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ON VARIATIONAL FORMULATIONS FOR THE STOKES EQUATIONS WITH NONSTANDARD BOUNDARY CONDITIONS (*)

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Abstract — *Variational formulations are proposed for the Stokes equations with nonstandard boundary conditions Due to the nonstandard boundary conditions, the pressure is decoupled from the system and the remaining velocity equation admits weak formulations with no divergence constraint on the spaces of test functions and yet the solutions of the variational problems are divergence free Finite element approximations are introduced based on these formulations and error estimates are proved*

Résumé — *Des formules variationnelles sont données pour l'équation de Stokes avec des conditions au bord non-standards Ces conditions font que l'équation de la pression est indépendante des autres variables, et que l'équation de la vitesse peut être formulée faiblement de telle sorte que la divergence des fonctions tests ne soit pas soumise a des conditions restrictives, et que la divergence des solutions au probleme variationnel soit nulle Nous donnons des approximations par éléments finis basés sur cette formulation, et des estimations d'erreur*

1. INTRODUCTION

We describe in this paper two formulations for solving the following Stokes equations with non-standard boundary conditions :

$$\begin{aligned} -\nu\,\Delta\mathbf{u}+\nabla p&=\mathbf{f} && \text{in } \Omega\\ \nabla\cdot\mathbf{u}&=0 && \text{in } \Omega\\ \mathbf{u}\cdot\mathbf{n}&=\mathbf{g}\cdot\mathbf{n} && \text{on } \Gamma\\ \nabla\times\mathbf{u}\times\mathbf{n}&=\mathbf{h}\times\mathbf{n} && \text{on } \Gamma\\ \int_\Omega p\,dx&=0\,, \end{aligned} \tag{1}$$

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where Ω is a bounded domain in $\mathbf{R}^3$ with boundary $\Gamma = \partial\Omega$, $\nu > 0$, and $\mathbf{n}$ is the outward normal vector on Γ and the variables $\mathbf{u}$ and p usually represent velocity and pressure in fluid mechanics. The function $\mathbf{g}$ satisfies the compatibility condition

$$\int_{\Gamma} \mathbf{g} \cdot \mathbf{n} \, ds = 0 . \tag{2}$$

Recently, there has been increasing interest in Stokes equations with nonstandard boundary conditions, *cf.* [2], [3], [7], [10], [13] and [16]. Such problems come up in many practical applications, e.g. fluid dynamics, electromagnetic field applications, and decompositions of vector fields. Unlike the Stokes equations with Dirichlet boundary conditions, the nonstandard boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n} \quad \text{and} \quad \nabla \times \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on} \quad \Gamma \tag{3}$$

allow one to decouple p from the equations. More precisely, one can find p independently of $\mathbf{u}$. Variational formulations can then be introduced for the reduced Stokes equations (without the p term).

It is well-known that one of the major difficulties in implementing numerical schemes for solving Stokes problems in general is how to impose the divergence free condition or sometimes how to avoid imposing the divergence free condition on the discrete spaces of test functions. In most cases, properly imposed divergence free conditions on properly chosen approximating spaces will directly lead to results of solvability and also error estimates. Various numerical methods have been developed for solving the Stokes problems in the literature to address this difficulty. Among those methods, perhaps the most frequently used and investigated are the mixed methods, *cf.* [11], which formulate the Stokes problems as saddle point problems. However, it is well documented that the analysis for such methods usually amounts to verification of the stability condition, usually referred to as the inf-sup condition or the Babuška-Brezzi condition which must be satisfied by the pair of discrete spaces chosen for velocity and pressure approximations. Many combinations of discrete approximation spaces which satisfy the stability condition are known to exist, although verification of the stability condition is often technical and delicate. There are also other methods which avoid imposing the divergence free condition by introducing the stream function in the two dimensional case or a vector potential φ in the three dimensional case satisfying $\nabla \times \varphi = \mathbf{u}$. In $\mathbf{R}^3$ the domain is required to be simply connected to insure uniqueness of the vector potential. Such methods generally lead to solutions of biharmonic or biharmonic-type equations.

The aforementioned studies [2], [3], [7], [10] and [16] on the Stokes equations with nonstandard boundary conditions all consider mixed formulations. In V. Girault [10], a mixed method using vector potential-vorticity formulation was proposed and analyzed. Similar studies can also be found in C. Begue, el. [2], A. Bendali, el. [3], J. M. Dominguez [7], and R. Verfürth [16]. In [15], a variational formulation similar to the first formulation of this paper was studied for the three dimensional Maxwell's equations. As we shall demonstrate in this paper, however, the nonstandard boundary conditions (3) are somewhat more natural to Stokes equations than the standard Dirichlet boundary conditions. They permit the pressure to be completely decoupled from the system. In fact, the pressure can be sought independently of the velocity as the solution of a certain Neumann problem. We can then show that the remaining velocity equation admits weak formulations in which the divergence constraint is not required for the spaces of test functions but the solutions of the variational problems indeed have zero divergence.

The remainder of this paper is outlined as follows. In Section 2 we will introduce some notation and preliminary results needed for the development in the remaining sections. Section 3 is devoted to the derivations of variational formulations and proofs of characterization results for the Stokes problems. Finite element approximation are then proposed in Section 4 and error estimates in the energy norm are proved. We will also give some brief comments on calculations of the bilinear forms in the weak formulations and preconditioning in Section 5.

Finally, we remark that time dependent problems can be treated similarly but they are omitted in this paper.

