Euclidean-group contraction. In the Abelian contraction, we recover some recently-constructed G_2 metrics with $S^3 \times T^3$ principal orbits. We obtain explicit solutions of these contracted equations in cases where there is an additional $U(1)$ isometry. We also demonstrate that the only solutions of the full system with $S^3 \times T^3$ principal orbits that are complete and non-singular are either flat $\mathbb{R}^4$ times T^3, or else the direct product of Eguchi-Hanson and T^3, which is asymptotic to $\mathbb{R}^4/\mathbb{Z}_2 \times T^3$. These examples are in accord with a general discussion of isometric fibrations by tori which, as we show, in general split off as direct products. We also give some (incomplete) examples of fibrations of G_2 manifolds by associative 3-tori with either T^4 or K3 as base.

[†] Research supported in part by DOE grant DE-FG02-95ER40893 and NATO grant 976951.

[⋆] Research supported in full by DOE grant DE-FG02-95ER40899

[‡] Research supported in part by DOE grant DE-FG03-95ER40917.

Contents

1 Introduction

Manifolds M_7 of G_2 holonomy are of considerable interest because they allow one to construct supersymmetric backgrounds of the form $(\text{Minkowski})_4 \times M_7$ in M-theory. Explicit examples of complete, regular, non-compact G_2 metrics exist. All the known examples are of cohomogeneity one, with principal orbits that are $S^3 \times S^3$, $\mathbb{CP}^3$ or $SU(3)/(U(1) \times U(1))$. The first regular examples were found in [1, 2]; these are asymptotically conical (AC). The most interesting case for physical purposes is when the principal orbits are $S^3 \times S^3$.[1] More general systems of equations for such metrics of G_2 holonomy were obtained in [4, 5], and an explicit new solution, which is asymptotically locally conical (ALC), was given in [5]. A rather general system of first-order equations for such metrics of G_2 holonomy was obtained in [6, 7]. Although the general solution was not found in [6, 7], it was shown that three types of regular metrics could arise, in which the orbits degenerate to S^2 [8], S^3 [5, 6, 7] or $T^{1,1}$ [7] at short distance. Classes of such metrics that are asymptotically locally conical were found; these were denoted as $\mathbb{D}_7$, $\mathbb{B}_7$ and $\widetilde{\mathbb{C}}_7$ respectively. They all have a non-trivial parameter (two for $\widetilde{\mathbb{C}}_7$) that adjusts the radius of the asymptotic circle relative to the overall scale-size of the metric. The Gromov-Hausdorff limits of $\mathbb{D}_7$ and $\mathbb{B}_7$ are the resolved and the deformed conifolds respectively, whilst the Gromov-Hausdorff limits of $\widetilde{\mathbb{C}}_7$ give the family of Ricci-flat Kähler metrics on the complex line bundle over $S^2 \times S^2$ [7].

In a recent paper, Hitchin has given a new construction of certain types of metrics of special holonomy, including seven-dimensional metrics of G_2 holonomy [9]. The procedure involves constructing diffeomorphism-invariant functionals on certain differential forms. By extremising the functionals, he obtains Hamiltonian flow equations that lead to metrics of G_2 holonomy. In [9], as an application of the method, a new derivation of a class of G_2 metrics previously obtained in [4, 5] was given.

In section 2 of this paper, we apply Hitchin's procedure with a somewhat more general starting point, and thereby we obtain a system of first-order equations for metrics of G_2 holonomy and $S^3 \times S^3$ principal orbits that is more general than any obtained hitherto.[2] As we show in section 3, it not only encompasses the system of first-order equations obtained in [4, 5], but also the inequivalent system obtained in [6, 7]. Additionally, it encompasses a

[1] The G_2 metrics with the other two types of principal orbits can be viewed as Gromov-Hausdorff limits of a new class of ALC Spin(7) metrics constructed in [3].

[2] The same generalisation of Hitchin's 3-form was considered also in [6], where the conditions for G_2 holonomy were expressed in the form of some coupled second-order equations that are presumably equivalent to our first-order system.

recently-obtained class of G_2 metrics with $S^3 \times T^3$ principal orbits [10]. We shall show how this first-order system arises as a contraction limit of our new results for $S^3 \times S^3$ orbits, in which the abelian limit of S^3 is taken.

We then turn in section 4 to a more general consideration of contraction limits. Specifically, we can apply various group contractions to the $S^3 = SU(2)$ factors. Some earlier discussion of this procedure in relation to metrics of special holonomy was given in [11].

In section 5 we study the solutions of the first-order equations that arise in the contraction limits. These equations are simpler than those associated with the uncontracted $S^3 \times S^3$ orbits, and so they are typically more tractable from the viewpoint of obtaining exact and explicit solutions. In particular, we study the solutions of the metrics with $S^3 \times T^3$ principal orbits in cases where there is an extra $U(1)$ isometry. We explicitly show that the metrics are all singular, except for flat $\mathbb{R}^4$ times T^3, or else Eguchi-Hanson times T^3. The latter can be viewed as a Gromov-Hausdorff limit of the generic singular metrics, in which the radius of one of the circles in T^3 goes to zero. The manifold in this case has the form $\mathbb{R}^4/\mathbb{Z}_2 \times T^3$ at large distance.

In section 6, we discuss the solutions of the more general triaxial case with $S^3 \times T^3$ orbits and no $U(1)$ isometry, and we show that again flat $\mathbb{R}^4$ times T^3, and Eguchi-Hanson times T^3, are the only complete and non-singular metrics.

In section 7 we give more general arguments to show that in isometric fibrations by tori, the toroidal directions always, in general, split off as a direct-product factor.

In section 8, we give some (incomplete) examples of fibrations of G_2 manifolds by associative 3-tori, with either T^4 or K3 as base.

Finally, we remark that since G_2 and Spin(7) manifolds provide natural compactifications in M-theory [12, 13, 14], there has been a considerable effort recently in constructing explicit non-compact G_2 and Spin(7) metrics (see also the additional references [15]-[20]).

2 New G_2 metrics with $S^3 \times S^3$ principal orbits

To obtain the new G_2 metrics we generalise the example considered by Hitchin in [9]. We take the 3-form and 4-form used in the general construction in [9] to be given by

$$\begin{aligned} \rho &= n\,\Sigma_1\,\Sigma_2\,\Sigma_3 - m\,\sigma_1\,\sigma_2\,\sigma_3 + x_1\,(\sigma_1\,\Sigma_2\,\Sigma_3 - \sigma_2\,\sigma_3\,\Sigma_1) + 2 \text{ cyclic terms}\,, \\ \sigma &= y_1\,\sigma_2\,\Sigma_2\,\sigma_3\,\Sigma_3 + y_2\,\sigma_3\,\Sigma_3\,\sigma_1\,\Sigma_1 + y_3\,\sigma_1\,\Sigma_1\,\sigma_2\,\Sigma_2\,, \end{aligned} \qquad (1)$$

where Σ_i and σ_i are two sets of left-invariant 1-forms of $SU(2)$. (The example considered in [9] had $m = n = 1$, and as we shall see, this choice significantly restricts the generality of the results.) From (1), the next step is to calculate the associated potentials $V(\rho)$ and $W(\sigma)$, whose general definitions were given in [9].

Since previous discussions of the approach in [9] have been somewhat abstract, it is perhaps worthwhile to present the key calculational steps here "with indices." To obtain $V(\rho)$, we first define

$$K_a{}^b \equiv \tfrac{1}{12}\rho_{c_1c_2c_3}\,\rho_{c_4c_5a}\,\varepsilon^{c_1c_2c_3c_4c_5b}\,, \tag{2}$$

where $\varepsilon^{c_1\cdots c_6}$ is the Levi-Civita tensor density in 6-dimensions with values ± 1 and 0. Then, $V(\rho)$ is given by

$$V(\rho) = \sqrt{-\tfrac{1}{6}K_a{}^b\,K_b{}^a}\,. \tag{3}$$

To calculate $W(\sigma)$, we first construct the dual tensor density

$$\tilde{\sigma}^{ab} \equiv \tfrac{1}{24}\,\varepsilon^{abc_1c_2c_3c_4}\,\sigma_{c_1c_2c_3c_4}\,. \tag{4}$$

From this, $W(\sigma)$ is calculated from

$$W(\sigma)^2 = \tfrac{1}{48}\,\varepsilon_{c_1\cdots c_6}\,\tilde{\sigma}^{c_1c_2}\,\tilde{\sigma}^{c_3c_4}\,\tilde{\sigma}^{c_5c_6}\,. \tag{5}$$

One now defines the Hamiltonian $H = V(\rho) - 2W(\sigma)$. It is shown in [9] that a metric of G_2 holonomy is obtained if the first-order equations following from the Hamiltonian flow

$$\dot{x}_i = -\frac{\partial H}{\partial y_i}\,, \qquad \dot{y}_i = \frac{\partial H}{\partial x_i}\,, \tag{6}$$

are satisfied,[3] where the dot denotes a derivative with respect to an additional "time" variable t. The metric itself is obtained as follows. First, one takes the "square root" of the 4-form σ, writing it as $\sigma = \frac{1}{2}\omega^2$. Then, the associative 3-form of the G_2 metric is given by [4]

$$\Phi_{(3)} = dt\wedge\omega + \rho\,. \tag{7}$$

From this, one calculates the G_2 metric as follows. First, we define the symmetric tensor density

$$B_{AB} = -\tfrac{1}{144}\Phi_{AC_1C_2}\,\Phi_{BC_3C_4}\,\Phi_{C_5C_6C_7}\,\varepsilon^{C_1\cdots C_7}\,, \tag{8}$$

where $\varepsilon^{C_1\cdots C_7}$ is the Levi-Civita tensor density in seven dimensions. The metric tensor is then given by

$$g_{AB} = \det(B)^{-1/9}\,B_{AB}\,. \tag{9}$$

[3] As in [9], there is a natural pairing between the invariant 3-form and 4-form that is non-degenerate, and the symplectic form is just a multiple of $dx_i \wedge dy_i$.

[4] This associative 3-form was also considered in [21].

