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Super-geometrodynamics

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Abstract: We present explicit solutions of the time-symmetric initial value constraints, expressed in terms of freely specifiable harmonic functions for examples of supergravity theories, which emerge as effective theories of compactified string theory. These results are a prerequisite for the study of the time-evolution of topologically non-trivial initial data for supergravity theories, thus generalising the "Geometrodynamics" program of Einstein-Maxwell theory to that of supergravity theories. Specifically, we focus on examples of multiple electric Maxwell and scalar fields, and analyse the initial data problem for the general Einstein-Maxwell-Dilaton theory both with one and two Maxwell fields, and the STU model. The solutions are given in terms of up to eight arbitrary harmonic functions in the STU model. As a by-product, in order compare our results with known static solutions, the metric in isotropic coordinates and all the sources of the non-extremal black holes are expressed entirely in terms of harmonic functions. We also comment on generalizations to time-nonsymmetric initial data and their relation to cosmological solutions of gauged so-called fake supergravities with positive cosmological constant.

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Contents

1 Introduction

The consequences of non-trivial spacetime topology for the laws of physics has been a topic of perennial interest for theoretical physicists [1]. In its most recent reincarnation [2–4], it is the relationship between non-trivial spatial topology, Einstein-Rosen bridges, wormholes, non-orientable spacetimes, and quantum-mechanical entanglement which has been at issue.

Not so long ago [5, 6], it was the question of whether such structures would give rise to closed timelike curves and the possibility of constructing time machines.

Such discussions are largely a matter of principle, since it is unlikely that either astronomical observations or laboratory experiments can shed light on them. It is important therefore to be sure that the range of such Gedanken experiments is restricted by the requirement that they be consistent with our best current knowledge of the laws of physics. Thus although the literature on time travel and wormholes is replete with models which violate the usual energy conditions of classical general relativity (cf. [6]), it is more informative to restrict attention to theories consistent with this principle, and our current understanding of quantum gravity. For these reasons, supergravity theories, in particular those arising as an effective theories of string and M-theory are especially attractive. Of course their equations of motion include Einstein's vacuum equations and the Einstein-Maxwell equations as special cases, and so their use does not invalidate existing work that takes those into account. Nevertheless, it is worthwhile to ask what additional features arise when specifically stringy aspects, such as the dilaton and axion fields, are taken into account. Moreover, while a great deal is now known about supergravity static and stationary solutions, such as black holes, rather less is known about time-dependent solutions.

In fact our best information about time-dependent wormholes and Einstein-Rosen bridges comes from a study of the initial value constraints, which place restrictions on the allowed topology and geometry of possible Cauchy surfaces. Interestingly, the first hint that Cauchy surfaces in General Relativity may be topologically non-trivial came just a year after the theory's inception, with Flamm's [7] well-known isometric embedding of the equatorial plane of the Droste-Schwarzschild solution into Euclidean space $\mathbb{E}^3$ as the paraboloid of revolution

$$\sqrt{x^2+y^2} = 2M + \frac{z^2}{8M} \,. \tag{1.1}$$

Flamm limited his consideration to the exterior, $z > 0$, of what we now call the event horizon, and his illustration shows only half of the full paraboloid. Einstein and Rosen [8] appear to have been the first to take seriously the universe on the other side of what has come to be called the Einstein-Rosen throat. Later, the study of the time development of topologically non-trivial initial data was taken up by Wheeler under the name "Geometrodynamics" [9]. In a landmark paper, Misner and Wheeler [10] provided examples of simply-connected initial data for both the vacuum Einstein and the Einstein-Maxwell equations, with arbitrarily many Einstein-Rosen throats connecting many universes to one another. Misner [11, 12], followed by Lindquist [13], constructed non-simply connected examples, called "wormholes," and Brill and Lindquist [14] studied their energetics. With the development of black hole theory, it was recognised that the minimal 2-surfaces arising as a consequence of the non-trivial topology provided, in the time-symmetric case, examples of marginally trapped surfaces, and that these could be used to study the Penrose conjecture $A \leq 16\pi M^2$ relating the area and mass, and to provide bounds on the amount of gravitational radiation emitted during the future evolution of the data [15–19].

A key notion of the Geometrodynamics programme was the idea of "Charge without Charge." The Maxwell field was taken to be source free, and so a non-vanishing charge could only arise from "electric flux lines trapped in the topology of space." With the construction of ungauged supergravity theories it was realised that the Abelian gauge fields in such theories were source-free, and so the charges arising therein were therefore "central charges" [20] and as a consequence satisfied a Bogomol'nyi-Prasad-Sommerfield (BPS) bound [21], where the embedding of Einstein-Maxwell theory into $N = 2$ supergravity theory was employed.

In this paper we set out to construct time-symmetric initial date sets for supergravity theories with multiple gauge fields and dilaton-axion fields, focusing on theories in $D = 4$ dimensions. These theories typically arise as a sector of an effective theory of compactified string theories. A specific "minimal example" in this class is the Einstein-Maxwell-Dilaton model, with a dilaton-Maxwell coupling constant $a = 1$. While time-symmetric initial data sets with two arbitrary harmonic functions were constructed by Ortin [22], in this paper we extend and generalise the analysis to Einstein-Maxwell-Dilaton models with an arbitrary dilaton-Maxwell coupling constant a, and obtain further initial data sets, now depending on three arbitrary harmonic functions. We compare these results with those of known static non-extremal black holes, which we express solely in terms harmonic functions. Furthermore we also generalise these results to the case of Einstein-Maxwell-Dilaton model with *two* Maxwell fields.

An important observation of Ortin [22], which remains true for our solutions, is that if scalars, such as the dilaton and hence the string coupling constant, are present, they cannot in general be globally defined if the the initial manifold is not simply connected, as it would be the case for wormhole topologies. This is because his explicit solutions for the scalars are not single-valued. This would seem to have important implications for the considerations of [2–4]. This problem may possibly be avoided by considering only initial data for which the scalars vanish. It would also not necessarily be a problem if the scalars were axions.

Our next focus is on the study of time-symmetric initial data for the STU supergravity theory, a sector of maximally supersymmetric ungauged supergravity (which is a sector of toroidally compactified string theories), specified by four Maxwell fields $F_I^{\mu\nu}$ (I=1,2,3,4) and three dilaton-axion fields $a_\alpha + \mathrm{i}\, e^{-\varphi_\alpha}$ ($\alpha = 1, 2, 3$). Our results are applicable for time-symmetric initial data with four electric fields and three dilation fields turned on, and depend on eight arbitrary harmonic functions. In order to compare the initial data problem with the known four-charge electric solutions we express the metric and all the sources of such black holes in terms of specific harmonic functions.

The analysis of the time-symmetric initial data of the Einstein-Maxwell-Dilaton models allows us to map the problem to that of multi-scalar systems coupled to gravity, which we generalise to the case of an arbitrary number N of scalar fields in section 5. In section 6 we study the Penrose inequality for the time-symmetric data of the Einstein-Scalar system, and obtain numerical evidence that it is always satisfied. We conclude the paper with remarks on interaction energies for time-symmetric initial data. We also comment on generalizations to time-dependent data and implications for the study of cosmological solutions of gauged supergravities with positive cosmological constant, i.e. so-called "fake supergravities."

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2 The initial value problem

The purpose of this section is to review the formalism for the study of the time-evolution problem for theories depending upon a metric $g_{\mu\nu}$, one or more scalars ϕ_α, and one or more closed two-forms, or Maxwell fields, $F^I = dA^I$ whose equations of motion may be obtained from an action functional $S[g_{\mu\nu}, \phi_\alpha, A^I_\mu]$ that is invariant under the semi-direct product of diffeomorphisms and gauge transformations. For the sake of simplicity of exposition, we assume that the Maxwell fields have no sources. Our intention here is merely to describe the general framework that we shall be working with. For more complete and more rigorous accounts the reader is directed to [23, 24].

In subsection 2.1 we present the evolution equations and derive constraints, and in subsection 2.2 give the explicit form of constraints. In the subsection 2.3 we address time-symmetric date and also present the well known explicit results for the vacuum Einstein gravity and Einstein-Maxwell gravity. In the subsequent sections we shall focus on new results for an Einstein-Maxwell-Dilaton gravity model and STU models with multiple scalars and Maxwell fields.

2.1 Evolution equations and constraints

Varying the action with respect to $g_{\mu\nu}$ gives field equations of the form[1]

$$\mathfrak{E}^{\mu\nu} = \sqrt{-g}\, E^{\mu\nu} = 2\frac{\delta S}{\delta g_{\mu\nu}} = 0\,. \tag{2.1}$$

Infinitesimal diffeomorphisms generated by an arbitrary smooth vector field V^μ of compact support induce a variation of the metric of the form

$$\delta g_{\mu\nu} = V_{\mu;\nu} + V_{\nu;\mu}\,, \tag{2.2}$$

where $V^\mu = g^{\mu\nu} V_\nu$, which leaves the action unchanged. As a consequence we have the Bianchi-identity

$$E^{\mu\nu}{}_{;\nu} = 0\,. \tag{2.3}$$

Similar identities hold for the Maxwell fields: the field equations take the form

$$K^{I\mu} = {}^*F^{I\mu\nu}{}_{;\nu} = 0\,, \qquad J_I{}^\mu = G_I{}^{\mu\nu}{}_{;\nu} = 0\,, \tag{2.4}$$

where

$$\mathfrak{G}_I{}^{\mu\nu} = \sqrt{-g}\, G_I{}^{\mu\nu} = -\frac{\delta S}{\delta F^{I\mu\nu}}\,. \tag{2.5}$$

The analogues of (2.3) are

$$K^{I\mu}{}_{;\mu} = 0\,, \qquad J_I{}^\mu{}_{;\mu} = 0\,, \tag{2.6}$$

which may, like (2.3), be regarded as the consequence of the invariance of the action under gauge transformations.

[1] We shall use units where $8\pi G = 1$.

