New Couplings of Six-Dimensional Supergravity

Hitoshi NISHINO¹

Department of Physics University of Maryland College Park, 20742-4111, USA

and

Ergin SEZGIN²

Department of Physics and Astronomy Texas A and M University College Station, TX 77843-4242, USA

Abstract

We describe the couplings of six-dimensional supergravity, which contain a self-dual tensor multiplet, to n_T anti-self-dual tensor matter multiplets, n_V vector multiplets and n_H hypermultiplets. The scalar fields of the tensor multiplets form a coset $SO(n_T, 1)/SO(n_T)$, while the scalars in the hypermultiplets form quaternionic Kähler symmetric spaces, the generic example being $Sp(n_H, 1)/Sp(n_H) \otimes Sp(1)$. The gauging of the compact subgroup $Sp(n_H) \times Sp(1)$ is also described. These results generalize previous ones in the literature on matter couplings of N = 1 supergravity in six dimensions.

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1. Introduction

Supersymmetric field theories in six dimensions (6D) are of considerable interest for various reasons. Those which admit chiral supersymmetry, namely the (1,0) and (2,0) supersymmetric cases, are especially interesting, because when anomaly free, they may hint at significant properties of various compactifications of a unifying theory in higher dimensions, such as M-theory.

While fairly general couplings of (1, 0) supergravity to matter were described sometime ago by the authors [1], recent investigations of M-theory compactifications have unsurfaced interesting generalizations of those couplings. For example, an anomaly free model that contained nine anti-self-dual tensor multiplets, eight vector multiplets and twenty hypermultiplets was found in [2] from M-theory on $(K_3 \times S_1)/Z_2$. A low energy field theory for this model does not exist in the literature at present. In this paper, we shall close this gap and provide the most general couplings to date of the (1,0) supergravity in 6D.

As is well-known, when the numbers of self-dual and anti-self-dual tensor multiplets are different, a manifestly Lorentz invariant lagrangian formulation no longer exists. This is because the non-paired self-dual or anti-self-dual components do not admit the usual kinetic term as the square of the third-rank antisymmetric tensor field strength. In the case of a single tensor matter multiplet coupling to the (1,0) supergravity, a lagrangian formulation does exist, because of the paring between the self-dual field strength of supergravity multiplet and the anti-self-dual field strength of the matter tensor multiplet.

The (1,0) supergravity will be alternatively referred to as N = 1 supergravity. The first model of N = 1 supergravity coupled to a single tensor multiplet was given in [3]. A more complicated system of N = 1 supergravity coupled to a single tensor multiplet, Yang-Mills vector multiplets, and hyper multiplets forming a quaternionic Kähler manifolds was accomplished in terms of lagrangian formulation in ref. [1]. (The model was called N = 2 supergravity in [1], according to an alternative convention for counting the number of supersymmetries.) The gauging of the scalar manifold isometries, as well as the Sp(1) automorphism group was also given in [1].

In a work by Romans [4], the multiple tensor multiplets were coupled to supergravity with no lagrangian formulation. It was found that the tensor multiplets form a coset $SO(n_T, 1)/SO(n_T)$ in order for the couplings to supergravity to be consistent. Afterwards it was found by Sagnotti [5] that vector multiplets can be further introduced to this system. In this case, an interesting Yang-Mills gauge anomaly structure emerges already at the level of classical field equations. This anomaly, and its relation to a supercurrent anomaly has been discussed in [6].

Globally supersymmetric limits of the models mentioned above turn out to be rather

subtle. For example, the sigma model describing the hypermultiplets is based on a hyper-Kähler manifold, rather than the quaternionic Kähler manifold that arises in coupling to supergravity. The coupling of a anti-self-dual tensor multiplet to Yang-Mills was worked out in [7] by direct construction rather than a rigid supersymmetry limit of the supergravity plus tensor multiplet system. Such a limiting procedure is not known yet. Further peculiarities arise in the tensor multiplet plus Yang-Mills system, for the description of which, we refer the reader to [7].

In this paper, we derive all the field equations N = 1 supergravity coupled to n_T tensor multiplets, n_H copies of hypermultiplets and n_V copies of vector multiplets. The n_T scalars of the tensor multiplet parametrize the coset $SO(n_T, 1)/SO(n_T)$, and the $4n_H$ scalars of the hypermultiplets parametrize the coset $SP(n_H, 1)/SP(n_H) \otimes SP(1)$. The choice of the latter coset is due to notational simplicity. Our formulae can straightforwardly be adapted to more general quaternionic symmetric spaces. In this paper, we also gauge the $Sp(n_H)$ subgroup of the hyperscalar manifold isometry group $Sp(n_H, 1)$ and the automorphism group Sp(1). The supersymmetry transformations provided here reduce to the full transformation rules of the single tensor multiplet case, and in that sense we expect our result to be exact. However, the fermionic field equations are given up to terms cubic in fermions, and bosonic field equations up to fermionic bilinears. In the interesting case of (anti) self-duality equations, the coefficients of the fermionic bilinears are determined as well.

An interesting feature that emerges in the coupling of Yang-Mills to multi-tensor multiplets is that the gauge kinetic term vanishes for certain expectation value of the scalar fields [5]. Here we also find the correlated singularities in the full supersymmetry transformation of the gravitino and the gaugino. It was proposed in [8] that these singularities signal a phase transition, and in [9, 10] this was attributed to tensionless strings. The couplings of the hypermatter constructed here do not exhibit singularities, provided that the group $Sp(n_H) \times Sp(1)$ is not gauged. The gauging of this group, however, gives rise to singularities in a number of hypermatter couplings at the point in the moduli space where the previously known gauge coupling singularities occur.

This paper is organized as follows. In the next section, we describe the geometrical aspects of the scalar manifolds $SO(n_T, 1)/SO(n_T)$, and $Sp(n_H, 1)/Sp(n_H) \otimes Sp(1)$, and set up the notation. In section 3, we give our results for all the field equations with supersymmetry transformation rules, with their derivations based on mutual consistency. In the same section, we also present the gauging of $Sp(n_H) \times Sp(1)$. The gauging of $SO(n_T)$ does not seem to be possible for reasons that will be explained in section 3. In section 4, the case of $n_T = 1$, namely a single tensor multiplet coupled to supergravity, which admits a lagrangian formulation is presented. Concluding remarks are given in section 5. The Appendix is devoted to useful notations and conventions crucial for our computations.

2. Preliminaries

We first fix the field contents of our total system. It consists of four kinds multiplets: the multiplet of N = 1 supergravity $(e_{\mu}{}^{m}, \psi_{\mu}{}^{A})$, n_{T} copies of anti-self-dual tensor multiplets plus one self-dual tensor multiplet denoted collectively as $(B_{\mu\nu}{}^{I}, \chi^{Ai}, \varphi^{\underline{\alpha}})$, n_{V} copies of Yang-Mills vector multiplets (A_{μ}, λ^{A}) , and n_{H} copies of hypermultiplets $(\phi^{\alpha}, \psi^{a})$. We use the world indices $\mu, \nu, \dots = 0, 1, \dots, 5$ and tangent space indices $m, n, \dots = (0), (1), \dots, (5)$. The indices $A, B, \dots = 1, 2$ label the fundamental representation of the automorphism group Sp(1). The scalar fields $\varphi^{\underline{\alpha}}(\underline{\alpha} = \underline{1}, \dots, \underline{n_{T}})$ parametrize the coset $SO(n_{T}, 1)/SO(n_{T})$. The indices $i, j, \dots = (1), \dots, n_{T}$ label the fundamental representation of $SO(n_{T}, 1)$, and the indices $i, j, \dots = (1), \dots, (n_{T})$ label the fundamental representation of $SO(n_{T})$. The hyperscalars $\phi^{\alpha}(\alpha = 1, \dots, 4n_{H})$ parametrize the coset $Sp(n_{H}, 1)/Sp(n_{H}) \otimes Sp(1)$, and the indices $a, b, \dots = 1, \dots, 2n_{H}$ label the fundamental representation of $Sp(n_{H})$.

The Yang-Mills multiplet fields are in the adjoint representation of a product group $G = G_1 \times G_2 \times G_3 \times \cdots \times G_p$. Some of these factors can be identified with any compact subgroups of the isometry groups $SO(n_T, 1)$ and $Sp(n_H, 1)$. In this paper we will consider the case when $G_1 = Sp(n_H)$ and $G_2 = Sp(1)$.

In describing the couplings of the tensor multiplet, it is useful to introduce the coset representatives L_I and L_I^i , which together form an $(n_T + 1) \times (n_T + 1)$ matrix which obeys the properties of an $SO(n_T, 1)$ group element [11]. Denoting the components of the inverse matrix by L^I and L_i^I , they obey the relations

$$L_I L^I = 1$$
 , $L_i^{\ I} L_I = 0$, $L_I^{\ i} L^I = 0$. (1)

The $SO(n_T, 1)$ invariant constant metric

$$\eta_{IJ} \equiv -L_I L_J + L_I^{\ i} L_{Ji} \quad , \tag{2}$$

can be used to raise and lower the $SO(n_T, 1)$ vector indices: $\eta_{IJ}L^J = -L_I$, $\eta_{IJ}L_i^J = L_{Ii}$. Another useful tensor G_{IJ} is defined by

$$G_{IJ} \equiv L_I L_J + L_I^{\ i} L_{Ji} \quad , \tag{3}$$

with the important distinction with the sign in the first term compared with (2). In contrast to the latter, G_{IJ} is not a constant tensor, but it depends on the coordinates $\varphi^{\underline{\alpha}}$.

Composite $SO(n_T)$ connection $A_{\underline{\alpha}}{}^{ij}$, and coset vielbeins $V_{\underline{\alpha}}{}^i$ can be defined by

$$\partial_{\underline{\alpha}} L_I{}^i = -A_{\underline{\alpha}{}^i j} L_I{}^j + V_{\underline{\alpha}{}^i} L_I .$$

$$\tag{4}$$

Thus we have the useful relations $D_{\underline{\alpha}}L_I = \partial_{\underline{\alpha}}L_I = L_I{}^iV_{\underline{\alpha}i}$, $D_{\underline{\alpha}}L_I{}^i = L_IV_{\underline{\alpha}}{}^i$, where $D_{\underline{\alpha}} = \partial_{\underline{\alpha}} + A_{\underline{\alpha}}$ which yields the commutator

$$\left[D_{\underline{\alpha}}, D_{\underline{\beta}}\right] L_{I}{}^{i} = \left(V_{\underline{\alpha}}{}^{j}V_{\underline{\beta}}{}^{i} - V_{\underline{\beta}}{}^{j}V_{\underline{\alpha}}{}^{i}\right) L_{Ij} .$$

$$(5)$$

The overall constant in the r.h.s., which is the square of the inverse radius of the hyperboloid $SO(n_T, 1)/SO(n_T)$, is fixed to be +1 by the $H\chi$ -terms in the closure of two supersymmetries on $B_{\mu\nu}{}^I$. Furthermore, the curvature tensor can be read off from (5), which shows that the manifold has a constant negative curvature.