Applying this construction to (1), we find that the 2-form ω can be taken to be

$$\omega = \sqrt{\frac{y_2\, y_3}{y_1}}\, \sigma_1 \wedge \Sigma_1 + \sqrt{\frac{y_3\, y_1}{y_2}}\, \sigma_2 \wedge \Sigma_2 + \sqrt{\frac{y_1\, y_1}{y_3}}\, \sigma_3 \wedge \Sigma_3\,, \tag{10}$$

and it is easily verified that this satisfies the criterion $\omega \wedge \rho = 0$ that is necessary for the $SL(3,\mathbb{C})$ reduction by ρ and the $Sp(6,\mathbb{R})$ reduction by σ to intersect in $SU(3)$ [9]. Writing $V(\rho) = \sqrt{-U}$, we find that the potentials are given by

$$\begin{aligned} U &= m^2\, n^2 - 2m\, n\, (x_1^2 + x_2^2 + x_3^2) - 4(m+n)\, x_1\, x_2\, x_3 \\ &\quad + x_1^4 + x_2^4 + x_3^4 - 2x_1^2\, x_2^2 - 2x_2^2\, x_3^2 - 2x_3^2\, x_1^2\,, \qquad (11) \\ W(\sigma) &= \sqrt{y_1\, y_2\, y_3}\,. \qquad (12) \end{aligned}$$

Thus the first-order equations following from (6) are

$$\dot x_1 = \sqrt{\frac{y_2\, y_3}{y_1}}\,, \qquad \dot y_1 = \frac{m\, n\, x_1 + (m+n)\, x_2\, x_3 + x_1\, (x_2^2 + x_3^2 - x_1^2)}{\sqrt{y_1\, y_2\, y_3}}\,, \tag{13}$$

and cyclically for the 2 and 3 directions. We have used the Hamiltonian constraint $H = 0$, i.e.

$$U = -4y_1\, y_2\, y_3\,, \tag{14}$$

in writing the $\dot y_i$ equations. Using (9), we find that the metric is given by

$$\begin{aligned} ds^2 &= dt^2 + \frac{1}{y_1}\Big[(n\, x_1 + x_2\, x_3)\, \Sigma_1^2 + (m\, n + x_1^2 - x_2^2 - x_3^2)\, \Sigma_1\, \sigma_1 + (m\, x_1 + x_2\, x_3)\, \sigma_1^2\Big] \\ &\quad + \frac{1}{y_2}\Big[(n\, x_2 + x_3\, x_1)\, \Sigma_2^2 + (m\, n + x_2^2 - x_3^2 - x_1^2)\, \Sigma_2\, \sigma_2 + (m\, x_2 + x_3\, x_1)\, \sigma_2^2\Big] \\ &\quad + \frac{1}{y_3}\Big[(n\, x_3 + x_1\, x_2)\, \Sigma_3^2 + (m\, n + x_3^2 - x_1^2 - x_2^2)\, \Sigma_3\, \sigma_3 + (m\, x_3 + x_1\, x_2)\, \sigma_3^2\Big]\,, \qquad (15) \end{aligned}$$

As an alternative demonstration that the first-order equations (13) do indeed imply G_2 holonomy, we can simply verify that $d\Phi_{(3)} = 0$ and $d{*\Phi_{(3)}} = 0$, where $\Phi_{(3)}$ is given by (7) and the Hodge dual is evaluated using the metric (15), which is derived from (8) and (9). After a mechanical calculation, one finds that the dual 4-form $\Psi_{(4)} \equiv {*\Phi_{(3)}}$ is given by

$$\begin{aligned} \Psi_{(4)} &= \Psi_{0123}\, dt\, \Sigma_1\, \Sigma_2\, \Sigma_3 + \Psi_{0456}\, dt\, \sigma_1\, \sigma_2\, \sigma_3 \\ &\quad + \Psi_{0156}\, dt\, \Sigma_1\, \sigma_2\, \sigma_3 + \Psi_{0234}\, dt\, \Sigma_2\, \Sigma_3\, \sigma_1 + \Psi_{2356}\, \Sigma_2\, \Sigma_3\, \sigma_2\, \sigma_3 + \text{cyclic}\,, \qquad (16) \end{aligned}$$

where two further sets of 3 terms are added in the second line, cycled on (1,2,3) and (4,5,6) simultaneously. The non-zero components of Ψ are given by

$$\Psi_{0123} = \frac{1}{\sqrt{-U}}\, [m\, n^2 - n\, (x_1^2 + x_2^2 + x_3^2) - 2x_1\, x_2\, x_3]\,,$$

$$
\begin{aligned}
\Psi_{0456} &= \frac{1}{\sqrt{-U}}\,[m^2\,n - m\,(x_1^2 + x_2^2 + x_3^2) - 2x_1\,x_2\,x_3]\,,\\
\Psi_{0156} &= \frac{1}{\sqrt{-U}}\,[m\,n\,x_1 + 2m\,x_2\,x_3 + x_1\,(x_2^2 + x_3^2 - x_1^2)]\,,\\
\Psi_{0234} &= \frac{1}{\sqrt{-U}}\,[m\,n\,x_1 + 2n\,x_2\,x_3 + x_1\,(x_2^2 + x_3^2 - x_1^2)]\,,\\
\Psi_{2356} &= -\tfrac{1}{2}\sqrt{-U}\,\sqrt{\frac{y_1}{y_2\,y_3}}\,,
\end{aligned}
\tag{17}
$$

together with those following from simultaneous cycling on (1,2,3) and (4,5,6), with U defined by (11).

It is clear that $d\Phi_{(3)} = 0$ immediately implies the equations for $\dot{x}_i$ in (13). From $d\Psi_{(4)} = 0$ we obtain

$$
\frac{d\Psi_{2356}}{dt} + \Psi_{0156} + \Psi_{0234} = 0\,,
\tag{18}
$$

and cyclic permutations. Taking linear combinations, and using the $\dot{x}_i$ equations, these imply

$$
\dot{y}_1 = \frac{\sqrt{y_1\,y_2\,y_3}}{U}\,[m\,n\,x_1 + (m+n)\,x_2\,x_3 + x_1\,(x_2^2 + x_3^2 - x_1^2)]\,,
\tag{19}
$$

and cyclic permutations. From these and the $\dot{x}_i$ equations we can then establish that $(y_1\,y_2\,y_3)/U$ is a constant, which without loss of generality may be chosen by scaling to be $-\frac{1}{4}$, as in (14). Using this, (19) can be reduced to the previous expressions given in (13). Thus we have confirmed, by directly requiring the closure and co-closure of $\Phi_{(3)}$, that the metrics (15) indeed have G_2 holonomy if the first-order equations (13) are satisfied.

3 Specialisations to previous results

In this section, we show how the G_2 metrics obtained above reduce to various previously-known cases.

3.1 Reduction to triaxial 6-function metrics

In [4, 5], a class of metrics given by

$$
ds^2 = dt^2 + a_i^2(\Sigma_i - \sigma_i)^2 + b_i^2\,(\Sigma_i + \sigma_i)^2
\tag{20}
$$

was considered. It was shown that the metrics have G_2 holonomy if the six functions a_i and b_i satisfy the first-order equations

$$
\begin{aligned}
\dot{a}_1 &= \frac{a_1^2}{4a_3\,b_2} + \frac{a_1^2}{4a_2\,b_3} - \frac{a_2}{4b_3} - \frac{a_3}{4b_2} - \frac{b_2}{4a_3} - \frac{b_3}{4a_2}\,,\\
\dot{b}_1 &= \frac{b_1^2}{4a_2\,a_3} - \frac{b_1^2}{4b_2\,b_3} - \frac{a_2}{4a_3} - \frac{a_3}{4a_2} + \frac{b_2}{4b_3} + \frac{b_3}{4b_2}\,,
\end{aligned}
\tag{21}
$$

and cyclically in $(1,2,3)$. These metrics have $SU(2)\times SU(2)$ isometry.

Comparing with our results in section 2, we can see that the triaxial six-function metrics arise from the specialisation in which we set $m=n$. This is the case that was analysed in [9]. The functions x_i and y_i in section 2 are then given by

$$
\begin{aligned}
x_1 &= a_1\, a_2\, a_3 + a_3\, b_1\, b_2 + a_2\, b_1\, b_3 - a_1\, b_2\, b_3\,,\\
x_2 &= a_1\, a_2\, a_3 + a_3\, b_1\, b_2 - a_2\, b_1\, b_3 + a_1\, b_2\, b_3\,,\\
x_3 &= a_1\, a_2\, a_3 - a_3\, b_1\, b_2 + a_2\, b_1\, b_3 + a_1\, b_2\, b_3\,,\\
y_1 &= 4a_2\, a_3\, b_2\, b_3\,,\qquad y_2 = 4a_1\, a_3\, b_1\, b_3\,,\qquad y_3 = 4a_1\, a_2\, b_1\, b_2\,,
\end{aligned}
\tag{22}
$$

and m and n, which are equal, are related to the a_i and b_i by

$$
m = n = -a_1\, a_2\, a_3 + a_3\, b_1\, b_2 + a_2\, b_1\, b_3 + a_1\, b_2\, b_3\,. \tag{23}
$$

It can be verified also that the Hamiltonian constraint (14) is identically satisfied. Note that after setting $m=n$ the expression (11) for U factorises, to give

$$
U = (m - x_1 - x_2 - x_3)\,(m + x_1 + x_2 - x_3)\,(m + x_1 - x_2 + x_3)\,(m - x_1 + x_2 + x_3)\,. \tag{24}
$$

(In this special case with $m=n$, the actual value of m is a trivial overall scale parameter, which can be set to $m=1$ as in [9].) The equation (23) can be understood directly from the six-function equations (21), which imply that the cubic function in (23) is a constant of the motion.

The associative 3-form now can be expressed as

$$
\Phi_{(3)} = e^{014} + e^{025} + e^{036} - e^{123} + e^{156} - e^{246} + e^{345}\,, \tag{25}
$$

where $e^{ijk}\equiv e^i\wedge e^j\wedge e^k$, and the vielbein is defined by

$$
e^0 = dt\,,\qquad e^i = a_i\,(\Sigma_i - \sigma_i)\,,\qquad e^{i+3} = b_i\,(\Sigma_i + \sigma_i)\,,\qquad i=1,2,3\,. \tag{26}
$$

A four-function specialisation of (20), in which $a_1 = a_2$ and $b_1 = b_2$, includes in its solutions a family of complete and regular ALC metrics with a minimal S^3. In the Gromov-Hausdorff limit this approaches the product of a circle and the deformed conifold.