2. NOTATION AND PRELIMINARIES

We will use the standard notation for gradient, divergence and curl operators, i.e. for a scalar function f and a vector function $\mathbf{v}$, we write ∇f, $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$ for the gradient, divergence and the curl respectively. We further recall that the vector-valued Laplace operator, Δ, is defined by

$$\Delta \mathbf{v} = (\Delta v_1, \Delta v_2, \Delta v_3) \tag{4}$$

and we have the vector identity

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \Delta \mathbf{v} .$$

Next, we let Ω be an open bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega = \Gamma$. The boundary of Ω is a 2-dimensional manifold without boundary. In general we shall assume a certain smoothness property for Γ, e.g. Γ is of $C^r (r \geq 2)$. We recall the definition of classical Sobolev space $H^m(\Omega)$ being the completion of smooth functions under the norm

$$\|\mathbf{v}\|_{m, \Omega} = \left\{ \sum_{k \leq m} |\mathbf{v}|_{k, \Omega}^2 \right\}^{1/2}, \tag{6}$$

where

$$|\mathbf{v}|_{k,\Omega} = \left\{ \sum_{|\alpha| = k} \int_{\Omega} |\partial^{\alpha} \mathbf{v}|^2 \, dx \right\}^{1/2} . \tag{7}$$

When $m = 0$ we have $L^2(\Omega) = H^0(\Omega)$. We will denote the inner products in $H^m(\Omega)$ by $(\cdot, \cdot)_{m,\Omega}$. In particular, we will use $(\cdot, \cdot)$ to denote the L^2 inner product. For $s > 0$, $H^s(\Omega)$ is defined by interpolation. The spaces $H^{\alpha}(\Gamma)$ for $\alpha \geqslant 0$ are defined in a similar fashion. We will use $\langle \cdot, \cdot \rangle_{\alpha,\Gamma}$ to denote the H^{α}-inner product and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{0,\Gamma}$.

For vector valued functions the above norms extend naturally as follows : for $\mathbf{v} = (v_1, v_2, v_3)$

$$\|\mathbf{v}\|_{s,\Omega} = \left\{ \sum_{j=1}^{3} \|\mathbf{v}_j\|_{s,\Omega}^2 \right\}^{1/2} . \tag{8}$$

The corresponding spaces will be denoted by $H^s(\Omega)^3$.

We shall now introduce some function spaces which are related to the study of equations (1). First, we denote the space of square integrable functions with zero mean value on Ω by

$$L_0^2(\Omega) = \left\{ \varphi \in L^2(\Omega) : \int_{\Omega} \varphi \, dx = 0 \right\} . \tag{9}$$

Next, for $0 \leqslant \alpha \leqslant 1/2$, we define the spaces

$$H_{\alpha}(\Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \nabla \times \mathbf{v} \in L^2(\Omega)^3, \nabla \cdot \mathbf{v} \in L^2(\Omega), \text{ and } \mathbf{v} \cdot \mathbf{n} \in H^{\alpha}(\Gamma)\} . \tag{10}$$

We further denote by $\mathbf{H}_{\mathrm{T}}(\Omega)$ the subspace of $\mathbf{H}_{\alpha}(\Omega)$ with vanishing normal components on Γ, i.e.

$$\mathbf{H}_{\mathrm{T}}(\Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \nabla \times \mathbf{v} \in L^2(\Omega)^3, \nabla \cdot \mathbf{v} \in L^2(\Omega), \text{ and } \mathbf{v} \cdot \mathbf{n} = 0\} . \tag{11}$$

The dual spaces of $\mathbf{H}_{\mathrm{T}}(\Omega)$ and $\mathbf{H}_{\alpha}(\Omega)$ are denoted by $\mathbf{H}'_{\mathrm{T}}(\Omega)$ and $\mathbf{H}'_{\alpha}(\Omega)$ respectively ; the dual of $H^{1/2}(\Gamma)$ is denoted by $H^{-1/2}(\Gamma)$.

The following preliminary results will be assumed in subsequent sections. Their proofs can be found in the references given below and therefore are omitted here. First we state a well-known result on orthogonal decompositions of functions in $L^2(\Omega)^3$, (*cf.* [11]).

LEMMA 2.1 : *For every functions* $v \in L^2(\Omega)^3$, *there exist unique* $\mathbf{w}$ *and* $\varphi \in H^1(\Omega)/\mathbb{R}$, *with* $\nabla \cdot \mathbf{w} = 0$ *and* $\mathbf{w} \times \mathbf{n} = 0$ *on* Γ, *such that the following orthogonal decomposition holds :*

$$\mathbf{v} = \nabla \times \mathbf{w} + \nabla \varphi .$$

The following lemma plays an important role in the variational formulation proposed in subsequent sections. it provides the positivity of the bilinear forms in the variational formulations to be introduced in the next section.

LEMMA 2.2 : *(1) The mapping* $\mathbf{v} \to \|\mathbf{v}\|_{\mathbf{H}_\alpha}$ *given by*

$$\|\mathbf{v}\|_{\mathbf{H}_\alpha} = (\|\nabla \times \mathbf{v}\|^2_{0,\Omega} + \|\nabla \cdot \mathbf{v}\|^2_{0,\Omega} + |\nabla \cdot \mathbf{n}|^2_{\alpha,\Gamma})^{1/2} \tag{12}$$

defines a norm on $\mathbf{H}_\alpha(\Omega)$.

(2) Moreover, when $\alpha = 1/2$, *the space* $\mathbf{H}_\alpha(\Omega)$ *is equal to* $H^1(\Omega)^3$ *algebraically and topologically.*

Proof : Statement (1) *is a direct consequence of Lemma* 2.1. *The proof for* (2) *can be found in* [3], [8] *and* [9].

As a consequence of the preceeding lemma, we have the following.

COROLLARY 2.1 : *The mapping* $\mathbf{v} \to \|\mathbf{v}\|_{\mathbf{H}_{T(\Omega)}}$ *given by*

$$\|\mathbf{v}\|_{\mathbf{H}_T} = \left\{\|\nabla \times \mathbf{v}\|^2_{0,\Omega} + \|\nabla \cdot \mathbf{v}\|^2_{0,\Omega}\right\}^{1/2} \tag{13}$$

defines a norm on $\mathbf{H}_T(\Omega)$.

In the remainder of this paper, we will use C to denote a generic constant which does not depend on functions in certain function spaces in context or the mesh size of finite element mesh domains.

We end this section by stating the following regularity result for (1). Its proof can be obtained following the general proof for elliptic systems, (*cf.* [1], [12], [17]).