The case $n_T = 1$ is special in the sense that the original coset $SO(n_T, 1)/SO(n_T)$ is reduced to a semi-simple group manifold SO(1, 1) with the non-positive definite metric. Moreover, since there is a pair of a self-dual and an anti-self-dual tensor multiplets forming a total field strength free of (anti)self-dual condition, we can construct an invariant lagrangian. We will discuss this particular case in section 4.

As for the coset $Sp(n_H, 1)/Sp(n_H) \otimes Sp(1)$, many of its properties have been exhibited in [1]. It is useful to recall that given a representative L of this coset, the Maurer-Cartan form decomposes as

$$L^{-1}\partial_{\alpha}L = A_{\alpha}{}^{ab}T_{ab} + A_{\alpha}{}^{AB}T_{AB} + V_{\alpha}{}^{aA}T_{aA} , \qquad (6)$$

where T_{ab} and T_{AB} are generators of $Sp(n_H)$ and Sp(1), T_{aA} are the coset generators, $A_{\alpha}{}^{ab}$ and $A_{\alpha}{}^{AB}$ are the $Sp(n_H)$ and Sp(1) composite connections, and $V_{\alpha}{}^{aA}$ are the coset vielbeins. It is convenient to define a triplet of complex structures $J_{\alpha\beta}{}^{AB}$ as ¹

$$J_{\alpha\beta}{}^{AB} = \left(V_{\alpha a}{}^{A}V_{\beta}{}^{aB} + V_{\alpha a}{}^{B}V_{\beta}{}^{aA}\right) , \qquad (7)$$

which obey the Sp(1) algebra. The Sp(1) curvature $F_{\alpha\beta}$ is related to the complex structures as

$$F_{\alpha\beta} = 2J_{\alpha\beta} \ . \tag{8}$$

For completeness, we also record the relations obeyed by the vielbeins:

$$g_{\alpha\beta}V_{aA}{}^{\alpha}V_{bB}{}^{\beta} = \epsilon_{ab}\epsilon_{AB} , \qquad V_{aA}{}^{\alpha}V^{aB\beta} + {}_{\alpha\leftrightarrow\beta} = g^{\alpha\beta}\delta_{A}{}^{B} , \qquad (9)$$

$$V_{aA}{}^{\alpha}V^{bA\beta} + {}_{\alpha\leftrightarrow\beta} = \frac{1}{n_H}g^{\alpha\beta}\delta_a{}^b , \qquad (10)$$

where ϵ^{ab} and ϵ^{AB} are the invariant tensors of $Sp(n_H)$ and Sp(1), respectively.

¹A correction to a misprint in [1]: The r.h.s. of eq. (2.7) should be multiplied with -2.

Let us next consider the local $Sp(n_H) \times Sp(1)$ gauge transformations

$$\delta\phi^{\alpha} = \xi^{\alpha}{}_{ab}\Lambda^{ab}(x) + \xi^{\alpha}{}_{AB}\Lambda^{AB}(x) , \qquad (11)$$

where $\xi_{\alpha}{}^{ab}$ and $\xi_{\alpha}{}^{AB}$ are Killing vectors in general, but for the case at hand they take the simple form $T^{ab}\phi_{\alpha}$ and $T^{AB}\phi_{\alpha}$, respectively. The covariant derivative of the hyperscalars are defined as

$$\mathcal{D}_{\mu}\phi^{\alpha} = \partial_{\mu}\phi^{\alpha} - gA_{\mu}{}^{ab}\xi^{\alpha}{}_{ab} - g'A_{\mu}{}^{AB}\xi^{\alpha}{}_{AB} , \qquad (12)$$

where g and g' are the gauge coupling constants for $Sp(n_H)$ and Sp(1), respectively.

The coupling of scalar field modifies the covariant derivatives of the fermionic fields in such a way that the following replacements have to be made

$$gA_{\mu}{}^{ab} \to gA_{\mu}{}^{ab} + (\mathcal{D}_{\mu}\phi^{\alpha})A_{\alpha}{}^{ab} \quad (\text{except in } D_{\mu}\lambda) \quad ,$$

$$g'A_{\mu}{}^{AB} \to g'A_{\mu}{}^{AB} + (\mathcal{D}_{\mu}\phi^{\alpha})A_{\alpha}{}^{AB} \quad . \tag{13}$$

The reason for the exception made for the covariant derivative of λ is a technical one, and it is explained in [1]. One consequence of the above replacements is that the occurrence of Yang-Mills field strength dependence terms in the following commutator

$$\left[\mathcal{D}_{\mu}, \mathcal{D}_{\nu}\right]\epsilon^{A} = \frac{1}{4} R_{\mu\nu}{}^{mn}\gamma_{mn}\epsilon^{A} + \left(\mathcal{D}_{\mu}\phi^{\alpha}\right)\left(\mathcal{D}_{\nu}\phi^{\beta}\right) F_{\alpha\beta}{}^{AB}\epsilon_{B} - \operatorname{tr}_{z}F_{\mu\nu} C^{AB}\epsilon_{B} , \qquad (14)$$

where the triplet of functions C^{AB} lie in the $Sp(n_H) \times Sp(1)$ algebra, and is given by

$$C^{AB} = gA_{\alpha}{}^{AB}\xi^{\alpha cd} T_{cd} + g' \left(A_{\alpha}{}^{AB}\xi^{\alpha CD} T_{CD} - T^{AB} \right) .$$

$$\tag{15}$$

This function arises in the field equations as well as the supersymmetry transformation rules, as we shall see in the next section. As discussed in detail in [1], this function satisfies

$$\mathcal{D}_{\mu}C^{AB} = \left(\mathcal{D}_{\mu}\phi^{\alpha}\right)\left(\mathcal{D}_{\alpha}C^{AB}\right) , \qquad (16)$$

and one can derive [1]

$$\mathcal{D}_{\alpha}C^{AB} = 2J_{\alpha\beta}{}^{AB}\xi^{\beta} , \qquad (17)$$

where

$$\xi^{\alpha} \equiv g A_{\alpha}{}^{AB} \xi^{\alpha c d} T_{c d} + g' A_{\alpha}{}^{AB} \xi^{\alpha C D} T_{C D} .$$
⁽¹⁸⁾

One of the peculiar features of the vector couplings to the tensor multiplet is the necessity of a constant matrix C^{Iz} , where the index ^z distinguishes the various factor groups in the total Yang-Mills gauge group, while $I = 1, 2, ..., n_T$ is for the local coordinate index for the coset $SO(n_T, 1)/SO(n_T)$ [5]. The constant coefficients C^{Iz} have been related to certain S-matrix elements in the conformal field theory an open superstring [5]. It is convenient to define the following quantities which arise frequently in our calculations and results:

$$C^{z} \equiv C^{Iz} L_{I} , \qquad C^{iz} \equiv C^{Iz} L_{I}^{i} , \qquad (19)$$

$$H_{\mu\nu\rho} \equiv H_{\mu\nu\rho}{}^{I}L_{I} , \qquad H_{\mu\nu\rho}{}^{i} \equiv H_{\mu\nu\rho}{}^{I}L_{I}{}^{i} . \qquad (20)$$

3. The Field Equations and Supersymmetry Transformations

Our strategy is to start with a general ansätze for the supersymmetry transformations and field equations with unknown coefficients. We then determine all the coefficients by the closure of supersymmetry transformations, modulo filed equations when necessary, and the requirement for the field equations to transform into each other. Although we will give the bosonic field equations up to fermionic bilinears and fermionic field equations up to cubic in fermion terms, it turns out that we can still fix the full transformation rules, as well as the full (anti)self-duality equations. We shall first give the results, and then explain the derivations. Several formulae that are useful in these derivations are provided in the Appendix.

The fermionic field equations are 2

$$\gamma^{\mu\nu\rho}\mathcal{D}_{\nu}\psi_{\rho}{}^{A} + \frac{1}{2}H^{\mu\nu\rho}\gamma_{\nu}\psi_{\rho}{}^{A} - \frac{1}{2}\gamma^{\nu}\gamma^{\mu}\chi^{Ai}V_{\underline{\alpha}}{}^{i}\partial_{\nu}\varphi^{\underline{\alpha}} - 2\gamma^{\nu}\gamma^{\mu}\psi_{a}V_{\alpha}{}^{aA}\mathcal{D}_{\nu}\phi^{\alpha} + \frac{1}{2}C^{z}\operatorname{tr}_{z}\left(\gamma^{\rho\sigma}\gamma^{\mu}\lambda^{A}F_{\rho\sigma}\right) + \frac{1}{4}H^{\mu\nu\rho}{}_{i}\gamma_{\nu\rho}\chi^{Ai} - \operatorname{tr}_{z}\left(\gamma^{\mu}C^{AB}\lambda_{B}\right) = 0 \quad , \tag{21}$$

$$\gamma^{\mu} \mathcal{D}_{\mu} \chi^{Ai} - \frac{1}{24} \gamma^{\mu\nu\rho} \chi^{Ai} H_{\mu\nu\rho} - \frac{1}{2} C^{iz} \operatorname{tr}_{z} \left(\gamma^{\mu\nu} \lambda^{A} F_{\mu\nu} \right) + \frac{1}{4} H^{\mu\nu\rho i} \gamma_{\mu\nu} \psi_{\rho}^{A}$$
(22)

$$-\frac{1}{2}\gamma^{\mu}\gamma^{\nu}\psi_{\mu}V_{\underline{\alpha}}^{\ i}\partial_{\nu}\varphi^{\underline{\alpha}} - C_{z}^{-1}C^{iz} \operatorname{tr}_{z}\left(C^{AB}\lambda_{B}\right) = 0 \quad , \tag{23}$$

$$\gamma^{\mu} \mathcal{D}_{\mu} \psi^{a} + \frac{1}{24} \gamma^{\mu\nu\rho} \psi^{a} H_{\mu\nu\rho} - \gamma^{\mu} \gamma^{\nu} \psi_{\mu A} V_{\alpha}{}^{aA} \mathcal{D}_{\nu} \phi^{\alpha} - 2\lambda_{A} \xi^{\alpha} V_{\alpha}{}^{aA} = 0 \quad , \tag{24}$$

$$C^{z}\gamma^{\mu}\mathcal{D}_{\mu}\lambda_{A} + \frac{1}{4}C^{iz}\gamma^{\mu\nu}\chi_{Ai}F_{\mu\nu} + \frac{1}{24}C^{iz}\gamma^{\mu\nu\rho}\lambda_{A}H_{\mu\nu\rho i} + \frac{1}{2}C^{iz}\gamma^{\mu}\lambda_{A}V_{\underline{\alpha}i}\partial_{\mu}\varphi^{\underline{\alpha}} + \frac{1}{4}\gamma^{\mu}\gamma^{\nu\rho}\psi_{\mu A}F_{\nu\rho} + \frac{1}{2}C_{AB}\gamma^{\mu}\psi_{\mu}{}^{B} - \frac{1}{2}C_{z}^{-1}C^{iz}C_{AB}\chi_{i}{}^{B} - 2\psi^{a}V_{aA}{}^{\alpha}\xi_{\alpha} = 0 \quad , \quad (25)$$

²A correction to a misprint in [1]: The gaugino field equation in eq. (4.3) should have the additional term: $\sqrt{2}e^{-\varphi/\sqrt{2}}\psi^a V_{\alpha aA}\tilde{\xi}^{\alpha \hat{I}}$, in notation of [1].

where T_z are the generators of the algebra in the adjoint representation of the gauge group labelled by $_z$ and the summation over $_z$ is always understood. All the terms which involve the gravitino field $\psi_{\mu}{}^A$ result from supercovariantizations.