3.2 Reduction to the conifold-unification metrics

In [8] a new class of G_2 metrics with $S^3\times S^3$ principal orbits was obtained, which includes regular solutions that describe the resolved conifold in the Gromov-Hausdorff limit. The class was extended further in [6, 7], to a system that encompasses [8] and also the previous four-function specialisation described in section 3.1. In particular, the system found in [6, 7] provides a unification, via M-theory, of the deformed and resolved conifolds [7].

For the present purposes it is best to use the metric parameterisation in eqn (15) of [7] (with the tildes omitted):

$$\begin{aligned} ds_7^2 &= dt^2 + a^2\,[(\Sigma_1 + g\,\sigma_1)^2 + (\Sigma_2 + g\,\sigma_2)^2] + b^2\,[(\Sigma_1 - g\,\sigma_1)^2 + (\Sigma_2 - g\,\sigma_2)^2] \\ &\quad + c^2\,(\Sigma_3 - \sigma_3)^2 + f^2\,(\Sigma_3 + g_3\,\sigma_3)^2\,. \end{aligned} \tag{27}$$

With respect to the vielbein basis

$$\begin{aligned} &e^0 = dt\,, \quad e^1 = a\,(\Sigma_1 + g\,\sigma_1)\,, \quad e^2 = a\,(\Sigma_2 + g\,\sigma_2)\,, \quad e^3 = c\,(\Sigma_3 - \sigma_3)\,, \\ &e^4 = b\,(\Sigma_1 - g\,\sigma_1)\,, \quad e^5 = b\,(\Sigma_2 - g\,\sigma_2)\,, \quad e^6 = f\,(\Sigma_3 + g_3\,\sigma_3)\,, \end{aligned} \tag{28}$$

the associative 3-form takes the same form as (25). The conditions for G_2 holonomy, $d\Phi_{(3)} = 0$ and $d{*}\Phi_{(3)} = 0$, then imply the algebraic relation

$$g_3 = g^2 - \frac{c\,(a^2 - b^2)(1 - g^2)}{2a\,b\,f}\,, \tag{29}$$

together with the first-order equations

$$\begin{aligned} \dot a &= \frac{c^2\,(a^2 - b^2) + [4a^2\,(a^2 - b^2) - c^2\,(5a^2 - b^2) - 4a\,b\,c\,f]\,g^2}{16a^2\,b\,c\,g^2}\,, \\ \dot b &= -\frac{c^2\,(a^2 - b^2) + [4b^2\,(a^2 - b^2) + c^2\,(5b^2 - a^2) - 4a\,b\,c\,f]\,g^2}{16a\,b^2\,c\,g^2}\,, \\ \dot c &= \frac{c^2 + (c^2 - 2a^2 - 2b^2)\,g^2}{4a\,b\,g^2}\,, \\ \dot f &= -\frac{(a^2 - b^2)\,[4a\,b\,f^2\,g^2 - c\,(4a\,b\,c + a^2\,f - b^2\,f)\,(1 - g^2)]}{16a^3\,b^3\,g^2}\,, \\ \dot g &= -\frac{c\,(1 - g^2)}{4a\,b\,g} \end{aligned} \tag{30}$$

for the five remaining metric functions.

Comparing with our results in section (2), we find that this conifold-unifying G_2 system arises by making the specialisation $x_1 = x_2$ and $y_1 = y_2$. The relations between the two sets of variables are given by

$$\begin{aligned} &x_1 = x_2 = -(a^2 + b^2)\,c\,g\,, \qquad x_3 = (a^2 - b^2)\,c + 2a\,b\,f\,g_3\,, \\ &y_1 = y_2 = -2a\,b\,c\,f\,g\,(1 + g_3)\,, \qquad y_3 = 4a^2\,b^2\,g^2\,. \end{aligned} \tag{31}$$

The first-order equations (30) have two simple integration constants m and n, given by

$$m = (b^2 - a^2)\,c\,g^2 + 2a\,b\,f\,g^2\,g_3\,, \qquad n = (b^2 - a^2)\,c + 2a\,b\,f\,. \tag{32}$$

(The integration constants m and n are called p and q in [6, 7].) Finally, the constraint (29) implies that the Hamiltonian constraint (14) is satisfied.

3.3 Reduction to metrics with $S^3 \times T^3$ principal orbits

Recently, a class of G_2 metrics with $S^3 \times T^3$ principal orbits was obtained [10]. This can be seen to arise as a specialisation of our results in section 2 in which a group contraction of one of the $S^3 \sim SU(2)$ factors in the principal orbits is performed. It can be seen from (1) that a regular limit of our 3-form ρ and 4-form σ will be obtained if we perform the rescalings

$$\sigma_i \longrightarrow \lambda\, \sigma_i\,, \quad x_i \longrightarrow \lambda^{-1}\, x_i\,, \quad y_i \longrightarrow \lambda^{-2}\, y_i\,, \quad m \longrightarrow \lambda^{-3}\, m\,, \tag{33}$$

and then send λ to zero. The metric (15) reduces to

$$\begin{aligned} ds^2 &= dt^2 + \frac{1}{y_1}\Big[x_2\, x_3\, \Sigma_1^2 + m\, n\, \Sigma_1\, \sigma_1 + m\, x_1\, \sigma_1^2\Big] \\ &\quad + \frac{1}{y_2}\Big[x_3\, x_1\, \Sigma_2^2 + m\, n\, \Sigma_2\, \sigma_2 + m\, x_2\, \sigma_2^2\Big] \\ &\quad + \frac{1}{y_3}\Big[x_1\, x_2\, \Sigma_3^2 + m\, n\, \Sigma_3\, \sigma_3 + m\, x_3\, \sigma_3^2\Big]\,, \end{aligned} \tag{34}$$

and the first-order equations (13) reduce to

$$\dot x_1 = \sqrt{\frac{y_2\, y_3}{y_1}}\,, \qquad \dot y_1 = \frac{m\, x_2\, x_3}{\sqrt{y_1\, y_2\, y_3}}\,, \tag{35}$$

and cyclically for the 2 and 3 directions. The Hamiltonian constraint $H = 0$ reduces to $m^2\, n^2 - 4m\, x_1\, x_2\, x_3 = -4y_1\, y_2\, y_3$. This reproduces the metric and first-order equations obtained in [10]. The 3-form and 4-form defined in (1) now become

$$\begin{aligned} \rho &= n\, \Sigma_1\, \Sigma_2\, \Sigma_3 - m\, \sigma_1\, \sigma_2\, \sigma_3 + [x_1\, \sigma_1\, \Sigma_2\, \Sigma_3 + 2 \text{ cyclic terms}]\,, \\ \sigma &= y_1\, \sigma_2\, \Sigma_2\, \sigma_3\, \Sigma_3 + y_2\, \sigma_3\, \Sigma_3\, \sigma_1\, \Sigma_1 + y_3\, \sigma_1\, \Sigma_1\, \sigma_2\, \Sigma_2\,, \end{aligned} \tag{36}$$

It was shown in [10] that the first-order equations for these metrics with $S^3 \times T^3$ principal orbits can be solved completely. Although, as we shall discuss later, no complete and regular metrics can be obtained (apart from the direct sum of Eguchi-Hanson and a 3-torus, or flat $\mathbb{R}^4$ and a 3-torus), it is nevertheless of considerable interest that one can solve the first-order equations fully [10] in this case. It provides a motivation for considering more general possibilities for group-contraction limits of the metrics obtained in section 2, and it is to this topic that we move next.

4 Group contraction limits

4.1 $SU(2)$ group contractions

There are three contractions of the $SU(2)$ algebra $d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\,\sigma_j \wedge \sigma_k$ of left-invariant differential forms. In increasing order of degeneracy, they are

<u>Euclidean group</u>:
$$\sigma_1 \longrightarrow \lambda\,\sigma_1\,, \qquad \sigma_2 \longrightarrow \lambda\,\sigma_2\,, \qquad \sigma_3 \longrightarrow \sigma_3\,,$$
$$d\sigma_1 = -\sigma_2 \wedge \sigma_3\,, \qquad d\sigma_2 = \sigma_1 \wedge \sigma_3\,, \qquad d\sigma_3 = 0\,, \tag{37}$$

<u>Heisenberg group</u>:
$$\sigma_1 \longrightarrow \lambda\,\sigma_1\,, \qquad \sigma_2 \longrightarrow \lambda\,\sigma_2\,, \qquad \sigma_3 \longrightarrow \lambda^2\,\sigma_3\,,$$
$$d\sigma_1 = 0\,, \qquad d\sigma_2 = 0\,, \qquad d\sigma_3 = -\sigma_1 \wedge \sigma_2\,, \tag{38}$$

<u>Abelian group</u>:
$$\sigma_1 \longrightarrow \lambda\,\sigma_1\,, \qquad \sigma_2 \longrightarrow \lambda\,\sigma_2\,, \qquad \sigma_3 \longrightarrow \lambda\,\sigma_3\,,$$
$$d\sigma_1 = 0\,, \qquad d\sigma_2 = 0\,, \qquad d\sigma_3 = 0\,, \tag{39}$$

where λ is sent to zero in each case. Note that the Heisenberg contraction can be viewed as a further contraction of the Euclidean group (with an appropriate relabelling of the indices), and the Abelian group is a further contraction of this.

In what follows, we shall consider various group contractions of G_2 metrics with $S^3 \times S^3$ principal orbits. First, we shall consider the Heisenberg and Euclidean-group contractions of the general class of G_2 metrics that we obtained in section (2). Then, in subsequent subsections, we shall consider in more explicit detail the group contractions of the metrics obtained in [7], in which there is an additional $U(1)$ factor in the isometry group. One can apply the group contractions to one or both of the S^3 factors in the $S^3 \times S^3$ principal orbits. We shall begin by considering the case where just one of the 3-spheres is contracted

4.2 Heisenberg and Euclidean-group contractions

In this section, we perform Heisenberg and Euclidean-group contractions of the new metrics found in section 2, analogous to the Abelian group contraction that we described in section 3.3.