Introducing coordinates (t, x^i), such that the spacetime $\{\mathcal{M} = \mathbb{R} \times \mathcal{N}, g_{\mu\nu}\}$ is foliated by spacelike hypersurfaces $\mathcal{N}$ given by $t =$ constant, we may write (2.3) as

$$\partial_t \mathfrak{E}^{\mu t} + \partial_i \mathfrak{E}^{\mu i} + \Gamma_i{}^\mu{}_j\, \mathfrak{E}^{ij} + 2\Gamma_t{}^\mu{}_j\, \mathfrak{E}^{tj} + \Gamma_t{}^\mu{}_t\, \mathfrak{E}^{tt} = 0\,, \tag{2.7}$$

where $\Gamma_\mu{}^\sigma{}_\nu$ are the Christoffel symbols of the metric $g_{\mu\nu}$. One sees from (2.7) that the equations (2.1) split into evolution equations

$$E^{ij} = 0 \tag{2.8}$$

and constraint equations

$$E^{\mu t} = 0\,, \tag{2.9}$$

such that if the evolution equations (2.8) hold for all times, and the constraint equations at some initial time, $t = 0$ say, then by (2.7) the constraint equations (2.9) will hold for *all* time.

A similar argument shows that the constraint equations for the Maxwell fields are given by

$$K^{It} = 0 = J^t_I\,. \tag{2.10}$$

The first equation in (2.10) expresses the absence of *local* magnetic charge densities and the second, usually called the Gauss constraint, expresses the absence of *local* electric charge densities. The constraint that $E^{tt} = 0$ is usually referred to as the *Hamiltonian constraint* and the constraint that $E^{ti} = 0$ as the *momentum constraint* or *diffeomorphism constraint.* For the systems of equations we are considering in this paper there are no further constraints arising from the scalars ϕ_α, since they are not subject to additional gauge invariances.

2.2 The explicit form of the constraints

To make progress we need to write out the constraints explicitly in terms of the metric g_{ij} induced on the initial surface and some further data including its time derivative $\partial_t g_{ij}$. In our chosen coordinate system $x^\mu = (t, x^i)$, often referred to as a *slicing* of spacetime, the four-dimensional metric takes the form

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^i + N^j dt)\,. \tag{2.11}$$

All quantities in (2.11) depend in general on all four coordinates. N is a function on $\mathcal{N}$ called the lapse and N^i is a vector field on $\mathcal{N}$ called the shift. The coordinates x^i are Lie dragged along the integral curves or time lines of the vector field $\frac{\partial}{\partial t}$. The inverse metric is given by

$$g^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} = \frac{1}{N^2}\left(\frac{\partial}{\partial t} - N^k \frac{\partial}{\partial x^k}\right) \otimes \left(\frac{\partial}{\partial t} - N^k \frac{\partial}{\partial x^k}\right) + g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}\,. \tag{2.12}$$

If the shift vector N_i is non-vanishing, the vector field $\frac{\partial}{\partial t}$ is not orthogonal to the slices $t =$ constant. The unit normal is given by

$$n = n^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{N}\left(\frac{\partial}{\partial t} - N^k \frac{\partial}{\partial x^k}\right). \tag{2.13}$$

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A full basis for the tangent bundle may be obtained by augmenting n with an orthonormal frame $e_{\hat{i}}$ for the Riemannian manifold $\{\mathcal{N}, g_{ij}\}$. The second fundamental form K_{ij} for the hypersurface $t =$ constant is defined by

$$K_{ij} = -\frac{1}{2}\pounds_n g_{ij}\,, \tag{2.14}$$

where $\pounds_n$ denotes the Lie derivative with respect to the hypersurface unit vector field n. For the case of interest to us we have

$$E^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} - T^{\mu\nu}\,, \tag{2.15}$$

where $T^{\mu\nu}$ is the symmetric energy-momentum tensor of the matter fields $(\phi_\alpha, F^I_{\mu\nu})$. The Hamiltonian and momentum constraints (2.9) thus take the form

$$R_{\hat{t}\hat{t}} + \frac{1}{2}R = T_{\mu\nu}n^\mu n^\nu\,, \qquad R_{\mu\nu}n^\mu e^\mu_{\hat{i}} = T_{\mu\nu}n^\mu e^\mu_{\hat{i}}\,. \tag{2.16}$$

The left-hand sides of (2.16) may be expressed entirely in terms of the the metric g_{ij}, its Ricci scalar ${}^{(3)}R$ and the second fundamental form K_{ij} and its covariant derivative ${}^{(3)}\nabla^k K_{ij}$, where ${}^{(3)}\nabla^k$ is the covariant derivative with respect to the metric g_{ij}. To do so one uses the Gauss-Coddazi equations, which relate the Riemann tensor of $g_{\mu\nu}$ to the Riemann tensor of g_{ij}, the second fundamental form K_{ij} and its covariant derivative. One finally obtains the usual form of the Hamiltonian and momentum constraints

$$\begin{aligned} {}^{(3)}R + K^2 - K_{ij}K^{ij} &= 2T_{\mu\nu}\,n^\mu n^\nu \\ {}^{(3)}\nabla^j\big(K_{ij} - K\,g_{ij}\big) &= T_{\mu i}\,n^\mu\,, \end{aligned} \tag{2.17}$$

where

$$K = g^{ij}\,K_{ij}\,. \tag{2.18}$$

The initial data for the scalars are simply $(\phi_\alpha, \dot{\phi}_\alpha)$ on the initial time slice, where we define $\dot{f} = n^\mu \frac{\partial f}{\partial x^\mu}$ for any function f. Those for the Maxwell fields are the magnetic fields $B^I{}_i = {}^*F^I_{\mu i}\,n^\mu$ and electric inductions $D_{I\mu i} = -G_{I\mu i}n^\mu$, subject to the constraints (2.10), which amount to the requirement that both are divergence free,

$${}^{(3)}\nabla^i B^I{}_i = 0 = {}^{(3)}\nabla^i D_{Ii}\,. \tag{2.19}$$

As an example, for the case of the Einstein-Maxwell-Dilaton theory an initial data set is a seven-tuple $\{\mathcal{N}, g_{ij}, K_{ij}, B_i, D_i, \phi, \dot{\phi}\}$ consisting of a Riemannian 3-manifold $\{\mathcal{N}, g_{ij}\}$ and a symmetric tensor field K_{ij}, two functions $(\phi, \dot{\phi})$ and two vector fields B_i and D_i, subject to the (2.17) and (2.10).

2.3 The time-symmetric case

A enormous simplification arises if one assumes that the second fundamental form of the initial surface, which we take to be at $t = 0$, vanishes. The shift vector N^i also vanishes. Thus the Hamiltonian and momentum constraints (2.9) reduce to

$${}^{(3)}R = 2T_{\hat{t}\hat{t}}\,, \qquad T_{\hat{t}\hat{i}} = 0\,. \tag{2.20}$$

In our case, the simplest way to arrange that the second equation of (2.20) holds is to assume that

$$B^I{}_i = 0 = \dot{\phi}_\alpha \,. \tag{2.21}$$

The time development of data of this sort will give rise to a solution which is invariant under $t \to -t$, and the spacetime is said to admit a moment of time symmetry. ¿From a dynamical point of view, the system is instantaneously at rest at $t = 0$.

One may now adopt a scheme first proposed by Lichnerowicz [25]. One assumes that the metric g_{ij} is conformal to some time-independent background metric $\bar{g}_{ij}$, with

$$g_{ij} = \Phi^4 \bar{g}_{ij} \,. \tag{2.22}$$

The first equation of (2.20) now becomes

$$\frac{1}{\Phi^5}\big(-8\bar{g}^{ij}\ {}^{(3)}\bar{\nabla}_i\ {}^{(3)}\bar{\nabla}_j + {}^{(3)}\bar{R}\big)\Phi = 2T_{\mu\nu}n^\mu n^\nu \,, \tag{2.23}$$

which is sometimes referred to as Lichnerowicz's equation.

In principle, Lichnerowicz's method works for any background manifold $\{\mathcal{N}, \bar{g}_{ij}\}$. In practice the most useful cases have been

- The flat metric on Euclidean space $\mathbb{E}^3$. This is typically used to give asymptotically flat data.
- The round metric on the 3-sphere S^3. This has been used to give initial data for an inhomogeneous closed universe.
- The standard product metric on $S^2 \times S^1$. This has been used to give initial data for wormholes.

2.3.1 Vacuum data

The simplest case is to set

$$g_{ij} = \Phi^4 \delta_{ij} \,, \qquad \partial_i \partial_i \Phi = 0 \,. \tag{2.24}$$

In other words Φ is a harmonic function on Euclidean space. We may take

$$\Phi = 1 + \sum_{n=1}^{N} \frac{m_n}{2|\mathbf{x} - \mathbf{x}_n|} \,, \tag{2.25}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and we assume $m_a > 0$. If $N = 1$, and setting $m_1 = M$, we obtain the initial data for the Schwarzschild solution. Taking $\mathbf{x}_1 = 0$ and writing $\rho = |\mathbf{x}|$, we can compare with the Schwarzschild metric in isotropic coordinates, for which $\{\mathcal{N}, g_{ij}\}$ is manifestly conformally flat:

$$ds^2 = -\frac{F^2}{\Phi^2} dT^2 + \Phi^4 \Big\{ d\rho^2 + \rho^2 \big(d\theta^2 + \sin^2\theta d\varphi^2\big) \Big\} \,, \tag{2.26}$$

with

$$F = 1 - \frac{M}{2\rho} \,. \tag{2.27}$$

Changing to the familiar area coordinate

$$R = \rho\,\Phi^2 \tag{2.28}$$

transforms the metric (2.26) to

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dT^2 + \frac{dR^2}{1 - \frac{2M}{R}} + R^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big)\,. \tag{2.29}$$

If we instead take the background 3-metric to be

$$\bar{g}_{ij} dx^i dx^j = d\chi^2 + \sin^2\chi\big(d\theta^2 + \sin^2\theta d\varphi^2\big) \tag{2.30}$$

which is the round metric on S^3, and solve for a spherically solution of (2.23) with a simple poles at the north and south poles of S^3, i.e. at $\chi = 0$ and $\chi = \pi$, we find

$$\Phi = \sqrt{M}\sqrt{\frac{1+\sin\chi}{\sin^2\chi}} = \frac{\sqrt{M}}{2}\left(\frac{1}{\sin\frac{\chi}{2}} + \frac{1}{\cos\frac{\chi}{2}}\right). \tag{2.31}$$

Now setting

$$R - M = \frac{M}{\sin\chi} = \rho + \frac{M^2}{4\rho}\,, \tag{2.32}$$

one finds that $\Phi^4\,\bar{g}_{ij}$ coincides with the Schwarzschild initial data, i.e. with (2.25) with $N = 1$ and $m_1 = M$. The event horizon is mapped to the equator of S^3, i.e. to $\chi = \frac{\pi}{2}$.