The bosonic field equations, up to fermionic bilinears, are

$$H^+_{\mu\nu\rho} = 0 \quad , \tag{26}$$

$$H^{-i}_{\mu\nu\rho} = 0 \quad , \tag{27}$$

$$R_{\mu\nu} = \frac{1}{4} G_{IJ} H_{\mu\rho\sigma}{}^{I} H_{\nu}{}^{\rho\sigma J} + g_{\underline{\alpha}\underline{\beta}} \partial_{\mu}\varphi^{\underline{\alpha}} \partial_{\nu}\varphi^{\underline{\beta}} + 4g_{\alpha\beta} \left(\mathcal{D}_{\mu}\phi^{\alpha}\right) \left(\mathcal{D}_{\nu}\phi^{\beta}\right) + 2C^{z} \operatorname{tr}_{z} \left(F_{\mu}{}^{\rho} F_{\nu\rho} - \frac{1}{8} g_{\mu\nu} F_{\rho\sigma}{}^{2}\right) + \frac{1}{4} g_{\mu\nu} C_{z}^{-1} \operatorname{tr}_{z} \left(C^{AB} C_{AB}\right) , \qquad (28)$$

$$e^{-1}D_{\mu}\left(eg^{\mu\nu}\partial_{\nu}\varphi^{\underline{\alpha}}\right) + \Gamma_{\underline{\beta\gamma}} \frac{\alpha}{2}\partial_{\mu}\varphi^{\underline{\beta}}\partial^{\mu}\varphi^{\underline{\gamma}} - \frac{1}{2}V_{i}^{\underline{\alpha}}C^{iz} \operatorname{tr}_{z}F_{\mu\nu}F^{\mu\nu} - \frac{1}{6}V_{i}^{\underline{\alpha}}H^{i}_{\mu\nu\rho}H^{\mu\nu\rho} - V_{i}^{\underline{\alpha}}C_{z}^{-2}C^{iz} \operatorname{tr}_{z}\left(C^{AB}C_{AB}\right) = 0 \quad ,$$

$$(29)$$

$$e^{-1}\mathcal{D}_{\mu}\left(eg^{\mu\nu}\mathcal{D}_{\nu}\phi^{\alpha}\right) + \Gamma_{\beta\gamma}{}^{\alpha} \left(\mathcal{D}_{\mu}\phi^{\beta}\right)\left(\mathcal{D}^{\mu}\phi^{\gamma}\right) - C_{z}^{-1}J^{\alpha}{}_{\beta AB} \operatorname{tr}_{z}\left(C^{AB}\xi^{\beta}\right) = 0 \quad , \quad (30)$$

$$\mathcal{D}_{\nu}\left(eC^{z}F^{\mu\nu}\right) + \frac{1}{2}e\left(C^{z}H_{-}^{\mu\rho\sigma} + C^{iz}H_{+}^{\mu\rho\sigma}{}_{i}\right)F_{\rho\sigma} - 2e\xi_{\alpha}\mathcal{D}_{\mu}\phi^{\alpha} = 0.$$
(31)

From experience with the $n_T = 1$ case, we expect that all the higher order fermion terms, except those which involve only matter fermions, can be determined by supercovariantization of the field strengths and covariant derivatives. In the interesting case of (anti) self-duality equations (26) and (27), we can actually determine them exactly. We find

$$\widehat{H}^{+}_{\mu\nu\rho} = \frac{1}{4} \left(\overline{\chi}^{i} \gamma_{\mu\nu\rho} \chi_{i} \right) + \frac{1}{2} \left(\overline{\psi}^{a} \gamma_{\mu\nu\rho} \psi_{a} \right) \quad , \tag{32}$$

$$\widehat{H}_{\mu\nu\rho}^{-i} = -\frac{1}{2} C^{iz} \operatorname{tr}_z \left(\overline{\lambda} \gamma_{\mu\nu\rho} \lambda \right) \quad , \tag{33}$$

where the (anti) self-dual field strengths are defined as the suitable projections of the supercovariantized field strength

$$\widehat{H}_{\mu\nu\rho}{}^{I} \equiv 3\partial_{\left[\mu}B_{\nu\rho\right]}{}^{I} + 3C^{Iz} \operatorname{tr}_{z} \left(F_{\left[\mu\nu\right.}A_{\rho\right]} - \frac{1}{3}\left[A_{\left[\mu\right.},A_{\nu}\right]A_{\rho}\right]\right) + 3\left(\overline{\psi}_{\left[\mu\right.}\gamma_{\nu}\psi_{\rho}\right]\right)L^{I} - 3\left(\overline{\psi}_{\left[\mu\right.}\gamma_{\nu\rho}\right]\chi^{i}\right)L_{i}^{I} \quad .$$

$$(34)$$

The gauge coupling constant is suppressed in the definition of the Chern-Simons form, for simplicity in notation. Despite the fact that the field equations given above are up to higher order fermionic terms, interestingly enough one can determine the full supersymmetry transformation rules. Using the previous works [1, 4, 5] as a guideline, we find the following generalized result:

$$\delta e_{\mu}{}^{m} = (\overline{\epsilon}\gamma^{m}\psi_{\mu}) \quad , \tag{35}$$

$$\delta\psi_{\mu} = \mathcal{D}_{\mu}(\widehat{\omega})\epsilon + \frac{1}{48}\gamma^{\rho\sigma\tau}\gamma_{\mu}\epsilon\widehat{H}_{\rho\sigma\tau} - (\delta\phi^{\alpha})(A_{\alpha}\psi_{\mu}) - \frac{1}{16}\gamma_{\mu}\chi^{i}(\overline{\epsilon}\chi_{i}) - \frac{3}{16}\gamma_{\nu}\chi^{i}(\overline{\epsilon}\gamma_{\mu\nu}\chi_{i}) + \frac{1}{32}\gamma_{\mu\nu\rho}\chi^{i}(\overline{\epsilon}\gamma^{\nu\rho}\chi_{i}) - \frac{9}{8}C^{z} \operatorname{tr}_{z}\left[\lambda(\overline{\epsilon}\gamma_{\mu}\lambda)\right] + \frac{1}{8}C^{z} \operatorname{tr}_{z}\left[\gamma_{\mu\nu}\lambda(\overline{\epsilon}\gamma^{\nu}\lambda)\right] - \frac{1}{16}C^{z} \operatorname{tr}_{z}\left[\gamma_{\rho\sigma}\lambda(\overline{\epsilon}\gamma_{\mu}^{\rho\sigma}\lambda)\right] + \frac{1}{16}\gamma^{\nu\rho}\epsilon\left(\overline{\psi}^{a}\gamma_{\mu\nu\rho}\psi_{a}\right) , \qquad (36)$$

$$\delta B_{\mu\nu}{}^{I} = 2C^{Iz} \operatorname{tr}_{z} \left(A_{[\mu} \delta A_{\nu]} \right) - 2 \left(\overline{\epsilon} \gamma_{[\mu} \psi_{\nu]} \right) L^{I} + \left(\overline{\epsilon} \gamma_{\mu\nu} \chi^{i} \right) L_{i}^{I} \quad , \tag{37}$$

$$\delta\varphi^{\underline{\alpha}} = (\overline{\epsilon}\chi^i) \, V_i^{\underline{\alpha}} \quad , \tag{38}$$

$$\delta\chi^{i} = \frac{1}{2}\gamma^{\mu}\epsilon\widehat{D}_{\mu}\varphi^{\underline{\alpha}}V_{\underline{\alpha}}{}^{i} - \frac{1}{24}\gamma^{\mu\nu\rho}\epsilon\widehat{H}_{\mu\nu\rho}{}^{i} - (\delta\phi^{\alpha})(A_{\alpha}\chi^{i}) - (\delta\varphi^{\underline{\alpha}})A_{\underline{\alpha}}{}^{ij}\chi_{j} + \frac{1}{2}C^{iz}\operatorname{tr}_{z}\left[\gamma^{\mu}\lambda\left(\overline{\epsilon}\gamma_{\mu}\lambda\right)\right] , \qquad (39)$$

$$\delta A_{\mu} = (\overline{\epsilon} \gamma_{\mu} \lambda) \quad , \tag{40}$$

$$\delta\lambda_A = -\frac{1}{4}\gamma^{\mu\nu}\epsilon_A \hat{F}_{\mu\nu} - (\delta\phi^{\alpha})(A_{\alpha}\lambda_A) - C_z^{-1}C^{iz}\left(\overline{\chi}_{i(A}\lambda_B)\right)\epsilon^B - \frac{1}{2}C_z^{-1}C_{AB}\epsilon^B \quad , \quad (41)$$

$$\delta\phi^{\alpha} = V_{aA}{}^{\alpha} \left(\overline{\epsilon}^{A}\psi^{a}\right) \quad , \tag{42}$$

$$\delta\psi^a = \gamma^\mu \epsilon_A \widehat{\mathcal{D}}_\mu \phi^\alpha \ V_\alpha{}^{aA} - (\delta\phi^\alpha) (A_\alpha \psi)^a \quad . \tag{43}$$

Here all the *hatted* field strengths and covariant derivatives are supercovariantizations of the *non-hatted* ones. Recalling that supercovariantization is achieved by replacing the parameter ϵ^A in the supersymmetry transformation by $(-\psi_{\mu}{}^A)$, in addition to (34), one finds

$$\widehat{F}_{\mu\nu} \equiv F_{\mu\nu} - 2\left(\overline{\psi}_{\left[\mu\gamma\nu\right]}\lambda\right) ,$$

$$\widehat{D}_{\mu\nu} = \partial_{\mu\nu} \left(\frac{\alpha}{2} - V_{\mu\nu} \left(\overline{\psi}_{\mu\nu}\gamma^{i}\right)\right)$$
(44)

The supercovariant derivative occurring in (36) is defined by

$$\mathcal{D}_{\mu}(\hat{\omega})\epsilon^{A} \equiv \left[\partial_{\mu}\varepsilon^{AB} + \frac{1}{4}\hat{\omega}_{\mu}{}^{rs}\gamma_{rs}\varepsilon^{AB} + g'A_{\mu}{}^{AB} + (\mathcal{D}_{\mu}\phi^{\alpha})A_{\alpha}{}^{AB}\right]\epsilon_{B} , \qquad (46)$$

where the supercovariantized spin connection is given by

$$\widehat{\omega}_{mrs} = \frac{1}{2} (\widehat{C}_{mrs} - \widehat{C}_{msr} + \widehat{C}_{srm}) , \qquad (47)$$

and \hat{C} is supercovariantized Ricci's rotation coefficient:

$$\widehat{C}_{\mu\nu m} \equiv \partial_{\mu} e_{\nu m} - \partial_{\nu} e_{\mu m} - (\bar{\psi}_{\mu} \gamma_m \psi_{\nu}) \quad .$$
(48)

Note that the gravitational constant has always been suppressed. It can easily be reintroduced by assigning mass dimension 1 to bosons, 3/2 to fermions and -1/2 to ϵ . Since the supersymmetry transformations determined here reduce to the *full* supersymmetry transformations given in [1] for the $n_T = 1$ case, we conjecture that they are the full supersymmetry transformations for all n_T .