<u>**Heisenberg contraction**</u>:

The Heisenberg contraction is given by (38). In order for the 3-form and 4-form in (1) and the metric in (15) to have non-singular limits, the following scalings should also be performed

$$
\begin{aligned}
&x_1 \longrightarrow \lambda^{-1}\, x_1\,, \quad x_2 \longrightarrow \lambda^{-1}\, x_2\,, \quad x_3 \longrightarrow \lambda^{-2}\, x_3\,, \quad m \longrightarrow \lambda^{-4}\, m\,,\\
&y_1 \longrightarrow \lambda^{-3}\, y_1\,, \qquad y_2 \longrightarrow \lambda^{-3}\, y_2\,, \qquad y_3 \longrightarrow \lambda^{-2}\, y_3\,,
\end{aligned} \tag{40}
$$

while n is unscaled. The forms defined in (1) now become

$$
\begin{aligned}
\rho &= n\,\Sigma_1\,\Sigma_2\,\Sigma_3 - m\,\sigma_1\,\sigma_2\,\sigma_3 + x_1\,\sigma_1\,\Sigma_2\,\Sigma_3 + +x_2\,\sigma_2\,\Sigma_3\,\Sigma_1 + x_3\,(\sigma_3\,\Sigma_1\,\Sigma_2 - \sigma_1\,\sigma_2\,\Sigma_3)\,,\\
\sigma &= y_1\,\sigma_2\,\Sigma_2\,\sigma_3\,\Sigma_3 + y_2\,\sigma_3\,\Sigma_3\,\sigma_1\,\Sigma_1 + y_3\,\sigma_1\,\Sigma_1\,\sigma_2\,\Sigma_2\,,
\end{aligned} \tag{41}
$$

and the metric (15) becomes

$$
\begin{aligned}
ds^2 &= dt^2 + \frac{1}{y_1}\Big[x_2\,x_3\,\Sigma_1^2 + (m\,n - x_3^2)\,\Sigma_1\,\sigma_1 + m\,x_1\,\sigma_1^2\Big]\\
&\quad + \frac{1}{y_2}\Big[x_3\,x_1\,\Sigma_2^2 + (m\,n - x_3^2)\,\Sigma_2\,\sigma_2 + m\,x_2\,\sigma_2^2\Big]\\
&\quad + \frac{1}{y_3}\Big[(n\,x_3 + x_1\,x_2)\,\Sigma_3^2 + (m\,n + x_3^2)\,\Sigma_3\,\sigma_3 + m\,x_3\,\sigma_3^2\Big]\,,
\end{aligned} \tag{42}
$$

The first-order equations after taking this Heisenberg scaling limit will be

$$
\begin{aligned}
&\dot{x}_1 = \sqrt{\frac{y_2\,y_3}{y_1}}\,, \qquad \text{cyclically for 2 and 3 directions}\,,\\
&\dot{y}_1 = \frac{m\,x_2\,x_3}{\sqrt{y_1\,y_2\,y_3}}\,, \qquad \dot{y}_2 = \frac{m\,x_1\,x_3}{\sqrt{y_1\,y_2\,y_3}}\,, \qquad \dot{y}_3 = \frac{m\,n\,x_3 + m\,x_1\,x_2 - x_3^3}{\sqrt{y_1\,y_2\,y_3}}\,,
\end{aligned} \tag{43}
$$

with the Hamiltonian constraint $H = 0$ giving

$$
m^2\,n^2\; - 2m\,n\,x_3^2 - 4m\,x_1\,x_2\,x_3 + x_3^4 + 4y_1\,y_2\,y_3 = 0\,. \tag{44}
$$

Euclidean contraction:

The Euclidean contraction is given by (37). To obtain a non-singular system in this limt, we need the following scalings:

$$
\begin{aligned}
&x_1 \longrightarrow \lambda^{-1}\, x_1\,, \quad x_2 \longrightarrow \lambda^{-1}\, x_2\,, \quad m \longrightarrow \lambda^{-2}\, m\,,\\
&y_1 \longrightarrow \lambda^{-1}\, y_1\,, \quad y_2 \longrightarrow \lambda^{-1}\, y_2\,, \quad y_3 \longrightarrow \lambda^{-2}\, y_3\,,
\end{aligned} \tag{45}
$$

while x_3 and n are unscaled. The forms ρ and σ defined in (1) now become

$$
\begin{aligned}
\rho &= n\,\Sigma_1\,\Sigma_2\,\Sigma_3 - m\,\sigma_1\,\sigma_2\,\sigma_3 + x_1\,(\sigma_1\,\Sigma_2\,\Sigma_3 - \sigma_2\,\sigma_3\,\Sigma_1)\\
&\quad + x_2\,(\sigma_2\,\Sigma_3\,\Sigma_1 - \sigma_3\,\sigma_1\,\Sigma_2) + x_3\,\sigma_3\,\Sigma_1\,\Sigma_2\,,\\
\sigma &= y_1\,\sigma_2\,\Sigma_2\,\sigma_3\,\Sigma_3 + y_2\,\sigma_3\,\Sigma_3\,\sigma_1\,\Sigma_1 + y_3\,\sigma_1\,\Sigma_1\,\sigma_2\,\Sigma_2\,,
\end{aligned} \tag{46}
$$

and the metric (15) becomes

$$
\begin{aligned}
ds^2 &= dt^2 + \frac{1}{y_1}\Big[(n\,x_1 + x_2\,x_3)\,\Sigma_1^2 + (m\,n + x_1^2 - x_2^2)\,\Sigma_1\,\sigma_1 + m\,x_1\,\sigma_1^2\Big] \\
&\quad + \frac{1}{y_2}\Big[(n\,x_2 + x_3\,x_1)\,\Sigma_2^2 + (m\,n + x_2^2 - x_1^2)\,\Sigma_2\,\sigma_2 + m\,x_2\,\sigma_2^2\Big] \\
&\quad + \frac{1}{y_3}\Big[x_1\,x_2\,\Sigma_3^2 + (m\,n - x_1^2 - x_2^2)\,\Sigma_3\,\sigma_3 + (m\,x_3 + x_1\,x_2)\,\sigma_3^2\Big]\,, \qquad (47)
\end{aligned}
$$

The first-order equations in this limit will be

$$
\begin{aligned}
&\dot{x}_1 = \sqrt{\frac{y_2\,y_3}{y_1}}\,, \qquad \text{and cyclically for 2 and 3 directions}\,, \\
&\dot{y}_1 = \frac{m\,n\,x_1 + m\,x_2\,x_3 + x_1\,(x_2^2 - x_1^2)}{\sqrt{y_1\,y_2\,y_3}}\,, \quad \dot{y}_2 = \frac{m\,n\,x_2 + m\,x_1\,x_3 + x_2\,(x_1^2 - x_2^2)}{\sqrt{y_1\,y_2\,y_3}}\,, \\
&\dot{y}_3 = \frac{m\,x_1\,x_2}{\sqrt{y_1\,y_2\,y_3}}\,, \qquad (48)
\end{aligned}
$$

with the Hamiltonian constraint $H = 0$ giving

$$
m^2\,n^2 - 2m\,n\,(x_1^2 + x_2^2) - 4m\,x_1\,x_2\,x_3 + (x_1^2 - x_2^2)^2 + 4y_1\,y_2\,y_3 = 0\,. \qquad (49)
$$

It should be noted that in all three of the three scalings of the metrics in section 2, i.e. the Abelian, Heisenberg and Euclidean cases, the coefficient m is rescaled while the coefficient n is not. Thus one cannot take such contraction limits in the case where $m = n$, which, as we showed in section 3.1, reduces to the six-function system found in [4, 5]. On the other hand, we *can* take these contraction limits in the other specialisation that we discussed in (3.2), where there is an additional $U(1)$ in the isometry group and the metrics are described by the first-order system obtained in [7].

We shall now turn to a more detailed investigation of the various group contractions for the G_2 metrics found in [7].

5 The contraction from $S^3 \times S^3$ with $U(1)$ isometry

5.1 $S^3 \times T^3$ principal orbits

In the case of the G_2 metrics with an additional $U(1)$ isometry in the principal $S^3 \times S^3$ orbits, it is possible to study the contraction limits in a more explicit manner. The extra $U(1)$ isometry arises if we set the functions x_i and y_i in two of the three S^3 directions equal, for example $x_1 = x_2$, $y_1 = y_2$. As we showed in section 3.2, the new G_2 metrics in section 2 are then equivalent to those found in [7], which are given in equations (27), together with the conditions (29) and (30) for G_2 holonomy. For our present purposes, we find it more convenient to work with the metrics written as in (27), which we therefore take as our starting point for studying the contraction limits. (The solutions that we obtain in this section are contained, albeit in a different parameterisation, within the general solutions constructed in [10].)

We now make the rescaling as in (39), at the same time rescaling the metric functions in (27) according to

$$g \longrightarrow \frac{g}{\lambda}\,, \qquad g_3 \longrightarrow \frac{g_3}{\lambda}\,. \tag{50}$$

The metric (27 then becomes

$$\begin{aligned} ds_7^2 &= dt^2 + a^2\,[(\Sigma_1 + g\,\alpha_1)^2 + (\Sigma_2 + g\,\alpha_2)^2] + b^2\,[(\Sigma_1 - g\,\alpha_1)^2 + (\Sigma_2 - g\,\alpha_2)^2] \\ &\quad + c^2\,\Sigma_3^2 + f^2\,(\Sigma_3 + g_3\,\alpha_3)^2\,. \end{aligned} \tag{51}$$

The algebraic constraint (29) becomes

$$2a\,b\,f = c\,(b^2 - a^2)\,, \tag{52}$$

while the five first-order equations (30) become

$$\begin{aligned} \dot a &= \frac{4a^4 - 4a^2\,b^2 - 3a^2\,c^2 - b^2\,c^2}{16a^2\,b\,c}\,, \\ \dot b &= \frac{4b^4 - 4a^2\,b^2 - a^2\,c^2 - 3b^2\,c^2}{16a\,b^2\,c}\,, \\ \dot c &= \frac{c^2 - 2a^2 - 2b^2}{4a\,b}\,, \\ \dot g &= \frac{c\,g}{4a\,b}\,, \qquad \dot g_3 = -\frac{c\,g_3}{2a\,b}\,. \end{aligned} \tag{53}$$

Note that the last two equations imply $g^2\,g_3 =$constant.