Since Lichnerowicz's equation (2.23) for Ψ is linear in the vacuum case, one may now superpose solutions, but centred on different points on S^3, as in [26], to obtain initial data for a time-symmetric closed universe of black holes (cf. [27, 28]).

Finally, if

$$\bar{g}_{ij} dx^i dx^j = a^2\big\{d\mu^2 + d\theta^2 + \sin^2\theta d\varphi^2\big\} \tag{2.33}$$

and the coordinate μ is taken to be periodic with period $2\mu_0$, we obtain the standard product metric on $S^1 \times S^2$. The function

$$\Phi = \frac{1}{\sqrt{\cosh\mu - \cos\theta}} \tag{2.34}$$

satisfies (2.23), and if

$$\begin{aligned} x &= a\frac{\sin\theta\cos\varphi}{\sqrt{\cosh\mu - \cos\theta}}\,, \\ y &= a\frac{\sin\theta\sin\varphi}{\sqrt{\cosh\mu - \cos\theta}}\,, \\ z &= a\frac{\sinh\mu}{\sqrt{\cosh\mu - \cos\theta}}\,, \end{aligned} \tag{2.35}$$

one finds that

$$\Phi^4 a^2\big(d\mu^2 + d\theta^2 + \sin^2\theta d\varphi^2\big) = dx^2 + dy^2 + dz^2\,. \tag{2.36}$$

In this case (2.23) is linear [11], and we may superpose solutions as

$$\Phi = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{\cosh(\mu + 2n\mu_0) - \cos\theta}}\,, \tag{2.37}$$

and we obtain Misner's asymptotically-flat wormhole data on $(S^1 \times S^2) \setminus \infty$. This example may also be obtained using the method of images. Misner showed also how to obtain more complicated non-simply connected examples using this method [12].

2.3.2 Einstein-Maxwell data

Since in this case we have no scalars, the electric field E_i is equal to the electric induction D_i. The initial-value constraints therefore reduce to

$$^{(3)}\bar{R} = 2g^{ij}E_iE_j\,, \qquad ^{(3)}\nabla^i E_i = 0\,. \tag{2.38}$$

For a flat background metric, $\bar{g}_{ij} = \delta_{ij}$, Misner and Wheeler [10] showed that if

$$\Phi = (CD)^{\frac{1}{2}}\,, \qquad E_i = \frac{D\partial_i C - C\partial_i D}{CD} = \partial_i \log\frac{C}{D}\,, \tag{2.39}$$

and C and D are two arbitrary harmonic functions on Euclidean space, then the initial-value constraints will be satisfied. Note that in fact E_i is curl free,

$$\partial_i E_j - \partial_j E_i = 0\,, \tag{2.40}$$

but this is irrelevant as far as the initial-value problem is concerned. Later we shall see that in more complicated examples it is not the case that E_{Ii} is curl-free.

To obtain regular initial data for N black holes one chooses

$$C = 1 + \sum_{n=1}^{N} \frac{m_n - q_n}{2|\mathbf{x} - \mathbf{x}_n|}\,, \qquad D = 1 + \sum_{n=1}^{N} \frac{m_n + q_n}{2|\mathbf{x} - \mathbf{x}_n|}\,, \tag{2.41}$$

with $m_n \geq |q_n|$. Taking $N = 1$, $\mathbf{x}_1 = 0$, $m_1 = M$, $q_1 = Q$ and $|\mathbf{x}| = \rho$, we obtain the initial data for the Reissner-Nordström metric in isotropic coordinates,

$$ds^2 = -\frac{E^2F^2}{C^2D^2}\,dT^2 + C^2D^2\Big\{d\rho^2 + \rho^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big)\Big\} \tag{2.42}$$

with

$$\begin{aligned} C &= 1 + \frac{M-Q}{2\rho}\,, & D &= 1 + \frac{M+Q}{2\rho} \\ E &= 1 + \frac{\sqrt{M^2-Q^2}}{2\rho}\,, & F &= 1 - \frac{\sqrt{M^2-Q^2}}{2\rho}\,. \end{aligned} \tag{2.43}$$

Using the Schwarzschild area coordinate R, given by (2.28), we find the metric takes the standard form

$$ds^2 = -\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)dT^2 + \frac{dR^2}{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} + R^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big)\,. \tag{2.44}$$

The initial data for the Majumdar-Papapetrou multi-black hole solutions

$$ds^2 = -H^{-2}dT^2 + H^2 d\mathbf{x}^2 \tag{2.45}$$

is obtained by setting $C = 1$, $D = H$. To obtain regular solutions one sets $m_a = q_a$ in (2.41).

It may be verified that for a non-flat background metric $\bar{g}_{ij}$, it suffices to replace C and D in (2.39) by solutions of

$$\left(-{}^{(3)}\bar{\nabla}_i\,{}^{(3)}\bar{\nabla}^i + \frac{1}{8}\,{}^{(3)}\bar{R}\right)C = 0 = \left(-{}^{(3)}\bar{\nabla}_i\,{}^{(3)}\bar{\nabla}^i + \frac{1}{8}\,{}^{(3)}\bar{R}\right)D\,. \tag{2.46}$$

For recent work on the numerical evolution of Einstein- Maxwell initial data the reader is directed to [29–31].

3 Einstein-Maxwell-dilaton theory

We consider the theory described by the Lagrangian

$$\mathcal{L} = \sqrt{-g}\,(R - 2(\partial\phi)^2 - e^{-2a\phi}\,F^2)\,, \tag{3.1}$$

where coupling a is arbitrary. The case with $a = 1$ is typically considered as a prototype of a sector of an effective theory arising from a compactification of string theory. It was this example which was first addressed for the time-symmetric initial data study in [22], with a two harmonic function Ansatz.

We shall take time-symmetric initial data constraints, with the magnetic field set to zero. The constants are therefore given by

$${}^{(3)}R = 2g^{ij}(\nabla_i\phi\nabla_j\phi + E_i D_j)\,, \tag{3.2}$$

$$\nabla^i D_i = 0\,, \tag{3.3}$$

where $D_i = e^{-2a\Phi}E_i$.

3.1 Ansatz for initial data using two harmonic functions

We shall first start with the two-harmonic function Ansatz. In particular, Case (1) generalizes results of [22] to an arbitrary coupling a. Case (2) is new, since it allows for electric field which is not a gradient of a potential. In the next subsection we will provide a further generalization to three harmonic functions.

The initial data Ansatz is

$$ds_3^2 = \Phi^4\,dx^i dx^i\,, \tag{3.4}$$

with

$$\Phi = C^{\frac{1}{4}\gamma}\,D^{\frac{1}{4}\delta}\,, \qquad e^{-2a\phi} = C^\mu\,D^\nu\,, \tag{3.5}$$

where C and D will be assumed to be harmonic functions in the flat 3-metric δ_{ij}, i.e $\partial_i\partial_i C = \partial_i\partial_i D = 0$. The exponents γ, δ, μ and ν will be determined below. Note that the Ricci scalar ${}^{(3)}R$ for the Ansatz (3.4) is given by

$${}^{(3)}R = -8\Phi^{-5}\,\partial_i\partial_i\Phi\,. \tag{3.6}$$

We shall deduce the required form for the initial data for the electric field by imposing the constraints (3.2) and (3.3).

Starting with (3.2), we have

$$-\,4\Phi^{-1}\,\partial_i\partial_i\Phi-(\partial_i\phi)(\partial_i\phi)=e^{-2a\phi}\,E_iE_i\,. \tag{3.7}$$

Substituting in the Ansatz (3.5), where C and D are harmonic, we seek to write the left-hand side of (3.7) as a perfect square,

$$\Big(x\,\frac{\partial_i C}{C}+y\,\frac{\partial_i D}{D}\Big)^2\,, \tag{3.8}$$

which implies the conditions

$$\gamma\Big(1-\frac{1}{4}\gamma\Big)-\frac{\mu^2}{4a^2}=x^2\,,\qquad \delta\Big(1-\frac{1}{4}\delta\Big)-\frac{\nu^2}{4a^2}=y^2\,,\qquad \gamma\delta+\frac{\mu\nu}{a^2}=-4xy\,. \tag{3.9}$$

We can then make the natural Ansatz

$$E_i=e^{a\phi}\,\Big(x\frac{\partial_i C}{C}+y\frac{\partial_i D}{D}\Big)\,. \tag{3.10}$$

The constraint (3.3), which is $\partial_i(\Phi^2\,e^{-2a\phi}\,E_i)=0$, implies

$$\partial_i\Big[C^{\frac12\gamma+\frac12\mu}\,D^{\frac12\delta+\frac12\nu}\,\Big(x\frac{\partial_i C}{C}+y\frac{\partial_i D}{D}\Big)\Big]=0\,. \tag{3.11}$$

For harmonic C and D, this will be satisfied provided the terms proportional to $(\partial_i C)^2$, $\partial_i D)^2$ and $\partial_i C\,\partial_i D$ vanish. This gives the conditions

$$(\gamma+\mu-2)\,x=0\,,\qquad (\delta+\nu-2)\,y=0\,,\qquad (\delta+\nu)\,x+(\gamma+\mu)\,y=0\,, \tag{3.12}$$

and hence

$$\mu=2-\gamma\,,\qquad \nu=2-\delta\,,\qquad y=-x\,. \tag{3.13}$$

The first two equations in (3.9) then imply

$$(\gamma-\delta)(\gamma+\delta-4)=0\,. \tag{3.14}$$

This has two possible solutions,

$$\gamma+\delta=4\,,\qquad \text{or}\qquad \gamma=\delta\,. \tag{3.15}$$

Let us call these Case (1) and Case (2) respectively.