The supersymmetry transformation rules presented above form a closed algebra with the composite parameters l_{mn} for the Lorentz transformation, ϵ_3 for the supersymmetry transformation, and ξ^{μ} for the general coordinate transformation given by

$$\begin{bmatrix} \delta(\epsilon_{1}), \delta(\epsilon_{2}) \end{bmatrix} e_{\mu m} = \xi^{\nu} \partial_{\nu} e_{\mu m} + (\partial_{\mu} \xi^{\nu}) e_{\nu m} + (\overline{\epsilon}_{3} \gamma_{m} \psi_{\mu}) + l_{m}^{n} e_{\mu n} ,$$

$$\xi^{\mu} \equiv (\overline{\epsilon}_{2} \gamma^{\mu} \epsilon_{1}) ,$$

$$\epsilon_{3}^{A} \equiv -\xi^{\mu} \psi_{\mu} + \begin{bmatrix} V_{bB}{}^{\alpha} (\overline{\epsilon}_{2}^{B} \psi^{b}) (A_{\alpha} \epsilon_{1})^{A} - (1 \leftrightarrow 2) \end{bmatrix} ,$$

$$l_{mn} \equiv \xi^{\mu} \widehat{\omega}_{\mu m n} + \begin{bmatrix} \frac{1}{24} \left(\overline{\epsilon}_{2} \gamma_{[m} \gamma^{\rho \sigma \tau} \gamma_{n]} \epsilon_{1} \right) H_{\rho \sigma \tau}^{-I} L_{I} - \frac{1}{8} \xi^{\mu} \left(\overline{\psi}^{a} \gamma_{\mu m n} \psi_{a} \right) \right)$$

$$- \frac{1}{4} \left(\overline{\epsilon}_{2} \gamma_{m n} \chi^{i} \right) (\overline{\epsilon}_{1} \chi_{i}) - \frac{1}{8} \left(\overline{\epsilon}_{2} \gamma_{[m} \rho \chi^{i} \right) \left(\overline{\epsilon}_{1} \gamma_{n] \rho} \chi_{i} \right)$$

$$- C^{z} \operatorname{tr}_{z} \left(\overline{\epsilon}_{2} \gamma_{[m} \lambda \right) \left(\overline{\epsilon}_{1} \gamma_{n]} \lambda \right) \end{bmatrix} - (1 \leftrightarrow 2) .$$
(49)

We now outline the steps we have followed to derive the equations of motion and supersymmetry transformations.

(1) We first parametrize the transformation rules and the (anti) self-duality equations, as dictated by the symmetries of the theory, the existing partial results for the multi-tensor multiplet couplings [4], and the full results for the $n_T = 1$ case [1]. At first, we assume that all matter fields are inert under the *local* Yang-Mills gauge transformations, as well as the *local* Sp(1) gauge transformations. It is convenient to perform the gauging process, after the ungauged results are obtained.

Note that factors of C_z^{-1} occur in a number of places in the equations of motion and the transformations rules. While these factors may seem unusual, it is easy to understand their origin, which has to do with the fact that we have parametrized the field equations for the gauge fermion and the Yang-Mills field (excluding the hypermatter contributions which are presented here) in such a way that they agree with those of ref. [5]. For example, once the C^z factor is introduced in (25), it is clear that the closure of the supersymmetry transformations (41) must include the $C^{-1}C^i\overline{\chi_i}\lambda$ -term in order to produce the $C^i\lambda H$ -term in the χ^i -field equation, upon the variation of χ^i . In comparing this term with the $n_T =$ 1 case [1], it is useful to note the identity (88) provided in the Appendix.

(2) Next, we require the closure of the supersymmetry transformations on the bosons. Normally, closure on the bosons does not require any field equations. However, as it has been known for some time [12], in the case of self-duality conditions which serve as equations of motion, the closure of supersymmetry algebra on the bosons does require the (anti) self-duality equations. Completing the closure calculation on the bosons, we are able to fix all the supersymmetry transformations, including the (fermion)²-terms, as well as the (anti) self-duality equations (32) and (33). In this context, (90) given in the Appendix is useful in establishing the closure on $e_{\mu}{}^{m}$ and $B_{\mu\nu}$.

(3) Next, we obtain the gravitino equation by supersymmetric variation of (32), and the χ -field equation by supersymmetry variation of (33).

(4) Varying the gravitino equation under supersymmetry, we obtain the Einstein equation (28). In doing so, (82) given in the Appendix is useful in handling the H^2 -terms. Note the occurrence of the trace $g_{\mu\nu}F^2$ -term in (28), which is absent in the case of $n_T = 1$, due to the use of dilaton equation of motion. Since, here we have multi dilatons, the trace term can no longer be absorbed into the the dilaton equation of motion.

(5) Varying the χ -field equation (23), we obtain the field equation (29) for the generalized dilatons $\varphi^{\underline{\alpha}}$ [5].

(6) Next, we obtain the hyperino field equation (24), by the requirement of the closure of two supersymmetries acting on the hyperino ψ^a . Varying this equation under supersymmetry, we obtain the field equation (30) for the hyperscalars ϕ^{α} .

(7) Finally the gaugino field equation (25) is also obtained by the closure of two supersymmetries on the gaugino λ . The variation of this equation under supersymmetry in turn yields the Yang-Mills field equation (31).

(8) Having determined the ungauged matter couplings and supersymmetry transformation rules, we now turn on the gauge coupling constants g, g', thereby gauging the group $Sp(n_H) \times Sp(1)$. To do this, we follow the following steps:

(a) We gauge covariantize the relevant derivatives in the supersymmetry transformation rules, as well as the field equations, according to the rules described in section 2.

(b) Next, we find that the closure calculation will require the introduction of only one new term to the transformation rules, namely the C^{AB} -dependent term in the gaugino transformation rule (41). This can be seen by examining the closure of supersymmetry on the gravitino, and by varying the new gauge coupling dependent terms in $D_{\mu}\epsilon$. Furthermore, we learn that we need to add the C^{AB} -dependent term in the gravitino field equation.

(c) Having determined the fact that the gaugino transformation rule is modified by the C^{AB} -dependent term that is proportional to the gauge coupling constant, we then examine systematically the effect of this new variation in all the closure calculations on the *fermions*. Thus we determine all the new, gauge coupling constant dependent modifications of the fermionic field equations, as given in [1].

(d) Finally, we vary the new terms in the fermionic equations of motion under the full transformation rules (old and new), as well as the old terms in the fermionic equations of motion under the new, gauge coupling constant dependent gaugino transformation rule, thereby obtaining all the modifications, up to the fermionic bilinear terms in the bosonic field equations of motion.

An important observation to be made here is that the gauging of the $SO(n_T, 1)$ or any of its subgroups does not work, because it is not known how, and it may as well be impossible, to write down a gauge covariant field strength for antisymmetric tensor fields.

4. Invariant Lagrangian for the Case of $n_T = 1$

When the number of self-dual tensor multiplets differs from that of anti-self-dual tensor multiplets, the system lacks invariant lagrangian, because we can not write down the kinetic term for purely (anti) self-dual third-rank tensor. However, we do have an invariant lagrangian for the case of $n_T = 1$ as in refs. [1, 4]. Since this particular case is also of another importance with the geometry SO(1,1) with no isotropy group, we give the details of the system.

First of all, the coset space is now reduced to a semi-simple group SO(1,1), and the coset representatives L_I^i and L_I can be parametrized as

$$\begin{pmatrix} L_0 & L_0^{(1)} \\ L_1 & L_1^{(1)} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} , \quad \begin{pmatrix} L^0 & L^1 \\ L_{(1)}^0 & L_{(1)}^1 \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}$$
(50)

for a rescaled field $\theta \equiv \varphi^{1}/\sqrt{2}$. Accordingly, we have

$$\eta_{11} = +1 , \quad \eta_{00} = -1 , \quad \eta_{10} = 0 ,$$
 (51)

$$V_{\underline{1}}^{(1)}\partial_{\mu}\varphi^{\underline{1}} = L^{I}\partial_{\mu}L_{I}^{(1)} = \partial_{\mu}\theta \quad .$$

$$\tag{52}$$

Following ref. [4], we define

$$a_{\mu\nu} \equiv \frac{1}{2} \left(B_{\mu\nu}{}^{0} - B_{\mu\nu}{}^{1} \right) , \quad b_{\mu\nu} \equiv \frac{1}{2} \left(B_{\mu\nu}{}^{0} + B_{\mu\nu}{}^{1} \right) , B_{\mu\nu}{}^{0} = b_{\mu\nu} + a_{\mu\nu} , \quad B_{\mu\nu}{}^{1} = b_{\mu\nu} - a_{\mu\nu} .$$
(53)

We define the field strengths of $a_{\mu\nu}$ and $b_{\mu\nu}$ as

$$f_{\mu\nu\rho} \equiv 3\partial_{[\mu}a_{\nu\rho]} + 3\tilde{v}^{z} \operatorname{tr}_{z} \left(F_{[\mu\nu}A_{\nu]} - \frac{2}{3}A_{[\mu}A_{\nu}A_{\rho]}\right) ,$$

$$g_{\mu\nu\rho} \equiv 3\partial_{[\mu}b_{\nu\rho]} + 3v^{z} \operatorname{tr}_{z} \left(F_{[\mu\nu}A_{\nu]} - \frac{2}{3}A_{[\mu}A_{\nu}A_{\rho]}\right) , \qquad (54)$$

where two *constants* v^z and \tilde{v}^z are defined by

$$v^{z} \equiv \frac{1}{2} \left(C^{0z} + C^{1z} \right) , \quad \tilde{v}^{z} \equiv \frac{1}{2} \left(C^{0z} - C^{1z} \right) .$$
 (55)

Accordingly we have $C^{(1)z} = v^z e^{\theta} - \tilde{v}^z e^{-\theta}$, and

$$C^z = v^z e^\theta + \tilde{v}^z e^{-\theta} . ag{56}$$

After some manipulations, we get

$$H_{\mu\nu\rho} \equiv H_{\mu\nu\rho}{}^{I}L_{I} = e^{-\theta}f_{\mu\nu\rho} + e^{+\theta}g_{\mu\nu\rho} = 2e^{+\theta}g_{\mu\nu\rho}^{-} , H_{\mu\nu\rho}{}^{(1)} \equiv H_{\mu\nu\rho}{}^{I}L_{I}{}^{(1)} = e^{+\theta}g_{\mu\nu\rho} - e^{-\theta}f_{\mu\nu\rho} = 2e^{+\theta}g_{\mu\nu\rho}^{+} .$$
(57)

Due to the (anti)self-dualities of $B_{\mu\nu}{}^{I}$, we have $f_{\mu\nu\rho} = -e^{2\theta}\tilde{g}_{\mu\nu\rho}$, $\tilde{f}_{\mu\nu\rho} = -e^{2\theta}g_{\mu\nu\rho}$, where $\tilde{g}^{mnr} \equiv (1/6)\epsilon^{mnrstu}g_{stu}$ and *idem* for \tilde{f}^{mnr} .