THEOREM 2.1 : *Let* Γ *be of class* $C^r (r \geq 2)$. *If* $(f, \mathbf{h} \times \mathbf{n}, \mathbf{g} \cdot \mathbf{n}) \in \mathbf{H}_T'(\Omega) \times H^{-1/2}(\Gamma)^3 \times H^{1/2}(\Gamma)$, *then there exists a unique solution* $(\mathbf{u}, p) \in H^1(\Omega)^3 \times L^2_0(\Omega)$ *of* (1) *such that*

$$\|\mathbf{u}\|_{1,\Omega} + \|p\|_{0,\Omega} \leq C \left\{\|\mathbf{f}\|_{\mathbf{H}_T'(\Omega)} + \|\mathbf{h} \times \mathbf{n}\|_{-1/2,\Gamma} + \|\mathbf{g} \cdot \mathbf{n}\|_{1/2,\Gamma}\right\} . \tag{14}$$

For the general shift theorem, we let Γ *be of class* $C^r (r \geq m + 2)$. *Then for* $(\mathbf{f}, \mathbf{h} \times \mathbf{n}, \mathbf{g} \cdot \mathbf{n}) \in H^{m-1}(\Omega)^3 \times H^{m-1/2}(\Gamma)^3 \times H^{m+1/2}(\Gamma)$, *there exists a unique solution* $(\mathbf{u}, p) \in H^{m+1}(\Omega)^3 \times (H^m(\Omega) \cap L^2_0(\Omega))$ *of* (1) *such that*

$$\|\mathbf{u}\|_{m+1,\Omega} + \|p\|_{m,\Omega} \leq$$

$$\leq C \left\{\|\mathbf{f}\|_{m-1,\Omega} + \|\mathbf{h} \times \mathbf{n}\|_{m-1/2,\Gamma} + \|\mathbf{g} \cdot \mathbf{n}\|_{m+1/2,\Gamma}\right\} .$$

3. VARIATIONAL FORMULATIONS

In this section we will derive two variational formulations for the Stokes problem (1) Without loss of generality, we assume from now on that $\nu = 1$. Using the nonstandard boundary conditions, we can solve for the pressure p independently. With p and hence ∇p known the resulting system contains only the velocity variable $\mathbf{u}$ and is self consistent. We therefore introduce variational formulations for the reduced velocity equations. To proceed, we let $(\mathbf{u}, p)$ be the solution of (1) and multiply the first equation by ∇q for any $q \in H^1\ \Omega$ and integrate over Ω. Using the vector identity (5), we find that

$$\begin{aligned} (-\Delta \mathbf{u}, \nabla q) &= (\nabla \times (\nabla \times \mathbf{u}), \nabla q) \\ &= -\langle \nabla \times \mathbf{u} \times \mathbf{n}, \nabla q \rangle \\ &= -\langle \mathbf{h} \times \mathbf{n}, \nabla q \rangle \end{aligned} \tag{15}$$

By the Stokes Theorem,

$$-\langle \mathbf{h} \times \mathbf{n}, \nabla q \rangle = \langle \nabla \times \mathbf{h} \cdot \mathbf{n}, q \rangle - \int_{\partial \Gamma} q \mathbf{h} \cdot d\sigma . \tag{16}$$

Since Γ is a two dimensional manifold without boundary, $\partial \Gamma$ is empty. Hence,

$$(-\Delta \mathbf{u}, \nabla q) = \langle \nabla \times \mathbf{h} \cdot \mathbf{n}, q \rangle \tag{17}$$

Consequently, for any $q \in H^1(\Omega)$, we have the equation

$$(\nabla p, \nabla q) = (\mathbf{f}, \nabla q) - \langle \nabla \times \mathbf{h} \cdot \mathbf{n}, q \rangle \tag{18}$$

On the other hand, if we apply the divergence operator to the both sides of first equation of (1), multiply by q for $q \in H^1(\Omega)$, integrate over Ω and then use the above equation, we obtain

$$\langle \nabla p \cdot \mathbf{n}, q \rangle = \langle \mathbf{f} \cdot \mathbf{n}, q \rangle - \langle \nabla \times \mathbf{h} \cdot \mathbf{n}, q \rangle , \quad \text{for all} \quad q \in H^1(\Omega) \tag{19}$$

From (18) and (19), we find that p is the solution of the Neumann problem,

$$\begin{aligned} -\Delta p &= -\nabla \cdot \mathbf{f} && \text{in } \Omega \\ \partial p / \partial n &= (\mathbf{f} - \nabla \times \mathbf{h}) \cdot \mathbf{n} && \text{on } \Gamma . \end{aligned} \tag{20}$$

Remark 3 1 (17) with $q = 1$ shows that $\int_\Gamma \nabla \times \mathbf{h} \cdot \mathbf{n}\, ds = 0$. This is the compatibility condition for (20).

Therefore p can be found indenpendently of $\mathbf{u}$ by solving the problem (20). With p known, we put $\mathbf{F} = \mathbf{f} - \nabla p$ and obtain a system of equations involving only the velocity variable $\mathbf{u}$. That is

$$\begin{aligned} -\Delta \mathbf{u} &= \mathbf{F} && \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u} \cdot \mathbf{n} &= \mathbf{g} \cdot \mathbf{n} && \text{on } \Gamma \\ \nabla \times \mathbf{u} \times n &= \mathbf{h} \times \mathbf{n} && \text{on } \Gamma . \end{aligned} \tag{21}$$

We note that by equations (1) $\mathbf{F}$ satisfies

$$\nabla \cdot \mathbf{F} = \nabla \cdot (-\Delta \mathbf{u}) = -\Delta(\nabla \cdot \mathbf{u}) = 0 , \tag{22}$$

and

$$\mathbf{F} \cdot \mathbf{n} = (\mathbf{f} - \nabla p) \cdot \mathbf{n} = \nabla \times \mathbf{h} \cdot \mathbf{n} . \tag{23}$$

Remark 3.2 : The above two conditions that $\mathbf{F}$ satisfies are not coincidental. They are in fact the compatibility conditions on $\mathbf{F}$ in order that the system (21) have a unique solution.

We shall now describe two variational formulations for the reduced velocity equations (21).