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3.1.1 Case (1)

For Case (1), where $\gamma+\delta=4$, the third equation in (3.9) is automatically consistent with $y=-x$, and hence we may write the various exponents in terms of a single parameter α, with

$$\gamma=2-2\alpha\,,\qquad \delta=2+2\alpha\,,\qquad \mu=2\alpha\,,\qquad \nu=-2\alpha\,. \tag{3.16}$$

Thus we have now satisfied the constraints (3.2) and (3.3), with the initial value data

$$\Phi^2=C^{1-\alpha}\,D^{1+\alpha}\,,\qquad e^{-2a\phi}=\Big(\frac{C}{D}\Big)^{2\alpha}\,,\qquad E_i=-\frac{x}{\alpha}\,\partial_i\Big(\frac{C}{D}\Big)^{-\alpha}\,, \tag{3.17}$$

with

$$x^2=1-\alpha^2-\frac{\alpha^2}{a^2}\,. \tag{3.18}$$

Note that in this Case (1) example, the electric field E_i can in fact be written as the gradient of a potential, as in (3.17). This is not a universal feature, or requirement, for initial value data, as we shall see in later examples. The special case when $a=1$ was obtained by Ortin [22]. The special case $\alpha=0$ implies $\phi=0$, and (3.17) reduces to the Einstein-Maxwell initial data discussed in subsection 2.3. The special case where $x=0$ reduces to the Einstein-Scalar initial data given in [22].

3.1.2 Case (2)

Turning now to the Case (2) example, where $\gamma=\delta$ as in the second option in (3.15), we may parameterise the indices in terms of a free constant λ, with

$$\gamma=\delta=2\lambda\,,\qquad \mu=\nu=2-2\lambda\,. \tag{3.19}$$

The condition that the third equation in (3.9) be consistent with $y=-x$ then implies either $\lambda=1$ (in which case the dilaton vanishes and we are back to the Einstein-Maxwell theory), or else

$$\lambda=\frac{1}{1+a^2}\,. \tag{3.20}$$

Thus in Case (2), we have the initial value data

$$\Phi^2=(CD)^{\frac{1}{1+a^2}}\,,\qquad e^{-2a\phi}=(CD)^{\frac{2a^2}{1+a^2}}\,,\qquad E_i=\frac{1}{\sqrt{1+a^2}}\,(CD)^{-\frac{a^2}{1+a^2}}\,\partial_i\log\frac{C}{D}\,. \tag{3.21}$$

Note that E_i is curl-free only if $a=0$, which reduces to the Einstein-Maxwell case. For all non-zero a, the Case (2) initial data uses an electric field that cannot be written as the gradient of a scalar potential. For this reason, it was not obtained in the analysis in [22].

3.2 A generalisation with three harmonic functions

Here, we construct an Ansatz for time-symmetric initial data that depends upon three independent harmonic functions, thus providing a further generalization of the Case (2), presented in the previous subsection. Our motivation for seeking this generalisation was provided by considering some known *non-extremal* static black hole solutions (to be discussed in the next subsection), and also by considering certain specialisations of the initial data for STU supergravity (to be discussed in section 4 below).

Our starting point, with the usual 3-metric $ds_3^2 = \Phi^4\, dx^i dx^i$, is the Ansatz

$$\Phi = C^{\gamma/4}\, D^{\delta/4}\, W^{\epsilon/4}\,, \qquad e^{-2a\phi} = C^{\mu}\, D^{\nu}\, W^{\sigma}\,, \tag{3.22}$$

where the exponents will be determined below. Assuming that C, D and W are harmonic, we substitute (3.22) into the left-hand side of (3.7), and seek to write it in the form

$$\Big(x\,\frac{\partial_i C}{C} + y\,\frac{\partial_i D}{D} + z\,\frac{\partial_i W}{W}\Big)^2\,. \tag{3.23}$$

This implies the conditions

$$\begin{aligned}
\gamma\Big(1-\frac{1}{4}\gamma\Big) - \frac{\mu^2}{4a^2} &= x^2\,, \qquad & \delta\epsilon + \frac{\nu\sigma}{a^2} &= -4yz\,,\\
\delta\Big(1-\frac{1}{4}\delta\Big) - \frac{\nu^2}{4a^2} &= y^2\,, \qquad & \gamma\epsilon + \frac{\mu\sigma}{a^2} &= -4xz\,,\\
\epsilon\Big(1-\frac{1}{4}\epsilon\Big) - \frac{\sigma^2}{4a^2} &= z^2\,, \qquad & \gamma\delta + \frac{\mu\nu}{a^2} &= -4xy\,.
\end{aligned} \tag{3.24}$$

From (3.7), this leads us to the Ansatz

$$E_i = e^{a\phi}\,\Big(x\,\frac{\partial_i C}{C} + y\,\frac{\partial_i D}{D} + z\,\frac{\partial_i W}{W}\Big) \tag{3.25}$$

for the electric field.

The constraint (3.3), which is $\partial_i(\Phi^2\, e^{-2a\phi}\, E_i) = 0$, then gives the conditions

$$\begin{aligned}
(\gamma+\mu-2)\,x &= 0\,, \qquad & (\epsilon+\sigma)\,y + (\delta+\nu)\,z &= 0\,,\\
(\delta+\nu-2)\,y &= 0\,, \qquad & (\epsilon+\sigma)\,x + (\gamma+\mu)\,z &= 0\,,\\
(\epsilon+\sigma-2)\,z &= 0\,, \qquad & (\delta+\nu)\,x + (\gamma+\mu)\,y &= 0\,.
\end{aligned} \tag{3.26}$$

It is easy to see that there is no solution where x, y and z are all non-zero. Without loss of generality, we may therefore proceed by taking $z=0$. The equations (3.24) and (3.26) then imply $y=-x$ and

$$\gamma=\delta=\frac{2}{1+a^2}\,, \qquad \mu=\nu=\frac{2a^2}{1+a^2}\,, \qquad \epsilon=-\sigma=\frac{4a^2}{1+a^2}\,. \tag{3.27}$$

Thus we arrive at the time-symmetric initial data

$$\begin{aligned}
\Phi^2 &= (CD)^{\frac{1}{1+a^2}}\, W^{\frac{2a^2}{1+a^2}}\,, \qquad e^{-2a\phi} = \Big(\frac{CD}{W^2}\Big)^{\frac{2a^2}{1+a^2}}\,,\\
E_i &= \frac{1}{\sqrt{1+a^2}}\, \Big(\frac{CD}{W^2}\Big)^{-\frac{a^2}{1+a^2}}\, \partial_i \log\frac{C}{D}\,,
\end{aligned} \tag{3.28}$$

where C, D and W are arbitrary harmonic functions. The electric field is not in general the gradient of a potential function. The expressions (3.28) reduce to those of the Case (2) initial data (3.21) if the function W is set equal to 1.

3.3 Some examples of known static solutions

Here we examine various examples of known static solutions, and show how their initial value data fit with the general classes that we obtained above.

3.3.1 Static multi-centre extremal solutions

Static multi-centre extremal solutions in the Einstein-Maxwell-Dilaton theory were constructed in [32], and are given by

$$\begin{aligned} ds^2 &= -C^{-\frac{2}{1+a^2}}\,dt^2 + C^{\frac{2}{1+a^2}}\,dx^i dx^i\,,\\ e^{-2a\phi} &= C^{\frac{2a^2}{1+a^2}}\,, \qquad A_\mu\,dx^\mu = \frac{1}{C}\,dt\,, \end{aligned} \tag{3.29}$$

where C is an arbitrary harmonic function in the flat metric $dx^i dx^i$. These solutions are extremal and saturate the BPS bound. The electric field in the initial data for this solution is therefore given by

$$E_i = -\frac{\sqrt{1+a^2}}{a^2}\,\partial_i C^{-\frac{a^2}{1+a^2}}\,. \tag{3.30}$$

Comparing with (3.21), we see that the multi-centre metrics correspond to a specialisation of the Case (2) initial data, in which the harmonic functions $D = 1$.

3.3.2 Non-extremal static black holes for general a

The theory described by the Lagrangian (3.1) has black hole solutions given by [33]

$$\begin{aligned} &ds^2 = -\Delta\,dt^2 + \Delta^{-1}\,dr^2 + R^2\,d\Omega_2^2\,,\\ &e^{-2a\phi} = F_-^{\frac{2a^2}{(1+a^2)}}\,, && A = q\cos\theta\,d\varphi\,,\\ &\Delta = F_+\,F_-^{\frac{(1-a^2)}{(1+a^2)}}\,, && R^2 = r^2\,F_-^{\frac{2a^2}{(1+a^2)}}\,,\\ &F_\pm = 1-\frac{r_\pm}{r}\,, && q = \sqrt{\frac{r_+\,r_-}{1+a^2}}\,. \end{aligned} \tag{3.31}$$

We can introduce the isotropic radial coordinate ρ, defined by

$$\log\rho = \int \frac{1}{r\,\sqrt{F_-\,F_+}}\,dr\,, \tag{3.32}$$

which implies that, with a convenient choice for the constant of integration,

$$r = \rho\left(1+\frac{u^2}{\rho}\right)\left(1+\frac{v^2}{\rho}\right), \tag{3.33}$$

where we have re-parameterised the constants $r_\pm$ as

$$r_+ = (u+v)^2\,, \qquad r_- = (u-v)^2\,. \tag{3.34}$$

In terms of the new quantities, we have

$$F_- = \frac{\left(1+\frac{uv}{\rho}\right)^2}{\left(1+\frac{u^2}{\rho}\right)\left(1+\frac{v^2}{\rho}\right)}\,, \qquad F_+ = \frac{\left(1-\frac{uv}{\rho}\right)^2}{\left(1+\frac{u^2}{\rho}\right)\left(1+\frac{v^2}{\rho}\right)}\,. \tag{3.35}$$

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The metric now takes the form

$$ds^2 = -\Delta\, dt^2 + \Phi^4\, (d\rho^2 + \rho^2\, d\Omega_2^2)\,, \tag{3.36}$$

where

$$\Phi^2 = \frac{R}{\rho} = \Big[\Big(1+\frac{u^2}{\rho}\Big)\Big(1+\frac{v^2}{\rho}\Big)\Big]^{\frac{1}{1+a^2}}\,\Big(1+\frac{uv}{\rho}\Big)^{\frac{2a^2}{1+a^2}}\,, \tag{3.37}$$

and with the dilaton given by

$$e^{2a\phi} = \Big[\Big(1+\frac{u^2}{\rho}\Big)\Big(1+\frac{v^2}{\rho}\Big)\Big]^{-\frac{2a^2}{1+a^2}}\,\Big(1+\frac{uv}{\rho}\Big)^{\frac{4a^2}{1+a^2}}\,. \tag{3.38}$$