We can obtain the field equations of this system from our general case, substituting above relations. The gravitino, dilatino, gaugino, and hyperino field equations thus obtained are

$$\gamma^{\mu\nu\rho}D_{\nu}\psi_{\rho}{}^{A} + \frac{1}{12}e^{\theta}\gamma_{[\mu}\gamma^{\rho\sigma\tau}\gamma_{\nu]}\psi^{\nu A}g_{\rho\sigma\tau} - \frac{1}{2}\gamma^{\nu}\gamma_{\mu}\chi^{A}\partial_{\nu}\theta - 2V_{\alpha}{}^{aA}\gamma^{\nu}\gamma_{\mu}\psi_{a}D_{\nu}\phi^{\alpha} + \frac{1}{2}C^{z}\gamma^{\rho\sigma}\gamma_{\mu}\operatorname{tr}_{z}\left(\lambda^{A}F_{\rho\sigma}\right) - \frac{1}{12}e^{\theta}\gamma^{\rho\sigma\tau}\gamma_{\mu}\chi^{A}g_{\rho\sigma\tau} - \operatorname{tr}_{z}\left(C^{AB}\gamma^{\mu}\lambda_{B}\right) = 0 \quad , \tag{58}$$

$$\gamma^{\mu} \left(D_{\mu} \chi - \frac{1}{2} \gamma^{\nu} \psi_{\mu} \partial_{\nu} \theta + \frac{1}{12} e^{\theta} \gamma^{\rho \sigma \tau} \psi_{\mu} g_{\rho \sigma \tau} \right) - C_{z}^{-1} \left(v^{z} e^{\theta} - \tilde{v}^{z} e^{-\theta} \right) \operatorname{tr}_{z} \left(C^{AB} \lambda_{B} \right) - \frac{1}{12} e^{\theta} \gamma^{\rho \sigma \tau} \chi g_{\rho \sigma \tau} - \frac{1}{2} \left(v^{z} e^{\theta} - \tilde{v}^{z} e^{-\theta} \right) \operatorname{tr}_{z} \left(\gamma^{\rho \sigma} \lambda F_{\rho \sigma} \right) = 0 \quad ,$$
(59)

$$C^{z}\gamma^{\mu}\left(D_{\mu}\lambda_{A}+\frac{1}{4}\gamma^{\rho\sigma}\psi_{\mu A}F_{\rho\sigma}\right)+\frac{1}{4}\gamma^{\rho\sigma}\chi_{A}(v^{z}e^{\theta}-\tilde{v}^{z}e^{-\theta})F_{\rho\sigma}$$

$$+\frac{1}{2}\left(v^{z}e^{\theta}-\tilde{v}^{z}e^{-\theta}\right)\gamma^{\mu}\lambda_{A}\partial_{\mu}\theta+\frac{1}{12}\left(v^{z}e^{\theta}-\tilde{v}^{z}e^{-\theta}\right)e^{\theta}\gamma_{\rho\sigma\tau}\lambda_{A}g_{\rho\sigma\tau}+\frac{1}{2}C_{AB}\gamma^{\mu}\psi_{\mu}{}^{B}-\frac{1}{2}C_{z}^{-1}\left(v^{z}e^{\theta}-\tilde{v}^{z}e^{-\theta}\right)C_{AB}\chi^{B}-2\psi^{a}V_{aA}{}^{\alpha}\xi_{\alpha}=0 \quad , \tag{60}$$

$$\gamma^{\mu} \left(D_{\mu} \psi^{a} - V_{\alpha}{}^{aA} \gamma^{\nu} \psi_{\mu A} D_{\nu} \phi^{\alpha} \right) + \frac{1}{12} \gamma^{\rho \sigma \tau} \psi^{a} e^{\theta} g_{\rho \sigma \tau} - 2\lambda_{A} \xi^{\alpha} V_{\alpha}{}^{aA} = 0 \quad , \tag{61}$$

where $\chi \equiv \chi^{(1)}$, and we recall that $C^z = v^z e^{\theta} + \tilde{v}^z e^{-\theta}$.

In a similar fashion, we can get all the bosonic field equations as

$$\frac{1}{4} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) - \frac{1}{12} e^{2\theta} \left(3g_{\mu\rho\sigma} g_{\nu\rho\sigma} - \frac{1}{2} g_{\mu\nu} g_{\rho\sigma\tau}^2 \right) - \frac{1}{4} \left(\partial_{\mu} \theta \right) (\partial_{\nu} \theta) + \frac{1}{8} g_{\mu\nu} (\partial_{\rho} \theta)^2 - g_{\alpha\beta} (\partial_{\mu} \phi^{\alpha}) (\partial_{\nu} \phi^{\beta}) + \frac{1}{2} g_{\mu\nu} g_{\gamma\delta} g^{\rho\sigma} (\partial_{\rho} \phi^{\gamma}) (\partial_{\sigma} \phi^{\delta}) + \frac{1}{8} g_{\mu\nu} C_z^{-1} \operatorname{tr}_z \left(C^{AB} C_{AB} \right) - \frac{1}{4} C^z \operatorname{tr}_z \left[2F_{\mu}{}^{\rho} F_{\nu\rho} - \frac{1}{2} g_{\mu\nu} (F_{\rho\sigma})^2 \right] = 0 \quad ,$$

$$\frac{1}{2}e^{-1}\partial_{\mu}\left(eg^{\mu\nu}\partial_{\nu}\theta\right) - \frac{1}{4}\left(v^{z}e^{\theta} - \tilde{v}^{z}e^{-\theta}\right)F_{\mu\nu}^{2} - \frac{1}{6}e^{2\theta}g_{\rho\sigma\tau}^{2} \\ -\frac{1}{4}C_{z}^{-2}\left(v^{z}e^{\theta} - \tilde{v}^{z}e^{-\theta}\right)\operatorname{tr}_{z}\left(C^{AB}C_{AB}\right) = 0 \quad ,$$

$$(62)$$

$$\frac{1}{2}D_{\nu}\left(eC^{z}F^{\mu\nu}\right) + \frac{1}{2}e\left(v^{z}e^{2\theta}g^{\mu\rho\sigma}F_{\rho\sigma} - \tilde{v}^{z}\tilde{g}^{\mu\rho\sigma}F_{\rho\sigma}\right) - e\xi_{\alpha}D_{\mu}\phi^{\alpha} = 0 , \qquad (63)$$

$$e^{-1}\partial_{\mu}\left(eg^{\mu\nu}\partial_{\nu}\phi^{\alpha}\right) + g^{\mu\nu}\Gamma_{\beta\gamma}{}^{\alpha}(\partial_{\mu}\phi^{\beta})(\partial_{\nu}\phi^{\gamma}) - C_{z}^{-1}J^{\alpha}{}_{\beta AB}\operatorname{tr}_{z}\left(C^{AB}\xi^{\beta}\right) = 0 \quad . \tag{64}$$

Now our antisymmetric tensor field equation is to be of the second order as a combination of the self-dual and anti-self-dual parts of $g_{\mu\nu\rho}$: To be more specific, we add the (anti)selfduality conditions $H^{-}_{\mu\nu\rho}{}^{I}L_{I}{}^{(1)} + \cdots = 0$ and $H^{+}_{\mu\nu\rho}{}^{I}L_{I} + \cdots = 0$ in (32) and (33), to get $e^{2\theta}g_{\mu\nu\rho} = -\tilde{f}_{\mu\nu\rho} + \cdots$. We next take its divergence to get

$$\frac{1}{2} D_{\mu} \left(e e^{2\theta} g^{\mu\nu\rho} \right) = -\frac{1}{8} \tilde{v}^{z} \epsilon^{\nu\rho\mu\sigma\tau\omega} \operatorname{tr}_{z} (F_{\mu\sigma}F_{\tau\omega}) \quad .$$
(65)

Similarly the supersymmetry transformation rule is also derived as

$$\begin{split} \delta e_{\mu}{}^{m} &= + (\bar{\epsilon}\gamma^{m}\psi_{\mu}) \quad , \\ \delta \psi_{\mu} &= + D_{\mu}(\hat{\omega}, A_{\underline{\alpha}})\epsilon + \frac{1}{24} e^{\theta}\gamma^{\rho\sigma\tau}\gamma_{\mu}\epsilon g_{\rho\sigma\tau} - (\delta\phi^{\alpha})(A_{\alpha}\psi_{\mu}) \\ &\quad - \frac{1}{16} \gamma_{\mu}\chi(\overline{\epsilon}\chi) - \frac{3}{16} \gamma_{\nu}\chi(\overline{\epsilon}\gamma_{\mu\nu}\chi) + \frac{1}{32} \gamma_{\mu\nu\rho}\chi(\overline{\epsilon}\gamma^{\nu\rho}\chi) + \frac{1}{16} \gamma^{\nu\rho}\epsilon(\overline{\psi}^{a}\gamma_{\mu\nu\rho}\psi_{a}) \\ &\quad - \frac{1}{16} C_{z} \operatorname{tr}_{z} \left[18\lambda(\overline{\epsilon}\gamma_{\mu}\lambda) - 2\gamma_{\mu\nu}\lambda(\overline{\epsilon}\gamma^{\nu}\lambda) + \gamma_{\rho\sigma}\lambda(\overline{\epsilon}\gamma_{\mu}^{\rho\sigma}\lambda) \right] \quad , \end{split}$$

$$\delta b_{\mu\nu} = +2v^{z} \operatorname{tr}_{z} \left(A_{[\mu} \delta A_{\nu]} \right) - e^{-\theta} (\overline{\epsilon} \gamma_{[\mu} \psi_{\nu]}) + \frac{1}{2} e^{-\theta} (\overline{\epsilon} \gamma_{\mu\nu} \chi) , \quad \delta \theta = +(\overline{\epsilon} \chi) ,$$

$$\delta \chi = +\frac{1}{2} \gamma^{\mu} \epsilon \partial_{\mu} \theta - (\delta \phi^{\alpha}) (A_{\alpha} \chi) - \frac{1}{12} e^{\theta} \gamma^{\mu\nu\rho} \epsilon g_{\mu\nu\rho} + \frac{1}{2} \left(v^{z} e^{\theta} - \widetilde{v}^{z} e^{-\theta} \right) \operatorname{tr}_{z} \left[\gamma^{\mu} \lambda (\overline{\epsilon} \gamma_{\mu} \lambda) \right] ,$$

$$\delta A_{\mu} = +(\overline{\epsilon}\gamma_{\mu}\lambda) ,$$

$$\delta \lambda_{A} = -\frac{1}{4}\gamma^{\mu\nu}\epsilon_{A}F_{\mu\nu} - (\delta\phi^{\alpha})(A_{\alpha}\lambda_{A}) - C_{z}^{-1}\left(v^{z}e^{\theta} - \tilde{v}^{z}e^{-\theta}\right)\left(\overline{\chi}_{(A}\lambda_{B)}\right)\epsilon^{B} - \frac{1}{2}C_{z}^{-1}C_{AB}\epsilon^{B} ,$$

$$\delta\phi^{\alpha} = +V_{A}^{\alpha}(\overline{\epsilon}^{A}\psi^{a})$$

$$\delta \psi^{a} = + V_{aA} \left(\epsilon^{a} \psi^{a} \right)^{a},$$

$$\delta \psi^{a} = + V_{\alpha}^{aA} \gamma^{\mu} \epsilon_{A} \widehat{D}_{\mu} \phi^{\alpha} - (\delta \phi^{\alpha}) (A_{\alpha} \psi)^{a} .$$
(66)