3.1. First Formulation

For simplicity, we will assume in this section that $\mathbf{g} \cdot \mathbf{n} = 0$ and $\mathbf{h} \times \mathbf{n} = 0$ on Γ. Nonhomogeneous boundary conditions are analyzed in the next section and the same argument could be used here to treat the case of nonhomogeneous boundary conditions. Let us recall in Section 2 that $\mathbf{H}_T(\Omega)$ is the subspace of $H^1(\Omega)^3$ whose normal components vanish on Γ. We define a bilinear form on $\mathbf{H}_T(\Omega) \times \mathbf{H}_T(\Omega)$ by

$$B_T(\mathbf{u}, \mathbf{v}) = (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) .$$

With the definition of the bilinear form $B_T(\mathbf{u}, \mathbf{v})$, the reduced Stokes equations (21) admits the following weak formulation :

(S_v^T) : For $\mathbf{F} \in \mathbf{H}_T'(\Omega)$, find $\mathbf{u} \in \mathbf{H}_T(\Omega)$ such that

$$B_T(\mathbf{u}, \mathbf{v}) = (\mathbf{F}, \mathbf{v})$$

for all $\mathbf{v} \in \mathbf{H}_T(\Omega)$. Clearly, by the Cauchy-Schwarz inequality and the Riesz Representation Theorem we have the following.

THEOREM 3.1 : *For every $\mathbf{F} \in \mathbf{H}_T'(\Omega)$ there exists a unique $\mathbf{u} \in \mathbf{H}_T(\Omega)$ such that $\mathbf{u}$ is a solution of problem (S_v^T). Furthermore,*

$$\|\mathbf{u}\|_{\mathbf{H}_T(\Omega)} \leq C \ \|\mathbf{F}\|_{\mathbf{H}_T'(\Omega)} ,$$

for some constant C.

The following theorem shows the equivalence of the reduced Stokes equations (21) and the variational problem (S_v^T) when $\mathbf{F}$ satisfies the compatibility conditions

THEOREM 3 2 *Assuming that the boundary Γ of Ω, the solutions of the reduced problem* (21) *and the variational problem* (S_v^T) *are sufficiently regular, then for* $\mathbf{F}$ *satisfying the compatibility conditions* (22) *and* (23), (S_v^T) *is equivalent to* (21) *with homogeneous boundary conditions, i e* $\mathbf{g} \cdot \mathbf{n} = 0$ *and* $\mathbf{h} \times \mathbf{n} = 0$

Proof It is straightforward to see that the solution $\mathbf{u}$ of (21) satisfies the variational equation of (S_v^T) For the converse, we sketch the proof as follows Let $\mathbf{u}$ be the solution of (S_v^T) and $\mathbf{v}$ an arbitrary smooth vector function We integrate by parts and use identity (5) to get

$$B_T(\mathbf{u}, \mathbf{v}) = (-\Delta \mathbf{u}, \mathbf{v}) + \langle \nabla \times \mathbf{u} \times \mathbf{n}, \mathbf{v} \rangle$$

Taking $\mathbf{v}$ vanishing on the boundary, we obtain

$$-\Delta \mathbf{u} = \mathbf{F} \quad \text{in} \quad \Omega .$$

Let $\mathbf{w}$ be arbitrary and choose $\mathbf{v}$ such that $\mathbf{v} = \mathbf{w} - (\mathbf{w} \cdot \mathbf{n})\, \mathbf{n}$ on Γ. Then

$$\langle \nabla \times \mathbf{u} \times \mathbf{n}, \mathbf{w} \rangle = \langle \nabla \times \mathbf{u} \times \mathbf{n}, \mathbf{v} \rangle = 0 .$$

We conclude that

$$\nabla \times \mathbf{u} \times \mathbf{n} = 0 \quad \text{on} \quad \Gamma .$$

It remains to show that the solution $\mathbf{u}$ of the variational problem satisfies $\nabla \cdot \mathbf{u} = 0$. For arbitrary $\varphi \in L^2(\Omega)$ and we let $\bar{\varphi}$ denote the mean value of φ, i.e

$$\bar{\varphi} = \frac{1}{|\Omega|} \int_\Omega \varphi \, dx ,$$

where $|\Omega|$ is the measure of Ω. We choose ψ such that

$$\begin{aligned} -\Delta \psi &= \varphi - \bar{\varphi} && \text{in} \quad \Omega \\ \partial \psi / \partial n &= 0 && \text{on} \quad \Gamma . \end{aligned}$$

Observe by the divergence theorem that

$$(\nabla \cdot \mathbf{u}, \bar{\varphi}) = \bar{\varphi} \int_\Gamma \mathbf{u} \cdot \mathbf{n} \, ds = 0 .$$

Then

$$\begin{aligned}(\nabla \cdot \mathbf{u}, \varphi) &= (\nabla \cdot \mathbf{u}, \varphi - \bar{\varphi}) \\ &= (\nabla \cdot \mathbf{u}, -\Delta\psi) \\ &= -(\nabla \cdot \mathbf{u}, \nabla \cdot \nabla\psi) \\ &= -B_T(\mathbf{u}, \nabla\psi) \\ &= -(\mathbf{F}, \nabla\psi) \\ &= (\nabla \cdot \mathbf{F}, \psi) - \langle \mathbf{F} \cdot \mathbf{n}, \psi \rangle \\ &= 0 .\end{aligned}$$

We remark that the last equality is the consequence of the compatibility conditions on **F**. Now it is clear that,

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega .$$

This completes the proof.

3.2. Second Formulation

To describe our second formulation, we define a bilinear form on $\mathbf{H}_\alpha(\Omega) \times \mathbf{H}_\alpha(\Omega)$ and for $0 \leqslant \alpha \leqslant 1/2$ and $\omega > 0$,

$$B_{\alpha,\omega}(\mathbf{u}, \mathbf{v}) = (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) + \omega \langle \mathbf{u} \cdot \mathbf{n}, \mathbf{v} \cdot \mathbf{n} \rangle_{\alpha,\Gamma}. \tag{25}$$

From Lemma 2.2 we know the above $B_{\alpha,\omega}$-form induces a norm which we shall denote by $|||\cdot|||_\alpha$. We therefore have the following.