The field strength $F = -q \sin\theta\, d\theta\wedge d\varphi$ has the Hodge dual

$$*F = -\frac{q\,\lambda}{f\,\rho^2}\, dt\wedge d\rho\,, \tag{3.39}$$

and hence we can define the dual electric field strength

$$\widetilde F \equiv e^{-2a\phi}\, *F = -\frac{q\sqrt{F_-\, F_+}}{\rho r}\, dt\wedge d\rho\,, \tag{3.40}$$

and so

$$\widetilde F_{t\rho} = -\frac{q}{\rho^2}\, \frac{\Big(1-\frac{uv}{\rho}\Big)\Big(1+\frac{uv}{\rho}\Big)}{\Big(1+\frac{u^2}{\rho}\Big)^2\Big(1+\frac{v^2}{\rho}\Big)^2}\,. \tag{3.41}$$

The electric field E_i in the initial data can be calculated from

$$\widetilde F^{\mu\nu}\, \widetilde F_{\mu\nu} = -2 g^{ij}\, E_i\, E_j\,, \tag{3.42}$$

and hence we find

$$E_\rho = \frac{q}{\rho^2}\, \frac{\Big(1+\frac{uv}{\rho}\Big)^{\frac{2a^2}{1+a^2}}}{\Big[\Big(1+\frac{u^2}{\rho}\Big)\Big(1+\frac{v^2}{\rho}\Big)\Big]^{\frac{1+2a^2}{1+a^2}}}\,. \tag{3.43}$$

Comparing with the general initial data sets that we derived in section 3.2, we see that (3.37), (3.38) and (3.43) correspond to the special case of (3.28) where the three harmonic functions are spherically symmetric, and given by

$$C = 1 + \frac{u^2}{\rho}\,, \qquad D = 1+\frac{v^2}{\rho}\,, \qquad W = 1+\frac{uv}{\rho}\,. \tag{3.44}$$

(Note that the sign of the dilaton in (3.38) is opposite to that in (3.28). This is because the spherically-symmetric solution (3.31) we are considering here is magnetic rather than electric. The sign of the dilaton reverses under dualisation.)

It is convenient to re-express the dilaton coupling a in terms of a parameter N, where

$$a^2 = \frac{4}{N} - 1\,. \tag{3.45}$$

The electric field is then given by

$$E_\rho = \frac{q}{\rho^2} \frac{\left(1+\frac{uv}{\rho}\right)^{2-N/2}}{\left[\left(1+\frac{u^2}{\rho}\right)\left(1+\frac{v^2}{\rho}\right)\right]^{2-N/4}}. \tag{3.46}$$

When $N = (1,2,3,4)$ we have $a = (\sqrt{3}, 1, \frac{1}{\sqrt{3}}, 0)$, corresponding to the dilaton couplings when N of the field strengths in the STU supergravity model are equated, with the remaining $4-N$ set to zero.

Because these solutions are spherically symmetric, the electric field in the initial data can always be written in terms of a potential, $E_i = -\partial_i Z$. For the $N = 2$ and $N = 4$ supergravity cases enumerated above, we have

$$\begin{aligned} N=2: \qquad & Z = -\frac{2q}{(u+v)^2} \frac{1-\frac{uv}{\rho}}{\left[\left(1+\frac{u^2}{\rho}\right)\left(1+\frac{v^2}{\rho}\right)\right]^{1/2}}, \\ N=4: \qquad & Z = \frac{q}{u^2-v^2} \log \frac{1+\frac{u^2}{\rho}}{1+\frac{v^2}{\rho}}. \end{aligned} \tag{3.47}$$

For general N (integer or non-integer) the potentials can also be found in closed form, but they involve the use of the hypergeometric function:.

$$Z = \frac{4q(u+v)^{N/2-3}\, u^{2-N/2}\; U^{N/4-1}}{(N-4)(u-v)} \; {}_2F_1\left[\frac{N}{2}-2, \frac{N}{4}-1, \frac{N}{4}; -\frac{v}{u} U\right], \tag{3.48}$$

where

$$U = \frac{1+\frac{u^2}{\rho}}{1+\frac{v^2}{\rho}}. \tag{3.49}$$

3.4 Time-symmetric initial data with two Maxwell fields

We conclude this section by pointing out that one can generalize the time-symmetric initial data results to the case of Einstein-Maxwell-Dilaton theory with two Maxwell fields. In this case the Lagrangian is of the form:

$$\mathcal{L} = \sqrt{-g}\left(R - 2(\partial\phi)^2 - e^{-2a\phi} F_1^2 - e^{-2b\phi} F_2^2\right), \quad b = -\frac{1}{a}. \tag{3.50}$$

The fixed choice of the dilation coupling b in terms of a is obtained by matching the Lagrangian to a consistent truncation of the STU model with two gauge fields which correspond to $a = 1$, $b = -1$, and $a = \sqrt{3}$, $b = -\frac{1}{\sqrt{3}}$, respectively.

The initial data Ansatz takes the form:

$$\begin{aligned} \Phi^2 &= (C_1 D_1)^{\frac{1}{1+a^2}} (C_2 D_2)^{\frac{a^2}{1+a^2}}, \qquad e^\phi = \left(\frac{C_1 D_1}{C_2 D_2}\right)^{-\frac{a}{1+a^2}}, \\ E_i^1 &= \frac{1}{\sqrt{1+a^2}} \left(\frac{C_1 D_1}{C_2 D_2}\right)^{-\frac{a^2}{1+a^2}} \partial_i \log\left(\frac{C_1}{D_1}\right), \end{aligned}$$

$$E_i^2 = \frac{a}{\sqrt{1+a^2}} \left(\frac{C_1 D_1}{C_2 D_2}\right)^{\frac{1}{1+a^2}} \partial_i \log\left(\frac{C_2}{D_2}\right). \tag{3.51}$$

It is interesting to note that if we set the two harmonic functions C_2 and D_2 equal in the above discussion, and, for convenience, define

$$C_1 = C\,, \qquad D_1 = D\,, \qquad C_2 = D_2 = W\,, \tag{3.52}$$

then the initial data given in (3.51) reduces precisely to the initial data (3.28) that we previously derived for the Einstein-Maxwell-Dilaton system, where $E_i^1 = E_i$ and $E_i^2 = 0$.

These data could in principle also be matched to examples of general static black hole solutions in this theory.

4 STU supergravity

Four-dimensional STU supergravity is comprised of the $\mathcal{N}=2$ pure supergravity multiplet coupled to three vector multiplets. Its gauged version can be obtained as the consistent truncation of $\mathcal{N}=8$ gauged SO(8) supergravity to its abelian $\mathrm{U}(1)^4$ subsector. It may also be viewed as $\mathcal{N}=2$ supergravity coupled to three vector multiplets. We shall set the gauge coupling constant to zero in our discussion, and focus just on the bosonic sector. There are six scalar fields in total, comprising a dilatonic and an axionic scalar in each of the three vector multiplets. We may consistently set the three axionic scalars to zero, provided at the same time we ensure that their sources, which are proportional to terms of the form $\epsilon^{\mu\nu\rho\sigma}\, F^I_{\mu\nu}\, F^J_{\rho\sigma}$, are vanishing. This can be achieved if we consider field configurations where the four field strengths have only electric, but not magnetic, components. The equations of motion for the remaining fields are then described by the Lagrangian[2]

$$\mathcal{L} = \sqrt{-g}\left[R - \frac{1}{2}\sum_{\alpha=1}^{3}(\partial\varphi_\alpha)^2 - \frac{1}{4}\sum_{I=1}^{4} X_I^{-2}\,(F^I)^2\right], \tag{4.1}$$

$$X_I = e^{-\frac{1}{2}\mathbf{a}_I\cdot\boldsymbol{\varphi}}\,, \qquad \boldsymbol{\varphi} = (\varphi_1,\varphi_2,\varphi_3)\,, \tag{4.2}$$

and we define

$$\mathbf{a}_1 = (1,1,1)\,, \qquad \mathbf{a}_2 = (1,-1,-1)\,, \qquad \mathbf{a}_3 = (-1,1,-1)\,, \qquad \mathbf{a}_4 = (-1,-1,1)\,. \tag{4.3}$$

The constraints for time-symmetric initial data will then be given by

$${}^{(3)}R = \frac{1}{2}g^{ij}\sum_\alpha \partial_i\varphi_\alpha\partial_j\varphi_\alpha + \frac{1}{2}g^{ij}\sum_I X_I^{-2}\,E_i^I E_j^I\,, \tag{4.4}$$

$${}^{(3)}\nabla^i(X_I^{-2}\,E_i^I) = 0\,. \tag{4.5}$$

[2]We are using the customary normalisations for the kinetic terms of STU supergravity here, which are smaller by a factor of 4 than those we have used for the other theories discussed in this paper.

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4.1 Time-symmetric initial data

We start with 8 arbitrary harmonic functions C_I, D_I, $I = 1,2,3,4$ on Euclidean space $\mathbb{E}^3$. The 3-metric is assumed to be given by

$$ds^2 = \Pi^{\frac{1}{2}} d\mathbf{x}^2 \tag{4.6}$$

where we have defined

$$\Pi \equiv \prod_{I=1}^{4} C_I D_I \,. \tag{4.7}$$

Thus we have

$$\Phi^2 = \Pi^{1/4} \,. \tag{4.8}$$

The scalars are given by

$$X_I = \frac{\Pi^{1/4}}{C_I D_I} \,. \tag{4.9}$$

In this example, we cannot in general express the electric fields E_i^I in terms of scalar potentials. A simple way to obtain expressions for E_i^I that are consistent with the constraints is first to substitute (4.8) and (4.9) into the constraint (4.4), since this leads us to a natural conjecture for E_i^I. Noting from (4.2) and (4.3) that

$$\partial_i\varphi_\alpha \partial_i\varphi_\alpha = \sum_{I=1}^{4} \left(\frac{\partial_i X_I\, \partial_i X_I}{X_I^2} \right) , \tag{4.10}$$

we find after a little algebra that

$$\Phi^4 \left({}^{(3)}R - \frac{1}{2} g^{ij}\, \partial_i\varphi_\alpha \partial_j\varphi_\alpha \right) = \frac{1}{2} \sum_{I=1}^{4} \left(\frac{\partial_i C_I}{C_I} - \frac{\partial D_I}{D_I} \right)^2 . \tag{4.11}$$

Thus the constraint (4.4) is satisfied if we take

$$E_i^I = X_I\, \partial_i \log \left(\frac{C_I}{D_I} \right) = \frac{\Pi^{1/4}}{C_I D_I}\, \partial_i \log \left(\frac{C_I}{D_I} \right) . \tag{4.12}$$

It remains to verify that the constraints (4.5) are satisfied. Thus we have

$$\partial_i(\Phi^2\, X_I^{-2}\, E_i^I) = \partial_i \left[C_I D_I\, \partial_i \log \left(\frac{C_I}{D_I} \right) \right] = D_I \nabla^2 C_I - C_I \nabla^2 D_I \,, \tag{4.13}$$

which indeed vanishes because C_I and D_I are harmonic. It is easy to verify that the curls of the electric fields E_i^I are non-vanishing, and so it is not possible to write them as the gradients of any potentials.