In this contraction limit the relations (31) that we obtained by matching the specialisation of section 2 where $x_1 = x_2$ and $y_1 = y_2$ to the metrics in [7] can be inverted simply, to give

$$\begin{aligned} &a^2 = \frac{2x_1\,x_3 + n\,\sqrt{-m\,x_3}}{4y_1}\,, \qquad b^2 = \frac{2x_1\,x_3 - n\,\sqrt{-m\,x_3}}{4y_1}\,, \qquad g^2 = -\frac{m}{x_3}\,, \\ &c^2 = \frac{4x_1^2\,x_3 + m\,n^2}{4x_3\,y_3}\,, \qquad f^2 = -\frac{m\,n^2}{4x_3\,y_3}\,, \qquad g_3 = -\frac{2x_3}{n}\,. \end{aligned} \tag{54}$$

Note that the constraint (52) is indeed satisfied, and also that

$$g^2\,g_3 = \frac{2m}{n}\,. \tag{55}$$

From (60), we also see that $c\,(b^2 - a^2)$ is a constant, and so from (52) we have $a\,b\,f =$constant. In terms of the metric variables in [10], this translates into $m^2\,n^2 + 4m\,x_1^2\,x^3 = -4y_1^2\,y_3$, which is indeed the condition of the vanishing of the Hamiltonian H given in their eq (4.32).

It should be remarked that whilst it is straightforward to take the limit $\lambda \longrightarrow 0$ in order to get the first-order equations for a, b, c and g, one has to be slightly more careful in order to get the algebraic constraint (52) and the first-order equation for g_3. Specifically, (52) is obtained by solving (29) for f and then taking the limit $\lambda \longrightarrow 0$. The first-order equation for g_3 is obtained by differentiating (29), and then using (29) itself to substitute for f in the resulting expression for $\dot{g}_3$. This gives

$$\dot{g}_3 = -\frac{c\,(g^2 - g_3)(1 + g_3)}{2a\,b\,g^2}\,. \tag{56}$$

Finally, after making the replacements $g \longrightarrow g/\lambda$ and $g_3 \longrightarrow g_3/\lambda$, we obtain the equation for $\dot{g}_3$ in (53). It is worth noting also that another way of obtaining (52) and (53) is by directly imposing $d\Phi_3 = 0$ and $d{*}\Phi_3 = 0$, where the vielbein in (25) is the natural one read off from (51).

5.1.1 Solving the equations for $S^3 \times T^3$ with $U(1)$ isometry

Introduce a new radial variable r by $dr = \frac{1}{2} a\, b\, c\, dt$, and define $A \equiv a^2$, $B \equiv b^2$, $C \equiv c^2$. The equations for $\dot{a}$, $\dot{b}$ and $\dot{c}$ in (53) become

$$\begin{aligned} A' &= \frac{A}{B\,C} - \frac{1}{C} - \frac{3}{4B} - \frac{1}{4A}\,,\\ B' &= \frac{B}{A\,C} - \frac{1}{C} - \frac{3}{4A} - \frac{1}{4B}\,,\\ C' &= -\frac{2}{A} - \frac{2}{B} + \frac{C}{A\,B}\,. \end{aligned} \tag{57}$$

Now define

$$A = X + Y\,, \qquad B = X - Y\,, \tag{58}$$

and introduce a new radial variable ρ such that $dr = -A\,B\,d\rho$. The equations (57) become

$$\frac{dX}{d\rho} = X - \frac{2Y^2}{C}\,, \qquad \frac{dY}{d\rho} = \tfrac{1}{2} Y - \frac{2X\,Y}{C}\,, \qquad \frac{dC}{d\rho} = 4X - C\,. \tag{59}$$

Note that we can deduce from these that

$$C\,Y^2 = k^2\,, \tag{60}$$

where k is a constant. (We must have k^2 non-negative, since $C = c^2$ is non-negative.)

Equation (60) implies that if k is non-vanishing, the metrics will be singular. This follows from (52), since we therefore have

$$2a\,b\,f = c\,(b^2 - a^2) = k\,. \tag{61}$$

This shows that at any putative short-distance endpoint, which would be characterised by a smooth degeneration of a circle or sphere to the origin of (spherical) polar coordinates, some other metric function would be diverging there. This is a rather common feature of Ricci-flat metrics with principal orbits that contain torus factors; various examples with special holonomy were discussed in [11]. Indeed, we shall show in section 6 that aside from the direct product of Eguchi-Hanson and T^3, there are no other non-trivial regular solutions to the G_2 metrics with $S^3 \times T^3$ principal orbits that were obtained in [10].

It is, nevertheless, of interest to study the explicit solutions for the G_2 metrics in section 5.1. In particular, we shall see how the Eguchi-Hanson times T^3 metric emerges as a non-singular Gromov-Hausdorff limit of a family of singular G_2 metrics, in which k goes to zero.

Solving the $dC/d\rho$ in equation (59 for X, and plugging into the $dX/d\rho$ equation, also making use of (60), we therefore obtain the following second-order equation for C:

$$\frac{d^2C}{d\rho^2} - C + \frac{8k}{C^2} = 0\,. \tag{62}$$

Multiplying by $dC/d\rho$, this can be integrated once, giving

$$\frac{dC}{d\rho} = \sqrt{C^2 + 16k^2\, C^{-1} + \mu}\,, \tag{63}$$

where μ is a constant. Note that we have chosen the positive square root here, because the $dC/d\rho$ equation in (59) gives $X = \frac{1}{4}(C + dC/d\rho)$, and hence

$$X = \tfrac{1}{4}C + \tfrac{1}{4}\sqrt{C^2 + 16k^2\, C^{-1} + \mu}\,. \tag{64}$$

This would have been negative at small C if we had chosen the other sign in (63), contradicting the fact that $X = \frac{1}{2}(A+B) = \frac{1}{2}(a^2+b^2)$ is non-negative. Our choice here is adapted to allowing c to become small.

Substituting from (64) and (60), we therefore find that

$$A = \tfrac{1}{4}C + \tfrac{1}{4}\sqrt{C^2 + 16k^2\, C^{-1} + \mu} + \frac{k}{\sqrt{C}}\,, \qquad B = \tfrac{1}{4}C + \tfrac{1}{4}\sqrt{C^2 + 16k^2\, C^{-1} + \mu} - \frac{k}{\sqrt{C}}\,. \tag{65}$$

Integrating (63), we obtain

$$\rho = \int_0^C \frac{dx}{\sqrt{x^2 + 16k^2\, x^{-1} + \mu}}\,. \tag{66}$$

We have chosen the integration limit so that C vanishes at $\rho = 0$.

The integral can be evaluated explicitly, if rather opaquely, in terms of elliptic functions. It is convenient to use C as the radial variable. We have $dt = 2/(a\,b\,c)\,dr = -2(a\,b/c)\,d\rho$, and so

$$dt^2 = \frac{4A\,B}{C}\,d\rho^2 = \frac{4A\,B\,dC^2}{C^2 + 16k^2\,C^{-1} + \mu}\,. \qquad (67)$$

Thus if we let $C \equiv z^2$, we get the metric in the form

$$\begin{aligned} ds^2 \;&=\; \frac{16A\,B\,dz^2}{z^4 + 16k^2/z^2 + \mu} + A\,[(\Sigma_1 + g\,\alpha_1)^2 + (\Sigma_2 + g\,\alpha_2)^2] \\ &\quad + B\,[(\Sigma_1 - g\,\alpha_1)^2 + (\Sigma_2 - g\,\alpha_2)^2] + z^2\,\Sigma_3^2 + \frac{4k^2}{A\,B}\,(\Sigma_3 - g_3\,\alpha_3)^2\,, \end{aligned} \qquad (68)$$

with

$$A = \tfrac14 z^2 + \tfrac14\sqrt{z^4 + 16k^2\,z^{-2} + \mu} + \frac{k}{z}\,, \qquad B = \tfrac14 z^2 + \tfrac14\sqrt{z^4 + 16k^2\,z^{-2} + \mu} - \frac{k}{z}\,. \qquad (69)$$

The functions g and g_3 are then given by

$$g = g_0\,e^{-\frac12\rho}\,, \qquad g_3 = \tilde g_0\,e^{\rho} \qquad (70)$$

where ρ is given by (66) with $C = z^2$, and g_0 and $\tilde g_0$ are constants. At $z \to \infty$, the metric becomes locally $\mathbb{R}^4 \times T^3$. For a non-vanishing value of k, the metric has a singularity at $z = 0$, where either A or B becomes divergent. If we had chosen the negative root of (62) for $dC/d\rho$, the singularity would occur at some finite z_0 where A or B vanishes.

Note that if we take k to vanish, then the integral in (66) becomes elementary, allowing us to obtain simple formulae for g and g_3. Letting $\mu = \ell^4$ and introducing a new radial variable y defined by $y^2 = z^2 + \sqrt{z^4 + \ell^4}$, we find $\rho = 2\log(y/\ell)$. The metric (68) then becomes

$$ds^2 = 2\Big(1 - \frac{\ell^4}{y^4}\Big)^{-1} dy^2 + \tfrac12 y^2\,\Big(1 - \frac{\ell^4}{y^4}\Big)\,\Sigma_3^2 + \tfrac12 y^2\,(\Sigma_1^2 + \Sigma_2^2) + g_0^2\,\ell^2\,(\alpha_1^2 + \alpha_2^2) + \hat g_0^2\,\ell^2\,\alpha_3^2\,, \quad (71)$$

where we have replaced the constant $\tilde g_0$ in (70) by $\hat g_0 = 8k\,\tilde g_0/\ell^3$ before sending k to zero. The metric (71), which is the Gromov-Hausdorff limit of the general solution (68) (with the radius of the circle described by α_3 sent to zero), is nothing but the direct sum of the Eguchi-Hanson metric and T^3. As was shown in [22], the Eguchi-Hanson metric is asymptotic to $\mathbb{R}^4/\mathbb{Z}_2$, and so the regular 7-metric we obtain here in the Gromov-Hausdorff limit $k = 0$ is asymptotic to $\mathbb{R}^4/\mathbb{Z}_2 \times T^3$. It should be emphasised, however, that before the Gromov-Hausdorff limit is taken, the metrics are singular.