Corresponding to the new bilinear form $B_{\alpha,\omega}(\cdot,\cdot)$. We introduce the following variational problem: (S_v): For $(\mathbf{F}, \mathbf{h} \times \mathbf{n}, \mathbf{g} \cdot \mathbf{n}) \in \mathbf{H}_T'(\Omega) \times H^{-1/2}(\Gamma)^3 \times H^\alpha(\Gamma)$, find $\mathbf{u} \in \mathbf{H}_\alpha(\Omega)$, such that

$$B_{\alpha,\omega}(\mathbf{u}, \mathbf{v}) = (\mathbf{F}, \mathbf{v}) + \langle \mathbf{h} \times \mathbf{n}, \mathbf{v} \rangle + \omega \langle \mathbf{g} \cdot \mathbf{n}, \mathbf{v} \cdot \mathbf{n} \rangle_{\alpha,\Gamma}, \quad \text{for all} \quad \mathbf{v} \in \mathbf{H}_\alpha(\Omega). \tag{26}$$

For the above variational problem (S_v), we have

THEOREM 3.3: *For every* $(\mathbf{F}, \mathbf{h} \times \mathbf{n}, \mathbf{g} \cdot \mathbf{n}) \in \mathbf{H}_T'(\Omega) \times H^{-1/2}(\Gamma)^3 \times H^\alpha(\Gamma)$, *there exists a unique solution* $\mathbf{u}$ *of problem* (S_v) *such that*

$$|||\mathbf{u}|||_\alpha \leqslant C \left\{ \|\mathbf{F}\|_{\mathbf{H}_T'(\Omega)} + \|\mathbf{h} \times \mathbf{n}\|_{-1/2,\Gamma} + \|\mathbf{g} \cdot \mathbf{n}\|_{\alpha,\Gamma} \right\}, \tag{27}$$

for some constant C.

Proof: Uniqueness follows directly from coercivity of $|||\cdot|||_\alpha$.. For existence, we split (S_v) into two parts, namely

(1) For $(\mathbf{F}, h \times \mathbf{n}) \in \mathbf{H}_T'(\Omega) \times H^{-1/2}(\Gamma)^3$, find $\mathbf{u}_1$ in $\mathbf{H}_T(\Omega)$ such that

$$B_{\alpha,\omega}(\mathbf{u}_1, \mathbf{v}) = (\mathbf{F}, \mathbf{v}) + \langle \mathbf{h} \times \mathbf{n}, \mathbf{v} \rangle, \quad \text{for all} \quad \mathbf{v} \in \mathbf{H}_\alpha(\Omega). \tag{28}$$

and

(2) for $\mathbf{g} \cdot \mathbf{n} \in H^{\alpha}(\Gamma)$, find $\mathbf{u}_2$ in $\mathbf{H}_{\alpha}(\Omega)$ such that

$$B_{\alpha, \omega}(\mathbf{u}_2, \mathbf{v}) = \omega \langle \mathbf{g} \cdot \mathbf{n}, \mathbf{v} \cdot \mathbf{n} \rangle_{\alpha, \Gamma}, \quad \text{for all } \mathbf{v} \in \mathbf{H}_{\alpha}(\Omega) \tag{29}$$

It is clear from the Cauchy-Schwarz inequality and the Riesz representation Theorem that both problems (1) and (2) admit unique solutions since $\mathbf{F}$ and $\mathbf{h} \times \mathbf{n}$ define bounded linear functionals on $\mathbf{H}_T(\Omega)$ and $\mathbf{g} \cdot \mathbf{n}$ gives rise to a bounded linear functional on $\mathbf{H}_{\alpha}(\Omega)$ Now by linearity, it is also easy to see that the solution $\mathbf{u}$ of (S_v) is the sum of $\mathbf{u}_1$ and $\mathbf{u}_2$ To prove estimate (27), we start with

$$|||\mathbf{u}_1|||_{\alpha}^2 = B_{\alpha, \omega}(\mathbf{u}_1, \mathbf{u}_1) = (\mathbf{F}, \mathbf{u}_1) + \langle \mathbf{h} \times \mathbf{n}, \mathbf{u}_1 \rangle$$

For the second term, we have

$$\begin{aligned} \langle \mathbf{h} \times \mathbf{n}, \mathbf{u}_1 \rangle &\leq \|\mathbf{h} \times \mathbf{n}\|_{-1/2, \Gamma} \|\mathbf{u}_1\|_{1/2, \Gamma} \\ &\leq C \|\mathbf{h} \times \mathbf{n}\|_{-1/2, \Gamma} \|\mathbf{u}_1\|_{1, \Omega} \\ &\leq C \|\mathbf{h} \times \mathbf{n}\|_{-1/2, \Gamma} \|\mathbf{u}_1\|_{\alpha}, \end{aligned}$$

since $\mathbf{u}_1 \cdot \mathbf{n} = 0$ on Γ In the second step we used the standard trace inequality $\|\mathbf{v}\|_{1/2, \Gamma} \leq C \|\mathbf{v}\|_{1, \Omega}$ for all $\mathbf{v} \in H^1(\Omega)^3$ Hence,

$$|||\mathbf{u}_1|||_{\alpha} \leq \|\mathbf{F}\|_{\mathbf{H}_T(\Omega)} + C \|\mathbf{h} \times \mathbf{n}\|_{-1/2, \Gamma} \tag{30}$$

Next, we have

$$\begin{aligned} |||\mathbf{u}_2|||_{\alpha}^2 &= B_{\alpha, \omega}(\mathbf{u}_2, \mathbf{u}_2) = \omega \langle \mathbf{g} \cdot \mathbf{n}, \mathbf{u}_2 \cdot \mathbf{n} \rangle_{\alpha, \Gamma} \\ &\leq \omega^{1/2} \|\mathbf{g} \cdot \mathbf{n}\|_{\alpha, \Gamma} |||\mathbf{u}_2|||_{\alpha} \end{aligned} \tag{31}$$

The estimate (27) then follows readily from (30) and (31)

We now prove the following characterization result for our second variational formulation