There are four special cases of the STU supergravity initial data that reduce to data for the Einstein-Maxwell-Dilaton system discussed in section 3; in particular, they all fit into the initial data with three harmonic functions, which we derived in section 3.2. They correspond to the truncations of the STU theory to the Einstein-Maxwell-Dilaton theory with $a = \sqrt{3}$, 1, $\frac{1}{\sqrt{3}}$ and 0. Modulo permutation choices, the specialisations of the initial data are:

$$\begin{aligned} a = \sqrt{3}: &\quad C_1 = C\,, \quad D_1 = D\,, \quad C_2 = C_3 = C_4 = D_2 = D_3 = D_4 = W\,, \\ a = 1: &\quad C_1 = C_2 = C\,, \quad D_1 = D_2 = D\,, \quad C_3 = C_4 = D_3 = D_4 = W\,, \\ a = \frac{1}{\sqrt{3}}: &\quad C_1 = C_2 = C_3 = C\,, \quad D_1 = D_2 = D_3 = D\,, \quad C_4 = D_4 = W\,, \\ a = 0: &\quad C_1 = C_2 = C_3 = C_4 = C\,, \quad D_1 = D_2 = D_3 = D_4 = D\,, \quad W = 1\,. \end{aligned} \tag{4.14}$$

There are two consistent truncations (modulo permutation) of the STU model to the Einstein-Maxwell-Dilaton theory with two Maxwell fields (3.50) and the following dilation couplings: $a = 1$, $b = -1$ and $a = \sqrt{3}$, $b = -\frac{1}{\sqrt{3}}$. These truncations result in four independent harmonic functions C_1, D_1, C_2 and D_2 remaining, namely

$$\begin{aligned} a = 1,\ b = -1: &\quad C_1 = C_3\,, \quad D_1 = D_3\,, \quad C_2 = C_4\,, \quad D_2 = D_4\,, \\ a = \sqrt{3},\ b = -\frac{1}{\sqrt{3}}: &\quad C_2 = C_3 = C_4\,, \quad D_2 = D_3 = D_4\,. \end{aligned} \tag{4.15}$$

4.2 Examples of known static solutions

Here we look at various examples of known static solutions in the STU supergravity theory, and show how their initial value data correspond to special cases of the above 8 harmonic function initial data. As a by-product we express the non-extremal static black hole spatial metric and all the sources in terms of eight specific harmonic functions.

4.2.1 Extremal multi-centre black holes

The general static extremal multi-centre black holes are given by

$$\begin{aligned} ds^2 &= -\Big(\prod_{I=1}^{4} C_I\Big)^{-1/2} dt^2 + \Big(\prod_{I=1}^{4} C_I\Big)^{1/2} dx^i dx^i\,, \\ X_I &= \Big(\prod_{I=1}^{4} C_I\Big)^{1/4} C_I^{-1}\,, \qquad A^I_\mu dx^\mu = -C_I^{-1}\, dt\,, \end{aligned} \tag{4.16}$$

where the C_I are arbitrary harmonic functions in the flat transverse metric $dx^i dx^i$. From this, we see that the electric fields E^I_i in the initial data are given by

$$E^I_i = \Big(\prod_{I=1}^{4} C_I\Big)^{1/4} C_I^{-2}\, \partial_i C_I\,. \tag{4.17}$$

These solutions saturate the BPS bound. The single centered solution was first obtained [34] in $N = 4$ supergravity and preserves $\frac{1}{4}$ of supersymmetry.

Comparing these solutions with the initial data (4.8), (4.9) and (4.12), we see that they correspond to the case $D_I = 1$.

4.2.2 Static spherically-symmetric non-extremal black holes

The static spherically symmetric solutions, first given in [35], take the form:

$$ds^2 = -\Big(\prod_{I=1}^4 H_I\Big)^{-\frac{1}{2}}\Big(1-\frac{2m}{r}\Big)dt^2 + \Big(\prod_{I=1}^4 H_I\Big)^{\frac{1}{2}}\Big\{\frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2\Big\},$$
$$H_I = 1+\frac{2m\sinh^2\delta_I}{r} \quad X_I = \Big(\prod_{I=1}^4 H_I\Big)^{1/4} H_I^{-1}, \quad A^I_\mu dx^\mu = (1-H_I^{-1})\coth\delta_I\, dt\,. \tag{4.18}$$

We define the isotropic radial coordinate ρ by $r = \rho + m + \frac{m^2}{4\rho}$ and find that

$$\frac{dr^2}{1-\frac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) = \Big(1+\frac{m}{2\rho}\Big)^4\Big\{d\rho^2 + \rho^2 d\Omega^2\Big\} \tag{4.19}$$

We now find that

$$\Big(1+\frac{m}{2\rho}\Big)^2 H_I = C_I D_I\,, \tag{4.20}$$

where C_I and D_I are spherically symmetric harmonic functions.[3]

$$C_I = 1+\frac{me^{2\delta_I}}{2\rho}\,, \qquad D_I = 1+\frac{me^{-2\delta_I}}{2\rho}\,. \tag{4.21}$$

Note that C_I and D_I, unlike H_I itself, are harmonic in the flat transverse 3-metric $d\rho^2 + \rho^2 d\Omega^2$.

In terms of the isotropic radial coordinate, the metric (4.18) becomes

$$ds^2 = -\Pi^{-1/2} f_+^2 f_-^2\, dt^2 + \Pi^{1/2}(d\rho^2 + \rho^2 d\Omega^2)\,, \tag{4.22}$$

where we have defined

$$\Pi = \prod_{1\le I\le 4} C_I D_I\,, \qquad f_\pm = 1 \pm \frac{m}{2\rho}\,. \tag{4.23}$$

The scalar fields and gauge potentials can be written as

$$X_I = \frac{\Pi^{1/4}}{C_I D_I}\,, \qquad A^I_\mu dx^\mu = \Big(-\frac{1}{C_I} + \frac{1}{D_I}\Big)dt\,. \tag{4.24}$$

The electric fields E^I_i on the initial data surface $t=$ constant are purely radial, and may be obtained by noting that

$$g^{ij}E^I_i E^I_j = -\frac{1}{2}F^I_{\mu\nu}F^{I\mu\nu}\,. \tag{4.25}$$

The result is that

$$E^I_\rho = -\frac{\Pi^{\frac{1}{4}} 2m\sinh\delta_I\cosh\delta_I}{\rho^2 C_I^2 D_i^2}\,. \tag{4.26}$$

It is now straightforward to verify that the initial data for this solution, given by $\Phi^2 = \Pi^{1/4}$ and $X_I = \Pi^{1/4}/(C_I D_I)$, and with E^I_i given by (4.26), corresponds to the special case of the general 8-function initial data (4.8), (4.9) and (4.12) where C_I and D_I are spherically symmetric and given by (4.21).

[3] Note that unless $m=0$, H_I are not harmonic functions with respect either of the metrics in braces in (4.18) or (4.19). In this paper we have used C and D, possibly subscripted, to denote generic harmonic functions.

5 Multi-scalar systems coupled to gravity

In this section we present some additional examples of time-symmetric initial for systems of scalar fields coupled to gravity. We begin by showing how all the cases we have discussed so far, involving one or more Maxwell fields, can be mapped into systems describing Einstein gravity coupled purely to scalar fields. The essential feature that allows this mapping is that in all the previous examples, the electric fields in the initial data are either expressible as the gradients of scalar functions, or else they are are proportional to the gradients of scalar functions. We also give a direct construction of a general new class of examples of Einstein gravity coupled to a system of scalar fields, and we show how the Einstein-Scalar systems obtained as mappings from systems with Maxwell fields are all special cases within this broader class.

5.1 Mapping Maxwell data to scalar data

It has been observed previously that the time-symmetric initial value problem for the Einstein-Maxwell system can be mapped into an equivalent initial value problem involving only scalar fields [22]. Let us consider for simplicity the case where the magnetic field vanishes, and hence the initial value constraints are given by (2.38). Since E_i is the gradient of a scalar in this case we can define $E_i = \partial_i \psi$, and so the Ricci constraint in (2.38) becomes

$$ {}^{(3)}R = 2g^{ij}\,\partial_i\psi\partial_j\psi\,. \tag{5.1}$$

This constraint is solved by the writing Φ and the scalar field ψ in terms of the two harmonic functions C and D as

$$\Phi^2 = CD\,, \qquad \psi = \log\frac{C}{D}\,. \tag{5.2}$$

We may now observe that a similar mapping of Maxwell data into data for a scalar field may be made in the more complicated theories that we have considered in this paper. For the Einstein-Maxwell-Dilaton system discussed in section 3, the Ricci constraint (3.2) for the Case (1) data in equation (3.17) may be rewritten as

$$ {}^{(3)}R = 2g^{ij}(\partial_i\phi\partial_j\phi + \partial_i\psi\partial_j\psi)\,, \tag{5.3}$$

where we have written

$$E_i = \Big(\frac{C}{D}\Big)^{-\alpha}\,\partial_i\psi \tag{5.4}$$

and the fields Φ, ϕ and ψ are expressed in terms of the harmonic functions C and D as

$$\Phi^2 = C^{1-\alpha}\,D^{1+\alpha}\,, \qquad \phi = -\frac{\alpha}{a}\log\frac{C}{D}\,, \qquad \psi = x\,\log\frac{C}{D}\,, \tag{5.5}$$

where $x = \sqrt{1-\alpha^2-\alpha^2/a^2}$. This provides initial data for the theory of Einstein gravity coupled to two scalar fields, described by the Lagrangian

$$\mathcal{L}_4 = \sqrt{-g}\,\Big(R - 2(\partial\phi)^2 - 2(\partial\psi)^2\Big)\,. \tag{5.6}$$

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The constants α and a in (5.5) are arbitrary parameters that may be chosen when specifying the initial data.