The integration (66) again becomes elementary if μ is set to zero instead of k, yielding

$$\rho = -\tfrac23\log(4k) + \tfrac23\log(z^3 + \sqrt{z^6 + 16k^2})\,. \qquad (72)$$

If we now introduce a new radial coordinate y such that $y^3 = z^3 + \sqrt{z^6 + 16k^2}$, the metric (68) becomes

$$\begin{aligned} ds^2 &= 2^{4/3}\Big(1-\frac{16k^2}{y^6}\Big)^{-1/3} dy^2 + 2^{-2/3} y^2 \Big(1-\frac{16k^2}{y^6}\Big)^{2/3} \Sigma_3^2 \\ &+322^{1/3} k^2\, y^{-4} \Big(1-\frac{16k^2}{y^6}\Big)^{-1/3} (\Sigma_3 + \tilde{g}_0\, y^2\, \alpha_3)^2 \\ &+2^{-5/3} y^2 \Big(1-\frac{16k^2}{y^6}\Big)^{-1/3} \Big(1+\frac{4k}{y^3}\Big) [(\Sigma_1 + g_0\, y^{-1}\, \alpha_1)^2 + (\Sigma_2 + g_0\, y^{-1}\, \alpha_2)^2] \\ &+2^{-5/3} y^2 \Big(1-\frac{16k^2}{y^6}\Big)^{-1/3} \Big(1-\frac{4k}{y^3}\Big) [(\Sigma_1 - g_0\, y^{-1}\, \alpha_1)^2 + (\Sigma_2 - g_0\, y^{-1}\, \alpha_2)^2], \end{aligned} \qquad (73)$$

after rescaling constants g_0 and $\tilde{g}_0$. This is, as expected, as singular metric. The metric runs from (locally) $\mathbb{R}^4 \times T^3$ at $y = \infty$ to a singularity at $y = (4k)^{1/3}$.

5.2 $S^3 \times$ (Heisenberg) principal orbits

Here, we instead perform the Heisenberg contraction given in (38). One must now perform the associated metric rescalings

$$g \longrightarrow \frac{g}{\lambda}\,, \qquad g_3 \longrightarrow \frac{g_3}{\lambda^2}\,, \qquad (74)$$

giving the metric

$$\begin{aligned} ds_7^2 &= dt^2 + a^2\,[(\Sigma_1 + g\,\beta_1)^2 + (\Sigma_2 + g\,\beta_2)^2] + b^2\,[(\Sigma_1 - g\,\beta_1)^2 + (\Sigma_2 - g\,\beta_2)^2] \\ &+c^2\,\Sigma_3^2 + f^2\,(\Sigma_3 + g_3\,\beta_3)^2\,. \end{aligned} \qquad (75)$$

It is evident from (29) and (30) that the constraint will now become

$$g_3 = \frac{[2a\,b\,f - c\,(b^2 - a^2)]\,g^2}{2a\,b\,f}\,, \qquad (76)$$

while the first-order equations for (a, b, c, f, g) will become

$$\begin{aligned} \dot{a} &= \frac{4a^2\,(a^2-b^2) - c^2\,(5a^2-b^2) - 4a\,b\,c\,f}{16a^2\,b\,c}\,, \\ \dot{b} &= -\frac{4b^2\,(a^2-b^2) + c^2\,(5b^2-a^2) - 4a\,b\,c\,f}{16a\,b^2\,c}\,, \\ \dot{c} &= \frac{c^2 - 2a^2 - 2b^2}{4a\,b}\,, \\ \dot{f} &= -\frac{(a^2-b^2)\,[4a\,b\,(c^2+f^2) + c\,f\,(a^2-b^2)]}{16a^3\,b^3}\,, \\ \dot{g} &= \frac{c\,g}{4a\,b}\,. \end{aligned} \qquad (77)$$

This first-order system has two integration constants $m = 2a\,b\,f\,g^2\,g_3$ and $n = (b^2 - a^2)\,c + 2a\,b\,f$. We can use the constant n to express the function f such that the functions a, b and c form a closed first-order system. We have not obtained the general solution of these equations.

5.3 $S^3 \times$ (Euclidean) principal orbits

Here, we instead perform the contraction (37) of the $SU(2)$ algebra for the left-invariant 1-forms σ_i in (27). Correspondingly, we now rescale only the metric function g, according to $g \longrightarrow g/\lambda$, giving

$$\begin{aligned} ds_7^2 &= dt^2 + a^2\,[(\Sigma_1 + g\,\gamma_1)^2 + (\Sigma_2 + g\,\gamma_2)^2] + b^2\,[(\Sigma_1 - g\,\gamma_1)^2 + (\Sigma_2 - g\,\gamma_2)^2] \\ &\quad + c^2\,(\Sigma_3 - \gamma_3)^2 + f^2\,(\Sigma_3 + g_3\,\gamma_3)^2\,. \end{aligned} \tag{78}$$

The first-order equations for G_2 holonomy now become

$$\begin{aligned} \dot a &= \frac{4a^4 - 4a^2\,b^2 - 3a^2\,c^2 - b^2\,c^2}{16a^2\,b\,c}\,, \\ \dot b &= \frac{4b^4 - 4a^2\,b^2 - a^2\,c^2 - 3b^2\,c^2}{16a\,b^2\,c}\,, \\ \dot c &= \frac{c^2 - 2a^2 - 2b^2}{4a\,b}\,, \\ \dot g &= \frac{c\,g}{4a\,b}\,, \qquad \dot g_3 = -\frac{c\,(g_3+1)}{2a\,b}\,, \end{aligned} \tag{79}$$

together with the same algebraic constraint (52) as in the Abelian contraction:

$$2a\,b\,f = c\,(b^2 - a^2)\,. \tag{80}$$

Note that the last two equations in (79) imply $g^2\,(g_3+1)$ =constant. Again, this first-order system has two integration constants m and n given by (32). The first three equations in (79) are the same as in the Abelian contraction. (The slightly delicate procedure for taking the limit to get the constraint (80) and the first-order equation for $\dot g_3$ goes in the same way as we described in section 3.3 for the Abelian limit. It can be seen that (56) now gives the expression for $\dot g_3$ appearing in (79).)

6 Global considerations in the $S^3 \times T^3$ metrics of G_2 holonomy

In this section, we make some observations about solutions of the system of G_2 metrics with $S^3 \times T^3$ principal orbits that was obtained in [10], and which is reproduced in section 3.3. In particular, we shall present simple arguments which show that there can be no complete and regular metrics within this class, other than flat $\mathbb{R}^4 \times T^3$, or the direct product of Eguchi-Hanson and T^3.

It was observed in [10] that measured in the G_2 metric (34), the volumes of the S^3 and T^3 factors in the principal orbits are bounded below, with

$$\mathrm{Vol}(S^3) \ge |n|\,, \qquad \mathrm{Vol}(T^3) \ge |m|\,. \tag{81}$$

(These results can easily be seen by using the Hamiltonian constraint in the expressions $(x_1x_2x_3)\,(y_1y_2y_3)^{-1/2}\,\Sigma_1\Sigma_2\Sigma_3$ and $m^{3/2}\,(x_1x_2x_3/(y_1y_2y_3))^{1/2}\,\sigma_1\sigma_2\sigma_3$ for the volume forms that one can read off from (34).) It follows, therefore, that if $m\,n\neq 0$, the principal orbits will never collapse, for any value of t. Under these circumstances, one can never obtain a complete and regular metric, since there will be no short-distance endpoint at which the metric "closes off." Thus the radial coordinate would be running between two endpoints corresponding to asymptotic infinities, but by a standard theorem one can have at most one asymptotic infinity in a complete regular Ricci-flat metric.

The only possibility for regular metrics, therefore, is to have $m\,n=0$, allowing one or other of the S^3 or T^3 to collapse on singular orbits. For example, the S^3 might degenerate to S^2 on such an orbit, which could then give rise to an $S^2\times T^3$ "bolt" at short distance, closing off the metric. However, from (34) we see that if $m\,n=0$, the metric becomes purely diagonal,

$$ds^2 = dt^2 + \frac{x_2\,x_3}{y_1}\,\Sigma_1^2 + \frac{x_3\,x_1}{y_2}\,\Sigma_2^2 + \frac{x_1\,x_2}{y_3}\,\Sigma_3^2 + \frac{m\,x_1}{y_1}\,\sigma_1^2 + \frac{m\,x_2}{y_2}\,\sigma_2^2 + \frac{m\,x_3}{y_3}\,\sigma_3^2\,, \qquad (82)$$

with the Hamiltonian constraint becoming $y_1y_2y_3 = m\,x_1x_2x_3$.

If the T^3 remains uncollapsed ($m\neq 0$), we can perform a Kaluza-Klein reduction on the T^3. The absence of off-diagonal terms in (82) means that there will be no Kaluza-Klein vectors, and so the Ricci-flatness of (82) will translate, in the reduced $D=4$ equations, to a system of equations that includes three "dilaton equations," each of the form

$$\Box\,\phi = 0\,, \qquad (83)$$

where we have a dilaton $\phi\sim\log(x_i/y_i)$ for each reduction circle. Since the metric is of cohomogeneity one, this means that ϕ is a function only of the radial coordinate t, and so (83) is just $d(\sqrt{g}\,\dot\phi)/dt=0$, where g is the determinant of the reduced 4-metric. The general solution is

$$\phi = c_1 + c_2\int^t \frac{dt'}{\sqrt{g(t')}}\,, \qquad (84)$$

where c_1 and c_2 are constants. In a putative regular 7-metric the radius of each circle within T^3 remains non-vanishing at short-distance, and so ϕ is finite there. If a q-sphere within the S^3 collapses, for any $1\le q\le 3$, we will have $\sqrt{g(t)}\sim t^q$, and so in order to have ϕ finite at short distance (i.e. at $t=0$) it must be that $c_2=0$, and hence we have $\phi=$constant. Repeating for all three dilatons, we conclude that x_i/y_i is a constant for each i. Without loss of generality we can rescale the lengths of the circles so that $x_i=y_i$ (and hence $m=1$), and so the metric (82) then becomes simply

$$ds^2 = dt^2 + \frac{y_2\, y_3}{y_1}\, \Sigma_1^2 + \frac{y_3\, y_1}{y_2}\, \Sigma_2^2 + \frac{y_1\, y_2}{y_3}\, \Sigma_3^2 + d\theta_1^2 + d\theta_2^2 + d\theta_3^2\,, \tag{85}$$

where we write the T^3 1-forms as $\sigma_i = d\theta_i$. The first-order equations (35) then become

$$\dot y_1 = \sqrt{\frac{y_2\, y_3}{y_1}}\,, \qquad \text{and cyclic}\,. \tag{86}$$

These are just one of the systems of first-order equations for Bianchi-IX 4-metrics of $SU(2)$ holonomy, which admit the Eguchi-Hanson metric as a complete regular solution if two of the three directions are set equal. If all directions are unequal, the solutions are incomplete, with curvature singularities [22]. Thus we have established that the only regular metrics within the class obtained in [10], with $m \neq 0$ so that the T^3 factor does not collapse, are either flat $\mathbb{R}^4$ times T^3, or else the product of Eguchi-Hanson and a flat 3-torus. In this latter case, the metric approaches $\mathbb{R}^4/\mathbb{Z}_2 \times T^3$ at large distance.