THEOREM 3 4 *Assuming that the boundary Γ of Ω, the solutions of the reduced problem* (21) *and the variational problem* (S_v) *are sufficiently regular, then for* $\mathbf{g}$, $\mathbf{h}$ *and* $\mathbf{F}$ *satisfying the compatibility conditions* (2), (22) *and* (23), (S_v) *is equivalent to* (21)

Proof Again it is straightforward to see that the solution $\mathbf{u}$ of (21) satisfies the variational equations (S_v) We will sketch the proof for the reverse direction Let $\mathbf{u}$ be the solution of (S_v) Then, after integrating by parts,

$$\begin{aligned} B_{\alpha, \omega}(\mathbf{u}, \mathbf{v}) = (-\Delta \mathbf{u}, \mathbf{v}) + \langle \nabla \times \mathbf{u} \times \mathbf{n}, \mathbf{v} \rangle &+ \\ + \langle \nabla \cdot \mathbf{u}, \mathbf{v} \cdot \mathbf{n} \rangle &+ \omega \langle \mathbf{u} \cdot \mathbf{n}, \mathbf{v} \cdot \mathbf{n} \rangle_{\alpha, \Gamma} \end{aligned}$$

Taking $\mathbf{v}$ vanishing on the boundary, we obtain

$$- \Delta \mathbf{u} = \mathbf{F} \quad \text{in} \quad \Omega .$$

Let $\mathbf{w}$ be arbitrary and choose $\mathbf{v}$ so that $\mathbf{v} = \mathbf{w} - (\mathbf{w} \cdot \mathbf{n})\mathbf{n}$ on Γ. Then

$$\langle \nabla \times \mathbf{u} \times \mathbf{n}, \mathbf{w} \rangle = \langle \nabla \times \mathbf{u} \times \mathbf{n}, \mathbf{v} \rangle = \langle \mathbf{h} \times \mathbf{n}, \mathbf{v} \rangle = \langle \mathbf{h} \times \mathbf{n}, \mathbf{w} \rangle .$$

We conclude that

$$\nabla \times \mathbf{u} \times \mathbf{n} = \mathbf{h} \times \mathbf{n} \quad \text{on} \quad \Gamma .$$

Therefore the remaining equation becomes

$$\langle \nabla \cdot \mathbf{u}, \boldsymbol{v} \cdot \mathbf{n} \rangle + \omega \, \langle \mathbf{u} \cdot \mathbf{n}, \boldsymbol{v} \cdot \mathbf{n} \rangle_{\alpha, \Gamma} = \omega \, \langle \mathbf{g} \cdot \mathbf{n}, \boldsymbol{v} \cdot \mathbf{n} \rangle_{\alpha, \Gamma} . \tag{32}$$

For arbitrary $\varphi \in L^2(\Omega)$, we choose ψ such that

$$\begin{aligned} - \Delta \psi &= \varphi - \bar{\varphi} && \text{in} \quad \Omega \\ \partial \psi / \partial n &= 0 && \text{on} \quad \Gamma . \end{aligned}$$

Now

$$\begin{aligned} \langle \nabla \cdot \mathbf{u}, \varphi - \bar{\varphi} \rangle &= (\nabla \cdot \mathbf{u}, - \Delta \psi) \\ &= (\nabla (\nabla \cdot \mathbf{u}), \nabla \psi) - \langle \nabla \cdot \mathbf{u}, \partial \psi / \partial n \rangle \\ &= (\nabla \times \nabla \times \mathbf{u} + \Delta \mathbf{u}, \nabla \psi) \\ &= (\nabla \times \mathbf{u}, \nabla \times (\nabla \psi)) - \langle \nabla \times \mathbf{u} \times \mathbf{n}, \nabla \psi \rangle - (\mathbf{F}, \nabla \psi) \\ &= - \langle \mathbf{h} \times \mathbf{n}, \nabla \psi \rangle + (\nabla \cdot \mathbf{F}, \psi) - \langle \mathbf{F} \cdot \mathbf{n}, \psi \rangle \\ &= \langle \nabla \times \mathbf{h} \cdot \mathbf{n}, \psi \rangle + (\nabla \cdot \mathbf{F}, \psi) - \langle \mathbf{F} \cdot \mathbf{n}, \psi \rangle \\ &= 0 . \end{aligned} \tag{33}$$

Again, the last equality results from the compatibility condition on $\mathbf{F}$. Equation (33) then implies

$$(\nabla \cdot \mathbf{u} - \overline{\nabla \cdot \mathbf{u}}, \varphi) = (\nabla \cdot \mathbf{u}, \varphi - \bar{\varphi}) = 0 .$$

Hence

$$\nabla \cdot \mathbf{u} = \overline{\nabla \cdot \mathbf{u}} .$$

Now from (2) we have that

$$\langle \nabla \cdot \mathbf{u}, \mathbf{g} \cdot \mathbf{n} \rangle = \overline{\nabla \cdot \mathbf{u}} \int_{\Gamma} \mathbf{g} \cdot \mathbf{n} \, ds = 0 .$$

Thus, choosing $\mathbf{v}$ in (32) such that $\mathbf{v} = \mathbf{u} - \mathbf{g}$ on Γ, it follows that

$$|\Omega| \, \overline{\nabla \cdot \mathbf{u}}^2 + \omega \, \langle (\mathbf{u} - \mathbf{g}) \cdot \mathbf{n}, (\mathbf{u} - \mathbf{g}) \cdot \mathbf{n} \rangle_{\alpha, \Gamma} = 0 .$$

Therefore

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega ,$$

and

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n} \quad \text{on} \quad \Gamma .$$

This completes the proof of Theorem 3.4.

Remark 3.3 : As we have seen, in both variational formulations, the divergence free condition is not imposed on the spaces of test functions. In fact, they are taken to be $\mathbf{H}_T(\Omega)$ and $\mathbf{H}_\alpha(\Omega)$, respectively. In particular when $\alpha = 1/2$ the space of test functions is all of $H^1(\Omega)^3$. And yet the solution $\mathbf{u}$ of our variational problem satisfies $\nabla \cdot \mathbf{u} = 0$. This is particularly useful in our approximate problem since the divergence free condition might be difficult to impose.