Note that in the absence of gauging, the function C^{AB} vanishes and singular behaviour in the couplings arises in the energy-momentum tensor for the Yang-Mills field in (26) and in the Yang-Mills equation (28). Furthermore, the $\lambda^2 \epsilon$ terms in the supersymmetry variation of the gravitino vanish and the $\chi \lambda \epsilon$ terms in the supersymmetry variation of the gaugino diverge, at the critical point where $C^z = v^z e^{\theta} + \tilde{v}^z e^{-\theta}$ vanishes. When the gauging of $Sp(n_H) \times Sp(1)$ is switched on, the divergent C_z^{-1} factors arise in χ , λ , Einstein, dilaton and hypermatter field equations, and the supersymmetry variation of the gaugino picks up another singular contribution.

When $\tilde{v}^z = 0$, this result agrees with ref. [4] as far as the supergravity and Yang-Mills multiplets are concerned, and also with ref. [1] with hypermultiplets. When $v^z = 0$, this result coincides with the system in ref. [14]. In the important cases of $v^z = 0$ and $\tilde{v}^z = 0$, this is reduced to the usual exponential factors. It is worthwhile to mention that the $\chi\lambda$ -terms in the λ -transformation rules have apparently different coefficients compared with [1], via (88). This is attributed to the fact that our gaugino λ -field is rescaled by an exponential function of the dilaton θ .

As will be discussed in the next section, since the conservation of the Yang-Mills current is satisfied only for $\eta_{IJ}C^{Iz}C^{Jz'} = 0$, the invariant lagrangian exists only for the two cases $v^z = 0$ or $\tilde{v}^z = 0$ for our $n_T = 1$ system. If we formally try to integrate the field equations for other cases of $v^z \tilde{v}^{z'} \neq 0$, we encounter gauge non-invariant terms in the lagrangian, that invalidate supersymmetry. This is because commutators of two supersymmetries will result in a gauge transformation. Another explicit way to see this is to take the variation of the $b \wedge F \wedge F$ term in the lagrangian under supersymmetry. This produces gauge non-invariant terms proportional to $v^z \tilde{v}^{z'}$ which can not be cancelled. Note also that this feature for the case $v^z \tilde{v}^{z'} \neq 0$ at the classical level does *not* necessarily mean the system is inconsistent, as will be discussed in the next section. In the case of $\tilde{v}^z = 0$, the invariant lagrangian by integrating the field equations is:

$$e^{-1}\mathcal{L}_{n=1}^{\widetilde{\nu}=0} = +\frac{1}{4}R(\omega) - \frac{1}{12}e^{2\theta}g_{\rho\sigma\tau}^{2} - \frac{1}{4}(\partial_{\mu}\theta)^{2} - \frac{1}{2}\left(\overline{\psi}_{\mu}\gamma^{\mu\nu\rho}D_{\nu}\psi_{\rho}\right) - \frac{1}{2}\left(\overline{\chi}\gamma^{\mu}D_{\mu}\chi\right) - \frac{1}{4}v^{z}e^{\theta}\operatorname{tr}_{z}(F_{\mu\nu})^{2} - v^{z}e^{\theta}\operatorname{tr}_{z}\left(\overline{\lambda}\gamma^{\mu}D_{\mu}\lambda\right) - g_{\alpha\beta}g^{\mu\nu}(\partial_{\mu}\phi^{\alpha})(\partial_{\nu}\phi^{\beta}) - \left(\overline{\psi}^{a}\gamma^{\mu}D_{\mu}\psi_{a}\right) + \frac{1}{2}v^{z}e^{\theta}\operatorname{tr}_{z}(\overline{\chi}\gamma^{\mu\nu}\lambda F_{\mu\nu}) + \frac{1}{2}\left(\overline{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\chi\right)\partial_{\nu}\theta + 2\left(\overline{\psi}_{\mu}A\gamma^{\nu}\gamma^{\mu}\psi_{a}\right)V_{\alpha}^{aA}\partial_{\nu}\phi^{\alpha} - \frac{1}{2}v^{z}e^{\theta}\operatorname{tr}_{z}\left(\overline{\psi}_{\mu}\gamma^{\rho\sigma}\gamma^{\mu}\lambda F_{\rho\sigma}\right) - \frac{1}{24}e^{\theta}g_{\mu\nu\rho}\left[\left(\overline{\psi}^{\lambda}\gamma_{[\lambda}\gamma^{\mu\nu\rho}\gamma_{\tau}]\psi^{\tau}\right) - 2\left(\overline{\psi}_{\lambda}\gamma^{\mu\nu\rho}\gamma^{\lambda}\chi\right) + 2\left(\overline{\psi}^{a}\gamma^{\mu\nu\rho}\psi_{a}\right) - (\overline{\chi}\gamma^{\mu\nu\rho}\chi) + 2v^{z}e^{\theta}\operatorname{tr}_{z}\left(\overline{\lambda}\gamma^{\mu\nu\rho}\lambda\right)\right] - \frac{1}{4}v_{z}^{-1}e^{-\theta}\operatorname{tr}_{z}\left(C^{AB}C_{AB}\right) - 4\operatorname{tr}_{z}\left(\overline{\psi}_{a}\lambda_{A}V_{\alpha}^{aA}\xi^{\alpha}\right) + \frac{1}{2}\operatorname{tr}_{z}\left(\overline{\psi}_{\mu}^{A}\gamma^{\mu}\lambda^{B}C_{AB}\right) - \operatorname{tr}_{z}\left(\overline{\chi}_{A}\lambda_{B}C^{AB}\right) \quad .$$
(67)

This lagrangian with the transformation rule (66) with $\tilde{v}^z = 0$ coincides with the system in refs. [4, 1] with the Chern-Simon modification in the $b_{\mu\nu}$ -transformation and its field strength. In 10D, this corresponds to the system in ref. [16].

In the case $v^z = 0$, the invariant lagrangian is

$$e^{-1}\mathcal{L}_{n=1}^{\nu=0} = +\frac{1}{4}R(\omega) - \frac{1}{12}e^{2\theta}g_{\rho\sigma\tau}^{2} - \frac{1}{4}(\partial_{\mu}\theta)^{2} - \frac{1}{2}\left(\overline{\psi}_{\mu}\gamma^{\mu\nu\rho}D_{\nu}\psi_{\rho}\right) - \frac{1}{2}\left(\overline{\chi}\gamma^{\mu}D_{\mu}\chi\right) - \frac{1}{4}\widetilde{v}^{z}e^{-\theta}\operatorname{tr}_{z}(F_{\mu\nu})^{2} - \widetilde{v}^{z}e^{-\theta}\operatorname{tr}_{z}\left(\overline{\lambda}\gamma^{\mu}D_{\mu}\lambda\right) - g_{\alpha\beta}g^{\mu\nu}(\partial_{\mu}\phi^{\alpha})(\partial_{\nu}\phi^{\beta}) - \left(\overline{\psi}^{a}\gamma^{\mu}D_{\mu}\psi_{a}\right) - \frac{1}{2}\widetilde{v}^{z}e^{-\theta}\operatorname{tr}_{z}(\overline{\chi}\gamma^{\mu\nu}\lambda F_{\mu\nu}) + \frac{1}{2}\left(\overline{\psi}_{\mu}\gamma^{\nu}\gamma^{\mu}\chi\right)\partial_{\nu}\theta + 2\left(\overline{\psi}_{\mu A}\gamma^{\nu}\gamma^{\mu}\psi_{a}\right)V_{\alpha}^{aA}\partial_{\nu}\phi^{\alpha} - \frac{1}{2}\widetilde{v}^{z}e^{-\theta}\operatorname{tr}_{z}\left(\overline{\psi}_{\mu}\gamma^{\rho\sigma}\gamma^{\mu}\lambda F_{\rho\sigma}\right) + \frac{1}{8}\widetilde{v}^{z}e^{-1}\epsilon^{\mu\nu\rho\sigma\tau\omega}b_{\mu\nu}\operatorname{tr}_{z}(F_{\rho\sigma}F_{\tau\omega}) - \frac{1}{24}e^{\theta}g_{\mu\nu\rho}\left[\left(\overline{\psi}^{\lambda}\gamma_{[\lambda}\gamma^{\mu\nu\rho}\gamma_{\tau]}\psi^{\tau}\right) - 2\left(\overline{\psi}_{\lambda}\gamma^{\mu\nu\rho}\gamma^{\lambda}\chi\right) + 2\left(\overline{\psi}^{a}\gamma^{\mu\nu\rho}\psi_{a}\right) - (\overline{\chi}\gamma^{\mu\nu\rho}\chi) - 2\widetilde{v}^{z}e^{-\theta}\operatorname{tr}_{z}\left(\overline{\lambda}\gamma^{\mu\nu\rho}\lambda\right)\right] - \frac{1}{4}\widetilde{v}_{z}^{-1}e^{-\theta}\operatorname{tr}_{z}\left(C^{AB}C_{AB}\right) - 4\operatorname{tr}_{z}\left(\overline{\psi}_{a}\lambda_{A}V_{\alpha}^{aA}\xi^{\alpha}\right) + \frac{1}{2}\operatorname{tr}_{z}\left(\overline{\psi}_{\mu}^{A}\gamma^{\mu}\lambda^{B}C_{AB}\right) + \operatorname{tr}_{z}\left(\overline{\chi}_{A}\lambda_{B}C^{AB}\right) \quad .$$
(68)

This lagrangian and the transformation with $v^z = 0$ correspond to the system in refs. [14, 4]

with the explicit $b \wedge F \wedge F$ -term in the lagrangian with no modification in the $b_{\mu\nu}$ -transformation rule or its field strength. In 10D, this corresponds to the dual formulation [17].