For the solution of the initial data for the Einstein-Maxwell-Dilaton system using three harmonic functions C, D and W, discussed in section 3.2, we may write

$$E_i = (CD)^{-\frac{a^2}{1+a^2}}\, W^{\frac{2a^2}{1+a^2}}\, \partial_i\psi\,, \tag{5.7}$$

and reinterpret the initial value problem as again being that for Einstein gravity coupled to two scalar fields, described by (5.6), with the initial constraint (5.3), and satisfied by the initial data

$$\Phi^2 = (CD)^{\frac{1}{1+a^2}}\, W^{\frac{2a^2}{1+a^2}}\,, \qquad \phi = -\frac{a}{1+a^2}\log\frac{CD}{W^2}\,, \qquad \psi = \frac{1}{\sqrt{1+a^2}}\log\frac{C}{D}\,. \tag{5.8}$$

Note that in both of the above examples the electric field E_i in the initial value data for the original Einstein-Maxwell-Dilaton theory is not curl-free, and thus cannot itself be written as the gradient of a scalar. Nonetheless, E_i is in each case proportional to a gradient, and that enables us map the initial value problem into one with a second scalar field instead of the electric field.

The initial value data that we obtained in section 4 for time-symmetric solutions of STU supergravity can also be mapped into data for an Einstein-Scalar system, this time with a total of seven scalar fields. We do this by noting from (4.12) that we may write

$$E_i^I = \frac{\Pi^{1/4}}{C_I D_I}\,\partial_i\psi_I\,, \tag{5.9}$$

for which the initial value Ricci constraint (4.4) becomes

$${}^{(3)}R = \frac{1}{2}g^{ij}(\partial_i\varphi_\alpha\partial_j\varphi_\alpha + \partial_i\psi_I\partial_j\psi_I)\,, \tag{5.10}$$

with the initial data being given by

$$\Phi^2 = \Pi^{1/4}\,, \qquad X_I = \frac{\Pi^{1/4}}{C_I D_I}\,, \qquad \psi_I = \log\frac{C_I}{D_I}\,, \tag{5.11}$$

where $\Phi = \prod_I (C_I D_I)$. Thus we have initial data in the form of eight arbitrary harmonic functions (C_I, D_I) for the system of seven scalar fields (φ_α, ψ_I) coupled to gravity, and described by the Lagrangian

$$\mathcal{L}_4 = \sqrt{-g}\Big(R - \frac{1}{2}(\partial\varphi_\alpha)^2 - \frac{1}{2}(\partial\psi_I)^2\Big)\,. \tag{5.12}$$

Finally, we may consider the theory of Einstein-Scalar gravity coupled to two gauge fields, which was described in section 3.4. We showed that the theory described by the Lagrangian (3.50), with $b = -1/a$, admits the time-symmetric initial data given in (3.51). We may therefore introduce two scalar fields ψ_1 and ψ_2, such that

$$E_i^1 = \Big(\frac{C_1 D_1}{C_2 D_2}\Big)^{-\frac{a^2}{1+a^2}}\,\partial_i\psi_1\,,$$

$$E_i^2 = \left(\frac{C_1 D_1}{C_2 D_2}\right)^{\frac{1}{1+a^2}} \partial_i \psi_2 \,, \tag{5.13}$$

and thus obtain the time-symmetric initial data

$$\begin{aligned} \Phi^2 &= (C_1 D_1)^{\frac{1}{1+a^2}} (C_2 D_2)^{\frac{a^2}{1+a^2}} \,, & \phi &= -\frac{a}{1+a^2} \log \frac{C_1 D_1}{C_2 D_2} \,, \\ \psi_1 &= \frac{1}{\sqrt{1+a^2}} \log \frac{C_1}{D_1} \,, & \psi_2 &= \frac{a}{\sqrt{1+a^2}} \log \frac{C_2}{D_2} \,, \end{aligned} \tag{5.14}$$

using the four harmonic functions C_1, D_1, C_2 and D_2, for the theory of three scalar fields coupled to gravity, described by the Lagrangian

$$\mathcal{L}_4 = \sqrt{-g}\left(R - 2(\partial\phi)^2 - 2(\partial\psi_1)^2 - 2(\partial\psi_2)^2\right). \tag{5.15}$$

5.2 Einstein gravity coupled to N scalar fields

Here we present a direct construction of time-symmetric initial data for a system of N scalar fields coupled to gravity, and described by the Lagrangian

$$\mathcal{L} = \sqrt{-g}\,(R - 2\partial\phi_I \partial\phi_I) \,, \tag{5.16}$$

where $1 \le I \le N$. The time-symmetric initial value constraint is then

$${}^{(3)}R = 2g^{ij} \partial_i \phi_I \partial_j \phi_I \,. \tag{5.17}$$

We make the Ansatz

$$\Phi = \prod_{a=1}^{M} C_a^{n_a} \,, \qquad \phi_I = 2\sum_{a=1}^{M} m_{aI} \log C_a \,, \tag{5.18}$$

where C_a for $1 \le a \le M$ are M harmonic functions.

Plugging into (5.17) implies the following constraints on the constants n_a and m_{aI}:

$$a \ne b: \quad n_a n_b + m_{aI} m_{bI} = 0 \,, \tag{5.19}$$

$$a = b: \quad n_a(n_a - 1) + m_{aI} m_{aI} = 0 \,. \tag{5.20}$$

(Summation over I is understood in each case.) This implies a total of $\frac{1}{2}M(M+1)$ constraints on the total of $M(N+1)$ constants. If we define the $(N+1)$-component vectors

$$\mathbf{m}_a = (m_{a1}, m_{a2}, \ldots, m_{aN}, n_a) \,, \tag{5.21}$$

then the conditions in (5.20) can be written as

$$\mathbf{m}_a \cdot \mathbf{m}_b = n_a\, \delta_{ab} \qquad (\text{no sum on } a) \,. \tag{5.22}$$

Defining $\mathbf{Q}_a = \mathbf{m}_a/\sqrt{n_a}$, we then have

$$\mathbf{Q}_a \cdot \mathbf{Q}_b = \delta_{ab} \,. \tag{5.23}$$

Thus we obtain a solution for every choice of orthonormal M-frame in $\mathbb{R}^{N+1}$. We must therefore have $M \leq N+1$.

Considering the maximal case $M = N+1$, $\mathrm{SO}(N+1)$ acts on the orthonormal bases for $\mathbb{R}^{N+1}$, but the $\mathrm{SO}(N)$ subgroup acting on the first N components merely rotates the N scalar fields into themselves, and produces a physically indistinguishable solution. If one acts with an element of $\mathrm{SO}(N+1)$ that is not contained in $\mathrm{SO}(N)$, the scalar fields ϕ_I will mix with the scalar $\sigma = 2\log\Phi$ and hence give rise to a geometrically distinct solution of the initial-value constraint (5.17). The space of inequivalent solutions is therefore given by the coset $\mathrm{SO}(N+1)/\mathrm{SO}(N)$, which is isomorphic to S^N.

The various Einstein-Scalar theories we obtained in section 5.1 are all examples encompassed within the above discussion. We have

- Einstein-Dilaton: $(M,N) = (2,1)$
- Einstein-Maxwell-Dilaton: $(M,N) = (3,2)$
- STU Supergravity: $(M,N) = (8,7)$
- Einstein-Dilaton+2 Maxwell: $(M,N) = (4,3)$

For example, in the case of STU supergravity, one finds, after defining

$$\phi_I = \frac{1}{2}(\psi_1,\psi_2,\psi_3,\psi_4,\varphi_1,\varphi_2,\varphi_3)\,, \qquad C_a = D_{a-4} \text{ for } 5 \leq a \leq 8\,, \tag{5.24}$$

that

$$\begin{aligned}
\mathbf{m}_1 &= \frac{1}{2}\Big(1,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\Big)\,, & \mathbf{m}_2 &= \frac{1}{2}\Big(0,1,0,0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\Big)\,,\\
\mathbf{m}_3 &= \frac{1}{2}\Big(0,0,1,0,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\Big)\,, & \mathbf{m}_4 &= \frac{1}{2}\Big(0,0,0,1,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\Big)\,,\\
\mathbf{m}_5 &= \frac{1}{2}\Big(-1,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\Big)\,, & \mathbf{m}_6 &= \frac{1}{2}\Big(0,-1,0,0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\Big)\,,\\
\mathbf{m}_7 &= \frac{1}{2}\Big(0,0,-1,0,-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\Big)\,, & \mathbf{m}_8 &= \frac{1}{2}\Big(0,0,0,-1,-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\Big)\,.
\end{aligned} \tag{5.25}$$

6 The Penrose inequality for time-symmetric data

In the time-symmetric case, a marginally closed outer trapped surface or (MCOTS) coincides with what mathematicians call a closed stable minimal surface, that is, one whose second variation is positive.[4] The apparent horizon is the outermost MCOTS, and coincides with the outermost stable minimal surface [15, 36]. The area A of an apparent horizon is usually taken as a lower bound for the area A_{initial} of the intersection of the event horizon with the initial surface. Assuming cosmic censorship is valid, then by Hawking's area theorem [36, 37] this should be no larger than the area A_{final} of the final black hole.

[4]For historical reasons mathematicians abandon their customary linguistic precision and refer to any critical point of the area functional, regardless of the nature of its Hessian as a "minimal surface." The adjective "stable" has no dynamical significance, but is taken to mean that the Hessian is positive definite.

If the final black hole is non-rotating and it and carries no electric charges, the final state should be a Schwarzschild black hole, whose mass M_{final} is given by

$$A_{\text{final}} = 16\pi M_{\text{final}}^2 \,. \tag{6.1}$$

We also have that the initial ADM mass M_{initial} of the data set should satisfy

$$M_{\text{final}} \leq M_{\text{initial}} \,. \tag{6.2}$$

Thus we expect that

$$A \leq A_{\text{initial}} \leq A_{\text{final}} = 16\pi M_{\text{final}}^2 \leq 16\pi M_{\text{initial}}^2 \,, \qquad \Rightarrow \qquad A \leq 16\pi M_{\text{initial}}^2 \,. \tag{6.3}$$

The last inequality in (6.3) is called the *Penrose Inequality*, or *Cosmic Censorship Inequality* [38]. Moreover, one obtains in this way an upper bound on the efficiency

$$\eta = \frac{M_{\text{initial}} - M_{\text{final}}}{M_{\text{initial}}} \leq 1 - \sqrt{\frac{A}{16\pi M_{\text{initial}}^2}} \tag{6.4}$$

with which the time development of the initial data converts rest mass to gravitational radiation. Thus although there now exist general proofs of the Penrose inequality in the general time-symmetric case due to Huisken and Ilmanen [39], based on the inverse curvature flow proposed for this purpose by Geroch, the value of η remains of interest.