4. FINITE ELEMENT APPROXIMATIONS

4.1. Approximating Spaces and Discrete Formulations

We will only consider the discrete approximations to the second variational formulation (S_v) introduced in Section 3. The discrete approximations corresponding to the first variational formulation can be given similarly and hence are omitted here.

We will need spaces of approximating functions for the pressure and approximating vectors for the velocity. To this end let $\{S^h\}$ be a family of finite dimensional subspaces of $H^1(\Omega) \cap L_0^2(\Omega)$ and or $m \geq 2$, an integer, we assume that for any $\varphi \in H^k(\Omega)$,

$$\inf_{\chi \in S^h} \|\varphi - \chi\|_{j, \Omega} \leq \mathrm{C}h^{k-j} \|\varphi\|_{k, \Omega} , \tag{34}$$

for $1 \leq j \leq k \leq m$. Furthermore, let $\{\mathscr{S}^h\} \subset \mathbf{H}_\alpha(\Omega)$ be a family of finite dimensional spaces of vector functions such that the following approximation property is satisfied. For $m \geq 2$ and $1 \leq s \leq m$, we have

$$\inf_{\mathbf{v}^h \in \mathscr{S}^h} (\|\mathbf{w} - \mathbf{v}^h\|_{0, \Omega}^2 + h^2 |\mathbf{w} - \mathbf{v}^h|_{1, \Omega}^2) \leq \mathrm{C}h^{2s} \|\mathbf{w}\|_{s, \Omega}^2 , \tag{35}$$

for any $\mathbf{w} \in H^s(\Omega)^3$.

Remark 4.1 : Finite element spaces satisfying (34) may be found in [6] for instance. Perhaps the simplest and most common example is that in which the elements in S^h are piecewise linear functions on a given triangulation of Ω. Similarly, the components of the vectors in $\mathcal{S}^h$ may be chosen in the same way.

Since $p \in L_0^2$ we will approximate p_h in $S_0^h = S^h \cap L_0^2$. Our finite element approximation to problem (1) is defined as follows.

Find $p^h \in S_0^h$ and $\mathbf{u}_h \in \mathcal{S}^h$ such that

$$(\nabla p^h, \nabla \chi) = (\mathbf{f}, \nabla \chi) - \langle \nabla \times \mathbf{h} \cdot \mathbf{n}, \chi \rangle, \quad \text{for all} \quad \chi \in S_0^h \tag{36}$$

and

$$B_{\alpha, h^{2\alpha-1}}(\mathbf{u}^h, \mathbf{v}^h) = (\mathbf{F}^h, \mathbf{v}^h) + h^{2\alpha-1} \langle \mathbf{g} \cdot \mathbf{n}, \mathbf{v}^h \cdot \mathbf{n} \rangle_{\alpha, \Gamma},$$
$$\text{for all} \quad \mathbf{v}^h \in \mathcal{S}^h, \tag{37}$$

where $\mathbf{F}^h = \mathbf{f} - \nabla p^h$.

Remark 4.2. the choice of $\omega = h^{2\alpha-1}$ is made for the purpose of balancing the norms and leads to optimal error estimates for the finite element solutions.

4.2. Error estimates

We will now prove the error estimates for the solutions of finite element approximations (36) and (37). In the following, $|||\cdot|||_{\alpha, h}$ will be used to denote the norm induced by the bilinear form $B_{\alpha, h^{2\alpha-1}}(\cdot, \cdot)$.

THEOREM 4.1. *Let* $\mathbf{f} \in L^2(\Omega)^3$, $\mathbf{g} \cdot \mathbf{n} \in H^{3/2}(\Gamma)$ *and* $\mathbf{h} \times \mathbf{n} \in H^{1/2}(\Gamma)^3$. *Let* $(\mathbf{u}, p)$ *be the solution of* (1) *and* $(\mathbf{u}^h, p^h)$ *be the solutions of* (37) *and* (36), *respectively. Then*

$$\|p - p^h\|_{0,\Omega} + |||\mathbf{u} - \mathbf{u}^h|||_{\alpha, h} \le C h (\|p\|_{1,\Omega} + \|\mathbf{u}\|_{2,\Omega}). \tag{38}$$

Proof. The estimates for $p - p_h$ are standard. We derive them here for completeness. As usual

$$(\nabla(p - p^h), \nabla \chi) = 0, \quad \textit{for all} \quad \chi \in S_0^h$$

implies that, for any $\chi \in S_0^h$,

$$\|\nabla(p - p^h)\|_{0,\Omega}^2 = (\nabla(p - p^h), \nabla(p - \chi))$$
$$\le \|\nabla(p - p^h)\|_{0,\Omega} \|\nabla(p - \chi)\|_{0,\Omega}.$$

Hence

$$\|\nabla(p - p^h)\|_{0,\Omega} \le \inf_{\chi \in S^h} \|\nabla(p - \chi)\|_{0,\Omega}.$$

This is the usual H^1 estimate for $p - p^h$. Now, to derive the L^2 estimate we use the standard duality argument. Let ω be the solution in $H^2(\Omega) \cap L_0^2$ of

$$\begin{aligned} -\Delta\omega &= p - p^h \\ \partial\omega/\partial n &= 0 . \end{aligned}$$

Note that $p - p^h \in L_0^2(\Omega)$, so that ω is well defined. Then we see that

$$\begin{aligned} \|p - p^h\|_{0,\Omega} &= (p - p^h, -\Delta\omega) \\ &= (\nabla(p - p^h), \nabla\omega) \\ &\leq \|\nabla(p - p^h)\|_{0,\Omega} \inf_{\chi \in S^h} \|\nabla(\omega - \chi)\|_{0,\Omega} \\ &\leq Ch\|\nabla(p - p^h)\|_{0,\Omega}\|\omega\|_{2,\Omega} . \end{aligned}$$

Using the *a priori* estimate, *cf.* [14],

$$\|\omega\|_{2,\Omega} \leq C\|p - p^h\|_{0,\Omega},$$

we conclude that

$$\|p - p^h\|_{0,\Omega} \leq Ch\|\nabla(p - p^h)\|_{0,\Omega} . \tag{39}$$

This is the standard L^2 estimate for $p - p^h$.