5. Discussion

In this paper we have constructed the field equations of the combined system of the N = 1 supergravity multiplet, n_T copies of anti-self-dual tensor multiplets with anti-self-dual tensor multiplet, forming the coset space $SO(n_T, 1)/SO(n_T)$, Yang-Mills multiplets, and hypermultiplets forming the coset $Sp(n_H, 1)/Sp(n_H) \otimes Sp(1)$. Furthermore we have gauged the group $Sp(n_H) \times Sp(1)$. These are the most general couplings of the six dimensional supergravity plus matter system to date. The resulting system exhibits some interesting features which we now comment on.

We have already commented on the singular behaviour of the couplings at a special point in moduli space, and the occurrence of new divergent couplings at the same point which proportional to the gauged $Sp(n_H) \times Sp(1)$ coupling constants. Another important feature, which was observed in [6], and which continues to hold in the more general system presented here, is the anomalous behaviour of the gauge couplings. Namely, writing the Yang-Mills equation (31) as

$$\mathcal{D}_{\nu}\left(eC^{z}F^{\mu\nu}\right) = eJ^{\mu} , \qquad (69)$$

we find that

$$\mathcal{D}_{\mu}(eJ^{\mu}) = \frac{1}{16} \epsilon^{\mu\nu\rho\sigma\tau\lambda} \eta_{IJ} C^{Iz} C^{Jz'} F_{\mu\nu} \operatorname{tr}_{z'} (F_{\rho\sigma} F_{\tau\lambda}) \quad .$$
(70)

Perhaps not too surprisingly, the hypermatter contributions to this anomaly equation have all canceled. Setting

$$\eta_{LI} C^{Iz} C^{Jz'} = 0 , \qquad (71)$$

eliminates the anomalous divergence. This corresponds to the familiar $n_T = 1$ case [1] for which a covariant action can be written down. Interestingly, there are two possibilities that have been treated simultaneously here. In notation of section 4, these correspond to the cases of $v^z = 0$ or $\tilde{v}^z = 0$. As discussed in section 4, the case of $\tilde{v}^z = 0$ is the familiar one constructed by the authors a long tome ago [1], and its ten dimensional analog is well-known [15, 16]. The case of $v^z = 0$ corresponds to the dual formulation, constructed also long ago [14], and its ten dimensional analog is the dual formulation of Chamseddine [17].

It should be emphasized that neither the elimination of the anomalous divergence (70) ensure anomaly freedom, nor does its nonvanishing mean that the theory is anomalous. The property of anomaly freedom solely depends on the choice of matter multiplets, and as is

well-known, there are many available anomaly free sets of such multiplets, some of which will be discussed below.

Considering the case of $n_T > 1$, eq. (69) represents the Bose non-symmetric covariant gauge anomaly, as observed in [6]. Its Bose symmetric covariant version, as well as the associated local supersymmetry anomaly can be determined by considerations of Wess-Zumino anomaly consistency conditions [6]. Although a covariant action can not be written down for $n_T > 1$, both the gauge as well as supersymmetry anomalies can nonetheless be associated with the gauge and supersymmetry variation, respectively, of the Green-Schwarz type term [6]

$$\mathcal{L}_{GS} = -\frac{1}{8} \epsilon^{\mu\nu\rho\sigma\tau\lambda} \left(\eta_{IJ} B_{\mu\nu}{}^{I} C^{Jz} \right) \operatorname{tr}_{z} \left(F_{\rho\sigma} F_{\tau\lambda} \right) - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma\tau\lambda} \eta_{IJ} C^{Iz} C^{Jz'} \omega^{z}_{\mu\nu\rho} \omega^{z'}_{\sigma\tau\lambda} , \qquad (72)$$

where $\omega_{\mu\nu\rho}^{z}$ is the Yang-Mills Chern-Simons form.

The results of [6] and our generalization which includes the hypermatter reveal therefore, a surprising situation in which an anomalous system of supermultiplets *can* be supersymmetrized, with the caveat that the integrability conditions for the equations of motion reflect the anomalies. The fact that this is possible at all may be due to the manifest gauge covariant nature of the field equations and supersymmetry transformation rules. Attempting to supersymmetrize a gauge non-invariant action, on the other hand, would run immediately into trouble with supersymmetrization.

The anomalies of the full system discussed here are to be cancelled by the quantum oneloop effects, so that the total effective action is gauge invariant and supersymmetric, provided that the right set of matter multiplets are included. The anomalies may cancel precisely, or Green-Schwarz cancellation mechanism may have to be employed for the cancellation [18]. The same mechanism works in the dual formulation as was shown in [19, 20, 14]. In the case of $n_T = 1$, a generalized version of the Green-Schwarz mechanism was found by Sagnotti [5] which involves the use of multi-tensor fields simultaneously. We conclude by mentioning some of the anomaly free matter contents for the $n_T = 1$ and $n_T > 1$ cases.

For $n_T = 1$ without gauging, a large number of anomaly free matter contents can be obtained by compactifying the well-known anomaly free ten dimensional supergravity-Yang-Mills systems that arise in string theory [21]. All of these models have a gauge group of rank ≤ 20 , and they arise from a perturbative treatment of string compactification. Witten [22] has discovered a new mechanism by which a nonperturbative symmetry enhancement occurs, and a new class of anomaly-free models, not realized in perturbative string theory, emerge in 6D. These can have rank greater than 20. Schwarz [23] has constructed new anomaly-free models in 6D, some of which may potentially arise in a similar nonperturbative scheme.

As for the gauged case with $n_T = 1$, an anomaly free model was found in [24], where the gauge group is $E_6 \times E_7 \times U(1)$. The U(1) factor is a subgroup of the automorphism group, and the hyper-fermions belong to the 912 dimensional representation of E_7 . The origin of this model still remains mysterious, and it would be very interesting to determine if it can be explained by a new kind of nonperturbative mechanism in M-theory.

In general, the necessary but not sufficient condition for the anomaly cancellation is [24]

$$n_H - n_V + 29 n_T = 273 . (73)$$

As mentioned in the introduction, an example of an anomaly free model with $n_T = 9$ has been found in [2] by considering a suitable M-theory compactification, and it has $n_V =$ 8, $n_H = 20$. The matter couplings of six dimensional supergravity constructed here provides the field theoretic description of this model.

Finally, we mention an example of an anomaly free matter content with $n_T > 1$ found sometime ago in [25]. It has:

$$n_T = 17$$
 , $n_H = 28$, $n_V = 248$. (74)

The vector fields fit into the adjoint representation of E_8 , and we can take the hyperscalar manifold to be $E_8/(E_7 \times Sp(1))$, in which case the hyperfermions transform in 56 dimensional representation of E_7 . As far as we know, this model, which has a rather simple field content, has not found an M-theory explanation so far, and it would be interesting to see if there is one.

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Appendix: Notations, Conventions and Lemmas

Our metric is $(\eta_{mn}) = \text{diag.} (-, +, +, +, +, +)$, while the Clifford algebra is generated by $\{\gamma_m, \gamma_n\} = 2\eta_{mn}$. Note that this signature differs from the one in [1]. The definition of the Ricci tensor is the same as in [1], however, namely: $R_{\mu}{}^a = R_{\mu\nu}{}^{ab} e_b{}^{\nu}$. We define $\gamma_7 \equiv \gamma_{(0)(1)\cdots(5)}, \ \epsilon^{012345} = +1$, such that $(\gamma_7)^2 = +1$. More generally we have

$$\gamma^{r_1 \cdots r_n} = \frac{(-1)^{[n/2]}}{(6-n)!} \epsilon^{r_1 \cdots r_n s_1 \cdots s_{6-n}} \gamma_{s_1 \cdots s_{6-n}} \gamma_7 \quad . \tag{75}$$

The basic gamma-matrix relations such as $\gamma_m \gamma^{rst} \gamma^m = 0$ stays the same as in ref. [1], as well as the conventions for the Sp(1) indices, *e.g.*,

$$\chi^{A}{}_{i} = \epsilon^{AB} \chi_{Bi} \quad , \qquad \chi_{Ai} = \chi^{B}{}_{i} \epsilon_{BA} \quad , \qquad \left(\epsilon_{AB}\right) = \left(\epsilon^{AB}\right) = \left(\begin{array}{cc} 0 & 1\\ -1 & 0 \end{array}\right) \quad , \tag{76}$$

$$\chi^{A}{}_{i} = \epsilon^{AB} \overline{\chi}_{Bi}{}^{T} \quad , \quad \overline{\chi}_{Ai} = (\chi^{A}{}_{i})^{\dagger} \gamma_{0} \quad , \tag{77}$$

$$\left(\overline{\chi}^{A}{}_{i}\gamma^{m_{1}\cdots m_{n}}\lambda^{B}\right) = (-1)^{n+1}\left(\overline{\lambda}^{B}\gamma^{m_{n}\cdots m_{1}}\chi^{A}{}_{i}\right) \quad .$$

$$\tag{78}$$

For inner products of Sp(1) (or $Sp(n_H)$) symplectic spinors [13], the contractions with ϵ_{AB} (or ϵ_{ab}) are always understood, *e.g.*, $(\overline{\chi}_i \gamma^{rs} \lambda) = (\overline{\chi}^A_i \gamma^{rs} \lambda_A)$ as in [1], *e.g.*,

$$\left(\overline{\chi}_{i}\gamma^{r_{1}\cdots r_{n}}\lambda\right) = (-1)^{n}\left(\overline{\lambda}\gamma^{r_{1}\cdots r_{n}}\chi_{i}\right) \quad .$$

$$(79)$$

Exactly as in ref. [1], for given four symplectic Majorana-Weyl spinors ψ_1, \dots, ψ_4 , where the labels 1, ..., 4 denote *all* the possible indices they may carry, including Sp(1), $Sp(n_H)$ or $SO(n_T)$ indices, the Fierz arrangement formula is

$$\left(\overline{\psi}_{1}\psi_{2}\right)\left(\overline{\psi}_{3}\psi_{4}\right) = -\frac{1}{8}\left(1+c_{2}c_{4}\right)\left[\left(\overline{\psi}_{1}\psi_{4}\right)\left(\overline{\psi}_{3}\psi_{2}\right)-\frac{1}{2}\left(\overline{\psi}_{1}\gamma^{rs}\psi_{4}\right)\left(\overline{\psi}_{3}\gamma_{rs}\psi_{2}\right)\right] \\ -\frac{1}{8}\left(1-c_{2}c_{4}\right)\left[\left(\overline{\psi}_{1}\gamma^{r}\psi_{4}\right)\left(\overline{\psi}_{3}\gamma_{r}\psi_{2}\right)-\frac{1}{12}\left(\overline{\psi}_{1}\gamma^{rst}\psi_{4}\right)\left(\overline{\psi}_{3}\gamma_{rst}\psi_{2}\right)\right]$$
(80)

where $\gamma_7(\psi_2, \psi_4) = (c_2\psi_2, c_4\psi_4).$

One of the most frequently used relationships related to the (anti)self-dual tensors is $S^{\mu\nu\rho}S_{\mu\nu\rho} \equiv A^{\mu\nu\rho}A_{\mu\nu\rho} \equiv 0$, where the third-rank tensors S and A are respectively self-dual and anti-self-dual tensors: $(1/6)\epsilon_{mnr}{}^{stu}S_{stu} = +S_{mnr}, (1/6)\epsilon_{mnr}{}^{stu}A_{stu} = -A_{mnr}$. For the tensor $H_{\mu\nu\rho}{}^{I}$ we use the symbols $H^{+}_{\rho\sigma\tau}{}^{I}$ (or $H^{-}_{\rho\sigma\tau}{}^{I}$) to distinguish their dual (or anti-self-dual) components. The important duality properties of the gamma-matrices multiplied

by fermions are summarized as follows. For fermions with the positive chirality such as $\psi_{\mu}{}^{A}$, λ^{Ar} or ϵ^{A} , or for fermions with negative chiralities such as χ^{Ai} , ψ^{a} , we have

$$\frac{1}{6} \epsilon_{mnr}{}^{stu} \left(\gamma_{stu} \psi_{\mu}\right) \equiv -\left(\gamma_{mnr} \psi_{\mu}\right) \quad , \qquad \frac{1}{6} \epsilon_{mnr}{}^{stu} \left(\gamma_{stu} \psi^{a}\right) \equiv +\left(\gamma_{mnr} \psi^{a}\right) \quad , \tag{81}$$