In the discussion above we have assumed that no electric or magnetic charges are carried by the final black hole. If that is not so, the bounds are modified.

As an illustration of the above idea, we shall now consider the example of time-symmetric initial data for an Einstein-Scalar system, for which the equations of motion are

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi \,. \tag{6.5}$$

The only time-symmetric initial value constraint is

$${}^{(3)}R = 2g^{ij}\partial_i\phi\partial_j\phi \,. \tag{6.6}$$

As shown by Ortin [22], a set of time-symmetric initial data depending upon one parameter α is given by

$$g_{ij} = C^{2(1-\alpha)} D^{2(1+\alpha)} \delta_{ij} \,, \qquad e^{\phi} = e^{\phi_0} \left(\frac{C}{D}\right)^{\pm\sqrt{1-\alpha^2}} \,, \tag{6.7}$$

with C and D harmonic and $-1 < \alpha < 1$. This can also be seen from our expressions for the Case (1) time-symmetric data for the Einstein-Maxwell-Dilaton system in section 3.1.1, by taking $a^2 = \alpha^2/(1-\alpha^2)$ so that the electric field vanishes. (Exchanging C and D sends $\phi \to -\phi$.) If

$$C = 1 + \frac{X}{2\rho} \,, \qquad D = 1 + \frac{Y}{2\rho} \,, \tag{6.8}$$

the initial ADM mass M_{initial} and initial (non-conserved) scalar charge Σ_{initial} are given by

$$M_{\text{initial}} = M = \frac{1}{2}(1-\alpha)X + \frac{1}{2}(1+\alpha)Y \,, \qquad \Sigma_{\text{initial}} = \Sigma = \frac{1}{2}\sqrt{1-\alpha^2}(Y - X) \,. \tag{6.9}$$

If $X > 0$ and $Y > 0$, the solution is regular for $0 < \rho < \infty$, and near $\rho = 0$ we have

$$ds^2 \approx X^{2(1-\alpha)}\, Y^{2(1+\alpha)} \frac{1}{16\rho^4} \Big(d\rho^2 + \rho^2 \big(d\theta^2 + \sin^2\theta d\varphi^2\big)\Big)\,. \tag{6.10}$$

If we set $R = 1/\rho$ we find

$$ds^2 \approx \frac{1}{16}\, X^{2(1-\alpha)}\, Y^{2(1+\alpha)} \Big(dR^2 + R^2 \big(d\theta^2 + \sin^2\theta d\varphi^2\big)\Big)\,. \tag{6.11}$$

Thus $\rho = 0$ corresponds to another asymptotically flat region. The two asymptotically flat regions are separated by an Einstein-Rosen bridge. If $X \neq Y$ the scalar charge is non-vanishing and there is, unlike in the Schwarzschild case, an asymmetry between the two asymptotic regions, and the values of the scalar fields at the two infinities will differ.

There is a unique totally geodesic two-sphere at

$$\rho = \rho_+ = \frac{\alpha(X-Y)}{4} + \sqrt{(\frac{\alpha(Y-X)}{4})^2 + \frac{XY}{4}}\,, \tag{6.12}$$

located between $\rho = 0$ and $\rho = \infty$, at which the area

$$A(\rho) = 4\pi\rho^2 \left(1 + \frac{X}{2\rho}\right)^{2(1-\alpha)} \left(1 + \frac{Y}{2\rho}\right)^{2(1+\alpha)} \tag{6.13}$$

attains an absolute minimum. The scalar no hair theorem [40–42] implies that there is no non-singular static black hole with non-constant scalar field, and the expected final state is a Schwarzschild black hole with mass M_{final} and vanishing scalar charge Σ_{final}.

We may now consider the Penrose inequality

$$W \equiv 16\pi M^2 - A(\rho_+) \geq 0\,. \tag{6.14}$$

It is helpful to parameterise X and Y in terms of new quantities q and s, such that

$$\alpha(X-Y) = q\, s\,, \qquad 2\sqrt{XY} = q\sqrt{1-s^2}\,, \tag{6.15}$$

where

$$q \geq 0\,, \qquad -1 \leq s \leq 1\,. \tag{6.16}$$

This gives

$$\rho_+ = \frac{1}{2}\alpha\, q\,(1-s) \tag{6.17}$$

when $\alpha > 0$, and $\rho_+ = -\frac{1}{2}\alpha\, q\,(1+s)$ when $\alpha < 0$. We can focus, without loss of generality, on the case $\alpha > 0$, since reversing the sign of α is equivalent to switching X and Y. We then have to prove that

$$W \equiv 16(\sqrt{\alpha^2(1-s^2)+s^2} - \alpha\, s)^2 - \alpha^2(1-s)^2\, Z_+^{2(1-\alpha)}\, Z_-^{2(1+\alpha)} \geq 0\,, \tag{6.18}$$

where

$$Z_\pm \equiv 1 + \frac{\sqrt{\alpha^2(1-s^2)+s^2} \pm s}{\alpha(1-s)}\,, \tag{6.19}$$

and where $0 < \alpha \le 1$ and $-1 \le s \le 1$. It is evident that we can take the square root of both sides in (6.18), and so we need to show that $H \ge 0$ for $0 \le \alpha \le 1$ and $-1 \le s \le 1$, where

$$H = 4\sqrt{\alpha^2 + (1-\alpha^2)\, s^2} - 4\alpha\, s - \alpha\,(1-s)\, Z_+\, Z_-\, R^\alpha\,, \tag{6.20}$$

and we have defined

$$R \equiv \frac{Z_-}{Z_+}\,. \tag{6.21}$$

It is rather straightforward to show analytically that R varies monotonically as a function of s, with

$$R(-1) = 1 + \frac{1}{\alpha}\,, \qquad R(0) = 1\,, \qquad R(1) = 0\,. \tag{6.22}$$

Z_+ and Z_- are both ≥ 1.

We have not found an analytic proof that H defined in (6.20) indeed satisfies $H \ge 0$ but it is evident from numerical analysis that this inequality is satisfied, and hence that the Penrose inequality holds for the Einstein-Scalar system with time-symmetric initial data.[5]

7 Concluding remarks

In this paper we have presented time-symmetric initial data results for the Einstein-Maxwell-Dilaton theory with a general dilation coupling a, which can be expressed in terms of three harmonic functions. We have also generalised the results to the Einstein-Maxwell-Dilaton theory with two Maxwell field and presented the result in terms of four harmonic functions. The initial data results for the STU model, with four electric fields and three dilation fields can be expressed in terms of eight harmonic functions. We also matched the results to know static black hole solutions, and as a by-product, we presented these metrics and all the sources in terms of specific harmonic functions.

The method can be in general applied to other supergravity models. We also showed how for all the theories with electric fields, the initial-data could be mapped into initial data for theories with scalar fields only. For example, the initial data for the Einstein-Maxwell-Dilaton theory of section 3 can be mapped onto the initial data problem of an Einstein-Scalar model with two scalars, and the initial data for the STU model map onto data for an Einstein-Scalar model with seven scalar fields. We then gave a rather general construction of time-symmetric initial data for a system of N scalar fields coupled to gravity.

While the work provides a prerequisite for study of the time evolution of initial data, there are also a number of physical properties one may explore without having to evolve the data with the full equations of motion. For example, for initial data for multi-black hole systems one may, following [14], associate masses and charges with the individual black holes and hence one may calculate binding energies. We have performed calculation of the interaction energies for the case of two multi-centered harmonic functions C and D, i.e. $\Phi^2 = C^\gamma D^\delta$. It turns out that for the general harmonic functions, the ADM mass M_∞, as measured at infinity and a constant part of the sum of constituent masses $\sum_{a=1}^n M_a$ do

[5] After circulating an initial version of this paper, David Chow showed us an analytic proof of the inequality $H \ge 0$.

not cancel, except for $\gamma = \delta = 1$ which is the Einstein-Maxwell case. It is also only in this case that the remaining interaction energies can be cast in a form that has a physical interpretation in terms of the gravitational and electric potential energies of the system. This may be due to the fact that in other examples the scalar field interactions modify both constituent mass contributions as well as the nature of the interaction potential energies. We note, however, that the initial data for the multi-centered static black hole solutions do have a cancellation of the asymptotic and constituent constant mass contributions, as expected. We have also carried out the calculation for the STU model, with parallel results.

Another example where one can study physical properties without needing to evolve the initial data is provided by the Penrose area inequality, given in (6.3). As an illustration we studied the specific example of a scalar field coupled to gravity, and we showed by numerical means that indeed the time-symmetric initial data necessarily gave rise to configurations that satisfy the area inequality. It would be of interest to generalise this result to other examples of time-symmetric initial data, and also to find an analytic proof that the inequality is obeyed.

We should like to conclude with remarks about results for the initial data, not necessarily-time symmetric, that lead to time-dependent solutions. Specifically, Nakao, Yamamoto and Maeda [43] pointed out that initial data for Einstein's vacuum equations with a positive cosmological constant $\Lambda = 3H^2$ can be constructed by starting with the time-symmetric initial data with a non-vanishing cosmological constant that is not time-symmetric. This result in turn also leads to time-dependent solutions, with a positive cosmological constant, such as the Kastor-Traschen multi-centered solutions of Einstein-Maxwell-de Sitter gravity [44]. There are by now large classes of multi-centered extremal cosmological black hole solutions known in fake gauged supergravity theories with a positive cosmological constant, such as those in the gauged STU model [45], as well as non-extremal cosmological black hole solutions in gauged Einstein-Maxwell-Dilaton models [46]. Further exploration of time-nonsymmetric initial data and the time-dependent solutions for general gauged supergravity theories, and especially the intriguing connection to the time-symmetric data of the corresponding ungauged supergravity theories, deserves further study.

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