7 A torus splitting theorem

The purpose of this section is to show that the problems found in section 6 when constructing a non-trivial and non-singular fibration by tori in the specific example of the metrics in [10] are in fact generic for Ricci-flat metrics, as long as one supposes the torus action to be by isometries. This has consequences for implementing at the level of concrete and explicit exact metrics some of the ideas of [23] and [10] on fibrations by special Lagrangian and associative tori respectively. These difficulties will be illustrated in the following section by means of concrete examples of exact but incomplete Calabi-Yau and G_2 metrics drawn from earlier work [11] on contractions. In the G_2 case, they are obtained by a more drastic contraction of the $SU(2) \times SU(2)$ isometry group than to $SU(2) \times T^3$.

7.1 Kaluza-Klein reductions

We shall begin by reviewing the standard toroidal reduction of a $(d + k)$-dimensional Riemannian manifold E to a d-dimensional Riemannian base manifold B with metric $g_{\mu\nu}$. Later, we shall replace the torus group fibres T^k by a general unimodular Lie group G. If the torus action is free, B will be a smooth manifold. If the action is not free, B may be singular as a manifold and/or its metric may be singular. The conclusion of our theorem is that if the action is free then the metric splits as an unwarped and untwisted product. The basic *local* formulae are the metric ansatz:

$$ds^2 = \exp\frac{2U}{k}\hat h_{mn}(dy^m + A^m_\mu\, dx^\mu)(dy^n + A^n_\nu\, dx^\nu)\, A\, B + \exp\frac{2U}{d-2}\, g_{\mu\nu}\, dx^\mu\, dx^\nu\,, \tag{87}$$

and the d-dimensional action from which the Ricci-flat conditions may be derived. The Lagrangian density is

$$\begin{aligned}\mathcal{L} &= \sqrt{-g}\Big(R - \tfrac{1}{4}\exp(\frac{2kU}{k-2})\,\hat{h}_{mn}\,F^m_{\mu\nu}\,F^{n\,\mu\nu} + \tfrac{1}{4}g^{\mu\nu}\,(\mathrm{Tr}\,\hat{h}^{-1}\partial_\mu\hat{h}\hat{h}^{-1}\partial_\nu\hat{h}) \\ &\quad +(\frac{1}{d-1}+\frac{1}{k^2})\,g^{\mu\nu}\,\partial_\mu U\partial_\nu U\Big).\end{aligned} \tag{88}$$

The matrix $\hat{h}_{mn}$ has unit determinant, and

$$F^m_{\mu\nu} = \partial_\mu A^m_\nu - \partial_\nu A^n_\mu. \tag{89}$$

Thus the quantity $\exp(U)$ is the volume of the toroidal fibres.

We now show that the vector fields must vanish under suitable global assumptions. To do so, we note that the equations of motion of the vectors imply that

$$\nabla_\mu\left(\exp(\frac{2kU}{k-2})\,\hat{h}_{mn}\,A^m_\nu\,F^{m\,\nu\mu}\right) = -\tfrac{1}{2}\exp(\frac{2kU}{k-2})\,\hat{h}_{mn}\,F^m_{\mu\nu}\,F^{n\,\mu\nu}. \tag{90}$$

Assuming that the vector potentials A^m_μ are globally defined, then integration over the base, together with the assumption that the boundary terms at infinity vanish (which is typically the case if the metric approaches the flat metric on $T^k \times \mathbb{E}^d$ sufficiently fast), shows that the vectors must vanish.

Having established the vanishing of the vectors, we can now show that the volume of the toroidal fibres must be constant. The quantity U satisfies

$$\nabla^2 U = \frac{k}{2\kappa^2(k-2)}\exp(\frac{2kU}{k-2})\,\hat{h}_{mn}\,F^m_{\mu\nu}\,F^{n\,\mu\nu}, \tag{91}$$

We see that even if the vectors are non-vanishing, the volume of the fibres can have no interior maximum. Moreover by integrating over the base B of the fibration we see that unless there is a boundary contribution from infinity, the field strengths must vanish. In that case, multiplication by U and integration over the base B then shows that U must be constant.

Now the action for the scalars $\hat{h}_{mn}$ reduces to a harmonic map into the space of unimodular symmetric matrices, i.e. into the non-compact Riemannian symmetric space $SL(k,\mathbb{R})/SO(k)$. This space is known to have negative sectional curvature.

7.1.1 A Bochner identity

To proceed, we need to apply a "Bochner Identity." To obtain it we suppose, more generally, that a field $\phi^A(x) : B \to N$ takes its values in some Riemannian target manifold N with metric $G_{AB}(\phi)$ and potential function $W(\phi)$. In our case ϕ corresponds to the field of unimodular matrices $\hat{h}_{mn}$, and thus N is $SL(k,\mathbb{R})/SO(k)$ with its $SL(k,\mathbb{R})$-invariant metric, and $W(\phi)$ vanishes. Actually we could include the field U and then $N = GL(k,\mathbb{R})/SO(k) = \mathbb{R}_+ \times SL(k,\mathbb{R})/SO(k)$ with the product metric. In what follows we retain $W(\phi)$ and assume that ϕ is coupled to the metric $g_{\mu\nu}$ on B in the standard fashion.

The Bochner identity tells us that:

$$
\begin{aligned}
&(\tfrac{1}{2} G_{AB} \frac{\partial \phi^A}{\partial x^\alpha} \frac{\partial \phi^B}{\partial x^\beta} g^{\alpha\beta})^{;\mu}{}_{;\mu} = \phi^{A;\alpha;\beta} \, \phi^B{}_{;\alpha;\beta} \, G_{AB} \\
&+\phi^{A;\alpha} \, (G_{AB} \, R_{\alpha\beta} - g_{\alpha\beta} \, R_{ACBD} \, \phi^C_{;\mu} \, \phi^D_{;\nu} \, g^{\mu\nu}) \, \phi^{B;\beta} \\
&+\phi^{A;\alpha} \, (\phi^{B;\beta}{}_{;\beta})_{;\alpha} \, G_{AB} \,,
\end{aligned} \tag{92}
$$

where all covariant derivatives are covariant with respect to the spacetime metric $g_{\alpha\beta}$ and the target-space metric G_{AB}, in the manner described by [24]. The field equations are:

$$
\phi^{A;\beta}{}_{;\beta} = G^{AB} \, \nabla_B \, W \,. \tag{93}
$$

It is important to realise that in (93), the semi-colon includes a contribution from the pull-back of the Levi-Civita connection of N to the spacetime manifold B . More precisely, $\partial_\mu \phi^A$ is a section of the bundle: $T^*B \otimes \phi^\star TN$ and ; denotes the connection on this bundle associated to the Levi-Civita connections on N and B. On the other hand, since it is acting on a scalar, the operator on the left hand side of (92) is the usual Laplacian on B with respect to the metric $g_{\alpha\beta}$.

The Einstein field equations read

$$
R_{\alpha\beta} = [G_{AB} \, \phi^A_{;\alpha} \, \phi^B_{;\beta} + g_{\alpha\beta} \, W(\phi)] \tag{94}
$$

Now if we integrate (92) over B, dropping the boundary term and assuming that $W(\phi) = 0$, and if we assume that the Ricci tensor of B, i.e. $R_{\alpha\beta}$, is non-negative and that the sectional curvatures of N are non-positive, we see that the map ϕ must be constant. This completes the proof of our splitting theorem.

7.2 Reduction on a unimodular Lie group

In this subsection turn briefly to case when the torus group T^k is replaced by a general unimodular group G of dimension k with left-invariant Cartan-Maurer forms λ^n and structure constants $C_l{}^m{}_n$. To say the group is unimodular is to say that the adjoint action preserves volume, or more concretely,

$$
C_l{}^m{}_m = 0 \,. \tag{95}
$$

In the context of the Bianchi classification of three-dimensional Lie algebras, the unimodular algebras are called Class A and the non-unimodular algebras are called Class B. It is known that Kaluza-Klein reduction on a unimodular group is consistent in the technical sense. It is useful to note that the contraction of a unimodular group is itself unimodular.

In the metric ansatz we replace $(dy^m + A^m)$ by $(\lambda^m + A^m)$, where now A^m are $\mathfrak{g}$-valued one forms. The Lagrangian must now be modified, since $F^m_{\mu\nu}$ is a non-abelian curvature, and $\partial_\mu \hat{h}_{mn}$ must be replaced by $\mathcal{D}_\mu \hat{h}_{mn}$ where $\mathcal{D}_\mu$ is the gauge-covariant derivative acting on the symmetric tensor representation $\mathfrak{g} \times^S \mathfrak{g}$. The quantity $\partial_\mu U$ appears unchanged, but a potential term now arises, of the form:

$$W = \tfrac{1}{4} \exp(\frac{-2U}{d-2})\, C^a{}_{bc}\, (2C^b{}_{ad}\, h^{cd} + C^e{}_{fd}\, h_{ad}\, h^{bf}\, h^{cd}), \tag{96}$$

where $h_{mn} = \exp \frac{2U}{k} \hat{h}_{mn}$.