In other words, the combination $\gamma^{rst}\psi_{\mu}$ behaves as an anti-self-dual tensor, while $\gamma^{rst}\psi^{a}$ behaves as a self-dual tensor, as far as the indices [rst] are concerned. It also follows that $\gamma^{\rho\sigma\tau}\lambda H^{-I}_{\rho\sigma\tau} \equiv 0$ or $\gamma^{\rho\sigma\tau}\chi H^{+I}_{\rho\sigma\tau} \equiv 0$.

In the remainder of this appendix, we shall list a number of lemmas that are useful in the derivation of field equations and supersymmetry transformation rules.

(1) For H^2 -term computation in gravitational equation the following lemma is useful:

$$H^{+ \ \tau i}_{[\mu\nu} H^{+}_{\rho]\sigma\tau i} \equiv 0 \quad . \tag{82}$$

This can be verified by using the duality property of H, and simple manipulations involving the Schouten identity, which in the present case means that an antisymmetrization of seven world indices vanishes identically.

(2) For χH -terms in the derivations of χ -field equation out of anti-self-duality condition, the following lemma is useful:

$$\left(\overline{\epsilon}\gamma^{\sigma}{}_{\left[\mu\right]}\chi^{i}\right)H^{-}_{\left[\nu\rho\right]\sigma} \equiv -\frac{1}{3}\left(\overline{\epsilon}\chi^{i}\right)H^{-}_{\mu\nu\rho} \quad .$$

$$(83)$$

This can be proven by the vanishing of $(\bar{\epsilon}\gamma_{\mu\nu\rho}\gamma^{\sigma\tau\omega}\chi^i) H^-_{\sigma\tau\omega} = 0$ with the γ -algebra for the l.h.s.

(3) In calculating the divergence of the Yang-Mills field equation, it is useful to note that

$$D_{\mu}C^{z} = \partial_{\mu}C^{z} = (\partial_{\mu}\varphi^{\underline{\alpha}})V_{\underline{\alpha}}^{\ i}C_{i}^{z} , \qquad D_{\mu}C^{zi} = (\partial_{\mu}\varphi^{\underline{\alpha}})V_{\underline{\alpha}}^{\ i}C^{z} .$$

$$(84)$$

(4) For the gravitino field equation out of self-duality condition, we use the lemma

$$\left(\overline{\epsilon}\gamma^{\sigma}\gamma_{\mu\nu\rho}\gamma^{\tau\omega}\chi_{i}\right)H_{\sigma\tau\omega}^{+\ i} = -\frac{1}{3}\left(\overline{\epsilon}\gamma^{\sigma\tau\omega}\gamma_{\mu\nu\rho}\chi_{i}\right)H_{\sigma\tau\omega}^{+\ i} - 16\left(\overline{\epsilon}\chi_{i}\right)H_{\mu\nu\rho}^{+\ i} , \qquad (85)$$

confirmed by γ -algebra as well as the self-duality of H^+ .

(5) The λ^2 and χ^2 -terms in the supersymmetry transformation of the gravitino can be rearranged as

$$\delta\psi_{\mu}|_{\lambda^{2}} = -C^{Iz}L_{I}\left(\frac{3}{4}\epsilon_{B}\lambda_{\mu}{}^{BA} + \frac{1}{4}\gamma_{\mu}{}^{\nu}\epsilon_{B}\lambda_{\nu}{}^{BA} + \frac{1}{16}\gamma^{\sigma\tau}\epsilon^{A}\lambda_{\mu\sigma\tau}\right) \quad , \tag{86}$$

$$\delta\psi_{\mu}|_{\chi^{2}} = -\frac{3}{8}\epsilon_{B}\chi_{\mu}{}^{BA} + \frac{1}{8}\gamma_{\mu}{}^{\nu}\epsilon_{B}\chi_{\nu}{}^{BA} \quad , \tag{87}$$

where $\lambda_{\mu}{}^{AB} = \operatorname{tr}_{z}\left(\overline{\lambda}{}^{A}\gamma_{\mu}\lambda^{B}\right), \ \lambda_{\rho\sigma\tau} \equiv \operatorname{tr}_{z}\left(\overline{\lambda}\gamma_{\rho\sigma\tau}\lambda\right), \ \chi_{\mu}{}^{AB} \equiv \left(\overline{\chi}{}^{A}\gamma_{\mu}\chi^{B}\right), \ \chi_{\rho\sigma\tau} \equiv (\overline{\chi}\gamma_{\rho\sigma\tau}\chi).$ (6) The $\lambda\chi$ -term in $\delta\lambda$ (41) can be rewritten by using the identity

$$\left(\overline{\chi}_{i(A}\lambda_{B)}\right)\epsilon^{B} = \frac{1}{4}\lambda_{A}\left(\overline{\epsilon}\chi_{i}\right) + \frac{1}{8}\gamma_{\rho\sigma}\lambda_{A}\left(\overline{\epsilon}\gamma^{\rho\sigma}\chi_{i}\right) + \frac{1}{2}\epsilon_{A}\left(\overline{\lambda}\chi_{i}\right) \quad , \tag{88}$$

obtained by Fierz rearrangement.

(7) In order to fix the λ^2 -terms in the supersymmetry transformation of the gravitino, we arrange the $\psi^a \lambda^2$ -terms in the commutator on ψ^a , which needs the lemma

$$\left(\overline{\epsilon}_{2}^{A}\gamma_{\mu\nu\rho}\epsilon_{1B}\right)\lambda^{\rho B}{}_{A} = -4 \operatorname{tr}_{z}\left(\overline{\epsilon}_{[1|}\gamma_{\mu}\lambda\right)\left(\overline{\lambda}\gamma_{\nu}\epsilon_{[2]}\right) + \frac{1}{2}\xi^{\rho}\lambda_{\mu\nu\rho} , \operatorname{tr}_{z}\left(\overline{\epsilon}_{[2|}\gamma^{mn}{}_{\rho}\lambda\right)\left(\overline{\epsilon}_{[1]}\gamma_{\rho}\lambda\right) = -\frac{1}{2}\xi^{\rho}\lambda_{\rho}{}^{mn} + 2 \operatorname{tr}_{z}\left(\overline{\epsilon}_{[2|}\gamma^{[m}\lambda\right)\left(\overline{\epsilon}_{[1]}\gamma^{n}\right)\lambda\right) ,$$

$$(89)$$

where $\xi^{\mu} \equiv (\overline{\epsilon}_2 \gamma^{\mu} \epsilon_1).$

(8) The following non-trivial lemmas are useful for the closure checks on $e_{\mu}{}^{m}$ or $B_{\mu\nu}$:

$$\operatorname{tr}_{z}\left(\overline{\epsilon}_{1}\gamma_{\rho\sigma}[_{\mu}\lambda\right)\left(\overline{\epsilon}_{2}\gamma_{\nu}\right]^{\rho\sigma}\lambda\right) - (1\leftrightarrow2) \equiv 0 \quad , \qquad \left(\overline{\epsilon}_{1}\gamma^{\mu\nu\rho\sigma}\chi^{(i)}\right)\left(\overline{\epsilon}_{2}\gamma_{\rho\sigma}\chi^{j)}\right) - (1\leftrightarrow2) \equiv 0 \quad ,$$
$$\left(\overline{\epsilon}_{2}\gamma^{\rho}\epsilon_{1}\right) \operatorname{tr}_{z}\left(\overline{\lambda}\gamma_{\mu\nu\rho}\lambda\right) \equiv \operatorname{tr}_{z}\left[2\left(\overline{\epsilon}_{2}\gamma_{\mu}\lambda\right)\left(\overline{\epsilon}_{1}\gamma_{\nu}\lambda\right) - \left(\overline{\epsilon}_{2}\gamma_{\mu\nu\rho}\lambda\right)\left(\overline{\epsilon}_{1}\gamma^{\rho}\lambda\right)\right] - (1\leftrightarrow2) \quad . \tag{90}$$

These are easily confirmed by appropriate Fierz arrangements as well as the duality properties we already know.

(9) It is useful to note the following relation for the closure check on λ :

$$\left(\overline{\epsilon}_{1}^{(A} \gamma_{\sigma\tau\rho} \epsilon_{2}^{B)} \right) \gamma^{\mu\nu\sigma\tau} \lambda_{B} H^{-}_{\mu\nu\rho} \equiv 0 \quad ,$$

$$\left(\overline{\epsilon}_{1}^{(A} \gamma^{\mu\rho\sigma} \epsilon_{2}^{B)} \right) \gamma^{\nu}_{\mu} \lambda_{B} H^{-}_{\nu\rho\sigma} \equiv 0 \quad ,$$

$$(91)$$

which can be proven by the relationship $\left(\overline{\epsilon}_{1}^{(A}\gamma_{\rho\sigma\tau}\epsilon_{2}^{B)}\right)\left[\gamma^{\rho\sigma\tau},\gamma^{\mu\nu\omega}\right]_{\pm}\lambda H^{-}_{\mu\nu\omega}\equiv 0, \ etc.$

due to the anti-self-duality of the combination $(\overline{\epsilon}_1^A \gamma_{\rho\sigma\tau} \epsilon_2^B)$ as well as the anti-self-duality of H^{-I} .

(10) In the arrangement of $\lambda \chi^2$ -terms in the commutator on λ , the following lemma for super-variation is useful:

$$\delta\left(C_z^{-1}C^{iz}\right) = C_z^{-2}\left(\overline{\epsilon}\chi^j\right)\left(\delta^{ij}C_z^2 - C^{iz}C^{jz}\right) \quad . \tag{92}$$

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