We next define the projection P_1^h of $H^1(\Omega)^3$ onto $\mathcal{S}^h$ by

$$B_{\alpha, h^{2\alpha-1}}(\mathbf{u} - P_1^h\mathbf{u}, \mathbf{v}^h) = 0, \quad \text{for all} \quad \mathbf{v}^h \in \mathcal{S}^h .$$

Then

$$|||\mathbf{u} - \mathbf{u}^h|||_{\alpha,h} \leq |||\mathbf{u} - P_1^h\mathbf{u}|||_{\alpha,h} + |||\mathbf{u}^h - P_1^h\mathbf{u}|||_{\alpha,h} .$$

The first term can be estimated by approximation property

$$|||\mathbf{u} - P_1^h\mathbf{u}|||_{\alpha,h} \leq \inf_{\mathbf{v}^h \in \mathcal{S}^h} |||\mathbf{u} - \mathbf{v}^h|||_{\alpha,h} . \tag{40}$$

For the second term, we have

$$\begin{aligned} &|||\mathbf{u}^h - P_1^h\mathbf{u}|||_{\alpha,h}^2 \\ &\quad = B_{\alpha, h^{2\alpha-1}}(\mathbf{u}^h - \mathbf{u}, \mathbf{u}^h - P_1^h\mathbf{u}) \\ &\quad = (\nabla(p - p^h), \mathbf{u}^h - P_1^h\mathbf{u}) \\ &\quad = -(p - p^h, \nabla\cdot(\mathbf{u}^h - P_1^h\mathbf{u})) + \langle p - p^h, (\mathbf{u}^h - P_1^h\mathbf{u})\cdot\mathbf{n}\rangle \end{aligned}$$

$$\leqslant \|p - p^h\|_{0,\Omega} \|\nabla \cdot (\mathbf{u}^h - P_1^h \mathbf{u})\|_{0,\Omega}$$
$$+ (h^{1/2-\alpha} |p - p^h|_{-\alpha,\Gamma})(h^{-1/2+\alpha} |(\mathbf{u}^h - P_1^h \mathbf{u}) \cdot \mathbf{n}|_{\alpha,\Gamma})$$
$$\leqslant (\|p - p^h\|_{0,\Omega}^2 + h^{1-2\alpha} |p - p^h|_{-\alpha,\Gamma}^2)^{1/2} |||\mathbf{u}^h - P_1^h \mathbf{u}|||_{\alpha,h} . \quad (41)$$

In (41),

$$|p - p^h|_{-\alpha,\Gamma} \equiv \sup_{\varphi \in H^\alpha(\Gamma)} \frac{\langle p - p^h, \varphi \rangle}{|\varphi|_{\alpha,\Gamma}} .$$

Hence

$$|||\mathbf{u}^h - P_1^h \mathbf{u}|||_{\alpha,h}^2 \leqslant (\|p - p^h\|_{0,\Omega}^2 + h^{1-2\alpha} |p - p^h|_{-\alpha,\Gamma}^2) .$$

Because of (39), we only need to show

$$h^{1/2-\alpha} |p - p^h|_{-\alpha,\Gamma} \leqslant C h \|\nabla(p - p^h)\|_{0,\Omega} . \quad (42)$$

For this purpose, we define ψ such that

$$\begin{aligned} -\Delta\psi + \psi &= 0 \quad \text{in} \quad \Omega \\ \partial\psi/\partial n &= \varphi \quad \text{on} \quad \Gamma . \end{aligned}$$

Then

$$\begin{aligned} \langle p - p^h, \varphi \rangle &= \langle p - p^h, \partial\psi/\partial n \rangle \\ &= (\nabla(p - p^h), \nabla\psi) + (p - p^h, \Delta\psi) \\ &= (\nabla(p - p^h), \nabla(\psi - \chi)) + (p - p^h, \psi) , \end{aligned}$$

for arbitrary χ in S_0^h. Therefore,

$$|p - p^h|_{-\alpha,\Gamma}$$
$$\leqslant \sup_{\varphi \in H^\alpha(\Gamma)} \left(\frac{\|\nabla(p - p^h)\|_{0,\Omega} \|\nabla(\psi - \chi)\|_{0,\Omega}}{|\varphi|_{\alpha,\Gamma}} + \frac{\|p - p^h\|_{0,\Omega} \|\psi\|_{0,\Omega}}{|\varphi|_{\alpha,\Gamma}} \right)$$
$$\leqslant C h^{1/2+\alpha} \|\nabla(p - p^h)\|_{0,\Omega} \sup_{\varphi \in H^\alpha(\Gamma)} \frac{\|\psi\|_{\alpha+3/2,\Omega}}{|\varphi|_{\alpha,\Gamma}}$$
$$+ C h \|\nabla(p - p^h)\|_{0,\Omega} \sup_{\varphi \in H^\alpha(\Gamma)} \frac{\|\psi\|_{0,\Omega}}{|\varphi|_{\alpha,\Gamma}}$$
$$\leqslant C h^{1/2+\alpha} \|\nabla(p - p^h)\|_{0,\Omega} ,$$

which completes the proof.

Similarly, we can show.

THEOREM 4.2 : *Let Γ be sufficiently smooth*, $\mathbf{f} \in H^r(\Omega)^3$, $\mathbf{g} \cdot \mathbf{n} \in H^{r+3/2}(\Gamma)$ *and* $\mathbf{h} \times \mathbf{n} \in H^{r+1/2}(\Gamma)^3$. *Let* $(\mathbf{u}, p)$ *be the solutions of* (1), $(\mathbf{u}^h, p^h)$ *be the solutions of* (37) *and* (36), *respectively. Then*

$$\|p - p^h\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}^h\|_{\alpha,h} \leq C h^{r+1}(\|p\|_{r+1,\Omega} + \|\mathbf{u}\|_{r+2,\Omega}), \quad (43)$$

for $0 \leq r \leq m - 1$.

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