Note that both (91) and (90) are modified, and so one cannot immediately draw the same conclusions as before. However, if the group G admits a suitable circle subgroup, as it does for the Heisenberg group or the Euclidean group, then more can be said, as we shall show in the next subsection.

7.3 Circle splitting; Heisenberg and Euclidean groups

It is worth remarking to begin with that the issue of boundary terms is a non-trivial one. Consider the case when the torus is a circle, i.e. $k = 1, T^1 = S^1 = SO(2) = U(1)$, for which there is a single Killing vector K. We might be tempted to use the identity for U 91, or use the covariant identity

$$\nabla^2 R^2 = |\nabla K|^2, \tag{97}$$

with $R^2 = |K|^2$, to attempt to prove that *any* circle which tends to constant length at infinity must split. But this is clearly not always true. Indeed it is not true even if the Killing vector has no fixed points and the length of the circle is bounded below by a positive constant. In previous work examples of such phenomena have already been exhibited (see, for example, [7]), in the case of asymptotically locally conical (ALC) metrics of G_2 holonomy. The circle in question could be identified with the M-theory circle, and R was related to the string coupling constant g by $R \propto g^{\frac{2}{3}}$. In these cases the boundary terms definitely do not vanish. However, the examples of interest in the present paper approach the flat metric at infinity at a fast enough rate that the boundary terms do vanish, and hence any circle must flat.

This means that we can extend our considerations to the case when we contract $SU(2)$ not to the abelian group T^3, but to a non-abelian group such as the Euclidean group $E(2)$ or the Heisenberg group. In both cases there are circle subgroups, and we can apply our results to any such circle subgroup, with the conclusion that they must split. But if they do, it means that in fact the group we thought was non-abelian is in fact abelian, i.e. is T^3, and moreover the torus must split. Thus the failure to find regular solutions with $SU(2) \times T^3$ isometry group is part of a more general phenomenon and not an artefact of performing too drastic a group contraction. In a later section we shall exhibit some explicit metrics illustrating what goes wrong.

8 Special Lagrangian and associative fibrations by tori

As an illustration of the above remarks, we can use some previous results on cohomogeneity one metrics with special holonomy, in the case of two-step nilpotent groups [11]. All the metrics are incomplete, and none is asymptotically Euclidean. Nevertheless, they provide instructive local models of the global problems with such fibrations. They may also provide local models of the situations envisaged in [10].

8.1 Special Lagrangian tori in $SU(3)$ manifolds

Here we refer to section(4.1.2) and the metric (56) of [11]:

$$\begin{aligned} ds_6^2 &= H^2\,dy^2 + H^{-1}\,(dz_1 + m\,z_4\,dz_3)^2 + H^{-1}\,(dz_2 + m\,z_5\,dz_3)^2 \\ &\quad + H^2\,dz_3^2 + H\,(dz_4^2 + dz_5^2)\,, \end{aligned} \tag{98}$$

where H is linear in y. The Killing vector fields $\frac{\partial}{\partial z_i}$ with $i = 1, 2, 3$ generate a torus action with metric $h_{mn} = \mathrm{diag}(H^{-1}, H^{-1}, H^2)$. The torus has constant volume and thus $U = \text{constant}$. The base B has coordinates (y, z_4, z_5) and has topology $\mathbb{R} \times \mathbf{R}^2$ if no identifications in z_4 and z_5 are made.

The torus is clearly Lagrangian since by (59) of [11] the symplectic form restricted to it vanishes. It is special because in terms of the complex coordinates $\zeta_1 \equiv z_1 + \mathrm{i}\,H\,z_4$, $\zeta_2 \equiv z_2 + \mathrm{i}\,H\,z_5$ and $\zeta_3 = y + \mathrm{i}\,z_3$, the first two are real and the last is pure imaginary along the torus, implying that the holomorphic 3-form restricted to the torus has a constant phase.

8.2 Associative tori in G_2 manifolds

Here we refer to section(4.2.2) and the metric (75) of [11]:

$$\begin{aligned} ds_7^2 &= H^3\,dy^2 + H^{-1}\,(dz_1 - m\,z_5\,dz_6)^2 + H^{-1}\,(dz_2 + m\,z_4\,dz_6)^2 \\ &\quad + H^{-1}\,(dz_3 + m\,z_5\,dz_4)^2 + H^2(dz_4^2 + dz_5^2 + dz_6^2)\,. \end{aligned} \tag{99}$$

The Killing vector fields $\frac{\partial}{\partial z_i}$ with $i = 1, 2, 6$ generate a torus action with metric $h_{mn} = \mathrm{diag}(H^{-1}, H^{-1}, H^2)$. The torus has constant volume and thus $U =$ constant. The base B has coordinates (y, z_3, z_4, z_5) and has topology $\mathbb{R} \times \mathbf{R}^3$ if no identifications in z_3, z_4 and z_5 are made.

The torus is associative since, by (76) of [11], the associative 3- form restricted to it gives its volume form. The base space B of the foliation is, by Hodge duality, a co-associative 4-fold. As explained in [11], these solutions are limiting forms of complete non-singular solutions with isometry group $SU(2) \times SU(2)$, and hence they are solutions of the equations obtained by contraction of the general set of equations given earlier in this paper.

8.3 Associative fibrations of G_2 manifolds constructed from K3 surfaces

Another G_2 metric was given in section 4.2.1 of [11], which was constructed from a nilpotent group which is a T^2 bundle over T^4. It was also pointed out in section 6.2 of [11] that it admits of an immediate generalisation in which the 4-torus is replaced by a K3 surface. The metric is given locally by (126) of [11] and is

$$ds^2 = H^4 dy^2 + H^{-1}(dx^1 + A^1)^2 + H^{-1}(dx^2 + A^2)^2 + H^2 ds_4^2, \tag{100}$$

where ds_4^2 is a K3 metric and $J^i = dA^i$, $i = 0, 1, 2$ are the three Kähler forms with vector potentials A^i on the K3 surface, and H is a harmonic function of y which may be taken without loss of generality as $H = y$. Globally we have a foliation by the sum of two line bundles over the K3 surface whose curvatures are given by the two Kähler forms J^1 and J^2. The global existence places some restrictions on the K3 surface. An alternative description is to say that we have a fibering over $K3$ by fibres F with coordinates (y, x^1, x^2) which are $\mathbb{R}_+ \times T^2$.

The associative 3-form $\psi_{(3)}$ was given in (128) of [11]:

$$\psi_{(3)} = \hat{e}^0 \wedge \hat{e}^1 \wedge \hat{e}^2 + H^2 \hat{e}^0 \wedge J^0 - \hat{e}^1 \wedge J^2 + \hat{e}^2 \wedge J^1. \tag{101}$$

Because the restriction of the associative 3-form to the fibres gives its volume form, it follows that the fibres F are associative and hence by Hodge-duality that the K^3's are co-associative.

References

[1] R.L. Bryant and S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58**, 829 (1989).

[2] G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on S^3, $\mathbb{R}^3$ and $\mathbb{R}^4$ bundles*, Commun. Math. Phys. **127**, 529 (1990).

[3] M. Cvetič, G. W. Gibbons, H. Lü and C. N. Pope, *New complete non-compact Spin(7) manifolds,* hep-th/0103155; *New cohomogeneity one metrics with Spin(7) holonomy,* math.DG/0105119.

[4] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Supersymmetry M3-branes and G_2 manifolds,* Nucl. Phys. **B620**, 3 (2002), hep-th/0106026.

[5] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, *Gauge theory at large N and new G(2) holonomy metrics,* Nucl. Phys. **B611**, 179 (2001), hep-th/0106034.

[6] A. Brandhuber, *G_2 holonomy spaces from invariant three-forms,* hep-th/0112113.

[7] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *A G_2 Unification of the Deformed and Resolved Conifolds*, hep-th/0112138, to appear in Phys. Lett. **B**.

[8] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *M-theory conifolds,* Phys. Rev. Lett. **88**, 121602 (2002), hep-th/0112098.

[9] N. Hitchin, *Stable forms and special metrics*, math.DG/0107101.

[10] S. Gukov, S-T. Yau and E. Zaslow, *Duality and fibrations on G_2 manifolds*, hep-th/0203217.

[11] G.W. Gibbons, H. Lü, C.N. Pope and K.S. Stelle, *Supersymmetric domain walls from metrics of special holonomy,* Nucl. Phys. **B623**, 3 (2002), hep-th/0108191.

[12] B. S. Acharya, *On realising $N = 1$ super Yang-Mills in M theory*, hep-th/0011089.

[13] M. Atiyah, J. Maldacena and C. Vafa, *An M-theory flop as a large n duality*, J. Math. Phys. **42**, 3209 (2001), hep-th/0011256.

[14] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of G_2 holonomy*, hep-th/0107177.

[15] H. Kanno and Y. Yasui, *On Spin(7) holonomy metric based on* $SU(3)/U(1)$, hep-th/0108226.

[16] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Cohomogeneity one manifolds of Spin(7) and* G_2 *holonomy,* hep-th/0108245.

[17] S. Gukov and J. Sparks, *M-theory on Spin(7) manifolds. I,* Nucl. Phys. **B625** (2002) 3, hep-th/0109025.

[18] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Orientifolds and slumps in* G_2 *and Spin(7) metrics,* hep-th/0111096.

[19] H. Kanno and Y. Yasui, *On Spin(7) holonomy metric based on SU(3)/U(1). II*, hep-th/0111198.

[20] R. Hernandez and K. Sfetsos, *An eight-dimensional approach to* G_2 *manifolds*, hep-th/0202135.

[21] S. Gukov, K Saraikin and A. Volovich, unpublished notes.

[22] V.A. Belinsky, G.W. Gibbons, D.N. Page and C.N. Pope, *Asymptotically Euclidean Bianchi IX metrics in quantum gravity*, Phys. Lett. **B76**, 433 (1978).

[23] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror Symmetry is T-Duality,* Nucl. Phys **B479**, 243-259, (1996), hep-th/9606040.

[24] C. W. Misner, *Harmonic maps as models for physical theories,* Phys. Rev. **D18** 4510-4524 (1978).